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An Iterative Method for Solving Nonlinear Navier-Stokes Equations in Complex Domains Taking into Account Boundary Conditions with Uniform Accuracy

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Abstract. In this paper, the Navier-Stokes equations describing the motion of viscous incompressible fluid in a bounded domain is considered. Method of fictitious domains is applied for approximate solution of the problem taking into account boundary conditions with uniform accuracy.

FORMULATION OF THE PROBLEM

In a bounded domain $\Omega \subset \mathbb{R}^2$, we consider the initial-boundary value problem for the non-stationary flow of a viscous incompressible fluid. The problem reduces to solving a system of nonlinear Navier-Stokes equations [1]

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)\, v = \mu \Delta v - \nabla p + f, \tag{1}$$

$$\operatorname{div} v = 0, \tag{2}$$

$$v\Big|_{t=0} = v_0\,(x)\,, \quad v\Big|_S = 0. \tag{3}$$

For simplicity, we assume $v_0\,(x) = 0$. The auxiliary problem corresponding to the method of fictitious domains reduces to solving a system of differential equations in $D = D_1 \cup \Omega$ [2]

$$\frac{\partial v^\varepsilon}{\partial t} + (v^\varepsilon \cdot \nabla)\, v^\varepsilon = \operatorname{div}\,(\mu^\varepsilon \nabla v^\varepsilon) - \nabla p^\varepsilon + f, \tag{4}$$

$$\operatorname{div} v^\varepsilon = 0, \tag{5}$$

$$v^\varepsilon\Big|_{t=0} = 0, \quad v^\varepsilon \cdot \tau\Big|_{S_1} = 0, \quad p^\varepsilon\Big|_{S_1} = 0, \tag{6}$$

$$\mu^\varepsilon = \begin{cases} \mu \text{ in } \Omega, \\ \frac{\mu}{\varepsilon} \text{ in } D_1, \end{cases}$$

$$\left[(\mu^\varepsilon \nabla v^\varepsilon - p^\varepsilon \cdot \delta)\, n\right]\Big|_S = 0, \quad [v^\varepsilon]\Big|_S = 0. \tag{7}$$

Here, n and τ are the normal and tangent vector to the boundary S_1, f is continued in D_1 with the preservation of the norm in $L_2\,(\Omega)$.

We introduce the set of infinitely differentiable vector-valued functions $v\,(x)$ solenoidal in D with tangential components vanishing on S:

$$M\,(D) = \{v\,(x) \in C^\infty\,(D)\,, \;\; \operatorname{div} v = 0, \;\; v \cdot \tau = 0, \;\; x \in S\}\,,$$

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where τ is the tangent vector to the boundary S. The spaces obtained by the closure of $M(D)$ in the norms in $L_2(D)$ and $\mathring{W}_2^1(D)$ are denoted by $V(D)$ and $V_1(D)$, respectively, and their conjugate spaces by $V^*(D)$ and $V_1^*(D)$, and $V(D)$ and $V^*(D)$ are identified.

Definition 1 *A generalized solution of problem (4)-(7) is a function $v^\varepsilon \in L_2(0,T;\,V_1(D)) \cap L_\infty(0,T;\,L_2(D))$ satisfying the integral identity*

$$-\int_0^T (v^\varepsilon, \Phi_t)_D\,dt - \int_0^T ((v^\varepsilon\cdot\nabla)\,\Phi, v^\varepsilon)_D\,dt + \int_0^T\int_{S_1}(v^\varepsilon\cdot\Phi)\,v^\varepsilon\cdot n\,ds\,dt$$

$$+\frac{\mu}{\varepsilon}\int_0^T\int_{S_1} k(x)\,(v\cdot\Phi)\,ds\,dt + \int_0^T(\mu^\varepsilon\nabla v^\varepsilon\cdot\nabla\Phi)_D\,dt = \int_0^T (f\cdot\Phi)_D\,dt \tag{8}$$

for any $\Phi \in C^1(0,T;\,V_1(D))$, $\Phi(T) = 0$, $(u,v)_D = \int_D u\cdot v\,dx$. It is assumed that $k(x)$ is a non-negative function.

Let $\omega_1, \omega_2, ..., \omega_N$ is a arbitrary basis in $V_1(D)$, and v_N^ε is an approximate solution of the problem (4)-(7):

$$v_N^\varepsilon = \sum_{m=1}^N \alpha_{Nm}(t)\,\omega_m, \tag{9}$$

$\alpha_{Nm}(t)$ is found from the system of ordinary differential equations

$$\frac{d}{dt}\left(v_N^\varepsilon, \omega_j\right)_D + \left(\left(v_N^\varepsilon\cdot\nabla\right)v_N^\varepsilon, \omega_j\right)_D + \frac{\mu}{\varepsilon}\int_{S_1} k(x)\cdot\left(v_N^\varepsilon,\omega_j\right)_D ds$$

$$+\left(\mu^\varepsilon\nabla v_N^\varepsilon, \omega_j\right)_D = \left(f, \omega_j\right)_D, \quad j = 1, 2, \cdots, N, \tag{10}$$

$$v_N^\varepsilon(t)\Big|_{t=0} = 0, \quad \alpha_{Nm}(t)\Big|_{t=0} = 0, \quad m = 1, 2, \cdots, N. \tag{11}$$

The solvability of (10)-(11) in a small time is known from the general theory of ordinary differential equations [3]. Global solvability follows from a priori estimates of the solution

$$\max_{0\le t\le T}\left\|v_N^\varepsilon(t)\right\|_{V(D)} \le C < \infty \tag{12}$$

which is obtained from system (10).

The following convergence theorem holds [1].

Theorem 1 *Let $f(t) \in L_2(0,T;\,V_1(D))$, and ε satisfies the condition*

$$\frac{\mu}{2\varepsilon} - C_0\int_0^T \|f(t)\|_{V_1^*(D)}\,dt \ge 0. \tag{13}$$

Then there exists at least one generalized solution of problem (4)-(7), and the following estimate holds for the solution

$$\max_{0\le t\le T}\left\|v_N^\varepsilon(t)\right\|^2_{L_2(D)} + \int_0^T\left\|\nabla v_N^\varepsilon(t)\right\|^2_\Omega dt + \frac{1}{\varepsilon}\int_0^T\left\|\nabla v_N^\varepsilon(t)\right\|^2_{D_1} dt$$

$$+\frac{1}{\varepsilon}\int_0^T\int_{S_1} k(x)\left|v_N^\varepsilon(t)\right|^2 ds\,dt \le C\int_0^T \|f(t)\|^2_{V_1^*(D)}\,dt \le C < \infty. \tag{14}$$

In addition, the solution of problem (4)-(7) converges to the solution of problem (1)-(3).

Next, a difference scheme of the second order of approximation is constructed for the problem (4)-(7). For a numerical solution of this difference problem, a special iterative method is constructed that determines approximate solutions on the boundary with uniform accuracy for a limited number of arithmetic operations.

To develop a new numerical implementation algorithm, the idea of the fictitious unknowns method with a two-step iterative process [4] and a method for solving the Poisson difference equation in a square with the right-hand side different from zero only at nodes that are a distance of the order of the grid distance from a given piecewise smooth curve are used [5].

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