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Citation: AIP Conference Proceedings 1880, 060001 (2017); doi: 10.1063/1.5000655
View online: http://dx.doi.org/10.1063/1.5000655
View Table of Contents: http://aip.scitation.org/toc/apc/1880/1
Published by the American Institute of Physics
Efficient Difference Schemes for the Three-phase Non-isothermal Flow Problem

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Abstract. The paper focuses on constructing and investigating of cost-effective difference schemes for the numerical solution of the two-dimensional three-phase non-isothermal flow problem without capillary and gravitational forces. In this paper, the finite difference method is used to solve the problem numerically. The method of energy inequalities is applied to examine the stability of the finite difference scheme with respect to the initial data and the right-hand sides of the equations. Three cost-effective difference schemes are constructed on the base of the studied scheme. The efficiency of the proposed algorithms is analyzed on the basis of comparing the average time spent on the numerical implementation of one time layer.

INTRODUCTION

The urgency of a rigorous theoretical justification of numerical methods for solving the problems of the three-phase non-isothermal flow theory is due to its practical importance in the oil industry when predicting the extraction of high-viscosity paraffin or resinous oil. This is due to the fact that currently the reserves of this category of oil are higher than the reserves of so-called light oils, which leads to the need to apply secondary or tertiary methods. However, due to the rather high cost of these methods, studies aimed at increasing its effectiveness are of great practical importance. At present, it is possible to reach this only using methods of mathematical modeling of hydrodynamic processes occurring in oil reservoirs during the development of deposits.

The model considered in the present paper consists of the mass conservation equation, equation of motion in the form of the linear Darcy’s law, energy equation, equation of state, and phase balance equation. The model with various assumptions about physical data was studied, for example, in [1, 2, 3]. In [4], a new “global” formulation of the three-phase non-isothermal flow problem was proposed, which is based on the introduction of a change of variables for the pressure, called “global” pressure, to eliminate the gradients of capillary pressures from the equations for pressure and temperature. As a result, the initial equations were reduced to a system of five partial differential equations with respect to pressure, temperature, velocity, and two saturations.

The main difficulty in the numerical solution of the obtained problem is connected with the complexity and strong non-linearity of the equations. Therefore, the issue of the development of computational algorithms for the numerical implementation of this problem, which requires less computational operations, becomes relevant. In the present work, the main attention is paid to the construction and investigation of cost-effective difference schemes for the three-phase non-isothermal flow problem. Under the cost-effective schemes we mean schemes for which the number of arithmetic operations for the transition from the $n$-th time layer to the $(n + 1)$-th one is proportional to the number of unknown values. In order to simplify the calculations when obtaining a priori estimates, the coefficients in the equations are linearized, or taken to be constant. A priori estimates are obtained using the method of energy inequalities which prove the stability of the constructed difference scheme. The proof of the stability is based on five preliminary lemmas. In the last section of the paper three efficient difference schemes are constructed on the base of...
with the following initial and boundary conditions:

\[ T (x, 0) = T_0, \quad p (x, 0) = p_0, \quad s_\alpha (x, 0) = s_{\alpha 0}, \]

\[ k_h \frac{\partial T}{\partial t} \bigg|_{\partial \Omega} = 0, \quad k_p \frac{\partial p}{\partial t} \bigg|_{\partial \Omega} = 0, \]

where subscripts \( w, o, g, r \) denote the phases of water, oil, steam, and rock; \( k \) is the absolute permeability, \( p \) is pressure, \( T \) is temperature, and \( s_\alpha \) is the saturation of the phase \( \alpha \); \( \vec{u} \) is velocity; \( \beta_T \) is some function, and \( v_\alpha \) is a constant.

Suppose that the functions \( k_p, k_h, \lambda \) are continuous in \( Q \) and the following conditions hold:

\[ k_p (x, t, p) \geq c_0 > 0, \quad k_h (x, t) \geq 4c_0, \quad \lambda (x, t) \leq c_1. \]

We note that in the multiphase flow theory the functions \( f_T \) and \( f_p \) usually have the form

\[ f_p = \sum_{j=1}^{W} \varphi_p^j (x, t) (p_{inj} - p) \delta (x - x_{Nj}^{(w)}), \quad f_T = \sum_{j=1}^{W} \varphi_T^j (x, t) (T_{inj} - T) \delta (x - x_{Nj}^{(w)}), \]

where \( x_1^{(w)}, x_2^{(w)}, \ldots, x_{W}^{(w)} \in \Omega \) are source coordinates; \( \varphi_p^j (x, t), \varphi_T^j (x, t) \) are some known functions; \( p_{inj} \) is the injection pressure; \( T_{inj} \) is the temperature corresponding to the pressure \( p_{inj} \) according to the table of thermophysical properties of water and steam; \( \delta (x) \) is a delta function. We assume that \( |\varphi_p^j (x, t)|, |\varphi_T^j (x, t)| \leq c_1. \)

### FORMULATION OF THE DIFFERENCE PROBLEM

Let us introduce the uniform grid \( \Omega_h \) in \( Q \) with spatial steps \( h_1, h_2 \) and time step \( \tau \) as follows:

\[ \Omega_{N\tau} = \overline{\Omega} \times \overline{\tau} = \{(x, t) : \quad x \in \overline{\Omega}, \quad t \in \overline{\tau} \}, \quad \Omega_h = \overline{\Omega}_{h,1} \times \overline{\Omega}_{h,2}, \]

\[ \Omega_{h, m} = \{i_m h_1 : \quad i_m = 0, 1, \ldots, N_m, \quad N_m h_1 = l \}, \quad \Omega_{\tau} = \{t_n \tau : \quad n = 0, 1, \ldots, N, \quad N \tau = t \}. \]

We also introduce the notations

\[ \Omega_{h, m} = \{i_m h_1 : \quad i_m = 1, \ldots, N_m - 1, \quad N_m h_1 = l \}, \]

\[ \Omega_{h, m}^* = \{i_m h_1 : \quad i_m = 1, \ldots, N_m, \quad N_m h_1 = l \}, \quad \Omega_{h, m}^- = \{i_m h_1 : \quad i_m = 0, \ldots, N_m - 1, \quad (N_m - 1) h_1 = l - h_1 \}. \]

\[ \Gamma_h = \bigcup_{m=1}^{N_h} \Omega_{h, m}, \quad \Gamma_h = \bigcup_{m=1}^{N_h} \Gamma_{h, m}^+, \quad \Gamma_{h, 0} = \Gamma_h \setminus \Gamma_h, \]

the efficiency of the proposed algorithms is analyzed on the basis of comparing the average time spent on the numerical implementation of one time layer.
\[ \Gamma^+_{h,m} = \{ x_m = N_m, \ 0 < x_{3-m} < N_{3-m} \}, \ \Gamma^-_{h,m} = \{ x_m = 0, \ 0 < x_{3-m} < N_{3-m} \}. \]

