MINISTRY OF SCIENCE AND HIGHER EDUCATION OF THE REPUBLIC OF KAZAKHSTAN

Түркі әлемі математиктерінің VII Дүниежүзілік Конгресі (TWMS Congress-2023) МАТЕРИАЛДАРЫ

(I бөлім)

MATERIALS

of the VII World Congress of Turkic World Mathematicians (TWMS Congress-2023)

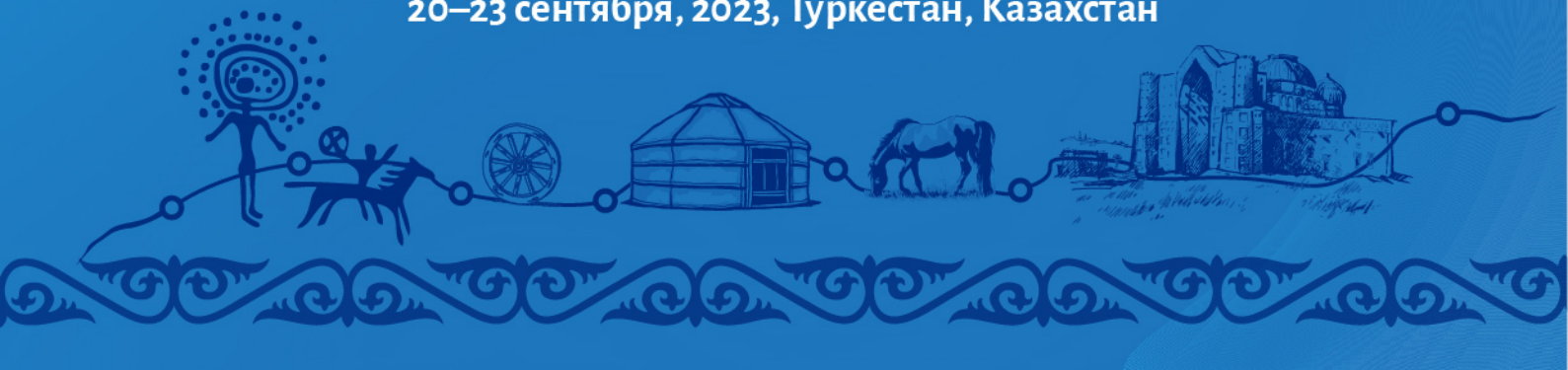
(Part I)

МАТЕРИАЛЫ

VII Всемирного Конгресса математиков тюркского мира (TWMS Congress-2023)

(Часть I)

20–23 қыркүйек, 2023, Түркістан, Қазақстан
September 20–23, 2023, Turkestan, Kazakhstan
20–23 сентября, 2023, Туркестан, Казахстан



Түркі әлемі математиктерінің
VII Дүниежүзілік Конгресі
(TWMS Congress-2023)
МАТЕРИАЛДАРЫ
(I бөлім)

MATERIALS
of the VII World Congress of Turkic
World Mathematicians
(TWMS Congress-2023)
(Part I)

МАТЕРИАЛЫ
VII Всемирного Конгресса математиков
тюркского мира
(TWMS Congress-2023)
(Часть I)

20–23 қыркүйек, 2023, Түркістан, Қазақстан
September 20–23, 2023, Turkestan, Kazakhstan
20–23 сентября, 2023, Туркестан, Казахстан

КОНГРЕСС ҰЙЫМДАСТЫРУШЫЛАРЫ

Түркі әлемінің математикалық қоғамы

Қазақстан Республикасының Ғылым және жоғары білім министрлігі

Түркістан облысының әкімдігі

Қазақстан Республикасының Ұлттық инженерлік академиясы

Қазақстан Республикасы Президентінің жанындағы Қазақстан Республикасының Ұлттық Ғылым Академиясы

Қожа Ахмет Ясауи атындағы Халықаралық қазақ-түрік университеті

Қазақстан инновациялық және телекоммуникациялық жүйелер университеті

Математика және математикалық модельдеу институты

Мұхтар Әуезов атындағы Оңтүстік Қазақстан университеті

Сәрсен Аманжолов атындағы Шығыс Қазақстан университеті

Оңтүстік Қазақстан педагогикалық университеті

М.Х.Дулати атындағы Тараз өңірлік университеті

Халықаралық инженерлік және технология университеті

«Халық» қайырымдылық қоры

Қорқыт-Ата атындағы Қызылорда университеті

Академик Ә. Қуатбеков атындағы Халықтар достығы университеті

БАҒДАРЛАМАЛЫҚ КОМИТЕТ

ТӨРАҒАЛАРЫ

Жұмағұлов Бақытжан Тұрсынұлы, ҚР ҰҒА академигі, TWMS Президенті, Қазақстан Республикасы Парламенті Сенатының депутаты, Қазақстан

Нұрбек Саясат, Қазақстан Республикасының Ғылым және жоғарғы білім министрі, Қазақстан

Сатыбалды Дархан Амангелдыулы, Түркістан облысының әкімі

ТЕҢ ТӨРАҒАЛАРЫ

Аюпов Шавкат Абдуллаұлы, Өзбекстан Ғылым академиясының академигі, Өзбекстан Ғылым академиясы В. И. Романовский атындағы математика институтының директоры, Өзбекстан

Борубаев Алтай Асылқанұлы, Қырғыз Республикасы ҰҒА академигі, Қырғыз Республикасы ҰҒА Математика институтының директоры, Қырғызстан

Әлиев Фикрет Ахмедали оглы, Әзірбайжан ҰҒА академигі, Баку мемлекеттік университетінің Қолданбалы математика институтының директоры, TWMS Journal of Pure and Applied Mathematics журналының редакторы, Әзірбайжан

Пенахов Этибар, профессор, Әзірбайжан

Темірбекова Жанар Амангелдіқызы, Х. А. Ясауи атындағы Халықаралық қазақ-түрік университетінің ректоры, Қазақстан

Хаджисалехоглы Хилми, профессор, Түркия

Шимшек Мухиттин, Х. А. Ясауи атындағы Халықаралық қазақ-түрік университетінің Өкілетті кеңесінің төрағасы

МҮШЕЛЕРІ:

Аббасов Әли, Әзірбайжан ҰҒА академигі, Әзірбайжан

Азамов Абдулла Адамович, Өзбекстан Ғылым академиясының академигі, Өзбекстан

Айдазаде Камил, Әзірбайжан ҰҒА корреспондент-мүшесі, Әзірбайжан

Айсағалиев Серікбай Әбдіғалиұлы, профессор, Қазақстан

Арипов Мерсаид, профессор, М. Ұлықбек атындағы Өзбекстан ұлттық университеті, Өзбекстан

Арсланов Марат Мирзаевич, Татарстан Республикасы Ғылым академиясының академигі, Ресей

Ахмед-Заки Дархан Жұмақанұлы, ҚР Ғылым және жоғары білім министрлігі Ғылым комитетінің төрағасы, Қазақстан

Ашыралыев Аллаберен, профессор, Бахчешехир университеті, Түркия

Багиров Адил, Австралия федералды университеті, Австралия

Бадаев Серікжан Ағыбайұлы, профессор, Қазақстан

Байжанов Бектұр Сембиұлы, ҚР ҰҒА корреспондент-мүшесі, Қазақстан

Башар Мұстафа Тамер, Иллинойс университетінің профессоры, Жетілдірілген зерттеулер орталығының директоры, АҚШ

Бектемесов Мақтағали Әбдімажитұлы, ҚР ҰИА корреспондент-мүшесі, Қазақ ұлттық аграрлық университеті, Қазақстан

Билялов Билял, Әзірбайжан ҰҒА корреспондент-мүшесі, Әзірбайжан

Блиев Назарбай Қадырұлы, ҚР ҰҒА академигі, Қазақстан

Боқаев Нұржан Әділханұлы, Гумилев атындағы Еуразия ұлттық университетінің профессоры, Қазақстан

Войцик Вальдемар, профессор, Люблин техникалық университеті, Польша

Гиви Пейман, профессор, Питтсбург университеті, АҚШ

Гогатишвили Амиран, Чехия ғылым академиясы Математика институтының бас ғылыми қызметкері

Гогташ Фахретдин, профессор, Фират университетінің ректоры, Түркия

Даирбеков Нұрлан Слямханұлы, профессор, Қазақстан

Дженалиев Мувашархан Танабайұлы, профессор, Қазақстан

Ебади Сеид, профессор, Иран

Егоров Иван Егорович, профессор, Саха Республикасы, Ресей

Ергут Махмут, профессор, Түркия

Жұмаділдаев Асқар Серқұлұлы, ҚР ҰҒА академигі, Қазақстан

Кабанихин Сергей Игоревич, РҒА корреспондент-мүшесі, РҒА Есептеу математикасы және математикалық геофизика институты, Ресей

Кальменов Тынысбек Шәріпұлы, ҚР ҰҒА академигі, Қазақстан

Кангужин Балтабек Есматұлы, профессор, Қазақстан

Канетов Бекболат Эменович, профессор, Қырғызстан

Караев Мубариз, профессор, Сауд Арабиясы

Карапинар Ердал, профессор, Тайвань

Кулиев Вагиф, Әзірбайжан ҰҒА корреспондент-мүшесі, Әзірбайжан

Қалимолдаев Мақсат Нұраділұлы, ҚР ҰҒА академигі, Ақпараттық және есептеу технологиялары институты, Қазақстан

Қалтаев Айдархан Жүсіпбекұлы, ҚР ҰИА академигі, Сатпаев атындағы Қазақ ұлттық техникалық университеті, Қазақстан

Маммадов Юсуф, Әзірбайжан ҰҒА академигі, Әзірбайжан

Марданов Мисир Джумаил оглы, Әзірбайжан ҰҒА корреспондент-мүшесі, Әзірбайжан ҰҒА математика және механика институтының директоры, Әзірбайжан

Матиева Гүлбадан, Қырғызстан, Қырғызстан ҰҒА корреспондент-мүшесі, Қырғызстан

Милошевич Хранислав, профессор, Приштина университеті, Сербия

Мусин Олег, профессор, АҚШ

Насыров Семен Рафаилович, Татарстан Республикасы Ғылым академиясының корреспондент-мүшесі, Ресей

Нигматулин Роберт Искандерович, Ресей Ғылым академиясының академигі, Ресей

Нұрсұлтанов Ерлан Даутбекұлы, Гумилев атындағы Еуразия ұлттық университетінің профессоры, Қазақстан

Өтелбаев Мұхтарбай Өтелбайұлы, ҚР ҰҒА академигі, Қазақстан

Панков Павел Сергеевич, Қырғызстан ҰҒА корреспондент-мүшесі, Қырғызстан

Ружанский Михаил Владимирович, профессор, Гент университеті, Бельгия

Саваш Екрем, профессор, Ушак университетінің ректоры, Түркия

Садыбеков Махмуд Әбдісаметұлы, ҚР ҰҒА корреспондент-мүшесі, ҚР ҒЖБМ математика институтының директоры, Қазақстан

Салимов Ариф, профессор, Әзірбайжан

Сұлтанаев Яудат, профессор, Башқұртстан Республикасы, Ресей

Тайманов Александр Асанович, РҒА академигі, РҒА СБ Соболев атындағы математика институты, Ресей

Тосун Мурат, Сакария университетінің профессоры, Түркия

Умирбаев Уалбай Утмахамбетович, ҚР ҰҒА академигі, профессор, АҚШ

Фазуллин Зиганур Юсупович, профессор, Башқұртстан Республикасы, Ресей

Хаджиева Назиле, Азербайджан

Харин Станислав Николаевич, ҚР ҰҒА академигі, Қазақстан

Хасиев Асеф, Әзірбайжан ҰҒА академигі, Әзірбайжан

Шокин Юрий Иванович, РҒА академигі, РҒА СБ Ақпараттық және есептеу технологияларының федералды зерттеу орталығы, Ресей

Шокри Әли, профессор, Иран
Эфендиев Месуд, профессор, Германия
Юксел Сердар, профессор, Канада

ҰЙЫМДАСТЫРУШЫ КОМИТЕТ

ТӨРАҒА
Темірбеков Нұрлан Мұқанұлы, ҚР ҰИА академигі, ҚР ҰҒА корреспондент-мүшесі, Қазақстан

МҮШЕЛЕРІ:
Адамов Әбілмәжін Әлирахымұлы, профессор, Л.Н.Гумилев атындағы Еуразия ұлттық университеті, Қазақстан
Акбаров Сурхай Джаббар оглы, профессор, Йылдыз техникалық университеті, Түркия
Акин Омер, профессор, Түркия
Байгереев Досан Рақымғалиұлы, қауымдастырылған профессор, ҚР ҰИА корреспондент-мүшесі, С. Аманжолов атындағы Шығыс Қазақстан университеті, Қазақстан
Байзақов Асан Байзақұлы, профессор, Б. Н. Ельцин атындағы Қырғыз-Ресей Славян университеті, Қырғызстан
Бақанов Галитдин Бақанұлы, Х.А.Ясауи атындағы Халықаралық қазақ-түрік университеті, Қазақстан
Бас Эрдал, профессор, Фират университеті, Түркия
Бахтизин Рамиль Назифович, профессор, Уфа мемлекеттік мұнай техникалық университеті, Башқұртстан
Бектемесов Жоламан Мақтағалиұлы, Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан
Бердышев Абдумаувлен Сүлейманұлы, профессор, Абай атындағы Қазақ ұлттық педагогикалық университеті, Қазақстан
Булут Хасан, профессор, Түркия
Гунгор Мехмет Али, профессор, Сакарья университеті, Турция
Гусейнзаде Сабир, профессор, Ресей
Ерсой Солей, профессор, Түркия
Ет Микаил, профессор, Түркия
Әбдібеков Уәлихан Сейділдайұлы, ҚР ҰИА корреспондент-мүшесі, әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан
Әлиев Араз Рафиг оглы, профессор, Баку мемлекеттік университеті, Әзірбайжан
Әлиев Зиятхан Сейфаддин оглы, профессор, Баку мемлекеттік университеті, Әзірбайжан
Әмірғалиев Еділхан Несіпханұлы, ҚР ҰИА академигі, Ақпараттық және есептеуіш технологиялар институты, Қазақстан
Жакебаев Дәурен Бақытбекұлы, ҚР ҰИА корреспондент-мүшесі, Астана IT университеті, Қазақстан
Зафер Агаджик, профессор, Таяу Шығыстағы Американдық университет, Кувейт
Ибрахимов Наби, профессор, Әзірбайжан
Иманғалиев Ернар Иманғалиұлы, ҚР ҰИА бас ғалым хатшысы, Қазақстан
Иманқұлов Тимур Сәкенұлы, Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан
Исахов Әлібек Әбдиашимұлы, профессор, ҚР ҰИА корреспондент-мүшесі, Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан

