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Numerical solution of an optimal control problem governed by three-phase non-isothermal flow equations

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Abstract. The paper focuses on the numerical implementation of a model optimal control problem governed by equations of three-phase non-isothermal flow in porous media. The objective is to achieve preassigned temperature distribution along the reservoir at a given time of development by controlling mass flow rate of heat transfer agent on the injection well. The problem of optimal control is formulated, the adjoint problem is presented, and an algorithm for the numerical solution is proposed. Results of computational experiments are presented for a test problem.

Keywords: Three-phase non-isothermal flow, Optimal control, Finite difference method, Numerical results
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INTRODUCTION

Optimal control problems governed by models describing multi-phase flows in porous media have received significant attention in applied mathematics recently. For example, in [1] an optimal control problem related to the maximization of the amount of trapped CO_2 after a finite time of injection is studied. The reference [2] studies optimal control for a system of Stokes equations coupled with a transport equation for the viscosity. In [3] an optimal control problem for the interface in a two-dimensional multi-phase fluid problem is considered. In [4] the problem of determining the rate of water injected into the reservoir to achieve the desired hydrodynamic state of the oil pattern was studied. In [5] numerical solution of the problem of identifying optimal pressure of steam on the injection well to achieve a predefined temperature along the reservoir was studied. The technique of solving the problem was based on the penalty method [6], and the gradient descent method was used to minimize the resulting extended ε-functional.

In this paper, a one-dimensional optimal control problem for three-phase non-isothermal flows in porous media is studied. The objective of the study is to approximate the temperature of the oil reservoir to the preassigned temperature distribution at a specified time of development by controlling mass flow rate of a heat transfer agent on the injection well. This is achieved by minimizing the cost functional, which expresses the deviation of the temperature distribution and a given function. To minimize the functional, the gradient projection method is used. We formulate the problem of optimal control, present the adjoint problem, and an algorithm for the numerical solution. Finally, results of computational experiments are presented for a test problem.

FORMULATION OF THE PROBLEM

We consider one-dimensional three-phase non-isothermal flow in porous media in a segment $\Omega = [0, 1]$. The model describing this process consists of conservation mass equation, generalized Darcy's law, energy equation, balance equation for saturations and fluid state equation [7]. Using the approach proposed in [8], the equations of the model can be rewritten as follows with minor simplifications with respect to the physical data

$$c_T T_t + u T_x - k_h T_{xx} = cq(x,t), \tag{1}$$

$$\beta p_t - (k_p p_x)_x = q(x,t), \tag{2}$$

$$s_{w,t} - (\nu_w p_x)_x = q_w, \;\; s_{g,t} - (\nu_g p_x)_x = q(x,t), \tag{3}$$

$$u = \rho_w c_w u_w + \rho_o c_o u_o + \rho_g c_g u_g, \tag{4}$$

where subscripts w, o, g denote the phases of water, oil, and heat transfer agent respectively, T is temperature, p is global pressure, s_α, ρ_α, c_α, u_α are the saturation, density, specific heat capacity and velocity of the phase α

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respectively, $s_o = 1 - s_w - s_g$, c_T, β, k_p, ν_α are some functions of $x \in \Omega$ and time, k_h, q_w, c are positive constants, $t \in [0, t_1]$ for $t_1 > 0$. The system (1)-(4) is complemented by boundary and initial conditions

$$p_x(0,t) = p_x(1,t) = 0,\ k_h T_x(0,t) = -k_h T_x(1,t) = 0, \tag{5}$$

$$p(x,0) = p_0,\ T(x,0) = T_0,\ s_\alpha(x,0) = s_{\alpha 0}. \tag{6}$$

It is required to minimize the functional

$$J(q) = \int_0^1 |T(x,t_1,q) - \varphi(x)|^2 dx, \tag{7}$$

where $T(x,t_1,q)$ is the solution of the problem (1)-(6) corresponding to the mass flow rate q at final time $t = t_1$, and φ is a given function. We assume that the following conditions hold for input data

$$c_* \le (\rho_\alpha, c_\alpha, \beta, c_T, k_p, \nu_\alpha) \le c^*. \tag{8}$$

The control $q(x,t)$ belongs to the set U consisting of functions $q \in L_2(Q)$, $Q = [0, 1] \times [0, t_1]$ such that

$$\|q\|^2 = \iint_Q |q(x,t)|^2 dx\,dt \le M,\ M > 0. \tag{9}$$

For brevity, we introduce the notation $\psi[x,t,q] = (T(x,t,q), p(x,t,q), s_w(x,t,q), s_g(x,t,q))$. The solution $\psi[x,t,q]$ of the problem (1)-(6) is uniquely determined for each fixed control q.