Let us associate the following finite difference scheme with the differential problem (1)-(6):

\[ BT^h + L \big( \tilde{v}^h, T^h \big) + \Lambda_1 T^h = f^h, \quad (9) \]
\[ BP^h + \Lambda_2 p^h = \beta^h T^h + f_p^h, \quad (10) \]
\[ BS^h + \Lambda_3 p^h = f^h, \quad \alpha = w, o, \quad (11) \]
\[ i^h = (u^h_1, u^h_2), \quad u^h_m = -k \lambda^h p^h_{x_m}, \quad m = 1, 2, \quad (12) \]
\[ T^h (0) = T_0, \quad p^h (0) = p_0, \quad x^h_0 (0) = x_0, \quad (13) \]

where

\[ B = E + \tau \omega A, \quad A = A_1 + A_2, \quad \omega > 0, \]
\[ A_m w = \left\{ -2 h^{-1}_m (w_{x_m} + w), \ x_m = 0; \ -w_{x_m}, \ x_m \in \Omega_{h,m}; \ 2 h^{-1}_m (w_{x_m} + w), \ x_m = l \right\}, \quad (14) \]

\[ L \big( \tilde{v}, \theta \big) = 0.5 \sum_{m = 1}^{2} \left( \beta^+_m (x) v^+_m \theta_{x_m} + \beta^-_m (x) v^-_m \theta_{x_m} \right). \]

\[ \beta^+_m (x) = \{ 2, \ x_m = 0; \ 1, \ x_m \in \Omega_{h,m}; \ 0, \ x_m = l \}, \ \beta^-_m (x) = 2 - \beta^+_m (x) \]

and the operators \( \Lambda_i \) are defined as follows:

\[ \Lambda_{\delta} = \sum_{m = 1}^{2} \Lambda_{\delta,m}, \quad \Lambda_2 w = \sum_{m = 1}^{2} \left( \chi^+_{m} (x) \Lambda_{2,m} w + \chi^-_{m} (x) \Lambda_{2,m} \tilde{w} w \right), \]

\[ \Lambda_{\delta,m} w = \left\{ -2 h^{-1}_m \eta w_{x_m}, \ x_m = 0; \ -\eta w_{x_m}, \ x_m \in \Omega_{h,m}; \ 2 h^{-1}_m \eta w_{x_m}, \ x_m = l \right\}, \]
\[ \Lambda_{2,m} w = \left\{ -2 h^{-1}_m \mu w_{x_m}, \ x_m = 0; \ - \mu w_{x_m}, \ x_m \in \Omega_{h,m}; \ 2 h^{-1}_m \mu w_{x_m}, \ x_m = l \right\}, \]
\[ \chi^+_{m} (x) = \{ 1, \ x_{3-m} = 0; \ 0.5, \ x_{3-m} \in \Omega_{h,3-m}; \ 0, \ x_{3-m} = l \}, \ \chi^-_{m} (x) = 1 - \chi^+_{m} (x), \]

where \( \eta = k^h_1 \) for \( \delta = 1; \ \eta = \nu_{x} \) for \( \delta = 3 \alpha \). The notations used above are defined in [5]. In the present paper we use the following scalar products and norms:

\[ (w, \tilde{w})_{\Omega_h} = \sum_{x \in \Omega_h} b_h (x) w (x) \tilde{w} (x) h_1 h_2, \quad (w, \tilde{w})_{\Omega_{h,m}} = \sum_{x \in \Omega_{h,m}} w (x) \tilde{w} (x) h_1 h_2, \quad (w, \tilde{w})_{\Omega_{h,m}^l} = \sum_{x \in \Omega_{h,m}^l} w (x) \tilde{w} (x) h_1 h_2, \]

\[ ||w||^2_0 = (w, w)_{\Omega_h}, \quad b_h (x) = \{ 1, \ x \in \Omega_1; \ 0.5, \ x \in \Gamma_1; \ 0.25, \ x \in \Gamma_{1,0} \}, \quad ||w||^2_1 = \frac{1}{2} \sum_{m = 1}^{2} \left( \sum_{x \in \Omega_{h,m}} \left( w^2_{x_m} + 1 \right) \right), \]

**STUDY OF THE STABILITY OF THE DIFFERENCE SCHEME**

Let us consider the problem for \( \theta = T^h - \tilde{T}^h, \ x = p^h - \tilde{p}^h, \ \sigma_x = s^h_{x_m} - \tilde{s}^h_{x_m}, \ \zeta_m = u_m - \tilde{u}_m \) to study the stability of the difference scheme (9)-(13):

\[ B \theta + L \big( \tilde{v}_h, T^h \big) - L \big( \tilde{v}^h, \tilde{T}^h \big) + \Lambda_1 \theta = \psi_T, \quad (15) \]
\[ B \pi^h + \Lambda_2 p^h - \Lambda_2 \tilde{p}^h = \beta^h T^h + \psi_p, \quad (16) \]
\[ B \sigma_{\alpha} + \Lambda_3 \pi = \psi_{\alpha}, \quad \alpha = w, o, \quad (17) \]
\[ \zeta_m = -k \lambda^h \pi_{x_m}, \quad m = 1, 2, \quad (18) \]
\[
\theta (0) = \theta^0, \quad \pi (0) = \pi^0, \quad \sigma_{\alpha} (0) = \sigma_{\alpha}^0, \quad \alpha = w, o,
\]
where \( \tilde{T}, \tilde{p}, \tilde{s}^h, \tilde{u}_m \) is the solution of the perturbed problem. Suppose that the following conditions hold for initial values of temperature and pressure:
\[
c_1 \eta - q_0 t_1 - \|T_0^0\|^2_B - c_5 \|p_0\|^2_B \geq 0, \quad c_1 \eta - q_0 t_1 - \|T_0^0\|^2_B - c_5 \|\tilde{p}_0\|^2_B \geq 0,
\]
where \( \eta > 0 \) is a real parameter, \( c_5 \) and \( q_0 \) are some constants. In addition, to obtain an a priori estimate of the difference solution, we make the assumption that for the dimensionless function \( \beta_T = \beta_T (x, t) \) and the dimensionless constant \( k \), the following inequalities hold:
\[
\beta_T \leq c_2 \tau, \quad c_2 > 0,
\]
\[
k \leq c_3 \tau, \quad c_3 > 0.
\]
Let us first formulate the main result of this chapter.

