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Abstract. The paper focuses on constructing and investigation of cost-effective difference schemes for the numerical solution of the two-dimensional three-phase non-isothermal flow problem without capillary and gravitational forces. In this paper, the finite difference method is used to solve the problem numerically. The method of energy inequalities is applied to examine the stability of the finite difference scheme with respect to the initial data and the right-hand sides of the equations. Three cost-effective difference schemes are constructed on the base of the studied scheme. The efficiency of the proposed algorithms is analyzed on the basis of comparing the average time spent on the numerical implementation of one time layer.

INTRODUCTION

The urgency of a rigorous theoretical justification of numerical methods for solving the problems of the three-phase non-isothermal flow theory is due to its practical importance in the oil industry when predicting the extraction of high-viscosity paraffin or resinous oil. This is due to the fact that currently the reserves of this category of oil are higher than the reserves of so-called light oils, which leads to the need to apply secondary or tertiary methods. However, due to the rather high cost of these methods, studies aimed at increasing its effectiveness are of great practical importance. At present, it is possible to reach this only using methods of mathematical modeling of hydrodynamic processes occurring in oil reservoirs during the development of deposits.

The model considered in the present paper consists of the mass conservation equation, equation of motion in the form of the linear Darcy's law, energy equation, equation of state, and phase balance equation. The model with various assumptions about physical data was studied, for example, in [1, 2, 3]. In [4], a new "global" formulation of the three-phase non-isothermal flow problem was proposed, which is based on the introduction of a change of variables for the pressure, called "global" pressure, to eliminate the gradients of capillary pressures from the equations for pressure and temperature. As a result, the initial equations were reduced to a system of five partial differential equations with respect to pressure, temperature, velocity, and two saturations.

The main difficulty in the numerical solution of the obtained problem is connected with the complexity and strong non-linearity of the equations. Therefore, the issue of the development of computational algorithms for the numerical implementation of this problem, which requires less computational operations, becomes relevant. In the present work, the main attention is paid to the construction and investigation of cost-effective difference schemes for the three-phase non-isothermal flow problem. Under the cost-effective schemes we mean schemes for which the number of arithmetic operations for the transition from the n-th time layer to the $(n+1)$-th one is proportional to the number of unknown values. In order to simplify the calculations when obtaining a priori estimates, the coefficients in the equations are linearized, or taken to be constant. A priori estimates are obtained using the method of energy inequalities which prove the stability of the constructed difference scheme. The proof of the stability is based on five preliminary lemmas. In the last section of the paper three efficient difference schemes are constructed on the base of

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the constructed scheme. The efficiency of the proposed algorithms is analyzed on the basis of comparing the average time spent on the numerical implementation of one time layer.

FORMULATION OF THE PROBLEM

Consider the following three-phase non-isothermal flow problem in a domain $Q \equiv \overline{\Omega} \times [0,\ t_1]$ where $\overline{\Omega} = [0,\ l] \times [0,\ l]$ [4]:

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T - k_h \nabla^2 T = f_T, \tag{1}$$

$$\frac{\partial p}{\partial t} - \nabla \cdot \left(k_p\left(x, t, p\right) \nabla p\right) - \beta_T \frac{\partial T}{\partial t} = f_p, \tag{2}$$

$$\frac{\partial s_\alpha}{\partial t} - \nu_\alpha \nabla^2 p = f_\alpha, \quad \alpha = w, o, \tag{3}$$

$$\vec{u} = -k\lambda \nabla p \tag{4}$$

with the following initial and boundary conditions:

$$T\left(x, 0\right) = T_0, \quad p\left(x, 0\right) = p_0, \quad s_\alpha\left(x, 0\right) = s_{\alpha 0}, \tag{5}$$

$$k_h \left.\frac{\partial T}{\partial \vec{n}}\right|_{\partial\overline{\Omega}} = 0, \quad k_p \left.\frac{\partial p}{\partial \vec{n}}\right|_{\partial\overline{\Omega}} = 0, \tag{6}$$

where subscripts w, o, g, r denote the phases of water, oil, steam, and rock; k is the absolute permeability, p is pressure, T is temperature, and s_α is the saturation of the phase α; $\vec{u}$ is velocity; β_T is some function, and ν_α is a constant.

Suppose that the functions k_p, k_h, λ are continuous in Q and the following conditions hold:

$$k_p\left(x, t, p\right) \geq c_0 > 0, \quad k_h\left(x, t\right) \geq 4c_0, \quad \lambda\left(x, t\right) \leq c_1. \tag{7}$$

We note that in the multiphase flow theory the functions f_T and f_p usually have the form

$$f_p = \sum_{j=1}^{W_n} \varphi_p^*\left(x, t\right)\left(p_{inj} - p\right)\delta\left(x - x_j^{(w)}\right), \quad f_T = \sum_{j=1}^{W_n} \varphi_T^*\left(x, t\right)\left(T_{inj} - T\right)\delta\left(x - x_j^{(w)}\right), \tag{8}$$

where $x_1^{(w)}, x_2^{(w)}, ..., x_{W_n}^{(w)} \in \overline{\Omega}$ are source coordinates; $\varphi_p^*\left(x, t\right)$, $\varphi_T^*\left(x, t\right)$ are some known functions; p_{inj} is the injection pressure; T_{inj} is the temperature corresponding to the pressure p_{inj} according to the table of thermophysical properties of water and steam; $\delta\left(x\right)$ is a delta function. We assume that $\left|\varphi_p^*\left(x, t\right)\right|, \left|\varphi_T^*\left(x, t\right)\right| \leq c_1$.

FORMULATION OF THE DIFFERENCE PROBLEM

Let us introduce the uniform grid $\Omega_{h\tau}$ in Q with spatial steps h_1, h_2 and time step τ as follows:

$$\overline{\Omega}_{h\tau} = \overline{\Omega}_h \times \overline{\Omega}_\tau = \left\{(x,\ t): \quad x \in \overline{\Omega}_h, \quad t \in \overline{\Omega}_\tau\right\}, \quad \overline{\Omega}_h = \overline{\Omega}_{h,1} \times \overline{\Omega}_{h,2},$$

$$\overline{\Omega}_{h,m} = \left\{i_m h_m: \quad i_m = 0,\ 1, ...,\ N_m, \quad N_m h_m = l\right\}, \quad \overline{\Omega}_\tau = \left\{t_n = n\tau: \quad n = 0,\ 1, ...,\ N_t, \quad N_t\tau = t_1\right\}.$$