Исгандаров Низамеддин, профессор, Баку мемлекеттік университеті, Әзірбайжан
Калантаров Варга, профессор, Коч университеті, Түркия
Қасенов Сырым Еркінұлы, Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан
Лакестани Мехридад, профессор, Иран
Мадияров Мұратқан Набенұлы, ҚР ҰИА академигі, С. Аманжолов атындағы Шығыс Қазақстан университеті, Қазақстан
Маммадова Гамар, профессор, Баку мемлекеттік университеті, Әзірбайжан
Мукуддинов Мукаддес, профессор, Татарстан
Муталимов Муталим, профессор, Әзірбайжан
Мұхамбетжанов Салтанбек Талапеденұлы, әл-Фараби Қазақ ұлттық университеті, Қазақстан
Намазова Гүлиза Өмірбекқызы, доцент, Қырғызстан
Озбай Хитай, профессор, Түркия
Озбеклер Абдуллах, профессор, Атилим университеті, Түркия
Панахов Гейлани, Әзірбайжан ҰҒА корреспондент-мүшесі, Әзірбайжан
Сабитов Иджок, профессор, Ресей
Сафарова Наргиз, профессор, Әзірбайжан
Сәрсенби Әбдіжахан Манапұлы, ҚР ҰИА корреспондент-мүшесі, М. Әуезов атындағы Оңтүстік Қазақстан университетінің профессоры
Сұлтанов Мұрат Әбдіқадырұлы, профессор, Х. А. Ясауи Халықаралық қазақ-түрік университеті, Қазақстан
Тамабай Динара Оразбекқызы, әл-Фараби Қазақ ұлттық университеті, Қазақстан
Темірбеков Алмас Нұрланұлы, қауымдастырылған профессор, әл-Фараби Қазақ ұлттық университеті, Қазақстан
Тұрметов Батырхан Құдайбергенұлы, профессор, Х. А. Ясауи Халықаралық қазақ-түрік университеті, Қазақстан
Урмашев Байдаулет Амантайұлы, Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан
Фараджов Араз, профессор, Әзірбайжан
Ханмаммедов Агил, профессор, Әзірбайжан
Хасаноглу Эльман, профессор, Исик университеті, Түркия
Хомпыш Хонатбек, әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан
Шахин Байрам, профессор, Эге университеті, Түркия
Шәкенов Қанат Қожахметұлы, әл-Фараби Қазақ ұлттық университеті, Қазақстан
Юстюнель Сүлеймен, профессор, Түркия
Ягубов Али, профессор, Әзірбайжан

CONGRESS ORGANIZERS

Turkic World Mathematical Society

Ministry of Science and Higher Education of the Republic of Kazakhstan

Akimat of Turkestan region

National Engineering Academy of the Republic of Kazakhstan

National Academy of Sciences of the Republic of Kazakhstan under the President of the Republic of Kazakhstan

Khoja Akhmet Yassawi International Kazakh-Turkish University

Kazakhstan University of Innovative and Telecommunication Systems

Institute of Mathematics and Mathematical Modeling

Mukhtar Auezov South Kazakhstan University

Sarsen Amanzholov East Kazakhstan University

South Kazakhstan Pedagogical University

M.H. Dulati Taraz Regional University

International University of Engineering and Technology

PF Charitable Foundation «Halyk»

Korkyt Ata Kyzylorda University

Peoples' Friendship University named after Academician A.Kuatbekov

PROGRAM COMMITTEE

CHAIRMEN

Bakytzhan T. Zhumagulov, Academician of NAS RK, President of TWMS, Deputy of the Senate of the Parliament of the Republic of Kazakhstan, Kazakhstan

Sayasat Nurbek, Minister of Science and Higher Education of the Republic of Kazakhstan, Kazakhstan

Darkhan Satybaldy, Akim of the Turkestan region

CO–CHAIRS

Altai A. Borubayev, Academician of NAS of the Kyrgyz Republic, Director of the Institute of Mathematics of NAS of the Kyrgyz Republic, Kyrgyzstan

Fikret A. Aliev, Academician of NAS of Azerbaijan, Director of the Institute of Applied Mathematics of Baku State University, Editor of TWMS Journal of Pure and Applied Mathematics, Azerbaijan

Etibar Panahov, Professor, Azerbaijan

Hilmi Hacisalihoğlu, Professor, Turkey

Muhittin Şimşek, Chairman of the Board of Trustees of Khoja Akhmet Yassawi International Kazakh-Turkish University

Shavkat A. Ayupov, Academician of the Academy of Sciences of the Republic of Uzbekistan, Director of the V.I. Romanovsky Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan, Uzbekistan

Zhanar A. Temirbekova, Rector of the Yassawi International Kazakh-Turkish University, Kazakhstan

MEMBERS:

Abdulla A. Azamov, Academician of the Academy of Sciences of the Republic of Uzbekistan, Uzbekistan

Adil Bagirov, Federation University Australia, Australia

Aidarkhan Zh. Kaltayev, Academician of NEA RK, Satpayev Kazakh National Technical University, Kazakhstan

Ali Şökri, Professor, University of Maragheh

Allaberen Ashyralyev, Professor, Bahçeşehir University, Turkey

Amiran Gogatishvili, Chief Researcher at the Institute of Mathematics of the Czech Academy of Sciences

Arif Salimov, Professor, Azerbaijan

Asaf Hacıyev, Academician of NAS of Azerbaijan, Azerbaijan

Askar S. Dzhumadildayev, Academician of NAS of the Republic of Kazakhstan, Kazakhstan

Ali Abbasov, Academic of NAS of Azerbaijan, Azerbaijan

Baltabek E. Kanguzhin, Professor, Kazakhstan

Bekbolat E. Kanetov, Professor, Kyrgyzstan

Bektur S. Baizhanov, Corresponding Member of NAS of the Republic of Kazakhstan, Kazakhstan

Bilal Bilalov, Corresponding Member of NAS of Azerbaijan, Azerbaijan

Darkhan Zh. Ahmed-Zaki, Chairman of the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan, Kazakhstan

Ekrem Savaş, Professor, rector of Usak University, Turkey

Erdal Karapınar, Professor, Taiwan

Fahrettin Göktaş, Professor, rector of Fırat University, Turkey

Gulbadan Matieva, Corresponding Member of the National Academy of Sciences of the Kyrgyz Republic, Kyrgyzstan
Hranislav Milošević, Professor, University of Pristina, Serbia
Iskander A. Taimanov, Academician of RAS, Sobolev Institute of Mathematics RAS, Russia
Ivan E. Egorov, Professor, Sakha Republic, Russia
Kamil Aydazade, Corresponding member of NAS of Azerbaijan, Azerbaijan
Mahmud A. Sadybekov, Corresponding Member of NAS of the Republic of Kazakhstan, Director of the Institute of Mathematics of NAS of the Republic of Kazakhstan, Kazakhstan
Mahmut Ergüt, Professor, Turkey
Maksat N. Kalimoldayev, Academician of NAS RK, Institute of Information and Computational Technologies, Kazakhstan
Maktagali A. Bektemessov, Corresponding Member of NEA RK, Kazakh National Agrarian University, Kazakhstan
Marat M. Arslanov, Academician of the Academy of Sciences of the Republic of Tatarstan, Russia
Mersaid Aripov, Professor, M. Ulugbek National University of Uzbekistan, Uzbekistan
Messoud Efendiev, Professor, Germany
Michael V. Ruzhansky, Professor, Ghent University, Belgium
Misir Mardanov, Corresponding Member of NAS of Azerbaijan, Director of the Institute of Mathematics and Mechanics of NAS of Azerbaijan, Azerbaijan
Mubariz Karaev, Professor, Saudi Arabia
Mukhtarbai O. Otelbayev, Academician of NAS of Kazakhstan, Kazakhstan
Murat Tosun, Professor at Sakarya University, Turkey
Mustafa Tamer Başar, Professor at the University of Illinois, Director of the Center for Advanced Studies, USA
Muvasharkhan T. Dzhenaliyev, Professor, Kazakhstan
Nazarbay K. Bliyev, Academician of NAS of Kazakhstan, Kazakhstan
Nazile Hajiyeva, Azerbaijan
Nurlan S. Dairbekov, Professor, Kazakhstan
Nurzhan A. Bokayev, Professor of the Gumilev Eurasian National University, Kazakhstan
Oleg Musin, Professor, USA
Pavel S. Pankov, Corresponding Member of NAS of the Kyrgyz Republic, Kyrgyzstan
Peyman Givi, Professor, University of Pittsburgh, USA
Robert I. Nigmatulin, Academician of RAS, Russia
Seid Ebadi, Professor, Iran
Semen R. Nasyrov, Corresponding Member of the Academy of Sciences of the Republic of Tatarstan, Russia
Serdar Yüksel, Professor, Canada
Sergey I. Kabanikhin, Corresponding Member of RAS, Institute of Computational Mathematics and Mathematical Geophysics SB RAS, Russia
Serikbay A. Aisagaliyev, Professor, Kazakhstan
Serikzhan A. Badayev, Professor, Kazakhstan
Stanislav N. Kharin, Academician of NAN RK, Kazakhstan
Tynysbek Sh. Kalmenov, Academician of NAS RK, Kazakhstan
Ualbai U. Umirbayev, Academician of NAS RK, Professor, USA
Vaqif Quliyev, Corresponding Member of NAS of Azerbaijan, Azerbaijan
Waldemar Wójcik, Professor, Lublin Technical University, Poland
Yaudat Sultanaev, Professor, Republic of Bashkortostan, Russia
Yerlan D. Nursultanov, Professor of the Gumilev Eurasian National University, Kazakhstan
Yuri I. Shokin, Academician of RAS, Federal Research Center for Information and Computing Technologies SB RAS, Russia

Yusif Mammadov, Academician of NAS of Azerbaijan, Azerbaijan
Ziganur Yu. Fazullin, Professor, Republic of Bashkortostan, Russia

ORGANIZING COMMITTEE

CHAIRMAN
Nurlan M. Temirbekov, Academician of NEA RK, corresponding member of NAS RK, Kazakhstan

MEMBERS:
Abdizhakhan M. Sarsenbi, Corresponding member of NEA RK, Professor of M. Auezov South Kazakhstan University
Abdullah Özbekler, Professor, Atilim University, Turkey
Abdumauvlen S. Berdyshev, Professor, Abay Kazakh National Pedagogical University, Kazakhstan
Abilmazhin A. Adamov, Professor, L.N. Gumilyov Eurasian National University, Kazakhstan
Ağacık Zafer, Professor, American University of the Middle East, Kuwait
Ali Yaqubov, Professor, Azerbaijan
Alibek A. Issakhov, Professor, Corresponding Member of NEA RK, Al-Farabi Kazakh National University, Kazakhstan
Almas N. Temirbekov, Associate Professor, Al-Farabi Kazakh National University, Kazakhstan
Aqil Khanmammadov, Professor, Azerbaijan
Araz Aliyev, Professor, Baku State University, Azerbaijan
Araz Faradzhov, Professor, Azerbaijan
Asan B. Bayzakov, Professor, Kyrgyz-Russian Slavic University named after B.N. Yeltsin, Kyrgyzstan
Batyrkhan Kh. Turmetov, Professor, Yassawi International Kazakh-Turkish University, Kazakhstan
Baydaulet A. Urmashev, Al-Farabi Kazakh National University, Kazakhstan
Bayram Shahin, Professor, Ege University, Turkey
Dauren B. Zhakebayev, Corresponding Member of NEA RK, Astana IT University, Kazakhstan
Dinara O. Tamabay, Al-Farabi Kazakh National University, Kazakhstan
Dossan R. Baigereyev, Associate Professor, Corresponding Member of NEA RK, S. Amanzholov East Kazakhstan University, Kazakhstan
Elman Hasanoğlu, Professor, Isik University, Turkey
Erdal Bas, Professor, Firat University, Turkey
Galitdin B. Bakanov, K.A. Yassawi International Kazakh-Turkish University, Kazakhstan
Guliza O. Namazova, Associate Professor, Kyrgyzstan
Hasan Bulut, Professor, Turkey
Hitay Özbay, Professor, Turkey
İdjok Sabitov, Professor, Russia
Kanat K. Shakenov, Al-Farabi Kazakh National University, Kazakhstan
Khompysh Honatbek, Al-Farabi Kazakh National University, Kazakhstan
Mehmet Ali Gungor, Professor, Sakarya University, Turkey
Mehridad Lakestani, Professor, Iran
Mikail Et, Professor, Turkey
Mukaddes Mukuddinov, Professor, Tatarstan

Murat A. Sultanov, Professor, Yassawi International Kazakh-Turkish University, Kazakhstan
Muratkan N. Madiyarov, Academician of NEA RK, S. Amanzholov East Kazakhstan University, Kazakhstan
Mutallim Mutallimov, Professor, Azerbaijan
Nabi İbrahimov, Professor, Azerbaijan
Nargiz Safarova, Professor, Azerbaijan
Nizamaddin İsgandarov, Professor, Baku State University, Azerbaijan
Ömer Akın, Professor, Turkey
Panahov Geylani, Corresponding member of NAS of Azerbaijan, Azerbaijan
Qamar Mammadova, Professor, Baku State University, Azerbaijan
Ramil N. Bakhtizin, Professor, Ufa State Petroleum Technological University, Bashkortostan
Sabir Guseinzade, Professor, Russia
Saltanbek T. Mukhambetzhanov, Al-Farabi Kazakh National University, Kazakhstan
Soley Ersoy, Professor, Turkey
Suleyman Üstünel, Professor, Turkey
Surkhay Akbarov, Professor, Yildiz Technical University, Turkey
Syrym Ye. Kassenov, Al-Farabi Kazakh National University, Kazakhstan
Timur S. Imankulov, Al-Farabi Kazakh National University, Kazakhstan
Ualikhan S. Abdibekov, Corresponding member of NEA RK, Al-Farabi Kazakh National University, Kazakhstan
Varga Kalantarov, Professor, Koc University, Turkey
Yedilkhan N. Amirgaliyev, Academician of NEA RK, Institute of Information and Computational Technologies, Kazakhstan
Yernar I. Imangaliyev, Chief Scientific Secretary of NEA RK, Kazakhstan
Zholaman M. Bektemessov, Al-Farabi Kazakh National University, Kazakhstan
Ziyatxan Aliyev, Professor, Baku State University, Azerbaijan