**Theorem 1**

Let the conditions (7), (20), (21), (22) and \( \omega > \omega_0, \omega_0 = \max_{i=1}^{14} \omega_i \) hold. Then the difference scheme (9)-(13) is stable with respect to the initial values and right-hand sides of the equations, and the following inequality holds:
\[
\|\tilde{\theta}\|^2_B + \|\tilde{\sigma}\|^2_B + \sum_{\alpha=w,o} \|\tilde{\sigma}_{\alpha}\|^2_B + \|\tilde{\sigma}_{\alpha}\|^2_B \
\]
\[
\leq d_T + d_0 \tau \|\tilde{\theta}\|^2_B + d_1 \|\tilde{\sigma}\|^2_B + \sum_{\alpha=w,o} d_0 \|\tilde{\sigma}_{\alpha}\|^2_B + d_1 \tau \left( \|\tilde{\psi}_{\alpha}\|^2_B + \|\tilde{\psi}_{\alpha}\|^2_B + \sum_{\alpha=w,o} \|\tilde{\psi}_{\alpha}\|^2_B \right).
\]

Before proving the theorem, we give the following auxiliary lemmas. Let us state without proof the following lemma, which is based on the results obtained in [6].

**Lemma 2**

Let the conditions (7) hold. Then the following inequalities hold:
\[
(A_1 w, w) \geq 4c_0 \|w\|^2_1, \quad (A_2 w - \Lambda_2 \tilde{w}, w - \tilde{w}) \geq c_0 \|w - \tilde{w}\|^2_1,
\]
\[
\tau (A_2 w - \Lambda_2 \tilde{w}, \omega) \leq c_1 \left( \|w - \tilde{w}\|^2_1 + \frac{\|z\|^2_1}{\varepsilon} + \frac{\tau^2}{\varepsilon \delta \omega} \|\tilde{c}\|^2_B \right),
\]
\[
2 \tau (A_1 w, \tilde{w}) \leq c_1 \left( \|w\|^2_1 + \frac{\|z\|^2_1}{\varepsilon} \|\tilde{w}\|^2_B + \frac{\tau}{\varepsilon \delta \omega} \|\tilde{w}\|^2_B \right), \quad \varepsilon > 0.
\]

**Lemma 3**

Let the conditions (7), (21) and \( \omega > \omega_1, \omega_1 = \frac{8 \varepsilon_1^2}{c_0} + \frac{c_1}{\delta_2} + \frac{2c_1}{c_0}, \quad \varepsilon > 0 \) hold. Then the following inequality holds:
\[
\|\tilde{T}\|^2_B + \tau^3 (\omega - \omega_1) \|\tilde{T}\|^2_B + c_0 \tau \|\tilde{T}\|^2_B \leq d_1 (\tau) \|\tilde{T}\|^2_B + \frac{2c_2 \tau^2}{\delta} \|\tilde{T}\|^2_B + \frac{c_0 \tau}{c_1} \|\tilde{p}\|^2_B.
\]

**Proof.** Multiply the equation (16) scalarly by \( 2\tau \tilde{T} \):
\[
\|\tilde{T}\|^2_B - \|\tilde{p}\|^2_B + \tau^2 \|\tilde{T}\|^2_B + 2 \tau (A_2 p - \Lambda_2 \tilde{p}, \tilde{T}) = 2 \tau \left( \beta_T \theta, \tilde{T} \right) + 2 \tau \left( \psi, \tilde{T} \right).
\]

Using Lemma 2, we obtain:
\[
2 \tau (A_2 p - \Lambda_2 \tilde{p}, \tilde{T}) \geq 2 \tau \left( c_0 - \frac{c_1}{\delta_1} \right) \|\tilde{T}\|^2_B - 2c_1 \tau \|\tilde{T}\|^2_B - \frac{2c_2 \varepsilon_1 \tau^2}{\omega_0} \|\tilde{T}\|^2_B.
\]

Using the condition (21) and Cauchy inequality, the rest of the terms in (24) can be estimated as follows:
\[
2 \tau \left( \beta_T \theta, \tilde{T} \right) \leq \frac{2c_2 \varepsilon_1 \tau^3}{\delta} \|\tilde{T}\|^2_B + \frac{c_2 \tau}{\delta_2} \|\tilde{T}\|^2_B + \frac{c_2 \tau^2}{\delta_2 \omega_0} \|\tilde{T}\|^2_B.
\]
Lemma 3. We obtain the inequality (29). Now multiply scalarly the equation (9) by 2

\[ \| \mathbf{\hat{u}} \|_{\alpha} + \| \mathbf{\hat{v}} \|_{\alpha} + 2 \| \mathbf{\hat{w}} \|_{\alpha} + \| \mathbf{\hat{z}} \|_{\alpha} + \| \mathbf{\hat{t}} \|_{\alpha} + \| \mathbf{\hat{h}} \|_{\alpha} \leq \left( 1 + \epsilon_1 \tau + \frac{\epsilon_2 \tau}{\delta} \right) \| \mathbf{\hat{u}} \|_{\alpha} + \| \mathbf{\hat{v}} \|_{\alpha} + \frac{2 \epsilon_2 \tau^3}{\delta} \| \mathbf{\hat{w}} \|_{\alpha} + \frac{2 \epsilon_1 \tau^3}{\delta} \| \mathbf{\hat{z}} \|_{\alpha} + \frac{2 \epsilon_1 \tau^3}{\delta} \| \mathbf{\hat{t}} \|_{\alpha} + \frac{2 \epsilon_1 \tau^3}{\delta} \| \mathbf{\hat{h}} \|_{\alpha} \].

(28)

Choosing \( \epsilon_1 = \frac{2 \epsilon_1}{\epsilon_0} \) and \( \epsilon \) from the condition \( \omega > \omega_1 \), we come to the assertion of the lemma.