We also introduce the notations

$$\Omega_{h,m} = \left\{i_m h_m, \quad i_m = 1, ...,\ N_m - 1, \quad N_m h_m = l\right\},$$

$$\Omega_{h,m}^+ = \left\{i_m h_m, \quad i_m = 1, ...,\ N_m, \quad N_m h_m = l\right\}, \quad \Omega_{h,m}^- = \left\{i_m h_m, \quad i_m = 0, ...,\ N_m - 1, \quad \left(N_m - 1\right) h_m = l - h_m\right\}.$$

$$\overline{\Gamma}_h = \overline{\Omega}_h \backslash \Omega_h, \quad \Omega_h = \bigcup_{m=1}^{2} \Omega_{h,m}, \quad \Gamma_h = \bigcup_{m=1}^{2} \Gamma_{h,m}^{\pm}, \quad \Gamma_{h,0} = \overline{\Gamma}_h \backslash \Gamma_h,$$

$$\Gamma^{+}_{h,m} = \{x_m = N_m,\ 0 < x_{3-m} < N_{3-m}\}, \quad \Gamma^{-}_{h,m} = \{x_m = 0,\ 0 < x_{3-m} < N_{3-m}\}.$$

Let us associate the following finite difference scheme with the differential problem (1)-(6):

$$BT^h_t + L\left(\vec{u}^h,\ T^h\right) + \Lambda_1 T^h = f^h_T, \tag{9}$$

$$Bp^h_t + \Lambda_2 p^h = \beta^h_T T^h_t + f^h_p, \tag{10}$$

$$Bs^h_{\alpha,t} + \Lambda_{3\alpha} p^h = f^h_\alpha, \quad \alpha = w,\ o, \tag{11}$$

$$\vec{u}^h = \left(u^h_1, u^h_2\right), \quad u^h_m = -k\lambda^h p^h_{\bar{x}_m}, \quad m = 1,\ 2, \tag{12}$$

$$T^h(0) = T_0, \quad p^h(0) = p_0, \quad s^h_\alpha(0) = s_{\alpha 0}, \tag{13}$$

where

$$B = E + \tau\omega A, \quad A = A_1 + A_2, \quad \omega > 0,$$

$$A_m w = \left\{-2h^{-1}_m\left(w_{x_m} + w\right),\ x_m = 0;\ -w_{\bar{x}_m x_m},\ x_m \in \Omega_{h,m};\ 2h^{-1}_m\left(w_{\bar{x}_m} + w\right),\ x_m = l\right\}, \tag{14}$$

$$L(\vec{v},\ \theta) = 0.5\sum_{m=1}^{2}\left(\beta^{+}_m(x)\, v^{+1_m}_m \theta_{x_m} + \beta^{-}_m(x)\, v_m \theta_{\bar{x}_m}\right),$$

$$\beta^{+}_m(x) = \{2,\ x_m = 0;\ 1,\ x_m \in \Omega_{h,m};\ 0,\ x_m = l\}, \quad \beta^{-}_m(x) = 2 - \beta^{+}_m(x)$$

and the operators Λ_i are defined as follows:

$$\Lambda_\delta = \sum_{m=1}^{2}\Lambda_{\delta,m}, \quad \Lambda_2 w = \sum_{m=1}^{2}\left(\chi^{+}_m(x)\,\Lambda^{+}_{2,m}w + \chi^{-}_m(x)\,\Lambda^{-}_{2,m}w\right),$$

$$\Lambda_{\delta,m}w = \left\{-2h^{-1}_m \eta w_{x_m},\ x_m = 0;\ -\eta w_{\bar{x}_m x_m},\ x_m \in \Omega_{h,m};\ 2h^{-1}_m \eta w_{\bar{x}_m},\ x_m = l\right\},$$

$$\Lambda^{+}_{2,m}w = \left\{-2h^{-1}_m \mu w_{x_m},\ x_m = 0;\ -\left(\mu w_{x_m}\right)_{\bar{x}_m},\ x_m \in \Omega_{h,m};\ 2h^{-1}_m\left(\mu w_{x_m}\right)^{-1_m},\ x_m = l\right\},$$

$$\Lambda^{-}_{2,m}w = \left\{-2h^{-1}_m\left(\mu w_{\bar{x}_m}\right)^{+1_m},\ x_m = 0;\ -\left(\mu w_{\bar{x}_m}\right)_{x_m},\ x_m \in \Omega_{h,m};\ 2h^{-1}_m \mu w_{\bar{x}_m},\ x_m = l\right\},$$

$$\chi^{+}_m(x) = \{1,\ x_{3-m} = 0;\ 0.5,\ x_{3-m} \in \Omega_{h,3-m};\ 0,\ x_{3-m} = l\}, \quad \chi^{-}_m(x) = 1 - \chi^{+}_m(x),$$

where $\eta = k^h_h$ for $\delta = 1$; $\eta = \nu_\alpha$ for $\delta = 3\alpha$. The notations used above are defined in [5]. In the present paper we use the following scalar products and norms:

$$(w, \tilde{w}) = (w,\ \tilde{w})_{\overline{\Omega}_h} = \sum_{x\in\overline{\Omega}_h} b_h(x)\, w(x)\, \tilde{w}(x)\, h_1 h_2, \quad (w,\ \tilde{w})_{\Omega_h} = \sum_{x\in\Omega_h} w(x)\, \tilde{w}(x)\, h_1 h_2, \quad (w,\ \tilde{w})_{\Omega^{\pm}_{h,m}} = \sum_{x\in\Omega^{\pm}_{h,m}} w(x)\, \tilde{w}(x)\, h_1 h_2,$$

$$\|w\|^2_0 = (w,\ w)_{\bar{\Omega}_h}, \quad b_h(x) = \{1,\ x \in \Omega_h;\ 0.5,\ x \in \Gamma_h;\ 0.25,\ x \in \Gamma_{h,0}\}, \quad \|w\|^2_1 = \frac{1}{2}\sum_{m=1}^{2}\left(\left(w^2_{\bar{x}_m},\ 1\right)_{\Omega^{+}_{h,m}} + \left(w^2_{x_m},\ 1\right)_{\Omega^{-}_{h,m}}\right).$$

STUDY OF THE STABILITY OF THE DIFFERENCE SCHEME

Let us consider the problem for $\theta = T^h - \tilde{T}^h$, $\pi = p^h - \tilde{p}^h$, $\sigma_\alpha = s^h_\alpha - \tilde{s}^h_\alpha$, $\zeta_m = u_m - \tilde{u}_m$ to study the stability of the difference scheme (9)-(13):

$$B\theta_t + L\left(\vec{u}^h,\ T^h\right) - L\left(\tilde{\vec{u}}^h,\ \tilde{T}^h\right) + \Lambda_1\theta = \psi_T, \tag{15}$$

$$B\pi_t + \Lambda_2 p^h - \Lambda_2\tilde{p}^h = \beta^h_T\theta_t + \psi_p, \tag{16}$$

$$B\sigma_{\alpha,t} + \Lambda_{3\alpha}\pi = \psi_\alpha, \quad \alpha = w,\ o, \tag{17}$$

$$\zeta_m = -k\lambda^h \pi_{\bar{x}_m}, \quad m = 1,\ 2, \tag{18}$$

$$\theta(0) = \theta^0, \quad \pi(0) = \pi^0, \quad \sigma_\alpha(0) = \sigma_\alpha^0, \quad \alpha = w,\, o, \tag{19}$$