ОРГАНИЗАТОРЫ КОНГРЕССА

Математическое общество тюркского мира

Министерство науки и высшего образования Республики Казахстан

Акимат Туркестанской области

Национальная инженерная академия Республики Казахстан

Национальная академия наук Республики Казахстан при Президенте Республики Казахстан

Международный казахско-турецкий университет имени Ходжи Ахмеда Ясави

Казахстанский университет инновационных и телекоммуникационных систем

Институт математики и математического моделирования

Южно-Казахстанский университет имени Ауэзова

Восточно-Казахстанский университет имени Сарсена Аманжолова

Южно-Казахстанский педагогический университет

Таразский региональный университет имени М.Х.Дулати

Международный инженерно-технологический университет

Благотворительный Фонд «Халык»

Кызылординский университет имени Коркыт-Ата

Университет дружбы народов имени академика А. Куатбекова

ПРОГРАММНЫЙ КОМИТЕТ

ПРЕДСЕДАТЕЛИ

Жумагулов Бакытжан Турсынович, академик НАН РК, Президент TWMS

Нурбек Саясат, Министр науки и высшего образования Республики Казахстан, Казахстан

Сатыбалды Дархан Амангелдыулы, Аким Туркестанской области

СОПРЕДСЕДАТЕЛИ

Алиев Фикрет Ахмедалиоглы, академик НАН Азербайджана, директор института прикладной математики Бакинского государственного университета, редактор журнала TWMS Journal of Pure and Applied Mathematics, Азербайджан

Аюпов Шавкат Абдуллаевич, академик АН Узбекистана, директор института математики им. В.И. Романовского АН Узбекистана, Узбекистан

Борубаев Алтай Асылканович, академик НАН Кыргызской Республики, директор Института математики НАН КР, Кыргызстан

Пенахов Этибар, профессор, Азербайджан

Темирбекова Жанар Амангельдиевна, ректор Международного казахско-турецкого университета им. Х.А. Ясави, Казахстан

Хаджисалехоглы Хилми, профессор, Турция

Шимшек Мухиттин, председатель Полномочного Совета Международного казахско-турецкого университета им. Ходжа Ахмета Яссави

ЧЛЕНЫ:

Аббасов Али, академик НАН Азербайджана, Азербайджан

Азамов Абдулла Азамович, академик АН Узбекистана, Узбекистан

Айдазаде Камил, член-корреспондент НАН Азербайджана, Азербайджан

Айсагалиев Серикбай Абдигалиевич, профессор, Казахстан

Арипов Мирсаид, профессор, Национальный университет Узбекистана им. М. Улугбека, Узбекистан

Арсланов Марат Мирзаевич, академик АН Республики Татарстан, Россия

Ахмед-Заки Дархан Жумаканович, Председатель Комитета науки Министерства науки и высшего образования РК, Казахстан

Ашыралыев Аллаберен, профессор, Бахчешехир университет, Турция

Багиров Адил, Федеративный университет Австралии, Австралия

Бадаев Серикжан Агыбаевич, профессор, Казахстан

Байжанов Бектур Сембиевич, член-корреспондент НАН РК, Казахстан

Башар Мустафа Тамер, профессор Университета Иллинойса, директор Центра перспективных исследований, США

Бектемесов Мактагали Абдимажитович, член-корреспондент НИА РК, Казахский национальный аграрный университет, Казахстан

Билялов Билял, член-корреспондент НАН Азербайджана, Азербайджан

Блиев Назарбай Кадырович, академик НАН РК, Казахстан

Бокаев Нуржан Адилханович, профессор Евразийского национального университета им. Гумилева, Казахстан

Войцик Вальдемар, профессор, Люблинский технический университет, Польша

Гиви Пейман, профессор, Университет Питтсбурга, США

Гогатишвили Амиран, главный научный сотрудник Института математики Чешской Академии наук

Гогташ Фахретдин, профессор, ректор Университета Фират, Турция

Даирбеков Нурлан Слямханович, профессор, Казахстан

Дженалиев Мувашархан Танабаевич, профессор, Казахстан
Джумадильдаев Аскар Серкулович, академик НАН РК, Казахстан
Ебади Сеид, профессор, Иран
Егоров Иван Егорович, профессор, Республика Саха, Россия
Ергут Махмут, профессор, Турция
Ефендиев Месуд, профессор, Германия
Кабанихин Сергей Игоревич, член-корреспондент РАН, Институт вычислительной математики и математической геофизики СО РАН, Россия
Калимолдаев Максат Нурадилович, академик НАН РК, Институт информационных и вычислительных технологий, Казахстан
Калтаев Айдархан Жусупбекович, академик НИА РК, Казахский национальный технический университет им. Сатпаева, Казахстан
Кальменов Тынысбек Шарипович, академик НАН РК, Казахстан
Кангужин Балтабек Есматович, профессор, Казахстан
Канетов Бекболат Эменович, профессор, Кыргызстан
Караев Мубариз, профессор, Саудовская Аравия
Карапинар Ердал, профессор, Тайвань
Кулиев Вагиф, член-корреспондент НАН Азербайджана, Азербайджан
Маммадов Юсуф, академик НАН Азербайджана, Азербайджан
Марданов Мисир Джумаилоглы, член-корреспондент НАН Азербайджана, директор Института математики и механики НАН Азербайджана, Азербайджан
Матиева Гульбадан, член-корреспондент НАН КР, Кыргызстан
Милошевич Хранислав, профессор, Университет Приштины, Сербия
Мусин Олег, профессор, США
Насыров Семен Рафаилович, член-корреспондент АН Республики Татарстан, Россия
Нигматулин Роберт Искандерович, академик РАН, Россия
Нурсултанов Ерлан Даутбекович, профессор Евразийского национального университета им. Гумилева, Казахстан
Отелбаев Мухтарбай Отелбаевич, академик НАН РК, Казахстан
Панков Павел Сергеевич, член-корреспондент НАН КР, Кыргызстан
Ружанский Михаил Владимирович, профессор, Университет Гент, Бельгия
Саваш Екрем, профессор, ректор Университета Ушак, Турция
Садыбеков Махмуд Абдысаметович, член-корреспондент НАН РК, директор института математики КН МНВО РК, Казахстан
Салимов Ариф, профессор, Азербайджан
Султанаев Яудат, профессор, Республика Башкортостан, Россия
Тайманов Искандер Асанович, академик РАН, Институт математики им. Соболева СО РАН, Россия
Тосун Мурат, профессор университета Сакария, Турция
Умирбаев Уалбай Утмахамбетович, академик НАН РК, профессор, США
Фазуллин Зиганур Юсупович, профессор, Республика Башкортостан, Россия
Хаджиева Назиле, Азербайджан
Харин Станислав Николаевич, академик НАН РК, Казахстан
Хасиев Асеф, академик НАН Азербайджана, Азербайджан
Шокин Юрий Иванович, академик РАН, Федеральный исследовательский центр информационных и вычислительных технологий СО РАН, Россия
Шокри Али, профессор, Иран
Юксел Сердар, профессор, Канада

ОРГАНИЗАЦИОННЫЙ КОМИТЕТ

ПРЕДСЕДАТЕЛЬ:

Темирбеков Нурлан Муханович, академик НИА РК, член-корреспондент НАН РК, Казахстан

ЧЛЕНЫ:

Абдибеков Уалихан Сейдильдаевич, член-корреспондент НИА РК, Казахский национальный университет им. аль-Фараби, Казахстан

Адамов Абилмажин Алирахымович, профессор, Евразийский национальный университет имени Л.Н. Гумилева, Казахстан

Акбаров Сурхай Джаббар оглы, профессор, Технический университет Йылдыз, Турция

Акин Омер, профессор, Турция

Алиев Араз Рафиг оглы, профессор, Бакинский государственный университет, Азербайджан

Алиев Зиятхан Сейфаддин оглы, профессор, Бакинский государственный университет, Азербайджан

Амиргалиев Едилхан Несипханович, академик НИА РК, Институт вычислительных и информационных технологий, Казахстан

Байгереев Досан Ракимгалиевич, ассоциированный профессор, член-корреспондент НИА РК, Восточно-Казахстанский университет им. С. Аманжолова, Казахстан

Байзаков Асан Байзакович, профессор, Кыргызско-Российский Славянский университет имени Б. Н. Ельцина, Кыргызстан

Баканов Галитдин Баканович, Международный казахско-турецкий университет им. Х.А. Ясави, Казахстан

Бас Эрдал, профессор, Университет Фират, Турция

Бахтизин Рамиль Назифович, профессор, Уфимский государственный нефтяной технический университет, Башкортостан

Бектемесов Жоламан Мактагалиевич, Казахский национальный университет им. аль-Фараби, Казахстан

Бердышев Абдумаувлен Сулейманович, профессор, Казахский национальный педагогический университет им. Абая, Казахстан

Булут Хасан, профессор, Турция

Гунгор Мехмет Али, профессор, Университет Сакарья, Турция

Гусейнзаде Сабир, профессор, Россия

Ерсой Солей, профессор, Турция

Ет Микаил, профессор, Турция

Жакебаев Даурен Бахытбекович, член-корреспондент НИА РК, Астана IT университет, Казахстан

Зафер Агаджик, профессор, Американский университет Ближнего Востока, Кувейт

Ибрахимов Наби, профессор, Азербайджан

Имангалиев Ернар Имангалиевич, Главный Ученый секретарь НИА РК, Казахстан

Иманкулов Тимур Сакенович, Казахский национальный университет им. аль-Фараби, Казахстан

Исахов Алибек Абдиашимович, профессор, член-корреспондент НИА РК Казахский национальный университет им. аль-Фараби, Казахстан

Исгандаров Низамеддин, профессор, Бакинский государственный университет, Азербайджан

Калантаров Варга, профессор, Коч университет, Турция

Касенов Сырым Еркинович, Казахский национальный университет им. аль-Фараби, Казахстан

Лакестани Мехридад, профессор, Иран

Мадияров Мураткан Набенович, академик НИА РК, Восточно-Казахстанский университет им. С. Аманжолова, Казахстан

Маммадова Гамар, профессор, Бакинский государственный университет, Азербайджан

Мукуддинов Мукаддес, профессор, Татарстан

Муталимов Муталим, профессор, Азербайджан

Мухамбетжанов Салтанбек Талапеденович, Казахский национальный университет им. аль-Фараби, Казахстан

Намазова Гулиза Омурбековна, доцент, Кыргызстан

Озбай Хитай, профессор, Турция

Озбеклер Абдуллах, профессор, Университет Атилим, Турция

Панахов Гейлани, член-корреспондент НАН Азербайджана, Азербайджан

Сабитов Иджок, профессор, Россия

Сарсенби Абжахан Манапулы, член-корреспондент НИА РК, профессор Южно-Казахстанского университета им. М.Ауезова

Сафарова Наргиз, профессор, Азербайджан

Султанов Мурат Абдукадырович, профессор, МКТУ им. Х.А. Ясави

Тамабай Динара Оразбекқызы, Казахский национальный университет им. аль-Фараби, Казахстан

Темирбеков Алмас Нурланович, ассоциированный профессор, Казахский национальный университет им. аль-Фараби, Казахстан

Турметов Батырхан Худайбергенович, профессор, МКТУ им. Х. А. Ясави, Казахстан

Урмашев Байдаулет Амантаевич, Казахский национальный университет им. аль-Фараби, Казахстан

Фараджов Араз, профессор, Азербайджан

Ханмаммадов Агил, профессор, Азербайджан

Хасаноглу Елман, профессор, Ишикский университет, Турция

Хомпыш Хонатбек, Казахский национальный университет им. аль-Фараби, Казахстан

Шакенов Канат Кожахметович, Казахский национальный университет им. аль-Фараби, Казахстан

Шахин Байрам, профессор, Эгейский университет, Турция

Юстюнел Сулеймен, профессор, Турция

Ягубов Али, профессор, Азербайджан

Дифференциалдық теңдеулер және математикалық физика теңдеулері

Differential equations and equations of mathematical physics

Дифференциальные уравнения и уравнения математической физики

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

INVESTIGATION OF SOLUTIONS OF PARTIAL INTEGRO-DIFFERENTIAL EQUATION OF FOURTH ORDER BY MEANS OF A NEW WAY

AIZHARKYN ASHIRBAEVA[1], CHEBIRE ZHOLDOSHOVA[2]

[1,2] Osh Technological University named after M. M.Adyshev, Osh, Kyrgyzstan
e-mail: aijarkyn.osh@mail.ru, chebire86@mail.ru

Abstract. This article discusses the partial integro-differential equation of fourth order. To solve the problem, a new method is used, with the help of which we will bring the partial integro-differential equation of fourth order to a form convenient for using the additional argument method. The initial problem for a fourth-order equation is first reduced to a system of second-order partial differential equations. And the system of equations in partial derivatives of the second order is reduced by the method of an additional argument to a system of integral equations, for which the principle of compressive reflections is applied.

Keywords: Integro-differential, partial derivatives, additional argument method, initial problem, integral equation, contraction mapping principle, fourth order

1. Introduction

Currently, this method of additional argument (MAA) is used to solve partial differential equations of various orders and systems of partial differential equations. Thus, MAA shows an advantage over the characteristic method in constructing a solution to high-order partial differential equations.

There is considered initial value problem for differential equation of the second order in [1,2]. There is considered a new way to build solutions of partial differential equations of the fourth order of hyperbolic type in [3]. We use results of this work.

2. Formulation of the problem

We use denotations, classes and spaces of functions [1]: $G_2(T) := [0,T] \times R$; $G := [0,T] \times [0,T]$; $\overline{C}^{(k)}$ be the class of functions being continuous and bounded together with their derivatives up to the k-th order.