**Lemma 4.** Let the conditions (22), (20) and \( \omega > \omega_2 \), \( \omega_2 = \frac{c_1^2(q+2)}{8 \delta} \left( 1 + \frac{1}{\delta} \right) + \frac{3 \epsilon_0 \lambda}{2 \delta q \eta^2} + \frac{4 \epsilon_0 c_1 (q+2)}{3 \delta q \eta^2} \) hold. Then the following inequalities hold:

\[ \| \mathbf{\hat{p}} \|_{\alpha} + \| \mathbf{\hat{q}} \|_{\alpha} + \| \mathbf{\hat{r}} \|_{\alpha} + \| \mathbf{\hat{s}} \|_{\alpha} + \| \mathbf{\hat{t}} \|_{\alpha} + \| \mathbf{\hat{u}} \|_{\alpha} \leq \frac{2 \epsilon_1 \tau}{\delta} \| \mathbf{\hat{p}} \|_{\alpha} + \frac{2 \epsilon_2 \tau^3}{\delta} \| \mathbf{\hat{q}} \|_{\alpha} + \frac{2 \epsilon_1 \tau^3}{\delta} \| \mathbf{\hat{r}} \|_{\alpha} + \frac{2 \epsilon_1 \tau^3}{\delta} \| \mathbf{\hat{s}} \|_{\alpha} + \frac{2 \epsilon_1 \tau^3}{\delta} \| \mathbf{\hat{t}} \|_{\alpha} + \frac{2 \epsilon_1 \tau^3}{\delta} \| \mathbf{\hat{u}} \|_{\alpha} \].

(29)

Proof. Multiplying the equation (10) scalarly by \( \| \mathbf{\hat{p}} \|_{\alpha} \), and carrying out calculations analogous to those made in Lemma 3, we obtain the inequality (29). Now multiply scalarly the equation (9) by \( 2 \| \mathbf{\hat{T}} \|_{\alpha} \):

\[ \| \mathbf{\hat{T}} \|_{\alpha} + \| \mathbf{\hat{P}} \|_{\alpha} + \| \mathbf{\hat{Q}} \|_{\alpha} + 2 \tau \left( \mathbf{L}(\mathbf{\hat{u}}, \mathbf{\hat{T}}) - \mathbf{L}(\mathbf{\hat{u}}, \mathbf{\hat{T}}) \right) \leq \frac{4 \epsilon_1 \eta \tau^2}{\delta} \| \mathbf{\hat{u}} \|_{\alpha} + \frac{4 \epsilon_1 \eta \tau^2}{\delta} \| \mathbf{\hat{u}} \|_{\alpha} + \epsilon > 0. \]

(33)

Proof. Multiplying the equation (10) scalarly by \( 2 \| \mathbf{\hat{T}} \|_{\alpha} \), and carrying out calculations analogous to those made in Lemma 3, we obtain the inequality (29). Now multiply scalarly the equation (9) by \( 2 \| \mathbf{\hat{T}} \|_{\alpha} \):

\[ \| \mathbf{\hat{T}} \|_{\alpha} \leq \| \mathbf{\hat{P}} \|_{\alpha} + \| \mathbf{\hat{Q}} \|_{\alpha} + 2 \tau \left( \mathbf{L}(\mathbf{\hat{u}}, \mathbf{\hat{T}}) - \mathbf{L}(\mathbf{\hat{u}}, \mathbf{\hat{T}}) \right) \]

(34)

Proof. Multiplying the equation (10) scalarly by \( 2 \| \mathbf{\hat{T}} \|_{\alpha} \), and carrying out calculations analogous to those made in Lemma 3, we obtain the inequality (29). Now multiply scalarly the equation (9) by \( 2 \| \mathbf{\hat{T}} \|_{\alpha} \):

\[ \| \mathbf{\hat{T}} \|_{\alpha} \leq \| \mathbf{\hat{P}} \|_{\alpha} + \| \mathbf{\hat{Q}} \|_{\alpha} + 2 \tau \left( \mathbf{L}(\mathbf{\hat{u}}, \mathbf{\hat{T}}) - \mathbf{L}(\mathbf{\hat{u}}, \mathbf{\hat{T}}) \right) \]

(34)

Let us estimate the scalar products in (34). We obtain a chain of inequalities by applying the Cauchy inequality to the term \( \zeta_1 = 2 \tau \left( \mathbf{L}(\mathbf{\hat{u}}, \mathbf{\hat{T}}) - \mathbf{L}(\mathbf{\hat{u}}, \mathbf{\hat{T}}) \right) \):

\[ \zeta_1 \leq 2 \tau \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{\| \mathbf{\hat{u}} \|_{\alpha} \| \mathbf{\hat{v}} \|_{\alpha} \| \mathbf{\hat{w}} \|_{\alpha} \| \mathbf{\hat{z}} \|_{\alpha} \| \mathbf{\hat{t}} \|_{\alpha} \| \mathbf{\hat{h}} \|_{\alpha} \right) + \]

\[ + 2 \tau \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{\| \mathbf{\hat{u}} \|_{\alpha} \| \mathbf{\hat{v}} \|_{\alpha} \| \mathbf{\hat{w}} \|_{\alpha} \| \mathbf{\hat{z}} \|_{\alpha} \| \mathbf{\hat{t}} \|_{\alpha} \| \mathbf{\hat{h}} \|_{\alpha} \right) \]

\[ \leq 2 \tau \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{\| \mathbf{\hat{u}} \|_{\alpha} \| \mathbf{\hat{v}} \|_{\alpha} \| \mathbf{\hat{w}} \|_{\alpha} \| \mathbf{\hat{z}} \|_{\alpha} \| \mathbf{\hat{t}} \|_{\alpha} \| \mathbf{\hat{h}} \|_{\alpha} \right) \]

\[ = 4 \epsilon_0 \sum_{m=1}^{\infty} \left( \frac{\| \mathbf{\hat{u}} \|_{\alpha} \| \mathbf{\hat{v}} \|_{\alpha} \| \mathbf{\hat{w}} \|_{\alpha} \| \mathbf{\hat{z}} \|_{\alpha} \| \mathbf{\hat{t}} \|_{\alpha} \| \mathbf{\hat{h}} \|_{\alpha} \right) \]