where $\tilde{T}^h$, $\tilde{p}^h$, $\tilde{s}_\alpha^h$, $\tilde{u}_m$ is the solution of the perturbed problem. Suppose that the following conditions hold for initial values of temperature and pressure:

$$c_1\eta - q_0 t_1 - \left\|T^0\right\|_B^2 - c_5\left\|p^0\right\|_B^2 \geq 0, \quad c_1\eta - q_0 t_1 - \left\|\tilde{T}^0\right\|_B^2 - c_5\left\|\tilde{p}^0\right\|_B^2 \geq 0, \tag{20}$$

where $\eta > 0$ is a real parameter, c_5 and q_0 are some constants. In addition, to obtain an a priori estimate of the difference solution, we make the assumption that for the dimensionless function $\beta_T = \beta_T(x,t)$ and the dimensionless constant k, the following inequalities hold:

$$\beta_T \leq c_2\tau, \quad c_2 > 0, \tag{21}$$

$$k \leq c_3\tau, \quad c_3 > 0. \tag{22}$$

Let us first formulate the main result of this chapter.

Theorem 1 *Let the conditions (7), (20), (21), (22) and $\omega > \omega_0$, $\omega_0 = \max\limits_{i=\overline{1,4}}\omega_i + \varepsilon$, $\varepsilon > 0$ hold. Then the difference scheme (9)-(13) is stable with respect to the initial values and right-hand sides of the equations, and the following inequality holds:*

$$\left\|\hat{\theta}\right\|_B^2 + \|\hat{\pi}\|_B^2 + \sum_{\alpha=w,o}\|\hat{\sigma}_\alpha\|_B^2 + 4c_0\tau\|\theta\|_1^2 + d_9\tau\|\pi\|_1^2 \leq$$

$$\leq d_7 + d_8(\tau)\|\theta\|_B^2 + d_1(\tau)\|\pi\|_B^2 + \sum_{\alpha=w,o} d_9\|\sigma_\alpha\|_B^2 + d_{10}\tau\left(\|\psi_T\|_{A^{-1}}^2 + \left\|\psi_p\right\|_{A^{-1}}^2 + \sum_{\alpha=w,o}\|\psi_\alpha\|_{A^{-1}}^2\right). \tag{23}$$

Before proving the theorem, we give the following auxiliary lemmas. Let us state without proof the following lemma, which is based on the results obtained in [6].

Lemma 2 *Let the conditions (7) hold. Then the following inequalities hold:*

$$(\Lambda_1 w,\, w) \geq 4c_0\|w\|_1^2, \quad (\Lambda_2 w - \Lambda_2\tilde{w},\, w - \tilde{w}) \geq c_0\|w - \tilde{w}\|_1^2,$$

$$\tau\left|(\Lambda_2 w - \Lambda_2\tilde{w},\, z)\right| \leq c_1\left(\varepsilon\|w - \tilde{w}\|_1^2 + \frac{\tau^2}{\varepsilon}\|z\|_A^2 + \frac{\tau}{\varepsilon\delta\omega}\|z\|_B^2\right),$$

$$2\tau(\Lambda_1 w, \tilde{w}) \leq c_1\left(\varepsilon\|w\|_1^2 + \frac{\tau^2}{\varepsilon}\|\tilde{w}\|_A^2 + \frac{\tau}{\varepsilon\delta\omega}\|\tilde{w}\|_B^2\right), \quad \varepsilon > 0.$$

Lemma 3 *Let the conditions (7), (21) and $\omega > \omega_1$, where $\omega_1 = \frac{8c_1^2}{c_0} + \frac{c_2}{\varepsilon\delta} + \frac{2c_1}{c_0\delta}$, $\varepsilon > 0$ hold. Then the following inequality holds:*

$$\|\hat{\pi}\|_B^2 + \tau^3(\omega - \omega_1)\|\pi_t\|_A^2 + c_0\tau\|\pi\|_1^2 \leq d_1(\tau)\|\pi\|_B^2 + \frac{2c_2\varepsilon\tau^3}{\delta}\|\theta_t\|_A^2 + \frac{c_0\tau}{c_1}\left\|\psi_p\right\|_{A^{-1}}^2. \tag{24}$$

Proof. Multiply the equation (16) scalarly by $2\tau\hat{\pi}$:

$$\|\hat{\pi}\|_B^2 - \|\pi\|_B^2 + \tau^2\|\pi_t\|_B^2 + 2\tau\left(\Lambda_2 p^h - \Lambda_2\tilde{p}^h,\, \hat{\pi}\right) = 2\tau\left(\beta_T^h\theta_t,\, \hat{\pi}\right) + 2\tau\left(\psi_p,\, \hat{\pi}\right). \tag{25}$$

Using Lemma 2, we obtain:

$$2\tau\left(\Lambda_2 p^h - \Lambda_2\tilde{p}^h,\, \hat{\pi}\right) \geq 2\tau\left(c_0 - \frac{c_1}{\varepsilon_1}\right)\|\pi\|_1^2 - 2c_1\varepsilon_1\tau^3\|\pi_t\|_A^2 - \frac{2c_1\varepsilon_1\tau^2}{\omega\delta}\|\pi_t\|_B^2. \tag{26}$$

Using the condition (21) and Cauchy inequality, the rest of the terms in (24) can be estimated as follows:

$$2\tau\left(\beta_T^h\theta_t, \hat{\pi}\right) \leq \frac{2c_2\varepsilon_2\tau^3}{\delta}\|\theta_t\|_A^2 + \frac{c_2\tau}{\varepsilon_2}\|\pi\|_B^2 + \frac{c_2\tau^2}{\varepsilon_2\omega\delta}\|\pi_t\|_B^2,$$

$$2\tau\left(\psi_p,\,\hat{\pi}\right)=2\tau\left(\psi_p,\,\pi\right)+2\tau^2\left(\psi_p,\,\pi_t\right)\le\varepsilon_1\tau\,\|\pi\|_B^2+\frac{\varepsilon_1\tau^3}{\delta}\,\|\pi_t\|_A^2+\frac{2\tau}{\varepsilon_1}\left\|\psi_p\right\|_{A^{-1}}^2. \tag{27}$$

Taking into account the obtained inequalities, we find from (25) that

$$\|\hat{\pi}\|_B^2+\tau^2\left(1-\frac{2c_1\varepsilon_1}{\omega\delta}-\frac{c_2}{\varepsilon_2\omega\delta}\right)\|\pi_t\|_B^2-\tau^3\left(2c_1\varepsilon_1+\frac{\varepsilon_1}{\delta}\right)\|\pi_t\|_A^2+2\tau\left(c_0-\frac{c_1}{\varepsilon_1}\right)\|\pi\|_1^2\le$$

$$\le\left(1+\varepsilon_1\tau+\frac{c_2\tau}{\varepsilon_2}\right)\|\pi\|_B^2+\frac{2c_2\varepsilon_2\tau^3}{\delta}\,\|\theta_t\|_A^2+\frac{2\tau}{\varepsilon_1}\left\|\psi_p\right\|_{A^{-1}}^2. \tag{28}$$

Choosing $\varepsilon_1=\frac{2c_1}{c_0}$, and ω from the condition $\omega>\omega_1$, we come to the assertion of the lemma.