THE METHOD OF SOLVING THE PROBLEM

To minimize the functional J, the gradient method is used in the present paper. Derivation of the gradient of functional J is close to the presentation given in [9]. Let us take arbitrary controls $q \in U$, $q + \tilde{q} \in U$. It follows from (1)-(6) that $\tilde{\psi} = \left(\tilde{T}, \tilde{p}, \tilde{s}_w, \tilde{s}_g\right) = \psi[x, t, q + \tilde{q}] - \psi[x, t, q]$ is a solution of the problem

$$c_T \tilde{T}_t + u\tilde{T}_x - k_h \tilde{T}_{xx} = c\tilde{q}, \tag{10}$$

$$\beta \tilde{p}_t - (k_p \tilde{p}_x)_x = \tilde{q}, \tag{11}$$

$$\tilde{s}_{w,t} - (\nu_w \tilde{p}_x)_x = 0,\ \tilde{s}_{g,t} - (\nu_g \tilde{p}_x)_x = \tilde{q}, \tag{12}$$

$$\tilde{p}_x(0,t) = \tilde{p}_x(1,t) = 0,\ \tilde{T}_x(0,t) = \tilde{T}_x(1,t) = 0, \tag{13}$$

$$\tilde{p}(x,0) = \tilde{T}(x,0) = \tilde{s}_\alpha(x,0) = 0. \tag{14}$$

Then, the increment of the functional (7) can be expressed as follows

$$\tilde{J} = J(q+\tilde{q}) - J(q) = 2\int_0^1 (T(x,t_1,q) - \varphi(x))\,\tilde{T}(x,t_1)\,dx + \int_0^1 \left|\tilde{T}(x,t_1)\right|^2 dx. \tag{15}$$

Using the technique presented in [9], one can easily show that the first term on the right-hand side of (15) can be expressed in the form

$$2\int_0^1 (T(x,t_1,q) - \varphi(x))\,\tilde{T}(x,t_1)\,dx = \int_0^{t_1}\int_0^1 c\theta\tilde{q}\,dx\,dt, \tag{16}$$

where $\theta = \theta(x,t,q)$ is determined from the solution of the auxiliary problem

$$c_T \theta_t - (u\theta)_x + k_h \theta_{xx} = 0, \tag{17}$$

$$\beta \pi_t + (k_p \pi_x)_x = 0 \tag{18}$$

with final and boundary conditions

$$k_h \theta_x(0,t) = -k_h \theta_x(1,t) = 0,\ k_p \pi_x(0,t) = -k_p \pi_x(1,t) = 0, \tag{19}$$

$$\theta(x,t_1) = 2c_T^{-1}(T(x,t_1,q) - \varphi(x)), \;\; \pi(x,t_1) = 0. \tag{20}$$

Indeed, using (10), (17) and (20) we have

$$2\int_0^1 (T(x,t_1,q) - \varphi(x))\tilde{T}(x,t_1)\,dx = \int_0^1 c_T\theta(x,t_1)\tilde{T}(x,t_1)\,dx$$

$$= \int_0^{t_1}\int_0^1 c_T\left(\tilde{T}(x,t)\frac{\partial}{\partial t}\theta(x,t) + \theta(x,t)\frac{\partial}{\partial t}\tilde{T}(x,t)\right)dx\,dt$$

$$= \int_0^{t_1}\int_0^1 \left(\tilde{T}\frac{\partial}{\partial x}(\theta u) - k_h\tilde{T}\frac{\partial^2\theta}{\partial x^2} + \theta u\frac{\partial\tilde{T}}{\partial x} + k_h\theta\frac{\partial^2\tilde{T}}{\partial x^2} + c\theta\tilde{q}\right)dx\,dt$$

$$= \int_0^{t_1}\int_0^1 \left(\tilde{T}\frac{\partial}{\partial x}(\theta u) + \theta u\frac{\partial\tilde{T}}{\partial x}\right)dx\,dt + \int_0^{t_1}\int_0^1 c\theta\tilde{q}\,dx\,dt.$$

The first integral vanishes after integration by parts, therefore, the desired relation (16) holds.