Using the equations (12), condition (22), and choosing \( \epsilon_0 = 4 \epsilon_1 \omega^{-1} \), we obtain

\[ \zeta_1 \leq 4 \epsilon_1 \omega^{-1} \frac{\| \mathbf{\hat{u}} \|_{\alpha} \| \mathbf{\hat{v}} \|_{\alpha} \| \mathbf{\hat{w}} \|_{\alpha} \| \mathbf{\hat{z}} \|_{\alpha} \| \mathbf{\hat{t}} \|_{\alpha} \| \mathbf{\hat{h}} \|_{\alpha} \]
Using a technique similar to that used in Lemma 3, we arrive at the following inequality from (34):

\[
\|\hat{T}^h\|_B^2 + \tau^2 \left(1 - \frac{c_1 e_1}{\omega \theta_2} \right) \|T^h\|_A^2 - c_1 e_1 \tau \|T^h\|_A^2 + \tau \left(4c_0 - \frac{c_1^2}{\omega \theta_1} \right) \|T^h\|_B^2 + \frac{c_1 c_2 e_1 \tau}{\omega} \|\hat{\rho}^h\|_{\lambda}^2 + 2\tau e_2 \|f^h\|_{A^{-1}}^2.
\]

Add the last inequality to the inequality (29) multiplied by some \(c_5 > 0\):

\[
\|\hat{T}^h\|_B^2 + c_5 \|\hat{\rho}^h\|_{\lambda}^2 + \tau^2 \left(\omega - \frac{c_1 e_1}{\delta} - \frac{1}{\delta c_2 e_2} - c_1 e_1 - \frac{2c_2 e_2 c_5}{\delta} \right) \|T^h\|_A^2 + c_5 \tau^3 (\omega - \omega_2) \|\rho^h\|_A^2 + \tau \left(4c_0 - \frac{c_1}{e_1} \right) \|\hat{T}^h\|_A^2 + \tau \left(2c_5 d_2 (e_3) - \frac{16c_1 c_2 e_1}{\omega} \right) \|\hat{\rho}^h\|_{\lambda}^2 \leq \|T^h\|_B^2 + c_5 \|\rho^h\|_B^2 + 2\tau e_2 \|f^h\|_{A^{-1}}^2 + \frac{6c_5 \tau}{c_0} \|\rho^h\|_{\lambda}^2.
\]

Using the relations (8), it is not difficult to show that

\[
d_5 \|\hat{\rho}^h\|_{A^{-1}}^2 + d_6 \|\rho^h\|_A^2 \leq c_4 \sum_{j=1}^{W_x} \sum_{x \in \Delta h} b_h (x) \left(\hat{\rho}^h_{w_{ij}} + T^2 \hat{\rho}^h_{w_{ij}} \right) \delta_h (x - x_{j}^{(w)}) h_1 h_2 = q_0.
\]

Let us choose \(e_1 = \frac{c_1 (\eta \tau^2)}{\delta c_5}, \quad e_2 = \frac{2r \eta}{3 \omega q} \quad \) and the constants \(c_5 \) and \(e_3 \) from the condition of non-negativity of the coefficient of \(\|\rho^h\|_A^2\). Then, under the conditions of the lemma, we arrive at the inequality

\[
\|\hat{T}^h\|_B^2 + c_5 \|\hat{\rho}^h\|_{\lambda}^2 + d_5 \tau \left(c_1 \eta - \|T^h\|_B^2 \right) \|\hat{T}^h\|_A^2 + d_4 \tau \|\hat{\rho}^h\|_{\lambda}^2 \leq \|T^h\|_B^2 + c_5 \|\rho^h\|_B^2 + \tau q_0.
\]  

(35)

To prove the lemma, we apply the method of mathematical induction. The inequality (35) for \(n' = 0\) yields

\[
\|T^{h,1}\|_B^2 + c_5 \|\rho^{h,1}\|_B^2 + d_3 \tau \left(c_1 \eta - \|T^{h,1}\|_B^2 \right) \|\hat{T}^{h,1}\|_A^2 + d_4 \tau \|\hat{\rho}^{h,1}\|_{\lambda}^2 \leq \|T^{h,1}\|_B^2 + c_5 \|\rho^{h,1}\|_B^2 + \tau q_0 \leq c_1 \eta.
\]

(36)

Since the inequality \(c_1 \eta - \|T^{h,0}\|_B^2 > 0\) holds under the condition (20), then it follows from (20) and (36) that

\[
\|T^{h,1}\|_B^2 + c_5 \|\rho^{h,1}\|_B^2 + d_3 \tau \|T^{h,0}\|_B^2 + d_4 \tau \|\hat{\rho}^{h,0}\|_{\lambda}^2 \leq \|T^{h,0}\|_B^2 + c_5 \|\rho^{h,0}\|_B^2 + \tau q_0 \leq c_1 \eta.
\]

(37)

Assuming that the inequality (31) holds for \(n' = n - 2\):

\[
\|T^{h,n-1}\|_B^2 + \|\rho^{h,n-1}\|_B^2 + d_4 \tau \|\rho^{h,n-2}\|_{\lambda}^2 \leq c_1 \eta,
\]

write the inequality (35) for \(n' = n - 1\):

\[
\|T^{h,n}\|_B^2 + c_5 \|\rho^{h,n}\|_B^2 + d_5 \tau \left(c_1 \eta - \|T^{h,n-1}\|_B^2 \right) \|\hat{T}^{h,n-1}\|_A^2 + d_4 \tau \|\hat{\rho}^{h,n-1}\|_{\lambda}^2 \leq \|T^{h,n-1}\|_B^2 + c_5 \|\rho^{h,n-1}\|_B^2 + \tau q_0.
\]

(38)

By virtue of the fulfillment of the inequality (38), the expression in brackets before \(\|T^{h,n}\|_B^2\) in the left-hand side of (39) is non-negative. Therefore it follows from (39) that

\[
\|T^{h,n}\|_B^2 + c_5 \|\rho^{h,n}\|_B^2 + d_3 \tau \left(c_1 \eta - \|T^{h,n-1}\|_B^2 \right) \|\hat{T}^{h,n-1}\|_A^2 + d_4 \tau \|\hat{\rho}^{h,n-1}\|_{\lambda}^2 \leq \|T^{h,n-1}\|_B^2 + c_5 \|\rho^{h,n-1}\|_B^2 + 2\tau q_0 \leq \|
\]