Lemma 4 *Let the conditions (22), (20) and $\omega>\omega_2$, $\omega_2=\frac{c_1^2(\eta+2)}{8c_0}\left(1+\frac{1}{\delta}\right)+\frac{3c_0\eta}{2\delta(\eta+2)}+\frac{4c_2c_5(\eta+2)}{3\delta c_0\eta}$ hold. Then the following inequalities hold:*

$$\left\|\hat{p}^h\right\|_B^2+\tau^3\left(\omega-\omega_2\right)\left\|p_t^h\right\|_A^2+2\tau d_2\left(\varepsilon_1\right)\left\|p^h\right\|_1^2\le\left\|p^h\right\|_B^2+\frac{2c_2\varepsilon_2\tau^3}{\delta}\left\|T_t^h\right\|_A^2+\frac{6\tau}{c_0}\left\|f_p\right\|_{A^{-1}}^2, \tag{29}$$

$$\left\|\hat{\tilde{p}}^h\right\|_B^2+\tau^3\left(\omega-\omega_2\right)\left\|\tilde{p}_t^h\right\|_A^2+2\tau d_2\left(\varepsilon_1\right)\left\|\tilde{p}^h\right\|_1^2\le\left\|\tilde{p}^h\right\|_B^2+\frac{2c_2\varepsilon_2\tau^3}{\delta}\left\|\tilde{T}_t^h\right\|_A^2+\frac{6\tau}{c_0}\left\|\tilde{f}_p\right\|_{A^{-1}}^2, \tag{30}$$

$$\left\|T^{h,n}\right\|_B^2+\left\|p^{h,n}\right\|_B^2+d_4\tau\left\|p^{h,n}\right\|_1^2\le c_1\eta, \tag{31}$$

$$\left\|\tilde{T}^{h,n}\right\|_B^2+\left\|\tilde{p}^{h,n}\right\|_B^2+d_4\tau\left\|\tilde{p}^{h,n}\right\|_1^2\le c_1\eta, \tag{32}$$

$$2\tau\left(L\left(u^h,T^h\right)-L\left(\tilde{u}^h,\tilde{T}^h\right),\hat{\theta}\right)\le\frac{4c_1^2c_3\varepsilon\eta}{d_4}+\frac{4c_1\eta\tau^2}{\varepsilon}\,\|\theta\|_B^2+\frac{4c_1\eta\tau^3}{\varepsilon\delta}\,\|\theta_t\|_A^2,\quad\varepsilon>0. \tag{33}$$

Proof. Multiplying the equation (10) scalarly by $2\tau\hat{p}^h$, and carrying out calculations analogous to those made in Lemma 3, we obtain the inequality (29) . Now multiply scalarly the equation (9) by $2\tau\hat{T}^h$:

$$\left\|\hat{T}^h\right\|_B^2-\left\|T^h\right\|_B^2+\tau^2\left\|T_t^h\right\|_B^2+2\tau\left(L\left(\vec{u}^h,\,T^h\right),\hat{T}^h\right)+2\tau\left(\Lambda_1T^h,\hat{T}^h\right)=2\tau\left(f_T^h,\hat{T}^h\right). \tag{34}$$

Let us estimate the scalar products in (34). We obtain a chain of inequalities by applying the Cauchy inequality to the term $\zeta_1\equiv2\tau\left(L\left(\vec{u}^h,\,T^h\right),\hat{T}^h\right)$:

$$\zeta_1\le2\tau\sum_{m=1}^{2}\left(\sum_{\Omega_{h,m}^-}\sum_{\bar{\Omega}_{h,3-m}}\left(u_m^{h,+1_m}\right)^2h_1h_2\right)^{\frac{1}{2}}\left(\sum_{\Omega_{h,m}^-}\sum_{\bar{\Omega}_{h,3-m}}\left(T_{x_m}^h\hat{T}^h\right)^2h_1h_2\right)^{\frac{1}{2}}+$$

$$+2\tau\sum_{m=1}^{2}\left(\sum_{\Omega_{h,m}^+}\sum_{\bar{\Omega}_{h,3-m}}\left(u_m^h\right)^2h_1h_2\right)^{\frac{1}{2}}\left(\sum_{\Omega_{h,m}^+}\sum_{\bar{\Omega}_{h,3-m}}\left(T_{\bar{x}_m}^h\hat{T}^h\right)^2h_1h_2\right)^{\frac{1}{2}}\le$$

$$\le2\tau\sum_{m=1}^{2}\left\|u_m^h\right\|_0\left\|\hat{T}^h\right\|_0\left(\sum_{\Omega_{h,m}^-}\sum_{\bar{\Omega}_{h,3-m}}\left(T_{x_m}^h\right)^2h_1h_2\right)^{\frac{1}{2}}+2\tau\sum_{m=1}^{2}\left\|u_m^h\right\|_0\left\|\hat{T}^h\right\|_0\left(\sum_{\Omega_{h,m}^+}\sum_{\bar{\Omega}_{h,3-m}}\left(T_{\bar{x}_m}^h\right)^2h_1h_2\right)^{\frac{1}{2}}\le$$

$$=4\varepsilon_0\sum_{m=1}^{2}\left\|u_m^h\right\|_0^2+\frac{\tau^2}{\varepsilon_0}\left\|\hat{T}^h\right\|_0^2\left\|T^h\right\|_1^2.$$

Using the equations (12), condition (22), and choosing $\varepsilon_0=4\varepsilon_1\omega^{-1}$, we obtain

$$\zeta_1\le4c_1c_3\varepsilon_0\tau\left\|p^h\right\|_1^2+\frac{2\tau^2}{\varepsilon_0}\left\|\hat{T}^h\right\|_1^2\left\|T^h\right\|_A^2-\frac{\tau^2}{\varepsilon_0}\left\|\hat{T}^h\right\|_1^2\left\|T^h\right\|_0^2.$$

Using a technique similar to that used in Lemma 3, we arrive at the following inequality from (34):

$$\left\|\hat{T}^h\right\|_B^2+\tau^2\left(1-\frac{c_1\varepsilon_1}{\omega\delta}-\frac{1}{\omega\delta\varepsilon_2}\right)\left\|T_t^h\right\|_B^2-c_1\varepsilon_1\tau^3\left\|T_t^h\right\|_A^2+\tau\left(4c_0-\frac{c_1}{\varepsilon_1}-\frac{1}{2\varepsilon_1}\left\|T^h\right\|_B^2\right)\left\|\hat{T}\right\|_1^2+$$

$$+\tau\left(4c_0-\frac{c_1}{\varepsilon_1}-\frac{2}{\varepsilon_2}\right)\left\|T^h\right\|_1^2\le\left\|T^h\right\|_B^2+\frac{4c_1c_3\varepsilon_1\tau}{\omega}\left\|p^h\right\|_1^2+2\tau\varepsilon_2\left\|f_T^h\right\|_{A^{-1}}^2.$$