In this work we consider the following task

$$u_{tttt}(t,x) - 2a^2 u_{ttxx}(t,x) + a^4 u_{xxxx}(t,x) = bu_{ttt}(t,x) + cu_{ttx}(t,x) + du_{txx}(t,x) + eu_{xxx}(t,x) + \int_0^t K(t,s)u(s,x)ds + f(t,x,u(t,x)), \quad (t,x) \in G_2(T) \tag{1}$$

with the initial conditions

$$\frac{\partial^k u(t,x)}{\partial t^k}|_{t=0} = u_k(x), \quad k = 0..3, \quad x \in R, \tag{2}$$

where

$$a,b,c,d = a^2 b, e = a^2 c \ - \ const, \ f(t,x,u) \in \overline{C}^4(G_2(T) \times R),$$

$$K(t,s) \in C(G), \int_0^T |K(t,s)|ds < \gamma = const.$$

Let in the equation (1):$d = a^2b, e = a^2c$.

In equation (1), we introduce the notation:

$\omega(t,x) = u_{tt}(t,x) - a^2u_{xx}(t,x) - bu_t(t,x) - cu_x(t,x)$,

we reduce equation (1) to a system of equations of the form:

$$\begin{cases} \omega_{tt}(t,x) - a^2u_{xx}(t,x) = \int\limits_0^t K(t,s) - u(s,x)ds + f(t,x,u), \\ u_{tt}(t,x) - a^2u_{xx}(t,x) = bu_t(t,x) + cu_x(t,x) + \omega(t,x). \end{cases} \tag{3}$$

From the first equation of system (3), using MAA, we express the function w in terms of u, then, substituting it into the second equation of the system, we find the required function $u(t,x)$.

Consider the first equation of system (3):

$$\frac{\partial^2\omega(t,x)}{\partial t^2} = a^2\frac{\partial^2\omega(t,x)}{\partial x^2} + \int\limits_0^t K(t,s)u(s,x)ds + f(t,x,u) \tag{4}$$

with the initial conditions (2).

Let introduce the notation into the integro-differential equation:

$p(s,t,x) = x + at - as$,

$q(s,t,x) = x - at + as$

We write (4) in the form:

$$\left(\frac{\partial\omega}{\partial t} - a\frac{\partial\omega}{\partial x}\right)_t + a\left(\frac{\partial\omega}{\partial t} - a(\frac{\partial\omega}{\partial x}\right)_x = \int\limits_0^t K(t,s)u(s,x)ds + f(t,x,u). \tag{5}$$

With the help of MAA from (5) we have:

$$\frac{\partial\omega}{\partial t} - a\frac{\partial\omega}{\partial t} = \omega_1(q(0,t,x)) + \int\limits_0^t\int\limits_0^\tau K(\tau,s)u(s,q(\tau,t,x))dsd\tau +$$

$$+ \int\limits_0^t f(\tau,q(\tau,t,x)),u(\tau,q(\tau,t,x))d\tau. \tag{6}$$

Where $\omega_1(x) = (\omega_t - a\omega_x)\ |_t = 0$, and is determined from the initial condition (2).

Now we apply MAA for (6) with (2).

Therefore, we get:

$$\omega(t,x) = \omega_0(p(0,t,x)) + \int\limits_0^t \omega_1(q(0,s,p(s,t,x))ds + \int\limits_0^t\int\limits_0^v\int\limits_0^\tau K(\tau,s)u(s,q(\tau,t,v,p(v,t,x)))dsd\tau dv +$$

$$+ \int\limits_0^t\int\limits_0^v f(\tau,q(\tau,v,p),u(\tau,q(\tau,v,p))d\tau dv = F(t,x,u) \tag{7}$$

Now we solve the second equation of system (3) using the results of [1].

$$u_{tt}(t,x) = a^2u_{xx}(t,x) + bu_t(t,x) + cu_x(t,x) + F(i,x,u). \tag{8}$$

Let's use the notation:

$$\vartheta(t,x) = D[-a]u(t,x) \tag{9}$$

$g = c/a, \beta_1 = b + g, \beta_2 = b - g$

Lemma 1. Equation (8) with initial condition (2) is equivalent to the system of integral equations (SIE):

$$\vartheta(t,x) = \frac{1}{2}\varphi_1(q(0,t,x)) + \frac{1}{2}\beta_1 u + \frac{\beta_2}{2}\int\limits_0^t \vartheta(s,q)ds + \int\limits_0^t F(s,q,u(s,q))ds, \tag{10}$$

$$u(t,x) = u_0(p(0,t,x)) + \int\limits_0^t \vartheta(s,p(s,t,x))ds, \tag{11}$$

where

$[2\vartheta(t,x) - \beta_1 u(t,x)] \mid_{t=0} = \varphi_1(x)$.

Proof of Lemma 1. Let $\vartheta(t,x), u(t,x)$ - solution SIE (10)-(11).

Finding partial derivatives up to the second order from the integral equation (10) and partial derivatives of the first order from (11), we obtain equation (8) and notation (9). The system satisfies the initial condition (2).

Now, with the help of MÀA from (3), (4) with (2), we must obtain (10), (11), for this we write equation (3) in a form that is convenient for using the indicated method:

$$\frac{\partial z}{\partial t} + a\frac{\partial z}{\partial x} + \beta_2\vartheta(t,x) + 2F(t,x,u), \tag{12}$$

where

$z(t,x) = 2\vartheta(t,x) - \beta_1 u(t,x)$.

From (12) with the initial condition (2) using MÀA, we obtain:

$2\vartheta(t,x) - \beta_1 u = \varphi_1(q(0,t,x)) + \beta_2\int\limits_0^t \vartheta(s,q)ds + 2\int\limits_0^t F(s,q,u(s,q))ds,$

Therefore, we obtain (10). The validity of (11) follows from the notation (9) with the help of MAA. We have proved Lemma 1.

Further, in (10), substituting (11), we obtain an integral equation with $\vartheta(t,x)$.

$$\vartheta(t,x) = A(t,x,\vartheta) = \frac{1}{2}\varphi_1(q(0,t,x)) + \frac{1}{2}\beta_1\left(u_0(p(0,t,x)) + \int\limits_0^t \vartheta(s,p(s,t,x))ds\right) +$$

$$+\frac{\beta_2}{2}\int\limits_0^t \vartheta(s,q)ds + \int\limits_0^t F(s,q)\left(u_0(p(0,s,q)) + \int\limits_0^s \vartheta(v,p(v,s,q))dv\right)ds. \tag{13}$$

Lemma 2. There exists $T^* > 0$, such that the integral equation (13) has a unique solution in $\overline{C}(G_2(T^*))$.

Proof of Lemma 2.

For equation (13), we apply the principle of compressed mappings. Let in the region $G_2(T)$ at $T < T_*$:

$\| \vartheta - \phi \| \leq M$,

where

$$\phi(x) = \frac{1}{2}\varphi_1(q(0,t,x)) + \frac{1}{2}\beta_1 u_0(p((0,t,x)) + \int\limits_0^t F(s,q)\left(u_0(p(0,s,q)) + \int\limits_0^s \vartheta(v,p(v,s,q))dv\right)ds +$$

$$+\int_0^t \omega_0(p(0,s,q))ds + \int_0^t \int_0^\tau \omega_1(q(0,s,p(s,\tau,q))dsd\tau +$$

$$+\int_0^t \int_0^\rho \int_0^v \int_0^\tau K(\tau,s)u(s,q(\tau,v,p(v,\rho,q)))dsd\tau dvd\rho$$

Fair ratings:

$\| A\vartheta - \phi \| \leq bKT + (\gamma K + \| f \|)\frac{T^2}{2} = \Omega_0(T)$,

$\| A_1\vartheta^1 - A_1\vartheta^2 \| \leq \Omega_0(T) \| \vartheta^1 - \vartheta^2 \|$,

where

$\| \vartheta \| \leq \| \phi \| + M = K$,

$\Omega_1(T) = bT + (\gamma + \| f \|)\frac{T^2}{2}$. Let T_0, T_1-are respectively positive roots of the equations: $\Omega_0(T) = M, \Omega_1(T) = 1$.

It follows from this that the operator A under $T < T^* = min\{T_0, T_1\}$ performs a compressed mapping. Therefore, the stated initial problem has a unique solution. This solution can be obtained by the method of successive approximations.

We found $\vartheta(t,x)$ Substituting it into (11), we find the unknown function $u(t,x)$. And the unknown function $\omega(t,x)$ is determined from (7).

3. Conclusions

In this work we do not reduce the initial value problem (1)-(2) to a canonical form, we reduce it to a system of integral equations. The developed scheme of using the method of additional argument for partial integro-differential equation of fourth order can be applied to partial integro-differential equations of higher order of other classes.

References

[1] Ashirbaeva A.J., Zholdoshova Ch.B. Solving of non-linear partial integro-differential equation of hyperbolic type (in Russian). *Bulletin of OshSU, series of natural and medical sciences*, 2012, No. 2, issue 1, pp. 144-149.

[2] Ashirbaeva A.J., Zholdoshova Ch.B. Invertigation of solutions of partial integro-differential equation (in Russian). *Bulletin of OshSU, series of natural and medical sciences*, 2012, No. 2, issue 1, pp. 150-153.

[3] Ashirbaeva A.J., Mamaziyaeva E.A. New way to build solutions of partial differential equations of the fourth order of hyperbolic type (in Russian). *Eurasian Scientific association*, 2019, No. 2-1(48), pp. 6-9.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

SOLVING OF SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER BY MEANS OF THE METHOD OF ADDITIONAL ARGUMENT

AIZHARKYN ASHIRBAEVA[1], MALIKA BEKIEVA[2]

[1]Osh Technological University named after M. M.Adyshev, Osh, Kyrgyzstan
[2]Osh State University, Osh, Kyrgyzstan
e-mail: aijarkyn.osh@mail.ru, malikabekieva@gmail.com

Abstract. The application of the method of additional argument to a system of high-order partial integro-differential equations is relevant. Kyrgyz scientists have considered applications of this method to a system of partial differential equations of first order. In this paper, the system of partial integro-differential equations of the second order with initial conditions is reduced to a system of integral equations.

Keywords: method of additional argument, system of equations, second order, partial derivatives, initial problem, integral equation, contraction.

Investigations of various classes of systems of partial differential equations of the first order on the base of the method of additional argument were considered in [1-3].

Let $\overline{C}^{(k)}(G_2(T))$ be the class of functions being continuous and bounded together with their derivatives up to the k-th order in $G_2(T) = [0,T] \times R$. There is considered the system of partial differential equations of the second order of type

$$\frac{\partial^2 u_i(t,x)}{\partial t^2} = k_i^2(t,x)\frac{\partial^2 u_i(t,x)}{\partial x^2} + a_i(t,x)\frac{\partial u_i(t,x)}{\partial t} + b_i(t,x)\frac{\partial u_i(t,x)}{\partial x} + f_i(t,x,u_1,u_2),$$
$$i = 1,2, (t,x) \in G_2(T) \tag{1}$$

with the initial conditions

$$\frac{\partial^k u_i(t,x)}{\partial t^k}|_{t=0} = u_i^k(x), i,k = 0,1, x \in R \tag{2}$$

where

$$f_i(t,x,u_1,u_2) = \psi_i(t,x,u_1,u_2) + \int_0^t K_i(t,s)u_i(s,x)ds,$$
$$u_i^k(x) \in \overline{C}^{(2-k)}(R), b_i(t,x), a_i(t,x) \in \overline{C}^{(2)}(G_2(T)),$$
$$\psi_i(t,x,u_1,u_2) \in \overline{C}^{(2)}(G_2(T) \times R^2),$$
$$K_i(t,s) \in C(G), \int_0^T \mid K_i(t,s) \mid ds \leq \gamma = const, (i,k = 0,1).$$

The paper proposes a new way of reducing the system of partial integro-differential equations to the system of integral equations. Let us introduce some notation. Using method of additional

argument.

In this paper, to reduce problem (1), (2) to a system of integral equations, the following notation is used:

1. Through $p_i(s,t,x), q_i(s,t,x), i = 1,2$ respectively denote the solutions of the following integral equations:

$$p_i(s,t,x) = x + \int_s^t k_i(v, p_i, (v,t,x))dv \tag{3}$$

$$q_i(s,t,x) = x - \int_s^t k_i(v, q_i, (v,t,x))dv \tag{4}$$

$i = 1,2, (s,t,x) \in Q_2(T), \quad Q_2(T) = \{(s,t,x) \mid 0 \le s \le t \le T, x \in R\}$; 2. Differential operator:

$D[\omega] = \frac{\partial}{\partial t} + \omega \frac{\partial}{\partial x}$; 3. New functions

$$\vartheta_i(t,x) = D[-k_i(t,x)]u_i(t,x), i = 1,2 \tag{5}$$

$g_i(t,x) = \frac{1}{k_i(t,x)}[b_i(t,x) - \frac{\partial k_i(t,x)}{\partial t} - k_i(t,x)\frac{\partial k_i(t,x)}{\partial x}], i = 1,2, \beta_i^1(t,x) = a_i(t,x) + g_i(t,x), \beta_i^2(t,x) = a_i(t,x) + g_i(t,x), \beta_i^3(t,x) = D[k_i(t,x)]\beta_i^1(t,x), i = 1,2$. **Lemma 1.** The system of partial integro-differential of second order (1) with the initial condition (2) is equivalent to the following system of integral equations:

$$\vartheta_i(t,x) = \frac{1}{2}\varphi_i(q_i(0,t,x)) + \frac{1}{2}\beta_i^1(t,x)u_i + \frac{1}{2}\int_0^t \beta_i^2(s,q_i)\vartheta_i(s,q_i)ds -$$

$$-\frac{1}{2}\int_0^t \beta_i^3(s,q_i)u_i(s,q_i)ds + \int_0^t f_i(s,q_i,u_1(s,q_i),u_2(s,q_i))ds, \tag{6}$$

$$u_i(t,x) = u_i^0(p_i(0,t,x)) + \int_0^t \vartheta_i(s,p_i(s,t,x))ds, i = 1,2, \tag{7}$$

where
$[2\vartheta_i(t,x) - \beta_i^1(t,x,u_i)u_i(t,x)] \mid_{t=0} = \varphi_i(x), i = 1,2.$

Proof. First, differentiating (6), (7) we prove that the system of integral equations (6), (7) satisfy equation (1) and initial condition (2).