(39)
Using the definition of $L$ and using the Cauchy inequality, it is easy to see that
\[
\zeta = 2\tau \left( L\left(\tilde{u}^e, T^h\right) - L\left(\bar{u}^0, \bar{T}^h\right), \bar{\theta} \right) \leq 2c_1 \sum_{m=1}^{2} \left( \|u_m\|^2 \|\bar{u}_m\|^2 \right) + \frac{2\tau^2}{\epsilon_1} \|\bar{\theta}\|^2 \left( \|T^h\|^2 + \|\bar{T}^h\|^2 \right).
\]

Using the equations (12) and the condition (22) yields
\[
\zeta \leq 2c_1\epsilon \epsilon_{11} \left( \|\bar{p}\|^2 + \|\bar{\tilde{p}}\|^2 \right) + \frac{2\tau^2}{\epsilon_1} \left( \|\bar{\theta}\|^2 + \tau \|\theta\|^2 \right) \left( \|T^h\|^2 + \|\bar{T}^h\|^2 \right).
\]

Finally, using the inequalities (31) and (32) to estimate the right-hand side of the last inequality, we obtain:
\[
\zeta \leq \frac{2c_1\epsilon \epsilon_{11}}{d_4} \times 2c_2 \eta + \frac{2\tau^2}{\epsilon_1} \left( \|\bar{\theta}\|^2 + \tau \|\theta\|^2 \right) \times 2c_4 \eta,
\]
from which the inequality (33) follows.

**Lemma 5** Let the conditions (7), (20) and $\omega > \omega_3$, $\omega_3 = \frac{c_7}{4c_0} \left( 1 + \frac{1}{\delta} \right) + \frac{c_{10}}{2c_2}\frac{\epsilon_{10}}{\epsilon_{10}}$ hold. Then the following inequality holds:
\[
\|\theta\|^2 \leq 2\tau \left( \omega - \omega_3 \right) \|\theta\|^2 + \frac{4c_0\tau}{\epsilon_1} \|\theta\|^2 \leq \tau \left( \frac{\epsilon_1}{\epsilon_3} \right) \left[ \frac{\tau}{\epsilon_1} \|\theta\|^2 + \frac{\epsilon_1}{\epsilon_3} \|\theta\|^2 \right] + \frac{8c_0\tau}{\epsilon_1} \|\psi_T\|^2 \theta_{\Lambda/h}.
\]

**Proof.** Multiply the equation (15) scalarly by $2\tilde{\theta}$:
\[
\|\tilde{\theta}\|^2 = \frac{\tau}{\epsilon_1} \|\tilde{\theta}\|^2 + \frac{\epsilon_1}{\epsilon_3} \|\tilde{\theta}\|^2 + 2\tau \left( L\left(\tilde{u}^e, \tilde{T}^h\right) - L\left(\bar{u}^0, \bar{T}^h\right), \tilde{\theta} \right) + 2\tau \left( \Lambda_1 \theta, \hat{\theta} \right) = 2\tau \left( \psi_T, \hat{\theta} \right).
\]

Estimate the scalar products in (41). Using the technique used above, we obtain
\[
2\tau \left( \Lambda_1 \theta, \hat{\theta} \right) = 2\tau \left( \Lambda_1 \theta, \theta \right) + 2\tau^2 \left( \Lambda_1 \theta, \theta \right) \geq 8\tau \|\theta\|^2 - \frac{c_1}{\epsilon_1} \left[ \frac{\tau}{\epsilon_1} \|\theta\|^2 + \frac{\epsilon_1}{\epsilon_3} \|\theta\|^2 \right] + \frac{\tau}{\epsilon_1} \|\theta\|^2 \theta_{\Lambda/h}.
\]
Using the inequalities (42), (43) and (33), we obtain from (41) that
\[
\|\tilde{\theta}\|^2 + \frac{\tau}{\epsilon_1} \|\theta\|^2 + \frac{\epsilon_1}{\epsilon_3} \|\theta\|^2 + \frac{2\tau}{\epsilon_1} \|\psi_T\|^2 \theta_{\Lambda/h} + \frac{1}{\epsilon_1} \|\theta\|^2 \theta_{\Lambda/h} \leq \frac{2c_1\epsilon \epsilon_{11}}{d_4} \times \frac{2c_2 \eta}{d_4} + \frac{\tau}{\epsilon_1} \|\theta\|^2 + \frac{\epsilon_1}{\epsilon_3} \|\theta\|^2 \theta_{\Lambda/h}.
\]

Choosing $\epsilon_1 = \frac{4c_0}{\tau}$, we obtain the assertion of the lemma.

The following lemma is proved similarly:

**Lemma 6** Let the condition $\omega > \omega_4$, $\omega_4 = \frac{2\tau e}{4e} - \frac{1}{\epsilon_4}$ hold. The following inequality holds:
\[
\|\sigma_{\omega}\|^2 + \frac{\tau}{\epsilon_1} \|\sigma_{\omega}\|^2 + \frac{4\tau}{\epsilon_1} \|\sigma_{\omega}\|^2 + \frac{2\tau}{\epsilon_1} \|\psi_{\omega}\|^2 \theta_{\Lambda/h} \leq \frac{2c_1\epsilon \epsilon_{11}}{d_4} \times \frac{2c_2 \eta}{d_4} + \frac{\tau}{\epsilon_1} \|\theta\|^2 + \frac{\epsilon_1}{\epsilon_3} \|\theta\|^2 \theta_{\Lambda/h}.
\]

Now let us prove Theorem 1. Combining the results of Lemma 3, 5 and 6, we arrive at the following inequality after obvious transformations:
\[
\|\bar{\theta}\|^2 + \|\bar{\tilde{\theta}}\|^2 + \|\bar{\psi}_T\|^2 \leq \frac{2\tau}{\epsilon_1} \|\theta\|^2 + \tau^2 \left( \omega - \omega_1 \right) \|\bar{\theta}\|^2 + \frac{2\tau^2}{\epsilon_1} \|\bar{\tilde{\theta}}\|^2 + \frac{\tau}{\epsilon_1} \|\bar{\psi}_T\|^2 \theta_{\Lambda/h}.
\]

Assuming that the conditions of the Theorem 1 are satisfied, and choosing $\epsilon_7$ from the condition $\epsilon_7 < \frac{c_{10}}{2c_2}$, we arrive at the assertion of the Theorem 1.
CONSTRUCTION OF COST-EFFECTIVE DIFFERENCE SCHEMES

Consider the following difference scheme

\[\hat{B}T^h_i + L\left(\hat{u}^h, T^h\right) + \Lambda_1 T^h = f^h_T,\] (44)

\[Bp^h_i + \Lambda_2 p^h = \beta^h_i T^h_i + f^h_p,\] (45)

\[\hat{B}s_{a,i} + \Lambda_3 s^h = f^h_a, \quad a = w, o,\] (46)

\[\hat{u}_m^h = -k^h p^h_{\alpha}, \quad m = 1, 2,\] (47)

\[T^h(0) = T_0, \quad p^h(0) = p_0, \quad s^h(0) = s_{a0}\] (48)

with some self-adjoint operator \(\hat{B}\) satisfying the condition \(\hat{B} \geq B\). It is known from the theory of two-layer difference schemes that if the difference scheme (9)-(13) is stable with respect to the initial data and the right-hand sides of the equations, then the scheme (44)-(48) also has the same property. Using this statement, we construct three cost-effective difference schemes for the problem (1)-(6) by choosing the operator \(\hat{B}\).