Substituting (16) into (15), we obtain

$$\tilde{J} = \int_0^{t_1}\int_0^1 c\theta\tilde{q}\,dx\,dt + \int_0^1 \left|\tilde{T}(x,t_1)\right|^2 dx. \tag{21}$$

Let us evaluate the last term in the right-hand side of (21). To this end, we multiply the equation (2) by p, then by p_{xx}, and then integrate the obtained equations over the rectangle Q. Using integration by parts and the Cauchy inequality, under conditions (8), we obtain

$$\frac{1}{2}\frac{d}{dt}\|p\|^2 + M_1\|p_x\|^2 \le M_2\|q\|^2, \tag{22}$$

$$\frac{1}{2}\frac{d}{dt}\|p_x\|^2 + M_3\|p_{xx}\|^2 \le M_4\|q\|^2 \tag{23}$$

where M_i denote some positive numbers. Using the Darcy's law and considering relations (22), (23), we have

$$\|u\|^2 + \|u_x\|^2 \le M_5\|q\|^2.$$

Further, we multiply the equation (10) by $\tilde{T}$, integrate the resulting equation over Q, and using the conditions (8), (9), we obtain

$$\int_0^1 \left|\tilde{T}(x,t_1)\right|^2 dx + \frac{c_*}{2}\left\|\tilde{T}_x\right\|^2 \le \int_0^{t_1}\int_0^1 |u|\,dx\,dt \int_0^{t_1}\int_0^1 \left|\tilde{T}\tilde{T}_x\right| dx\,dt$$

$$+c\int_0^{t_1}\int_0^1 \tilde{q}\tilde{T}\,dx\,dt \le M_6\left\|\tilde{T}_x\right\|^2 + M_7\|\tilde{q}\|^2,$$

and therefore,

$$\int_0^1 \left|\tilde{T}(x,t_1)\right|^2 dx + M_8\left\|\tilde{T}_x\right\|^2 \le M_7\|\tilde{q}\|^2.$$

Thus,

$$J(q+\tilde{q}) - J(q) = \int_0^{t_1}\int_0^1 c\theta\tilde{q}\,dx\,dt + C\|\tilde{q}\|^2,$$

where $C > 0$ does not depend on the choice of $q \in U$ and $q+\tilde{q} \in U$. It follows from this view that the value of the gradient of functional J at q has the form

$$J'(q) = c\theta(x,t), \tag{24}$$

where θ is determined from (17)-(20).

In the present paper, the gradient projection method is used for the numerical solution of the problem (1)-(7). The idea of the method is based on constructing a sequence $\{q_k\}$ according to the rule

$$q_{k+1} = P_U\left(q_k - \alpha_k J'(q_k)\right), \;\; k = 0, 1, ..., \;\; \alpha_k > 0,$$

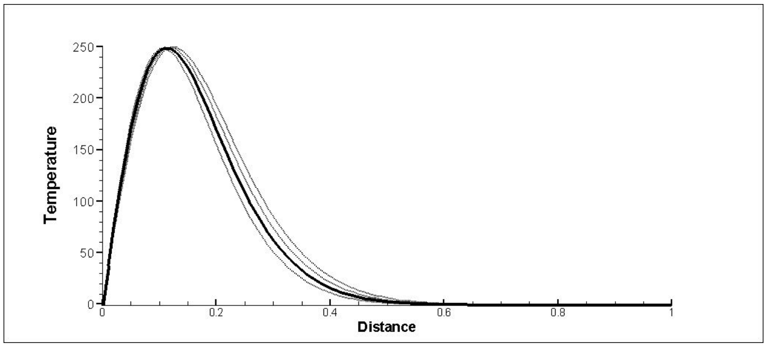

FIGURE 1. Reference temperature (solid line) and solutions of the optimization problem (dotted lines)