From (6) we have:

$$\frac{\partial \vartheta_i(t,x)}{\partial t} + k_i(t,x)\frac{\partial \vartheta_i(t,x)}{\partial x} = a_i(t,x)\frac{\partial u_i(t,x)}{\partial t} + k_i(t,x)g_i(t,x)\frac{\partial u_i(t,x)}{\partial x} + f_i(t,x,u_1,u_2), \tag{8}$$

$i = 1,2,$
Using the new functions (5), from (8) we obtain (1). Therefore, (2) also holds.

Let us now show that the solution of problem (1), (2) is a solution of system of integral equations (6), (7). To do this, we write equation (1) in the form

$$D[k_i(t,x)]z_i(t,x,u_i) = \beta_i^2(t,x)\vartheta_i(t,x) - \beta_i^3(t,x)u_i(t,x) + 2f(t,x,u_1,u_2), i = 1,2, \tag{9}$$

where
$z_i(t,x,u_i) = 2\vartheta_i(t,x) - \beta_i^1(t,x)u_i(t,x), i - 1,2.$

The solution of problem (9), (2) by the method of an additional argument is reduced to the integral equation (6). It follows from the notation (3) that (7)

In equation (6), substituting (7), we obtain an integral equation with
$\vartheta_i(t,x), i=1,2.$

$$\vartheta_i(t,x)=A_i(t,x;\vartheta_1,\vartheta_2)\equiv\frac{1}{2}\varphi_i(q_i(0,t,x))+\frac{1}{2}\beta_i^1(t,x)\Bigg(u_i^0(p_i(0,t,x))+$$

$$+\int_0^t\vartheta_i(s,p_i(s,t,x))ds\Bigg)+\frac{1}{2}\int_0^t\beta_i^2(s,q_i)\vartheta_i(s,q_i)ds-\frac{1}{2}\int_0^t\beta_i^3(s,q_i)\Bigg(u_i^0(p_i(0,s,q_i))+$$

$$+\int_0^s\vartheta_i(v,p_i(v,s,q_i))dv\Bigg)ds+\int_0^t f_i(s,q_i,u_1^0(p_1(0,s,q_1))+\int_0^s\vartheta_1(v,p_i(v,s,q_i))dv,$$

$$u_2^0(p_2(0,s,q_2))+\int_0^s\vartheta_2(v,p_2(v,s,q_2))dv))ds, i=1,2. \tag{10}$$

Lemma 2. There is such a $T^*>0$, that the integral equation (10) has a unique solution in $\overline{C}(G_2(T^*))$.

Proof. Let us show that Eq. (10) has a unique, continuous solution in the domain $G_2(T)$ at $T<T_*$ that satisfies the inequality
$\| \vartheta_i-\phi_i \|\leq M,$

$$\phi_i\equiv\frac{1}{2}\varphi_i(q_i(0,t,x))+\frac{1}{2}\beta_i^1(t,x)u_i^0(p_i(0,t,x))-\frac{1}{2}\int_0^t\beta_i^3(s,q_i)u_i^0(p_i(0,s,q_i))ds, i=1,2. \tag{11}$$

Let us show that, for $T<T_*$, the operators Ai are contraction operators
$\| A_i\vartheta_i-\phi_i \|\leq (M_0K+\|f\|)T+NK\frac{T^2}{2}=\Omega_0(T),$
where
$\| \vartheta_i \|\leq\| \phi_i \|+M=K, i=1,2.$
$| \beta_i^k(t,x) |\leq M_0=const, i,k=1,2, \quad \beta_i^3(t,x)\leq N_0=const, i=1,2.$
Denote by T_0- the positive root of the equation $\Omega_0(T)=M$. It remains for us to show that the operators Ai compresses the distance between elements.
The following estimate is valid
$\| A_i\vartheta^1-A_i\vartheta^2 \|\leq\Omega_1(T)\| \vartheta^1-\vartheta^2 \|,$
where
$\Omega_1(T)=M_0T+[\frac{N}{2}+L_1+L_2+\gamma]\frac{T^2}{2}$
$| \psi_i(t,x,u_1^1,u_2^1)-\psi_i(t,x,u_1^2,u_2^2) |\leq L_1 | u_1^1-u_2^1 |+L_2 | u_1^1-u_2^1 | L_i\geq 0, L_i-const, i=1,2.$
Denote by the positive root of the equation $\Omega_1(T)=1$.

It follows from this that the operators A_i under $T<T^*=min\{T_0,T_1\}$ perform a contraction mapping. Then the equation defines a unique solution that belongs to this ball. This solution can be obtained by the method of successive approximations.

Conclusion

We can construct solutions of linear partial differential equations of the second order with given initial conditions using the proposed scheme of reduction to an integral equation.

References

[1] Imanaliev M.I., Alekseenko S.N. Contribution to theory of systems of partial differential equations of Whitham type, *Russian Academy of Sciences. Doklady. Mathematics*, 1992, vol. 325, No. 6, pp. 1111-1115.

[2] Ashirbaeva A.J., Mambetov Zh.I. Solving of integro-differential equations by the method of additional argument (in Russian). *Bulletin of OshSU, special issue*, 2013, No. 1, pp. 91-94.

[3] Ashirbaeva A.J., Mambetov Zh.I. Method of additional argument for system of non-linear partial differential equations of the first order with many variables (in Russian). *Science, new technologies and innovations of Kyrgyzstan*. Bishkek, 2017, No. 5, pp. 87-90.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

INTEGRATION OF THE MODIFIED KORTEWEG-DE VRIES-LIOUVILLE-SINE-GORDON EQUATION (MKDF-L-SG) IN THE CLASS OF PERIODIC INFINITE-GAP FUNCTIONS

AKNAZAR KHASANOV[1], KHODJIMUROD NORMURODOV[2], ULUGBEK XUDAYOROV[3]

[1,2] Samarkand State University named after Sharof Rashidov, Samarkand, Uzbekistan,
[3] Samarkand State University of Architecture and Construction, Samarkand, Uzbekistan
e-mail: xasanovaknazar@samdu.uz,
e-mail: normurodov@samdu.uz,
e-mail: xudayorov.2022@bk.ru

Abstract. In this paper, the inverse spectral problem method is used to integrate the nonlinear equation mKdF-L-sG in the class of periodic infinite-gap functions. The solvability of the Cauchy problem for an infinite system of Dubrovin differential equations in the class of sixfold continuously differentiable periodic infinite-gap functions is proved. It is shown that the sum of a uniformly convergent functional series constructed by solving the system of Dubrovin equations and the first trace formula satisfies the mKdF-L-sG equations. Moreover, it is proved that if the initial function is a real, π-periodic analytic function, then the solution of the Cauchy problem for the mKdF-L-sG equation is also a real, π-periodic analytic function with respect to x, and if the number $\frac{\pi}{2}$ is the period (antiperiod) of the initial function, then the number $\frac{\pi}{2}$ is also the period (antiperiod) with respect to x of the solution of the Cauchy problem for mKdF-L-sG equations.

Keywords: modified Korteweg-de Vries-Liouville-sine Gordon equation (mKdF-L-sG), Dirac operator, spectral data, Dubrovin system of equations, trace formulas.

AMS Subject Classification: 35J10, 34A55.

1. Introduction

One of the representatives of integrable nonlinear partial differential equations, which has the largest number of applications, is the nonlinear modified Korteweg-de Vries (mKdV) equations. It is used as a model for the nonlinear evolution of plasma waves and is a physical model that includes the propagation of transverse waves in a molecular chain model and in a generalized elastic solid.

The Liouville equation and the Liouville-type system occur in nonlinear optics.

Sine-Gordon equation arise in a variety of areas, such as the description of pseudospherical surfaces in differential geometry or the propagation of a magnetic vortex in an infinitely long Joseph junction.

In this paper, we consider the Cauchy problem for a combination of the modified Korteweg-de Vries, Liouville equation and the sine-Gordon (mKdF-L-sG) of the following form:

$$q_{xt} = a(t)\left\{q_{xxxx} - \frac{3}{2}q_x^2 q_{xx}\right\} + b(t)e^{q} + c(t)e^{-q}, \quad q = q(x,t),\ x \in R,\ t > 0 \tag{1}$$

with initial conditions

$$q(x,t)|_{t=0} = q_0(x), \qquad q_0(x+\pi) = q_0(x) \in C^6(R) \tag{2}$$

in the class of real infinite-gap π-periodic functions with respect to x:

$$q(x+\pi,t)=q(x,t), \quad q(x,t)\in C^{4,1}_{x,t}(t>0)\cap C(t\geq 0). \tag{3}$$

Here $a(t),\ b(t)\in C([0,\infty))$ are given continuous bounded functions.

It is easy to see that the conditions for the compatibility of linear equations

$$y_x=\begin{pmatrix} \frac{1}{2}q_x & -\lambda \\ \lambda & -\frac{1}{2}q_x \end{pmatrix} y, y_t=\left\{\frac{1}{2\lambda}\begin{pmatrix} 0 & b(t)e^{q} \\ c(t)e^{-q} & 0 \end{pmatrix}+a(t)\begin{pmatrix} A & B \\ C & -A \end{pmatrix}\right\} y, y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

is equivalent to the equation (1) for the function $q=q(x,t), x\in R, t>0$. Here

$$A=-2\lambda^2 q_x-\frac{1}{4}q_x^3+\frac{1}{2}q_{xxx}, B=\lambda q_{xx}+4\lambda^3+\frac{\lambda}{2}q_x^2, C=\lambda q_{xx}-4\lambda^3-\frac{\lambda}{2}q_x^2.$$

Note that from the equation (1), for the case $a(t)=1$, $b(t)=0$, $c(t)=0$ we obtain the modified Korteweg-de Vries equation (mKdV) [1-2], for $a(t)=0$, $b(t)=1$, $c(t)=0$ we have the Liouville equation [see [3, p. 14], [4]], and for $a(t)=0$, $b(t)=c(t)=\frac{1}{2}$ we have the sine-Gordon(sG) type equation[3]. Modified KdV equation of the following form:

$$u_t=u_{xxx}-6u^2u_x,\ u=u(x,t),$$

was integrated in [1]-[2], (see also [5]-[8]) in the class of rapidly decreasing and finite-gap functions. In addition, for finite-gap solutions, an explicit formula was derived in terms of the Riemann theta functions. Thus, the solvability of the Cauchy problem for the mKdV equation was established (see [5–8]) for any finite-gap initial data. This theory is described in more detail in monographs [9–10], as well as in papers [11–13]. It should be noted that the solvability of the Cauchy problem for the mKdV equation in the class of five times continuously differentiable periodic infinite-gap functions was proved in [14], i.e., $u(x+\pi,0)=u(x,0)\in C^5(R)$. Sine-Gordon (sG) equations of the form

$$u_{xt}=\sin u,\ u=u\,(x,t)\,,\ x\in R,\ t>0$$

was integrated in [15], [16], and also [17]–[20], in the class of rapidly decreasing and finite-gap functions. In addition, an explicit formula was derived for finite-gap solutions in terms of the Riemann theta functions. Thus, they (see [17]-[20]) proved the solvability of the Cauchy problem for the sine-Gordon equation for any finite-gap initial data.

It is known [21] that if $q(x)=2a\cos 2x,\ a\neq 0$, then in the spectrum of the Sturm-Liouville operator $Ly\equiv -y''+q(x)y,\ x\in R$ all gaps are open, in other words, $q(x)$ is a periodic infinite-gap potential. There are similar examples for the periodic Dirac operator [22].

We note that the Cauchy problem in the class of periodic and almost periodic infinite-gap functions for nonlinear evolution equations without a source and with a source, as well as with an additional term in various formulations, was studied in [23]–[35].

In this paper, we propose an algorithm for constructing a solution $q(x,t),\ x\in R,\ t>0$, of the problem (1)-(3) by reducing to the inverse spectral problem for the Dirac operator:

$$L(\tau,t)y\equiv B\frac{dy}{dx}+\Omega(x+\tau,t)y=\lambda y,\ x\in R,\ t>0,\ \tau\in R \tag{4}$$

where

$$B=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \Omega(x,t)=\begin{pmatrix} P(x,t) & Q(x,t) \\ Q(x,t) & -P(x,t) \end{pmatrix}, P(x,t)=0, Q(x,t)=\frac{1}{2}q_x'(x,t).$$

2. Evolution of spectral data

Denote by

$c(x,\lambda,\tau,t)=(c_1(x,\lambda,\tau,t),c_2(x,\lambda,\tau,t))^T$ and $s(x,\lambda,\tau,t)=(s_1(x,\lambda,\tau,t),s_2(x,\lambda,\tau,t))^T$ solutions of the equation (4) with initial conditions $c(0,\lambda,\tau,t)=(1,0)^T$ and $s(0,\lambda,\tau,t)=(0,1)^T$. Function

$$\Delta(\lambda,\tau,t)=c_1(\pi,\lambda,\tau,t)+s_2(\pi,\lambda,\tau,t)$$

is called the Lyapunov function for the equation (4).