Factorization scheme

Let us take \(\hat{B} = (E + \tau \omega A_1)(E + \tau \omega A_2)\), where the operators \(A_m\) are defined as \(A_m = A^+_m + A^-_m\) \[7\],

\[A^+_m w = \{h^+_m w + h^+_m w, \quad x_m = 0; \quad h^+_m w, \quad x_m \in \Omega_0; \quad h^+_m w + h^+_m w - 2h^+_m w^{1/2}, \quad x_m = 1\},\]

\[A^-_m w = \{h^-_m w + h^-_m w - 2h^-_m w^{1/2}, \quad x_m = 0; \quad -h^-_m w, \quad x_m \in \Omega_0; \quad h^-_m w + h^-_m w, \quad x_m = 1\}.\]

It can be seen by direct verification that \((A^+_m w, \hat{w}) = (A^-_m \hat{w}, w)\), i.e. the operators \(A^+_m, A^-_m\) are mutually conjugate. In this case the operator \(A\) decomposes into the sum of two operators \(A = A^+ + A^-\), where \(A^+ = A^+_1 + A^+_2\). With this choice of the operator \(\hat{B}\), the factorization scheme is written as follows:

\[(E + \tau \omega A^+)(E + \tau \omega A^+) T^h_i + L\left(\hat{u}^h, T^h\right) + \Lambda_1 T^h = f^h_T,\] (49)

\[(E + \tau \omega A^-)(E + \tau \omega A^-) p^h_i + \Lambda_2 p^h = \beta^h_i T^h_i + f^h_p,\] (50)

\[(E + \tau \omega A^+)(E + \tau \omega A^-) s^h_{a,i} + \Lambda_3 s^h = f^h_a,\] (51)

\[\hat{u}_m^h = -k^h p^h_{\alpha}, \quad m = 1, 2,\] (52)

\[T^h(0) = T_0, \quad p^h(0) = p_0, \quad s^h(0) = s_{a0}\] (53)

It is easy to verify \[7\] that the operator \(\hat{B} = (E + \tau \omega A^+)(E + \tau \omega A^-)\) is self-adjoint and the operator inequality \(\hat{B} \geq B\) holds. Thus, the difference scheme (49)-(53) is stable with respect to the initial data and the right-hand sides of the equations under the condition \(\omega \geq \omega_0\). Construct the following algorithm to implement the scheme (49)-(53):

\[(E + \tau \omega A^+)^{\theta^{h+1}} + L\left(\hat{u}^{h+1}, T^{h+1}\right) + \Lambda_1 T^{h+1} = f^{h+1},\] (54)

\[(E + \tau \omega A^-)^{\theta^{h-1}} = \theta^{h+1}, \quad T^{h+1} + \tau \theta^{h+1},\] (55)

\[(E + \tau \omega A^+) \rho^{h+1} + \Lambda_2 \rho^{h+1} = \rho^h \theta^{h+1} + \theta^{h+1},\] (56)

\[(E + \tau \omega A^-) \rho^{h+1} = \rho^{h+1}, \quad \rho^{h+1} = \rho^h + \tau \rho^{h+1},\] (57)

\[(E + \tau \omega A^+) \sigma^a_{a+1} + \Lambda_3 \sigma^a_{a+1} = f^{a+1}_a,\] (58)

\[(E + \tau \omega A^-) \sigma^a_{a+1} = \sigma^a_{a+1}, \quad \sigma^a_{a+1} = \sigma^a_{a+1} + \tau \sigma^a_{a+1},\] (59)

\[\hat{u}^{h+1} = -k^h p^h_{\alpha}, \quad m = 1, 2,\] (60)

where \(\theta^{h+1}, \theta^{h-1}, \rho^{h+1}, \sigma^a_{a+1}, \sigma^a_{a+1}\) are auxiliary functions. Determination of auxiliary functions and solutions by algorithm (54)-(54) is carried out by explicit formulas, therefore the difference scheme (49)-(53) is cost-effective.
**Alternating directions scheme**

Choosing the operator $\tilde{B}$ in the form $\tilde{B} = (E + \tau \omega A_1)(E + \tau \omega A_2)$, we arrive at the scheme

$$
(E + \tau \omega A_1)(E + \tau \omega A_2) T^{h,n+1} + \nu N T^{h,n} + \Lambda A_1 T^{h,n} = f_T^{h,n+\frac{1}{2}},
$$

(61)

$$
(E + \tau \omega A_1)(E + \tau \omega A_2) p^{h,n} + \Lambda A_2 p^{h,n} = f_p^{h,n+\frac{1}{2}},
$$

(62)

$$
(E + \tau \omega A_1)(E + \tau \omega A_2) s_{n+1}^{h,n} + \Lambda A_3 p^{h,n} = f_s^{h,n+\frac{1}{2}}, \quad \alpha = w, \sigma,
$$

(63)

$$
\nu_m^{h,n} = -k^h \nu_m^{h,n}, \quad m = 1, 2,
$$

(64)

$$
T^{h,n} = T_0, \quad p^{h,0} = p_0, \quad s_{n+1}^{h,0} = s_0,
$$

(65)

where the operators $A_m$ are defined in (14). In this case $\tilde{B} = \tilde{B}^* \geq B$, since the operators $A_1$ and $A_2$ are self-adjoint and commute. Thus, the scheme (61)-(65) is stable with respect to the initial data and the right-hand sides of the equations under the condition $\omega \geq \omega_0$. Construct the following algorithm for the numerical implementation of the scheme:

$$
(E + \tau \omega A_1) T^{h,n+\frac{1}{2}} + \nu N T^{h,n} + \Lambda A_1 T^{h,n} = f_T^{h,n+\frac{1}{2}},
$$