Add the last inequality to the inequality (29) multiplied by some $c_5>0$:

$$\left\|\hat{T}^h\right\|_B^2+c_5\left\|\hat{p}^h\right\|_B^2+\tau^3\left(\omega-\frac{c_1\varepsilon_1}{\delta}-\frac{1}{\delta\varepsilon_2}-c_1\varepsilon_1-\frac{2c_2\varepsilon_2c_5}{\delta}\right)\left\|T_t^h\right\|_A^2+c_5\tau^3\left(\omega-\omega_2\right)\left\|p_t^h\right\|_A^2+$$

$$+\tau\left(4c_0-\frac{c_1}{\varepsilon_1}-\frac{1}{2\varepsilon_1}\left\|T^h\right\|_B^2\right)\left\|\hat{T}\right\|_1^2+\tau\left(4c_0-\frac{c_1}{\varepsilon_1}-\frac{2}{\varepsilon_2}\right)\left\|T^h\right\|_1^2+\tau\left(2c_5d_2\left(\varepsilon_3\right)-\frac{16c_1c_3\varepsilon_1}{\omega}\right)\left\|p^h\right\|_1^2\le$$

$$\le\left\|T^h\right\|_B^2+c_5\left\|p^h\right\|_B^2+2\tau\varepsilon_2\left\|f_T^h\right\|_{A^{-1}}^2+\frac{6\tau c_5}{c_0}\left\|f_p^h\right\|_{A^{-1}}^2.$$

Using the relations (8), it is not difficult to show that

$$d_5\left\|f_T^h\right\|_{A^{-1}}^2+d_6\left\|f_p^h\right\|_{A^{-1}}^2\le c_4\sum_{j=1}^{W_n}\sum_{x\in\bar{\Omega}_h}b_h\left(x\right)\left(p_{inj}^2+T_{inj}^2\right)\delta_h\left(x-x_j^{(w)}\right)h_1h_2=q_0.$$

Let us choose $\varepsilon_1=\frac{c_1(\eta+2)}{8c_0}$, $\varepsilon_2=\frac{2(\eta+2)}{3c_0\eta}$ and the constants c_5 and ε_3 from the condition of non-negativity of the coefficient of $\left\|p^h\right\|_1^2$. Then, under the conditions of the lemma, we arrive at the inequality

$$\left\|\hat{T}^h\right\|_B^2+c_5\left\|\hat{p}^h\right\|_B^2+d_3\tau\left(c_1\eta-\left\|T^h\right\|_B^2\right)\left\|\hat{T}^h\right\|_1^2+d_4\tau\left\|p^h\right\|_1^2\le\left\|T^h\right\|_B^2+c_5\left\|p^h\right\|_B^2+\tau q_0. \tag{35}$$

To prove the lemma, we apply the method of mathematical induction. The inequality (35) for $n'=0$ yields

$$\left\|T^{h,1}\right\|_B^2+c_5\left\|p^{h,1}\right\|_B^2+d_3\tau\left(c_1\eta-\left\|T^{h,0}\right\|_B^2\right)\left\|T^{h,1}\right\|_1^2+d_4\tau\left\|p^{h,0}\right\|_1^2\le\left\|T^{h,0}\right\|_B^2+c_5\left\|p^{h,0}\right\|_B^2+\tau q_0. \tag{36}$$

Since the inequality $c_1\eta-\left\|T^{h,0}\right\|_B^2>0$ holds under the condition (20), then it follows from (20) and (36) that

$$\left\|T^{h,1}\right\|_B^2+c_5\left\|p^{h,1}\right\|_B^2+d_4\tau\left\|p^{h,0}\right\|_1^2\le\left\|T^{h,0}\right\|_B^2+c_5\left\|p^{h,0}\right\|_B^2+\tau q_0\le c_1\eta. \tag{37}$$

Assuming that the inequality (31) holds for $n'=n-2$:

$$\left\|T^{h,n-1}\right\|_B^2+\left\|p^{h,n-1}\right\|_B^2+d_4\tau\left\|p^{h,n-2}\right\|_1^2\le c_1\eta, \tag{38}$$

write the ineqality (35) for $n'=n-1$:

$$\left\|T^{h,n}\right\|_B^2+c_5\left\|p^{h,n}\right\|_B^2+d_3\tau\left(c_1\eta-\left\|T^{h,n-1}\right\|_B^2\right)\left\|T^{h,n}\right\|_1^2+d_4\tau\left\|p^{h,n-1}\right\|_1^2\le\left\|T^{h,n-1}\right\|_B^2+c_5\left\|p^{h,n-1}\right\|_B^2+\tau q_0. \tag{39}$$

By virtue of the fulfillment of the inequality (38), the expression in brackets before $\left\|T^{h,n}\right\|_1^2$ in the left-hand side of (39) is non-negative. Therefore it follows from (39) that

$$\left\|T^{h,n}\right\|_B^2+c_5\left\|p^{h,n}\right\|_B^2+d_3\tau\left(c_1\eta-\left\|T^{h,n-1}\right\|_B^2\right)\left\|T^{h,n}\right\|_1^2+d_4\tau\left\|p^{h,n-1}\right\|_1^2\le$$

$$\le\left\|T^{h,n-1}\right\|_B^2+c_5\left\|p^{h,n-1}\right\|_B^2+\tau q_0\le\left\|T^{h,n-2}\right\|_B^2+c_5\left\|p^{h,n-2}\right\|_B^2+2\tau q_0\le\ldots\le\left\|T^{h,0}\right\|_B^2+c_5\left\|p^{h,0}\right\|_B^2+n\tau q_0\le c_1\eta$$

since $n\tau\le t_1$ holds for $0\le n\le N_t$. Similarly, the inequality (32) is proved.

Using the definition of L and using the Cauchy inequality, it is easy to see that

$$\zeta = 2\tau\left(L\left(u^h, T^h\right) - L\left(\tilde{u}^h, \tilde{T}^h\right), \hat{\theta}\right) \leq 2\varepsilon_1 \sum_{m=1}^{2}\left(\|u_m\|_0^2 + \|\tilde{u}_m\|_0^2\right) + \frac{2\tau^2}{\varepsilon_1}\left\|\hat{\theta}\right\|_0^2\left(\left\|T^h\right\|_1^2 + \left\|\tilde{T}^h\right\|_1^2\right).$$

Using the equations (12) and the condition (22) yields

$$\zeta \leq 2c_1c_3\varepsilon_1\tau\left(\|p\|_1^2 + \|\tilde{p}\|_1^2\right) + \frac{2\tau^2}{\varepsilon_1}\left(\|\theta\|^2 + \tau\|\theta_t\|^2\right)\left(\left\|T^h\right\|_1^2 + \left\|\tilde{T}^h\right\|_1^2\right).$$

Finally, using the inequalities (31) and (32) to estimate the right-hand side of the last inequality, we obtain:

$$\zeta \leq \frac{2c_1c_3\varepsilon_1}{d_4}\cdot 2c_1\eta + \frac{2\tau^2}{\varepsilon_1}\left(\|\theta\|^2 + \tau\|\theta_t\|^2\right)\cdot 2c_1\eta,$$

from which the inequality (33) follows.