Moreover, for solutions $c(x,\lambda,\tau,t)$ and $s(x,\lambda,\tau,t)$ for large $|\lambda|$ the following asymptotic formulas hold:

$$c(x,\lambda,\tau,t)=\begin{pmatrix}\cos\lambda x\\ \sin\lambda x\end{pmatrix}+O\left(\frac{e^{|Im\lambda|x}}{\lambda}\right),\ |\lambda|\to\infty,$$

$$s(x,\lambda,\tau,t)=\begin{pmatrix}-\sin\lambda x\\ \cos\lambda x\end{pmatrix}+O\left(\frac{e^{|Im\lambda|x}}{\lambda}\right),\ |\lambda|\to\infty.$$

From these asymptotics for real λ, we have

$$\Delta(\lambda,\tau,t)=2\cos\lambda\pi+O\left(\frac{1}{\lambda}\right),\ |\lambda|\to\infty.$$

Vector function

$$\psi^{\pm}(x,\lambda,\tau,t)=(\psi_1^{\pm}(x,\lambda,\tau,t),\psi_2^{\pm}(x,\lambda,\tau,t))^T=c(x,\lambda,\tau,t)+m^{\pm}(\lambda,\tau,t)s(x,\lambda,\tau,t)$$

are called Floquet solutions for the equation (4). The Weil-Titchmarsh functions are defined by the following formulas [36]

$$m^{\pm}(\lambda,\tau,t)=\frac{s_2(\pi,\lambda,\tau,t)-c_1(\pi,\lambda,\tau,t)\mp\sqrt{\Delta^2(\lambda,\tau,t)-4}}{2s_1(\pi,\lambda,\tau,t)}.$$

The spectrum of the Dirac operator $L(\tau,t)$ is purely continuous and consists of the set

$$\sigma(L)=\{\lambda\in R:\ |\Delta(\lambda)|\le 2\}=R\backslash\left(\bigcup_{n=-\infty}^{\infty}(\lambda_{2n-1},\lambda_{2n})\right).$$

The intervals $(\lambda_{2n-1},\lambda_{2n})$, $n\in Z\backslash\{0\}$, are called gaps, where λ_n, the roots of the equation $\Delta(\lambda)\mp 2=0$. They coincide with the eigenvalues of the periodic or antiperiodic $(y(0)=\pm y(\pi))$ problem for the equation (4). It is easy to prove that $\lambda_{-1}=\lambda_0=0$, i.e. $\lambda=0$ is the double eigenvalue of the periodic problem for the equation (4).

The roots of the equation $s_1(\pi,\lambda,\tau,t)=0$ will be denoted by $\xi_n(\tau,t), n\in Z\backslash\{0\}$, while $\xi_n(\tau,t)\in[\lambda_{2n-1},\lambda_{2n}]$, $n\in Z\backslash\{0\}$. Since the coefficient in the equation (4) has the form $P(x,t)\equiv 0$, $Q(x,t)=\frac{1}{2}q_x'(x,t)$, then $\lambda_{-1}=\lambda_0=\xi_0=0$, i.e. $\xi=0$ is an eigenvalue of the Dirichlet problem.

Numbers $\xi_n(\tau,t), n\in Z\backslash\{0\}$, and signs $\sigma_n(\tau,t)=sign\{s_2(\pi,\xi_n,\tau,t)-c_1(\pi,\xi_n,\tau,t)\}$, $n\in Z\backslash\{0\}$, are called the spectral parameters of the operator $L(\tau,t)$. Spectral parameters $\xi_n(\tau,t)$, $\sigma_n(\tau,t)=\pm 1$, $n\in Z\backslash\{0\}$, and the spectrum boundaries $\lambda_n(\tau,t), n\in Z\backslash\{0\}$, are called the spectral data of the Dirac operator $L(\tau,t)$.

The problem of recovering the coefficient $\Omega(x,t)$ of the operator $L(\tau,t)$ from spectral data is called the inverse problem. The coefficients $p(x+\tau,t)$ and $q(x+\tau,t)$ of the operator $L(\tau,t)$ are determined uniquely from the spectral data $\lambda_n(\tau,t),\xi_n(\tau,t),\sigma_n(\tau,t)=\pm 1$, $n\in Z$.

If using the initial function $q_0(x+\tau)$, $\tau\in R$, we construct the Dirac operator $L(\tau,0)$ of the form

$$L(\tau,0)y\equiv B\frac{dy}{dx}+\Omega_0(x+\tau)y=\lambda y,\ x\in R,\ \tau\in R$$

$$\Omega_0(x+\tau)=\begin{pmatrix}0 & \frac{1}{2}q_0'(x+\tau)\\ \frac{1}{2}q_0'(x+\tau) & 0\end{pmatrix},\ y=\begin{pmatrix}y_1\\ y_2\end{pmatrix},$$

then we will see that the boundaries of the spectrum $\lambda_n(\tau), n \in Z$, of the resulting problem do not depend on the parameter $\tau \in R$, i.e. $\lambda_n(\tau) = \lambda_n, n \in Z$,. The spectral parameters depend on the parameter τ: $\xi_n^0 = \xi_n^0(\tau), \sigma_n^0 = \sigma_n^0(\tau) = \pm 1, n \in Z$, and they are periodic functions: $\xi_n^0(\tau + \pi) = \xi_n^0(\tau), \sigma_n^0(\tau + \pi) = \sigma_n^0(\tau), \tau \in R, n \in Z$.

Solving the direct problem, we find the spectral data $\left\{\lambda_n,\ \xi_n^0(\tau),\ \sigma_n^0(\tau),\ n \in Z \setminus \{0\}\right\}$ of the operator $L(\tau, 0)$.

The inverse problem for the Dirac operator of the form

$$Ly \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda y, x \in R$$

with periodic coefficients $p(x) = p(x + \pi), q(x) = q(x + \pi)$, in various formulations were studied in [37],[38] and [39-45] . It should be noted that the inverse problem in terms of spectral data $\{\lambda_n,\ \xi_n,\ \sigma_n,\ n \geq 1\}$ for the Hill operator was studied in [13], [39] and [46], [48].

The main result of this paper is contained in the following theorem:

Theorem 1. Let $q(x, t),\ x \in R,\ t > 0$, be the solution to the Cauchy problem (1)-(3). Then the limits of the spectrum $\lambda_n(\tau, t), n \in Z \setminus \{0\}, \tau \in R$, of the operator $L(\tau, t)$ do not depend on the parameters τ and t, i.e. $\lambda_n(\tau, t) = \lambda_n, n \in Z \setminus \{0\}$, and the spectral parameters $\xi_n(\tau, t),\ \sigma_n(\tau, t) = \pm 1, n \in Z \setminus \{0\}$, satisfy the first and second systems of Dubrovin's differential equations, respectively :

$$1. \begin{cases} \frac{\partial \lambda_n(\tau,t)}{\partial \tau} = 0, \quad n \in Z \setminus \{0\}, \\ \\ \frac{\partial \xi_n(\tau,t)}{\partial \tau} = 2(-1)^{n-1} \sigma_n(\tau, t) h_n\left(\xi(\tau, t)\right) \xi_n(\tau, t), \quad n \in Z \setminus \{0\}; \end{cases} \tag{5}$$

$$2. \begin{cases} \frac{\partial \lambda_n(\tau,t)}{\partial t} = 0, \quad n \in Z \setminus \{0\}, \\ \\ \frac{\partial \xi_n(\tau,t)}{\partial t} = 2(-1)^{n} \sigma_n(\tau, t) h_n(\xi(\tau, t)) g_n(\xi(\tau, t)),\ n \in Z \setminus \{0\}. \end{cases} \tag{6}$$

Here the sign of $\sigma_n(\tau, t) = \pm 1,\ n \in Z \setminus \{0\}$, is reversed for each collision of point $\xi_n(\tau, t), n \in Z \setminus \{0\}$, with the boundaries of its gap $[\lambda_{2n-1}, \lambda_{2n}]$. In addition, the following initial conditions are satisfied

$$\xi_n(\tau, t)|_{t=0} = \xi_n^0(\tau), \quad \sigma_n(\tau, t)|_{t=0} = \sigma_n^0(\tau),\ n \in Z \setminus \{0\} \tag{7}$$

where $\xi_n^0(\tau),\ \sigma_n^0(\tau) = \pm 1,\ n \in Z \setminus \{0\}$ are spectral parametres of the operator Dirac $L(\tau, 0)$. The sequence $h_n(\xi)$ and $g_n(\xi),\ n \in Z \setminus \{0\}$, involved in the equation (6) is determined by the following formulas:

$$h_n(\xi) = \sqrt{(\xi_n(\tau, t) - \lambda_{2n-1})(\lambda_{2n} - \xi_n(t, \tau))} \times f_n(\xi),$$

$$f_n(\xi) = \sqrt{\prod_{k=-\infty k \neq n}^{\infty} \frac{(\lambda_{2k-1} - \xi_n(\tau, t))(\lambda_{2k} - \xi_n(\tau, t))}{(\xi_k(\tau, t) - \xi_n(\tau, t))^2}}, \tag{8}$$

$$g_n(\xi) = a(t) \left\{ 4\xi_n^3(\tau, t) + \xi_n(\tau, t) \left[\frac{1}{2} q_\tau^2 + q_{\tau\tau} \right] \right\} + \frac{b(t)}{2\xi_n(\tau, t)} \exp\left\{ q(\tau, t) \right\}.$$

Proof. Let $q(x, t),\ x \in R,\ t > 0$ is π is periodic function with respect to x satisfy the equation (1). Denote by $y_n = (y_{n,1}(x, \tau, t), y_{n,2}(x, \tau, t))^T,\ n \in Z \setminus \{0\}$ orthonormal eigenfunctions of the operator $L(\tau, t)$ considered on the interval $[0, \pi]$, with Dirichlet boundary conditions

$$y_1(0, \tau, t) = 0, y_1(\pi, \tau, t) = 0, \tag{9}$$

corresponding to the eigenvalues $\xi_n = \xi_n(\tau, t),\ n \in Z \setminus \{0\}$. Differentiating with respect to t the identity

$$\xi_n(\tau, t) = (L(\tau, t) y_n, y_n),\ n \in Z \setminus \{0\},$$

and using the symmetry of the operator $L(\tau,t)$, we have

$$\frac{\partial\xi_n(\tau,t)}{\partial t}=(\dot{\Omega}(x+\tau,t)y_n,y_n),\ n\in Z\backslash\{0\}. \tag{10}$$

Using the explicit form of the scalar product

$$(y,z)=\int_0^\pi[y_1(x)\overline{z_1(x)}+y_2(x)\overline{z_2(x)}]dx,\ y=\begin{pmatrix}y_1(x)\\y_2(x)\end{pmatrix},\ z=\begin{pmatrix}z_1(x)\\z_2(x)\end{pmatrix},$$

the equality (10) rewrite in the following form

$$\frac{\partial\xi_n(\tau,t)}{\partial t}=\int_0^\pi y_{n,1}y_{n,2}q_{xt}dx. \tag{11}$$

Substituting the expressions (1) into (11), we obtain the following equality

$$\frac{\partial\xi_n(\tau,t)}{\partial t}=a(t)I_1(\tau,t)+b(t)I_2(\tau,t)+c(t)I_3(\tau,t) \tag{12}$$

where

$$I_1(\tau,t)=\int_0^\pi\left[y_{n,1}y_{n,2}\left\{q_{xxxx}(x+\tau,t)-\frac{3}{2}q_x^2(x+\tau,t)q_{xx}(x+\tau,t)\right\}\right]dx, \tag{13}$$

$$I_2(\tau,t)=\int_0^\pi\left[y_{n,1}y_{n,2}e^{q(x+\tau,t)}\right]dx. \tag{14}$$

$$I_3(\tau,t)=\int_0^\pi\left[y_{n,1}y_{n,2}e^{-q(x+\tau,t)}\right]dx \tag{15}$$

Applying the following equations

$$\begin{cases}y_{n,1}(x,\tau,t)=\frac{1}{\xi_n(\tau,t)}\left(y'_{n,2}(x,\tau,t)+\frac{1}{2}q'_x(x+\tau,t)y_{n,2}(x,\tau,t)\right),\\ y_{n,2}(x,\tau,t)=\frac{1}{\xi_n(\tau,t)}\left(-y'_{n,1}(x,\tau,t)+\frac{1}{2}q'_x(x+\tau,t)y_{n,1}(x,\tau,t)\right)\end{cases}$$

it is easy to derive the following equality

$$y_{n,1}y_{n,2}\left\{q_{xxxx}(x+\tau,t)-\frac{3}{2}q_x^2(x+\tau,t)q_{xx}(x+\tau,t)\right\}=\left\{\left(\frac{1}{2}\xi_nq_x^2+4\xi_n^3-\xi_nq_{xx}\right)y_{n,1}^2+\right.$$

$$\left.+\left(q_{xxx}-\frac{1}{2}q_x^3-4\xi_n^2q_x\right)y_{n,1}y_{n,2}+\left(\frac{1}{2}\xi_nq_x^2+4\xi_n^3+\xi_nq_{xx}\right)y_{n,2}^2\right\}'.$$

Using this formula, it is easy to calculate $I_1(\tau,t)$:

$$I_1(\tau,t)=\left(\frac{1}{2}\xi_nq_\tau^2(\tau,t)+4\xi_n^3(\tau,t)+\xi_nq_{\tau\tau}(\tau,t)\right)\left[y_{n,2}^2(\pi,\tau,t)-y_{n,2}^2(0,\tau,t)\right]. \tag{16}$$

Now, let's calculate $I_2(\tau,t)$ and $I_3(\tau,t)$:

$$I_2(\tau,t)=\int_0^\pi\left[y_{n,1}y_{n,2}e^{q(x+\tau,t)}\right]dx=$$

$$=\frac{1}{\xi_n(\tau,t)}\int_0^\pi\left[y_{n,2}e^{q(x+\tau,t)}\left(y'_{n,2}(x,\tau,t)+\frac{1}{2}q'_x(x+\tau,t)y_{n,2}(x,\tau,t)\right)\right]dx=$$

$$=\frac{1}{2\xi_n(\tau,t)}\left(\int_0^\pi\left[y_{n,2}^2(x,\tau,t)e^{q(x+\tau,t)}\right]'dx\right)=$$

$$=\frac{1}{2\xi_n(\tau,t)}e^{q(\tau,t)}\left[y_{n,2}^2(\pi,\tau,t)-y_{n,2}^2(0,\tau,t)\right],$$

$$I_3(\tau,t)=\int_0^\pi\left[y_{n,1}y_{n,2}e^{-q(x+\tau,t)}\right]dx=$$

$$=\frac{1}{\xi_n(\tau,t)}\int_0^\pi\left[y_{n,1}e^{-q(x+\tau,t)}\left(-y'_{n,1}(x,\tau,t)+\frac{1}{2}q'_x(x+\tau,t)y_{n,1}(x,\tau,t)\right)\right]dx=$$

$$= \frac{1}{2\xi_n(\tau,t)} \left(\int_0^\pi \left[-y_{n,1}^2(x,\tau,t) e^{-q(x+\tau,t)} \right]' dx \right) = 0$$

i.e.