where $P_U(q)$ is the projection of the point q on the set U. For the problem (1)-(7), it reduces to constructing a sequence by formulas [9]

$$q_{k+1} = q_k - c\alpha_k\theta(x,t,q_k) \ if \ \iint_Q |q_k - c\alpha_k\theta(x,t,q_k)|^2 dxdt \le M \tag{25}$$

or

$$q_{k+1} = \frac{\sqrt{M}(q_k - c\alpha_k\theta(x,t,q_k))}{\left(\iint_Q |q_k - c\alpha_k\theta(x,t,q_k)|^2 dxdt\right)^{1/2}} \ if \ \iint_Q |q_k - c\alpha_k\theta(x,t,q_k)|^2 dxdt > M, \tag{26}$$

where α_k is chosen from the condition

$$\zeta_k(\alpha_k) = \inf_{\alpha\ge 0} \zeta_k(\alpha), \ \ \zeta_k(\alpha) = J\left(P_U\left(q_k - \alpha J'(q_k)\right)\right).$$

The numerical implementation of the problem is defined as follows. Let us set an initial control $q = q_0$. The iterative process is started from solving the direct problem (1)-(6). Using the solution of the latter at final time t_1, the adjoint problem (17)-(20) is solved. Further, the gradient of the functional J is calculated by (24). Finally, the value of the control is determined using (25), (26). The iterative process is performed until the stopping criterion

$$\max_{t\in[0,t_1]} \max_{x\in[0,1]} |q_{k+1}(x,t) - q_k(x,t)| < \varepsilon$$

is met.

NUMERICAL SOLUTION OF A TEST PROBLEM

The finite difference method is used for the numerical implementation of the problems (1)-(6) and (17)-(20). The equations are solved on a uniform grid $\Omega_h = \left\{x_i = i\Delta x : \ i = \overline{0,N}, \ x_0 = 0, \ x_N = 1\right\}$. The equations (1)-(3) were discretized as follows:

$$c_{T,i}T_{t,i} + b_i^+\hat{T}_{x,i} + b_i^-\hat{T}_{\bar{x},i} - k_h\hat{T}_{\bar{x}x,i} = cq_i,$$

$$\beta_i p_{t,i} - (k_p\hat{p}_{\bar{x}})_{x,i} = q_i,$$

$$s_{w,t,i} - (\nu_w\hat{p}_{\bar{x}})_{x,i} = q_w,$$

$$s_{g,t,i} - (\nu_g\hat{p}_{\bar{x}})_{x,i} = q_i,$$

where

$$b_i^+ = 0.5(u_i + |u_i|), \ \ b_i^- = 0.5(u_i - |u_i|),$$

$$k_{p,i} = k_p(x_i - 0.5\Delta x), \ \nu_{\alpha,i} = \nu_\alpha(x_i - 0.5\Delta x).$$

The equations (17)-(20) were discretized similarly.

The computational experiment was performed with the following values of parameters: $t_1 = 200\Delta t$, $\Delta t = 0.0002$, $\Delta x = 0.001$. The initial control was selected in the form $q_0 = \chi_1 \cdot \exp\left(-a^2/(a^2 - x^2)\right)$ if $|x - 0.12| < a$, $a = 0.09$, and 0 otherwise. As the reference temperature, we selected a temperature distribution corresponding to the solution of the problem (1)-(6) with $\chi_1 = 2.0$ at time $t_1 = 200\tau$ shown in Fig. 1 (solid line). The solutions corresponding to different values of q_k (dotted lines) are approaching the reference mode with the increase of number of iterations.

CONCLUSION

Thus, a method of solving an optimal control problem is considered in the present paper. To minimize the cost functional, the gradient projection method is used. The explicit form of the gradient of the functional is derived, which is expressed through the adjoint problem. Finite difference scheme for numerical implementation of direct and adjoint problems are proposed, an algorithm for the numerical solution of the problem is presented. The approach considered in the paper can be used to predict the effectiveness of the operational methods of oil field development.

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