Lemma 5 *Let the conditions (7), (20) and* $\omega > \omega_3$, $\omega_3 = \frac{c_1^2}{4c_0}\left(1 + \frac{1}{\delta}\right) + \frac{c_1}{2c_0} + \frac{c_1^2\eta}{2c_0\delta}$ *hold. Then the following inequality holds:*

$$\left\|\hat{\theta}\right\|_B^2 + \tau^3(\omega - \omega_3)\|\theta_t\|_A^2 + 4c_0\tau\|\theta\|_1^2 \leq d_7 + d_8(\tau)\|\theta\|_B^2 + \frac{8c_0\tau}{c_1}\|\psi_T\|_{A^{-1}}^2. \qquad (40)$$

Proof. Multiply the equation (15) scalarly by $2\tau\hat{\theta}$:

$$\left\|\hat{\theta}\right\|_B^2 - \|\theta\|_B^2 + \tau^2\|\theta_t\|_B^2 + 2\tau\left(L\left(\vec{u}^h, T^h\right) - L\left(\tilde{\vec{u}}^h, \tilde{T}^h\right), \hat{\theta}\right) + 2\tau\left(\Lambda_1\theta, \hat{\theta}\right) = 2\tau\left(\psi_T, \hat{\theta}\right). \qquad (41)$$

Estimate the scalar products in (41). Using the technique used above, we obtain

$$2\tau\left(\Lambda_1\theta, \hat{\theta}\right) = 2\tau(\Lambda_1\theta, \theta) + 2\tau^2(\Lambda_1\theta, \theta_t) \geq 8\tau c_0\|\theta\|_1^2 - c_1\tau\left(\varepsilon_1\|\theta\|_1^2 + \frac{\tau^2}{\varepsilon_1}\|\theta_t\|_A^2 + \frac{\tau}{\varepsilon_1\delta\omega}\|\theta_t\|_B^2\right), \qquad (42)$$

$$2\tau\left(\psi_T, \hat{\theta}\right) \leq \frac{\tau}{\varepsilon_1}\|\theta\|_B^2 + \frac{\tau^3}{\varepsilon_1}\|\theta_t\|_A^2 + 2\tau\varepsilon_1\|\psi_T\|_{A^{-1}}^2. \qquad (43)$$

Using the inequalities (42), (43) and (33), we obtain from (41) that

$$\left\|\hat{\theta}\right\|_B^2 + \tau^3\left(\omega - \frac{c_1}{\varepsilon_1\delta} - \frac{2}{\varepsilon_1} - \frac{2c_1\eta}{\varepsilon_1\delta} - \frac{c_1}{\varepsilon_1}\right)\|\theta_t\|_A^2 + \tau(8c_0 - c_1\varepsilon_1)\|\theta\|_1^2 \leq \frac{2c_1^2c_3\varepsilon_1\eta}{d_4} + \left(1 + \frac{2c_1\eta\tau^2}{\varepsilon_1}\right)\|\theta\|_B^2 + 2\tau\varepsilon_1\|\psi_T\|_{A^{-1}}^2.$$

Choosing $\varepsilon_1 = \frac{4c_0}{c_1}$, we obtain the assertion of the lemma.

The following lemma is proved similarly:

Lemma 6 *Let the condition* $\omega > \omega_4$, $\omega_4 = \frac{2\nu_\alpha}{\varepsilon\delta} - \frac{\nu_\alpha}{\varepsilon}$ *hold. The the following inequality holds:*

$$\|\hat{\sigma}_\alpha\|_B^2 + \tau^3(\omega - \omega_4)\left\|\sigma_{\alpha,t}\right\|_A^2 \leq d_9\|\sigma_\alpha\|_B^2 + 2\tau\nu_\alpha\varepsilon\|\pi\|_1^2 + \frac{2\tau}{\varepsilon}\|\psi_\alpha\|_{A^{-1}}^2, \ \varepsilon > 0.$$

Now let us prove Theorem 1. Combining the results of Lemma 3, 5 and 6, we arrive at the following inequality after obvious transformations:

$$\left\|\hat{\theta}\right\|_B^2 + \|\hat{\pi}\|_B^2 + \sum_{\alpha=w,o}\|\hat{\sigma}_\alpha\|_B^2 + \tau^3\left(\omega - \omega_4 - \frac{2c_2\varepsilon_6}{\delta}\right)\|\theta_t\|_A^2 + \tau^3(\omega - \omega_1)\|\pi_t\|_A^2 +$$

$$+\sum_{\alpha=w,o}\tau^3(\omega - \omega_5)\left\|\sigma_{\alpha,t}\right\|_A^2 + 4c_0\tau\|\theta\|_1^2 + \tau\left(c_0 - \sum_{\alpha=w,o}4\nu_\alpha\varepsilon_7\right)\|\pi\|_1^2 \leq d_7 + d_8(\tau)\|\theta\|_B^2 +$$

$$+d_1(\tau)\|\pi\|_B^2 + \sum_{\alpha=w,o}d_9\|\sigma_\alpha\|_B^2 + \frac{8c_0\tau}{c_1}\|\psi_T\|_{A^{-1}}^2 + \frac{c_0\tau}{c_1}\left\|\psi_p\right\|_{A^{-1}}^2 + \sum_{\alpha=w,o}\frac{2\tau}{\varepsilon_7}\|\psi_\alpha\|_{A^{-1}}^2.$$

Assuming that the conditions of the Theorem 1 are satisfied, and choosing ε_7 from the condition $\varepsilon_7 < \frac{c_0}{4\nu_\alpha}$, we arrive at the assertion of the Theorem 1.

CONSTRUCTION OF COST-EFFECTIVE DIFFERENCE SCHEMES

Consider the following difference scheme

$$\tilde{B}T_t^h + L\left(\vec{u}^h,\, T^h\right) + \Lambda_1 T^h = f_T^h, \tag{44}$$

$$\tilde{B}p_t^h + \Lambda_2 p^h = \beta_T^h T_t^h + f_p^h, \tag{45}$$

$$\tilde{B}s_{\alpha,t}^h + \Lambda_{3\alpha} p^h = f_\alpha^h, \quad \alpha = w,\, o, \tag{46}$$

$$u_m^h = -k\lambda^h p_{\bar{x}_m}^h, \quad m = 1,\, 2, \tag{47}$$

$$T^h(0) = T_0, \quad p^h(0) = p_0, \quad s_\alpha^h(0) = s_{\alpha 0} \tag{48}$$

with some self-adjoint operator $\tilde{B}$ satisfying the condition $\tilde{B} \geq B$. It is known from the theory of two-layer difference schemes that if the difference scheme (9)-(13) is stable with respect to the initial data and the right-hand sides of the equations, then the scheme (44)-(48) also has the same property. Using this statement, we construct three cost-effective difference schemes for the problem (1)-(6) by choosing the operator $\tilde{B}$.