$$I_2(\tau,t) = \frac{1}{2\xi_n(\tau,t)} e^{q(\tau,t)} \left[y_{n,2}^2(\pi,\tau,t) - y_{n,2}^2(0,\tau,t) \right], \tag{17}$$

$$I_3(\tau,t) = 0. \tag{18}$$

Substituting (16),(17) and (18) into (12), we have

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = \left[a(t) \left\{ 4\xi_n^3(\tau,t) + \xi_n(\tau,t) \left[\frac{1}{2} q_\tau^2 + q_{\tau\tau} \right] \right\} + \frac{b(t)}{2\xi_n(\tau,t)} \exp\{q(\tau,t)\} \right] \times$$

$$\times \left[y_{n,2}^2(\pi,\tau,t) - y_{n,2}^2(0,\tau,t) \right]. \tag{19}$$

Since the eigenvalues $\xi_n(\tau,t)$ of the Dirichlet problem for the equation (4) are simple, the following equality holds

$$y_n(x,\tau,t) = \frac{1}{c_n(\tau,t)} s(x,\tau,\xi_n(t),t),$$

where

$$c_n^2(\tau,t) = \int_0^\pi [s_1^2(x,\xi_n(\tau,t),\tau,t) + s_2^2(x,\xi_n(\tau,t),\tau,t)]dx \ = -\frac{\partial s_1(\pi,\xi_n(\tau,t),\tau,t)}{\partial \lambda} \cdot s_2(\pi,\xi_n(\tau,t),\tau,t).$$

Using these equalities, we have

$$y_{n,2}^2(\pi,\tau,t) - y_{n,2}^2(0,\tau,t) = -\frac{s_2(\pi,\xi_n(\tau,t),\tau,t) - \frac{1}{s_2(\pi,\xi_n(\tau,t),\tau,t)}}{\frac{\partial s_1(\pi,\xi_n(\tau,t),\tau,t)}{\partial \lambda}}.$$

Substituting into this relation the equality

$$s_2(\pi,\xi_n(\tau,t),\tau,t) - c_1(\pi,\xi_n(\tau,t),\tau,t) = s_2(\pi,\xi_n(\tau,t),\tau,t) - \frac{1}{s_2(\pi,\xi_n(\tau,t),\tau,t)} =$$

$$= \sigma_n(\tau,t)\sqrt{\Delta^2(\xi_n(\tau,t)) - 4}$$

we have

$$y_{n,2}^2(\pi,\tau,t) - y_{n,2}^2(0,\tau,t) = -\frac{\sigma_n(\tau,t)\sqrt{\Delta^2(\xi_n(\tau,t)) - 4}}{\frac{\partial s_1(\pi,\xi_n(\tau,t),\tau,t)}{\partial \lambda}}. \tag{20}$$

Given expansions

$$\Delta^2(\lambda) - 4 = -4\pi^2 \prod_{k=-\infty}^{\infty} \frac{(\lambda - \lambda_{2k-1})(\lambda - \lambda_{2k})}{a_k^2},$$

$$s_1(\pi,\lambda,t) = \pi \prod_{k=-\infty}^{\infty} \frac{\xi_k - \lambda}{a_k}$$

where $a_0 = 1$ and $a_k = k$ at $k \neq 0$, rewrite the equation (20) of the following form:

$$y_{n,2}^2(\pi,t) - y_{n,2}^2(0,t) = 2(-1)^n \sigma_n(\tau,t) h_n(\xi).$$

Substituting this expression into the identity (19), we have (6). Similarly to the previous one, (5) can be proved.

If we replace the Dirichlet boundary conditions with periodic ($y(0,t) = y(\pi,t)$) or antiperiodic ($y(0,t) = -y(\pi,t)$) boundary conditions, then instead of (19) equation, we have

$$\frac{\partial \lambda_n(\tau,t)}{\partial t} = 0, \text{ i.e. } \lambda_n(\tau,t) = \lambda_n(\tau,0), \ n \in Z \setminus \{0\}.$$

Now in the equation $L(\tau,t)\nu_n = \lambda_n(\tau,t)\nu_n,\ \ n \in Z\setminus\{0\}$, put $t=0$. Since the eigenvalues $\lambda_n(\tau) = \lambda_n(\tau,0),\ \ n \in Z\setminus\{0\}$, periodic or antiperiodic problem for the equation $L(\tau,0)\nu_n = \lambda_n(\tau)\nu_n,\ \ n \in Z\setminus\{0\}$ do not depend on the parameter $\tau \in R$, we have $\lambda_n(\tau,t) = \lambda_n(\tau) = \lambda_n,\ \ n \in Z\setminus\{0\}$. The theorem has been proven. ■

Further, taking into account the trace formulas

$$q'_\tau(\tau,t) = 2\sum_{k=-\infty,k\neq 0}^{\infty}(-1)^{k-1}\sigma_k(\tau,t)\,h_k(\xi(\tau,t)), \tag{21}$$

$$q(\tau,t) = C(t) + 2\int_0^\tau\left(\sum_{k=-\infty,\,k\neq 0}^{\infty}(-1)^{k-1}\sigma_k(s,t)\,h_k(\xi(s,t))\right)ds, \tag{22}$$

$$\left(\frac{1}{2}q_\tau(\tau,t)\right)^2 + \frac{1}{2}q_{\tau\tau}(\tau,t) = \sum_{k=-\infty,k\neq 0}^{\infty}\left(\frac{\lambda_{2k-1}^2+\lambda_{2k}^2}{2} - \xi_k^2(\tau,t)\right) \tag{23}$$

where $C(t)$ is given continuously bounded function, the system (6) can be rewritten in closed form:

$$\frac{\partial\xi_n(\tau,t)}{\partial t} = 2(-1)^n\sigma_n(\tau,t)\sqrt{(\xi_n(\tau,t)-\lambda_{2n-1})(\lambda_{2n}-\xi_n(t,\tau))}\cdot f_n(\xi)\cdot g_n(\xi(\tau,t)) \tag{24}$$

where

$$g_n(\xi) = a(t)\left[4\xi_n^3(\tau,t) + 2\xi_n(\tau,t)\sum_{k=-\infty,k\neq 0}^{\infty}\left(\frac{\lambda_{2k-1}^2+\lambda_{2k}^2}{2} - \xi_k^2(\tau,t)\right)\right] +$$

$$+\frac{b(t)}{2\xi_n(\tau,t)}\exp\left\{C(t) + 2\int_0^\tau\left(\sum_{k=-\infty,\,k\neq 0}^{\infty}(-1)^{k-1}\sigma_k(s,t)\,h_k(\xi(s,t))\right)ds\right\}. \tag{25}$$

The trace formula for $q'_\tau(\tau,t)$ can be proved by applying the Mittag-Leffler theorem to the function:

$$\frac{s_2(\pi,\lambda,\tau,t) - c_1(\pi,\lambda,\tau,t)}{s_1(\pi,\lambda,\tau,t)}.$$

As a result of the change of variables

$$\xi_n(\tau,t) = \lambda_{2n-1} + (\lambda_{2n}-\lambda_{2n-1})\sin^2 x_n(\tau,t),\ \ n \in Z\setminus\{0\} \tag{26}$$

Dubrovin's system of differential equations (24) and the initial conditions (7) can be rewritten as a single equation in the Banach space K:

$$\frac{dx(\tau,t)}{dt} = H(x(\tau,t)),\quad x(\tau,t)|_{t=0} = x^0(\tau) \in K \tag{27}$$

where

$$K = \left\{x(\tau,t) = (...,x_{-1}(\tau,t),x_1(\tau,t),...) : \|x(\tau,t)\| = \sum_{n=-\infty,n\neq 0}^{\infty}(1+|n|)\,\gamma_n\,|x_n(\tau,t)| < \infty\right\},$$

$$H(x) = (...,\,H_{-1}(x),\,H_1(x),...),\quad H_n(x) = (-1)^n\sigma_n(0)\cdot g_n(...,\,\lambda_1+(\lambda_2-\lambda_1)\sin^2 x_1(\tau,t),\,...)\times$$
$$\times f_n(...,\,\lambda_1+(\lambda_2-\lambda_1)\sin^2 x_1(\tau,t),\,,...) = (-1)^n\sigma_n(0)g_n(x(\tau,t))\,f_n(x(\tau,t)).$$

It is known that if $q_0(x+\pi) = q_0(x) \in C^6(R)$ then $(q_0(x))' \in C^5(R)$. Therefore, for the gap length of the operator $L(\tau,0)$, we have the equality (see [36], p. 98):

$$\gamma_k \equiv \lambda_{2k} - \lambda_{2k-1} = \frac{|q_{2k}^5|}{2^4\,|k|^5} + \frac{\delta_k}{|k|^6}, \tag{28}$$

where

$$\lambda_{2k},\lambda_{2k-1}=k+\sum_{j=1}^{6}c_jk^{-j}\pm 2^{-5}\left|k\right|^{-5}\left|q_{2k}^{5}\right|+\left|k\right|^{-6}\varepsilon_k^{\pm},$$

$$\sum_{k=-\infty}^{\infty}\left|q_{2k}^{5}\right|^2<\infty,\quad \sum_{k=-\infty}^{\infty}(\varepsilon_k^{\pm})^2<\infty,\quad \delta_k=\varepsilon_k^{+}-\varepsilon_k^{-}.$$

Hence, taking into account $\xi_n(\tau,t)\in[\lambda_{2n-1},\lambda_{2n}]$, we get

$$\inf_{k\neq n}\left|\xi_n(\tau,t)-\xi_k(\tau,t)\right|\geq a>0.$$

Now, using this inequality and (28), we estimate the functions

$$\left|f_n(x(\tau,t))\right|,\ \left|\tfrac{\partial f_n(x(\tau,t))}{\partial x_m}\right|,\ \left|g_n(x(\tau,t))\right|,\ \left|\tfrac{\partial g_n(x(\tau,t))}{\partial x_m}\right|.$$

LEMMA 1. *The following estimates hold:*

$$1.C_1\leq\left|f_n(x)\right|\leq C_2,\left|\frac{\partial f_n(x)}{\partial x_m}\right|\leq C_3\gamma_m, \tag{29}$$

$$2.\left|g_n(x)\right|\leq C_4\left|n\right|^3,\left|\frac{\partial g_n(x)}{\partial x_m}\right|\leq C_5\gamma_m\left|m\right|\left|n\right|,m,\ n\in Z\backslash\{0\} \tag{30}$$

where $C_j>0$, $j=1,2,3,4,5$, do not depend on the parameters m and n.

Proof. The inequality (29) was proved in [30]. Therefore, we prove the inequalities (30). Since the functions $a(t)$ and $b(t)$ are bounded, then $\exists M_j>0, j=1,2$, such that the inequalities $|a(t)|\leq M_1,\ \ |b(t)|\leq M_2$. Using this inequality and (26), as well as (29), we obtain the following estimates:

$$\left|g_n(x(\tau,t)\right|\leq 4M\left|\lambda_{2n-1}+\gamma_n\sin^2x_n(\tau,t)\right|^3+M|\lambda_{2n-1}+\gamma_n\sin^2x_n(\tau,t)|\times$$

$$\times\left|\sum_{k=-\infty,k\neq0}^{\infty}\left\{\left[\lambda_{2k}+\lambda_{2k-1}+\gamma_k\sin^2x_k(\tau,t)\right]\gamma_k\cos^2x_k(\tau,t)\right\}\right|+$$

$$+M|\lambda_{2n-1}+\gamma_n\sin^2x_n(\tau,t)|\left|\sum_{k=-\infty,k\neq0}^{\infty}\left\{\left[2\lambda_{2k-1}+\gamma_k\sin^2x_k(\tau,t)\right]\gamma_k\sin^2x_k(\tau,t)\right\}\right|+$$

$$+\frac{M}{2|\lambda_{2n-1}+\gamma_n\sin^2x_n(\tau,t)|}\times$$

$$\times\exp\left\{|C(t)|+\int_0^{\tau}\left|\sum_{k=-\infty,\,k\neq0}^{\infty}(-1)^{k-1}\gamma_k\sigma_k^0(s)\sin(2x_k(s,t))f_k\left(x\left(s,t\right)\right)\right|ds\right\}\leq$$

$$\leq 4M(A_1|n|)^3+MA_1|n|\sum_{k=-\infty,k\neq0}^{\infty}\{A_2|n|\gamma_k\}+MA_1|n|\sum_{k=-\infty,k\neq0}^{\infty}\{A_3|n|\gamma_k\}$$

$$+\frac{M}{2A_4|n|}\exp\left\{M_1+\int_0^{\tau}\sum_{k=-\infty,\,k\neq0}^{\infty}C_2\gamma_kds\right\}\leq C_4|n|^3,$$

where $M=max\left\{M_1,M_2\right\}$.

Now, let's evaluate the function $\left|\frac{\partial g_n(x(\tau,t))}{\partial x_m}\right|$:

$$\left|\frac{\partial g_n(x(\tau,t))}{\partial x_m}\right|\leq|a(t)|\left|-2[\lambda_{2n-1}+\gamma_n\sin^2x_n(\tau,t)](\lambda_{2m-1}+\gamma_m\sin^2x_m(\tau,t))\right|\times$$

$$\times\gamma_m \sin^2 x_m(\tau,t) + \frac{|b(t)|}{2|\lambda_{2n-1}+\gamma_n \sin^2 x_n(\tau,t)|}\times$$

$$\times \exp\left\{|C(t)| + \int_0^\tau \left|\sum_{k=-\infty,\,k\neq 0}^{\infty} (-1)^{k-1}\gamma_k \sigma_k^0(s)\sin(2x_k(s,t)) f_k\left(x\left(s,t\right)\right)\right| ds\right\}\times$$

$$\times[\left|\int_0^\tau (-1)^{m-1}\gamma_m\sigma_m^0(s)\left[2\cos(2x_m(s,t)) f_m\left(x(s,t)\right) + \sin(2x_m(s,t))\frac{\partial f_m\left(x(s,t)\right)}{\partial x_m(s,t)}\right] ds\right| +$$

$$+\int_0^\tau \left|\sum_{k=-\infty,\,k\neq m}^{\infty} (-1)^{k-1}\gamma_k\sigma_k^0(s)\sin(2x_k(s,t))\frac{\partial f_k\left(x(s,t)\right)}{\partial x_m(s,t)}\right| ds] \leq$$

$$\leq MB_1|n||m|\gamma_m + \frac{M}{2A_4|n|}\exp\left\{M_1 + \int_0^\tau \sum_{k=-\infty,\,k\neq 0}^{\infty} C_2\gamma_k ds\right\}\times$$

$$\times\left\{B_2\gamma_m|\tau| + B_3\gamma_m^2|\tau| + \gamma_m\int_0^\tau \sum_{k=-\infty,\,k\neq m}^{\infty} C_3\gamma_k ds\right\} \leq C_5\gamma_m|n||m|.$$

Lemma 1 is proved.