(66)

$$
(E + \tau \omega A_2) T^{h,n+\frac{1}{2}} = T^{h,n+\frac{1}{2}}, \quad T^{h,n+1} = T^{h,n} + \tau T^{h,n+\frac{1}{2}},
$$

(67)

$$
(E + \tau \omega A_1) p^{h,n+\frac{1}{2}} + \Lambda A_2 p^{h,n} = f_p^{h,n},
$$

(68)

$$
(E + \tau \omega A_2) p^{h,n+\frac{1}{2}} = p^{h,n}, \quad p^{h,n+1} = p^{h,n} + \tau p^{h,n+\frac{1}{2}},
$$

(69)

$$
(E + \tau \omega A_1) s_{n+1}^{h,n+\frac{1}{2}} + \Lambda A_3 p^{h,n} = f_s^{h,n},
$$

(70)

$$
(E + \tau \omega A_2) s_{n+1}^{h,n+\frac{1}{2}} = s_{n+1}^{h,n}, \quad s_{n+1}^{h,n+1} = s_{n}^{h,n} + \tau s_{n+1}^{h,n+\frac{1}{2}},
$$

(71)

$$
\nu_m^{h,n+1} = -k^h \nu_m^{h,n+1}.
$$

(72)

Auxiliary functions $T^{h,n+\frac{1}{2}}, T^{h,n+1}, p^{h,n+\frac{1}{2}}, p^{h,n+1}, s_{n+1}^{h,n+\frac{1}{2}}, s_{n+1}^{h,n+1}$ are defined from the equations (66)-(71) by the scalar sweep method, and the solution $T^{h,n+1}, p^{h,n+1}, s_{n+1}^{h,n+1}$ on the $(n+1)$-th layer is determined by explicit formulas. Therefore, the scheme (61)-(65) is cost-effective.

**Stabilizing correction scheme**

Introduce the intermediate solutions $T^{h,n+\frac{1}{2}}, p^{h,n+\frac{1}{2}}, s_{n+1}^{h,n+\frac{1}{2}}$ instead of $\theta^{n+\frac{1}{2}}, \pi^{n+\frac{1}{2}}, \sigma^{n+\frac{1}{2}}$ in the difference scheme (49)-(52) by formulas

$$
\theta^{n+\frac{1}{2}} = \frac{T^{h,n+\frac{1}{2}} - T^{h,n}}{\tau}, \quad \pi^{n+\frac{1}{2}} = \frac{p^{h,n+\frac{1}{2}} - p^{h,n}}{\tau}, \quad \sigma^{n+\frac{1}{2}} = \frac{s_{n+1}^{h,n+\frac{1}{2}} - s_{n}^{h,n}}{\tau}.
$$

Then the algorithm (54)-(60) is replaced with the following one:

$$
\frac{T^{h,n+\frac{1}{2}} - T^{h,n}}{\tau} + \omega A^+ T^{h,n+\frac{1}{2}} + \nu N T^{h,n} + \Lambda A_1 T^{h,n} = f_T^{h,n+\frac{1}{2}},
$$

(73)

$$
\frac{T^{h,n+1} - T^{h,n}}{\tau} + \omega A^+ T^{h,n+1} + \Lambda A_1 T^{h,n} = \frac{T^{h,n+\frac{1}{2}} - T^{h,n}}{\tau},
$$

(74)

$$
\frac{p^{h,n+\frac{1}{2}} - p^{h,n}}{\tau} + \omega A^+ (p^{h,n+\frac{1}{2}} - p^{h,n}) + \Lambda A_2 p^{h,n} = \frac{p^{h,n+\frac{1}{2}} - p^{h,n}}{\tau},
$$

(75)

$$
\frac{p^{h,n+1} - p^{h,n}}{\tau} + \omega A^+ (p^{h,n+1} - p^{h,n}) = \frac{p^{h,n+\frac{1}{2}} - p^{h,n}}{\tau},
$$

(76)

$$
\frac{s_{n+1}^{h,n+\frac{1}{2}} - s_{n+1}^{h,n}}{\tau} + \omega A^+ (s_{n+1}^{h,n+\frac{1}{2}} - s_{n+1}^{h,n}) + \Lambda A_3 s_{n+1}^{h,n} = \frac{s_{n+1}^{h,n+\frac{1}{2}} - s_{n+1}^{h,n}}{\tau},
$$

(77)

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\[
(E + \tau \omega A^-)^{\frac{s_{h,n+1} - s_{h,n}}{\tau}} = \frac{h_{n+1}^\tau}{\tau} - \frac{h_{n}^\tau}{\tau},
\]
(78)

\[
u_m^{h_{n+1}} = -k_1 \lambda \rho_m^{h_{n+1}}, \quad m = 1, 2.
\]
(79)

The equations (73)-(78) are solved by the scalar sweep method, and the values \(u_m^{h_{n+1}}\) are determined by explicit formulas, therefore the algorithm (73)-(79) is cost-effective.

**Results of the computational experiments**

Let us analyze the effectiveness of the proposed algorithms by comparing the mean time spent on computing one time layer. Calculations were carried out on a uniform grid. The grid step was varied between 0.02 and 0.002, and the time step was chosen equal to \(\tau = 0.001\). Calculations for each of the three algorithms were performed until \(t = 1000\tau\). The results of the analysis are given in Table 1.

<table>
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<th>TABLE 1. Mean time (in milliseconds) required to calculate one time layer</th>
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It can be seen that the differences in time are minor, however, since calculations by the factorization scheme are carried out using explicit formulas, this algorithm requires less time to calculate one time layer.

**CONCLUSION**

Thus, the following results were obtained in the paper. A two-layer finite-difference scheme is constructed for the model two-dimensional three-phase non-isothermal flow problem without allowance for capillary forces. Using the method of energy inequalities, an a priori estimate is obtained for the solution of the difference scheme which proves the stability of the scheme with respect to the initial data and the right-hand sides of the equations. On the basis of the studied difference scheme, three cost-effective difference schemes are constructed. The efficiency of the schemes is verified by comparing the average time spent on calculating one time layer. The results obtained in this paper can be used to carry out further research on the numerical solving the problems of the theory of multiphase non-isothermal flows using difference methods.

**REFERENCES**