Factorization scheme

Let us take $\tilde{B} = (E + \tau\omega A_1)(E + \tau\omega A_2)$, where the operators A_m are defined as $A_m = A_m^+ + A_m^-$ [7],

$$A_m^+ w = \left\{h_m^{-1} w + h_m^{-2} w,\ x_m = 0;\ h_m^{-1} w_{\bar{x}_m},\ x_m \in \Omega_{h,m};\ h_m^{-2} w + h_m^{-1} w - 2h_m^{-2} w^{-1_m},\ x_m = 1\right\},$$

$$A_m^- w = \left\{h_m^{-2} w + h_m^{-1} w - 2h_m^{-2} w^{+1_m},\ x_m = 0;\ -h_m^{-1} w_{x_m},\ x_m \in \Omega_{h,m};\ h_m^{-1} w + h_m^{-2} w,\ x_m = 1\right\}.$$

It can be seen by direct verification that $(A_m^+ w,\, \tilde{w}) = (A_m^- \tilde{w},\, w)$, i. e. the operators A_m^+, A_m^- are mutually conjugate. In this case the operator A decomposes into the sum of two operators $A = A^+ + A^-$, where $A^\pm = A_1^\pm + A_2^\pm$. With this choice of the operator $\tilde{B}$, the factorization scheme is written as follows:

$$(E + \tau\omega A^+)(E + \tau\omega A^-)\, T_t^{h,n} + L\left(\vec{u}^{h,n}, T^{h,n}\right) + \Lambda_1 T^{h,n} = f_T^{h,n}, \tag{49}$$

$$(E + \tau\omega A^+)(E + \tau\omega A^-)\, p_t^{h,n} + \Lambda_2 p^{h,n} = \beta_T^h T_t^{h,n} + f_p^{h,n}, \tag{50}$$

$$(E + \tau\omega A^+)(E + \tau\omega A^-)\, s_{\alpha,t}^{h,n} + \Lambda_{3\alpha} p^{h,n} = f_\alpha^{h,n}, \tag{51}$$

$$u_m^h = -k\lambda^h p_{\bar{x}_m}^{h,n}, \quad m = 1,\, 2, \tag{52}$$

$$T^{h,0} = T_0, \quad p^{h,0} = p_0, \quad s_\alpha^{h,0} = s_{\alpha 0}. \tag{53}$$

It is easy to verify [7] that the operator $\tilde{B} = (E + \tau\omega A^+)(E + \tau\omega A^-)$ is self-adjoint and the operator inequality $\tilde{B} \geq B$ holds. Thus, the difference scheme (49)-(53) is stable with respect to the initial data and the right-hand sides of the equations under the condition $\omega \geq \omega_0$. Construct the following algorithm to implement the scheme (49)-(53):

$$(E + \tau\omega A^+)\, \theta^{n+\frac{1}{2}} + L\left(\vec{u}^{h,n}, T^{h,n}\right) + \Lambda_1 T^{h,n} = f_T^{h,n+\frac{1}{2}}, \tag{54}$$

$$(E + \tau\omega A^-)\, \theta^{n+1} = \theta^{n+\frac{1}{2}}, \quad T^{h,n+1} = T^{h,n} + \tau\theta^{n+1}, \tag{55}$$

$$(E + \tau\omega A^+)\, \pi^{n+\frac{1}{2}} + \Lambda_2 p^{h,n} = \beta_T^h T_t^{h,h} + f_p^{h,n+\frac{1}{2}}, \tag{56}$$

$$(E + \tau\omega A^-)\, \pi^{n+1} = \pi^{n+\frac{1}{2}}, \quad p^{n+1} = p^n + \tau\pi^{n+1}, \tag{57}$$

$$(E + \tau\omega A^+)\, \sigma_\alpha^{n+\frac{1}{2}} + \Lambda_{3\alpha} s_\alpha^{h,n} = f_\alpha^{h,n+\frac{1}{2}}, \tag{58}$$

$$(E + \tau\omega A^-)\, \sigma_\alpha^{n+1} = \sigma_\alpha^{n+\frac{1}{2}}, \quad s_\alpha^{h,n+1} = s_\alpha^{h,n} + \tau\sigma_\alpha^{n+1}, \tag{59}$$

$$u_m^{h,n+1} = -k\lambda^h p_{\bar{x}_m}^{h,n+1}, \quad m = 1,\, 2, \tag{60}$$

where $\theta^{n+\frac{1}{2}}$, θ^{n+1}, $\pi^{n+\frac{1}{2}}$, π^{n+1}, $\sigma_\alpha^{n+\frac{1}{2}}$, σ_α^{n+1} are auxiliary functions. Determination of auxiliary functions and solutions by algorithm (54)-(54) is carried out by explicit formulas, therefore the difference scheme (49)-(53) is cost-effective.

Alternating directions scheme

Choosing the operator $\tilde{B}$ in the form $\tilde{B} = (E + \tau\omega A_1)(E + \tau\omega A_2)$, we arrive at the scheme

$$(E + \tau\omega A_1)(E + \tau\omega A_2)\, T_t^{h,n} + L\left(\vec{u}^{h,n},\, T^{h,n}\right) + \Lambda_1 T^{h,n} = f_T^{h,n+\frac{1}{2}}, \tag{61}$$

$$(E + \tau\omega A_1)(E + \tau\omega A_2)\, p_t^{h,n} + \Lambda_2 p^{h,n} = f_p^{h,n+\frac{1}{2}}, \tag{62}$$

$$(E + \tau\omega A_1)(E + \tau\omega A_2)\, s_{\alpha,t}^{h,n} + \Lambda_{3\alpha} p^{h,n} = f_\alpha^{h,n+\frac{1}{2}}, \quad \alpha = w,\, o, \tag{63}$$

$$u_m^{h,n} = -k\lambda^h p_{\bar{x}_m}^{h,n}, \quad m = 1,\, 2, \tag{64}$$

$$T^{h,0} = T_0, \quad p^{h,0} = p_0, \quad s_\alpha^{h,0} = s_{\alpha 0}, \tag{65}$$

where the operators A_m are defined in (14). In this case $\tilde{B} = \tilde{B}^* \geq B$, since the operators A_1 and A_2 are self-adjoint and commute. Thus, the scheme (61)-(65) is stable with respect to the initial data and the right-hand sides of the equations under the condition $\omega \geq \omega_0$. Construct the following algorithm for the numerical implementation of the scheme:

$$(E + \tau\omega A_1)\, T^{h,n+\frac{1}{3}} + L\left(\vec{u}^{h,n},\, T^{h,n}\right) + \Lambda_1 T^{h,n} = f_T^h, \tag{66}$$

$$(E + \tau\omega A_2)\, T^{h,n+\frac{2}{3}} = T^{h,n+\frac{1}{3}}, \quad T^{h,n+1} = T^{h,n} + \tau T^{h,n+\frac{2}{3}}, \tag{67}$$

$$(E + \tau\omega A_1)\, p^{h,n+\frac{1}{3}} + \Lambda_2 p^{h,n} = f_p^h, \tag{68}$$

$$(E + \tau\omega A_2)\, p^{h,n+\frac{2}{3}} = p^{h,n+\frac{1}{3}}, \quad p^{h,n+1} = p^{h,n} + \tau p^{h,n+\frac{2}{3}}, \tag{69}$$

$$(E + \tau\omega A_1)\, s_\alpha^{h,n+\frac{1}{3}} + \Lambda_{3\alpha} s_\alpha^{h,n} = f_\alpha^h, \tag{70}$$

$$(E + \tau\omega A_2)\, s_\alpha^{h,n+\frac{2}{3}} = s_\alpha^{h,n+\frac{1}{3}}, \quad s_\alpha^{h,n+1} = s_\alpha^{h,n} + \tau s_\alpha^{h,n+\frac{2}{3}}, \tag{71}$$

$$u_m^{h,n+1} = -k\lambda^h p_{\bar{x}_m}^{h,n+1}. \tag{72}$$

Auxiliary functions $T^{h,n+\frac{1}{3}}$, $T^{h,n+\frac{2}{3}}$, $p^{h,n+\frac{1}{3}}$, $p^{h,n+\frac{2}{3}}$, $s_\alpha^{h,n+\frac{1}{3}}$, $s_\alpha^{h,n+\frac{2}{3}}$ are defined from the equations (66)-(71) by the scalar sweep method, and the solution $T^{h,n+1}$, $p^{h,n+1}$, $s_\alpha^{h,n+1}$ on the $(n+1)$-th layer is determed by explicit formulas. Therefore, the scheme (61)-(65) is cost-effective.

Stabilizing correction scheme

Introduce the intermediate solutions $T^{h,n+\frac{1}{2}}$, $p^{h,n+\frac{1}{2}}$, $s_\alpha^{h,n+\frac{1}{2}}$ instead of $\theta^{n+\frac{1}{2}}$, $\pi^{n+\frac{1}{2}}$, $\sigma_\alpha^{n+\frac{1}{2}}$ in the difference scheme (49)-(52) by formulas

$$\theta^{n+\frac{1}{2}} = \frac{T^{h,n+\frac{1}{2}} - T^{h,n}}{\tau}, \quad \pi^{n+\frac{1}{2}} = \frac{p^{h,n+\frac{1}{2}} - p^{h,n}}{\tau}, \quad \sigma_\alpha^{n+\frac{1}{2}} = \frac{s_\alpha^{h,n+\frac{1}{2}} - s_\alpha^{h,n}}{\tau}.$$

Then the algorithm (54)-(60) is replaced with the following one:

$$\frac{T^{h,n+\frac{1}{2}} - T^{h,n}}{\tau} + \omega A^+\left(T^{h,n+\frac{1}{2}} - T^{h,n}\right) + L\left(\vec{u}^{h,n}, T^{h,n}\right) + \Lambda_1 T^{h,n} = f_T^{h,n+\frac{1}{2}}, \tag{73}$$

$$\frac{T^{h,n+1} - T^{h,n}}{\tau} + \omega A^-\left(T^{h,n+1} - T^{h,n}\right) = \frac{T^{h,n+\frac{1}{2}} - T^{h,n}}{\tau}, \tag{74}$$

$$\frac{p^{h,n+\frac{1}{2}} - p^{h,n}}{\tau} + \omega A^+\left(p^{h,n+\frac{1}{2}} - p^{h,n}\right) + \Lambda_2 p^{h,n} = \beta_T^h T_t^{h,h} + f_p^{h,n+\frac{1}{2}}, \tag{75}$$

$$\frac{p^{h,n+1} - p^{h,n}}{\tau} + \omega A^-\left(p^{h,n+1} - p^{h,n}\right) = \frac{p^{h,n+\frac{1}{2}} - p^{h,n}}{\tau}, \tag{76}$$

$$\frac{s_\alpha^{h,n+\frac{1}{2}} - s_\alpha^{h,n}}{\tau} + \omega A^+\left(s_\alpha^{h,n+\frac{1}{2}} - s_\alpha^{h,n}\right) + \Lambda_{3\alpha} s_\alpha^{h,n} = f_\alpha^{h,n+\frac{1}{2}}, \tag{77}$$

$$(E + \tau\omega A^-) \frac{s_\alpha^{h,n+1} - s_\alpha^{h,n}}{\tau} = \frac{s_\alpha^{h,n+\frac{1}{2}} - s_\alpha^{h,n}}{\tau}, \tag{78}$$

$$u_m^{h,n+1} = -k\lambda^h p_{\bar{x}_m}^{h,n+1}, \quad m = 1, 2. \tag{79}$$

The equations (73)-(78) are solved by the scalar sweep method, and the values $u_m^{h,n+1}$ are determined by explicit formulas, therefore the algorithm (73)-(79) is cost-effective.

Results of the computational experiments

Let us analyze the effectiveness of the proposed algorithms by comparing the mean time spent on computing one time layer. Calculations were carried out on a uniform grid. The grid step was varied between 0.02 and 0.002, and the time step was chosen equal to $\tau = 0.001$. Calculations for each of the three algorithms were performed until $t = 1000\tau$. The results of the analysis are given in Table 1.

TABLE 1. Mean time (in milliseconds) required to calculate one time layer

Algorithm / Grid step	51x51	101x101	201x201	501x501
Factorization scheme	22	86.4	354.2	2213.8
Alternating directions scheme	24.9	98.3	396	2481.2
Stabilizing correction scheme	26	103.1	407.3	2553.1

It can be seen that the differences in time are minor, however, since calculations by the factorization scheme are carried out using explicit formulas, this algorithm requires less time to calculate one time layer.

CONCLUSION

Thus, the following results were obtained in the paper. A two-layer finite-difference scheme is constructed for the model two-dimensional three-phase non-isothermal flow problem without allowance for capillary forces. Using the method of energy inequalities, an a priori estimate is obtained for the solution of the difference scheme which proves the stability of the scheme with respect to the initial data and the right-hand sides of the equations. On the basis of the studied difference scheme, three cost-effective difference schemes are constructed. The efficiency of the schemes is verified by comparing the average time spent on calculating one time layer. The results obtained in this paper can be used to carry out further research on the numerical solving the problems of the theory of multiphase non-isothermal flows using difference methods.

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