Lemma 2. If $q_0(x+\pi) = q_0(x) \in C^6(R)$, then the vector function $H(x(\tau,t))$ satisfies the Lipschitz condition in the Banach space K, i.e. there exists a constant $L > 0$ such that the following inequality holds for arbitrary elements $x(\tau,t)$, $y(\tau,t) \in K$

$$\|H(x(\tau,t)) - H(y(\tau,t))\| \leq L\,\|x(\tau,t) - y(\tau,t)\|$$

where

$$L = C\sum_{n=-\infty,n\neq 0}^{\infty} |n|^3\left(|n|+1\right)\left(\lambda_{2n}-\lambda_{2n-1}\right) < \infty. \tag{31}$$

Proof. First, using Lemma 1, we estimate the derivative with respect to $x_m(\tau,t)$ of the function $F_n(x) = \bar{g}_n(x) f_n(x), n \in Z\setminus\{0\}$:

$$\left|\frac{\partial F_n(x)}{\partial x_m}\right| = \left|\frac{\partial g_n(x)}{\partial x_m} f_n(x) + \frac{\partial f_n(x)}{\partial x_m} g_n(x)\right| \leq$$

$$\leq \left|\frac{\partial f_n(x)}{\partial x_m}\right| |g_n(x)| + \left|\frac{\partial g_n(x)}{\partial x_m}\right| |f_n(x)| \leq$$

$$\leq C_3C_4\,|n|^3\,\gamma_m + C_2C_5\,|m|\,|n|\,\gamma_m \leq C\,|n|^3\left(|m|+1\right)\gamma_m$$

where $C = const > 0$ do not depend on the parameters m and n.

Next, using the expression

$$H_n(x(\tau,t)) = (-1)^n\sigma_n^0(\tau)F_n(x(\tau,t)), n \in Z\setminus\{0\},$$

we get the equality

$$|H_n(x(\tau,t)) - H_n(y(\tau,t))| = |F_n(x(\tau,t)) - F_n(y(\tau,t))|.$$

Now, let's apply Lagrange's finite increment theorem to the function

$$\varphi(t) = F_n(x + t(y-x)),$$

on the interval $t \in [0,1]$. Then we get the equality

$$\varphi(1) - \varphi(0) = \varphi'(t^*),$$

i.e.

$$F_n(x) - F_n(y) = \sum_{m=-\infty,m\neq 0}^{\infty} \frac{\partial F_n(\theta)}{\partial x_m}(x_m - y_m),$$

where $\theta = x + t^*(y - x)$. Hence it follows that

$$|H_n(x(\tau,t)) - H_n(y(\tau,t))| = |F_n(x(\tau,t)) - F_n(x(\tau,t))| \leq$$

$$\leq \sum_{m=-\infty, m\neq 0}^{\infty} \left| \frac{\partial F_n(\theta)}{\partial x_m} \right| \cdot |x_m(\tau,t) - y_m(\tau,t)| \leq$$

$$\leq C|n|^3 \sum_{m=-\infty, m\neq 0}^{\infty} (|m|+1) \cdot |\lambda_{2m} - \lambda_{2m-1}| \cdot |x_m(\tau,t) - y_m(\tau,t)| = C|n|^3 \|x - y\|$$

Now estimate the norm $\|H(x(\tau,t)) - H(y(\tau,t))\|$:

$$\|H(x) - H(y)\| = \sum_{n=-\infty, n\neq 0}^{\infty} (|n|+1)(\lambda_{2n} - \lambda_{2n-1}) |H_n(x) - H_n(y)| \leq$$

$$\leq \sum_{n=-\infty, n\neq 0}^{\infty} C|n|^3 (|n|+1)(\lambda_{2n} - \lambda_{2n-1}) \cdot \|x - y\| = L\|x - y\|.$$

where

$$L = \sum_{n=-\infty, n\neq 0}^{\infty} C|n|^3 (|n|+1)(\lambda_{2n} - \lambda_{2n-1}) = C \sum_{n=-\infty, n\neq 0}^{\infty} |n|^3 (|n|+1)\gamma_n < \infty.$$

Thus, the Lipschitz condition is satisfied. Therefore, the solution of the Cauchy problem (6), (7) for all $t > 0$ and $\tau \in R$ exists and is unique. Lemma 2. is proved. ■

Remark 1. Theorem 1 and Lemma 2 give a method for solving the problem (1)-(3).

Proof. First find the spectral data λ_n, $\xi_n^0(\tau)$, $\sigma_n^0(\tau) = \pm 1$, $n \in Z\setminus\{0\}$ of the Dirac operator $L(\tau, 0)$. Denote the spectral data of the operator $L(\tau,t)$ by λ_n, $\xi_n(\tau,t)$, $\sigma_n(\tau,t) = \pm 1$, $n \in Z\setminus\{0\}$. Then solving the Cauchy problem (24), (7) for an arbitrary value of τ, we find $\xi_n(\tau,t)$, $\sigma_n(\tau,t)$, $n \in Z\setminus\{0\}$. From the trace formula (21) we define the function $q_\tau(\tau,t)$, i.e. solving the problem (1)-(3). ■

So far, we have assumed that the (1)-(3) Cauchy problem has a solution. It is not difficult to get rid of this assumption by directly verifying that the function $q_\tau(\tau,t)$, $\tau \in R$, $t > 0$ obtained in this way actually satisfies the equations (1).

Remark 2. The function $q_\tau(\tau,t)$ constructed using the Dubrovin system of equations (6), (7) and the trace formula (21) satisfies the equation (1).

Proof. To do this, we use the second trace formula

$$\left(\frac{1}{2} q_\tau(\tau,t)\right)^2 + \frac{1}{2} q_{\tau\tau}(\tau,t) = \sum_{k=-\infty, k\neq 0}^{\infty} \left(\frac{\lambda_{2k-1}^2 + \lambda_{2k}^2}{2} - \xi_k^2(\tau,t) \right). \tag{32}$$

Differentiating the formula (32) with respect to t, we have

$$\frac{1}{2} q_\tau(\tau,t) q_{\tau t}(\tau,t) + \frac{1}{2} (q_{\tau t}(\tau,t))_\tau = -\sum_{k=-\infty, k\neq 0}^{\infty} 2\xi_k(\tau,t) \frac{\partial \xi_k(\tau,t)}{\partial t}. \tag{33}$$

Here, we used the equality $(q_\tau(\tau,t))_{t\tau} = (q_\tau(\tau,t))_{\tau t}$. Further, taking into account the Dubrovin system of equations (6), from (33), we obtain

$$q_\tau(\tau,t) z(\tau,t) + z_\tau(\tau,t) = -8 \sum_{k=-\infty, k\neq 0}^{\infty} (-1)^k \sigma_k(\tau,t) h_k(\xi) \xi_k(\tau,t) \times$$

$$\times \left[a(t) \left\{ 4\xi_k^3(\tau,t) + \xi_k(\tau,t) \left[\frac{1}{2} q_\tau^2 + q_{\tau\tau} \right] \right\} + \frac{b(t)}{2\xi_k(\tau,t)} \exp\{q(\tau,t)\} \right] \tag{34}$$

where

$$z(\tau,t) = q_{\tau t}(\tau,t). \tag{35}$$

Now, we will use the Dubrovin system of differential equations corresponding to the equation (4):

$$\frac{\partial \xi_n(\tau,t)}{\partial \tau} = 2(-1)^{n-1}\sigma_n(\tau,t)\, h_n(\xi(\tau,t))\,\xi_n(\tau,t), \tag{36}$$

as well as the trace formula (21). Then from (34), we get the following equality:

$$q_\tau(\tau,t)\,z(\tau,t) + z_\tau(\tau,t) = 4a(t)\sum_{k=-\infty,k\neq 0}^{\infty}\frac{\partial \xi_k^4}{\partial \tau} +$$

$$+2a(t)\left[\frac{1}{2}q_\tau^2 + q_{\tau\tau}\right]\sum_{k=-\infty,k\neq 0}^{\infty}\frac{\partial \xi_k^2}{\partial \tau} + 2b(t)e^{q(\tau,t)}q_\tau(\tau,t). \tag{37}$$

Next, we calculate the sums of the following functional series:

$$\sum_{k=-\infty,k\neq 0}^{\infty}\frac{\partial \xi_k^4}{\partial \tau}, \sum_{k=-\infty,k\neq 0}^{\infty}\frac{\partial \xi_k^2}{\partial \tau}.$$

To do this, we differentiate with respect to τ the trace formulas (32) and

$$-q_{\tau\tau\tau\tau}(\tau,t) - q_\tau(\tau,t)q_{\tau\tau\tau}(\tau,t) + q_{\tau\tau}(\tau,t)q_\tau^2(\tau,t) + \frac{1}{4}q_\tau^4(\tau,t) =$$

$$= 4\sum_{k=-\infty,k\neq 0}^{\infty}\left(\frac{\lambda_{2k}^4 + \lambda_{2k-1}^4}{2} - \xi_k^4(\tau,t)\right) \tag{38}$$

we have

$$q_{\tau\tau\tau\tau\tau} + q_{\tau\tau}q_{\tau\tau\tau} + q_\tau q_{\tau\tau\tau\tau} - q_{\tau\tau\tau}q_\tau^2 - 2q_{\tau\tau}^2 q_\tau - q_\tau^3 q_{\tau\tau} = 4\sum_{k=-\infty,k\neq 0}^{\infty}\frac{\partial \xi_k^4}{\partial \tau},$$

$$q_\tau(\tau,t)\,q_{\tau\tau}(\tau,t) + q_{\tau\tau\tau}(\tau,t) = -2\sum_{k=-\infty,k\neq 0}^{\infty}\frac{\partial \xi_k^2(\tau,t)}{\partial \tau}.$$

Using these formulas from (37), we find

$$q_\tau(\tau,t)\,z(\tau,t) + z_\tau(\tau,t) = a(t)\left(q_{\tau\tau\tau\tau\tau} + q_{\tau\tau}q_{\tau\tau\tau} + q_\tau q_{\tau\tau\tau\tau} - q_{\tau\tau\tau}q_\tau^2 - 2q_{\tau\tau}^2 q_\tau - q_\tau^3 q_{\tau\tau}\right) -$$

$$-a(t)\left[\frac{1}{2}q_\tau^2 + q_{\tau\tau}\right](q_\tau q_{\tau\tau} + q_{\tau\tau\tau}) + 2b(t)e^{q(\tau,t)}q_\tau(\tau,t),$$

i.e.

$$q_\tau(\tau,t)\,z(\tau,t) + z_\tau(\tau,t) = a(t)\left[q_{\tau\tau\tau\tau\tau} + q_\tau q_{\tau\tau\tau\tau} - \frac{3}{2}q_{\tau\tau\tau}q_\tau^2 - 3q_{\tau\tau}^2 q_\tau - \frac{3}{2}q_\tau^3 q_{\tau\tau}\right] +$$

$$+2b(t)e^{q(\tau,t)}q_\tau(\tau,t). \tag{39}$$

It is easy to check that the following function

$$z(\tau,t) = a(t)\left(q_{\tau\tau\tau\tau} - \frac{3}{2}q_\tau^2 q_{\tau\tau}\right) + b(t)e^{q(\tau,t)} + C_1(t)e^{-q(\tau,t)}$$

is a solution to the linear equation (39). Choosing $C_1(t) = c(t)$, we have

$$z(\tau,t) = a(t)\left(q_{\tau\tau\tau\tau} - \frac{3}{2}q_\tau^2 q_{\tau\tau}\right) + b(t)e^{q} + c(t)e^{-q(\tau,t)}.$$

From here and from the notation (35), we get the equations (1):

$$q_{\tau t} = a(t)\left\{q_{\tau\tau\tau\tau} - \frac{3}{2}q_\tau^2 q_{\tau\tau}\right\} + b(t)e^{q} + c(t)e^{-q(\tau,t)}.$$

Remark 3. The uniform convergence of the series in the above formulas (21)-(23) and (31), as well as (38) follows from the equality (28) and the estimate (29).

Thus, we have proved the following theorem.

Theorem 2. If the initial function $q_0(x)$ satisfies the condition

$$q_0(x+\pi) = q_0(x) \in C^6(R),$$

then there exists a uniquely determined solution $q'_x(x,t),\ x \in R,\ t > 0$ of the problem (1)-(3), which is determined by the formula (21).

Result 1. Using the results of [38] and [39], we deduce that if the initial function $q_0(x)$ is real analytic π periodic function, then the solution $q(x,t)$ of the problem (1)-(3) is a real analytic function with respect to x.

Result 2. If the number $\frac{\pi}{2}$ is a period (antiperiod) for the initial function $q_0(x)$, then all roots of the equation $\Delta(\lambda) + 2 = 0$ $(\Delta(\lambda) - 2 = 0)$ are twofold. Since the Lyapunov function corresponding to the coefficient $q(x,t)$ coincides with $\Delta(\lambda)$, according to the results of [40] and [41] the number $\frac{\pi}{2}$ is also a period (antiperiod) for the solution $q(x,t)$ with respect to the variable x.

References

[1] Wadati M. The exact solution of the modified Korteweg-de Vries equation. J.Phys.Soc.Japn., 32:6,44-47(1972).

[2] Khasanov A. B., Hoitmetov U. A. On Integration of the Loaded mKdV Equation in the Class of Rapidly Decreasing Functions. Bulletin of Irkutsk State University, Series Mathematicsthis link is disabled, 2021, 38, pp. 19–35

[3] Zhiber A.V., Murtozina R.D., Habibullin I.T., Shabat A.B. Haraktericheskie kol'co Li i nelinejnye integriruemye uravnenija. Moskva, Izhevsk 2012.

[4] Zhiber A. V., Ibragimov N. H., Shabat A. B., "Uravnenija tipa Liuvillja", Dokl. AN SSSR, 249:1 (1979), 26–29

[5] Its A.R. Obrashhenie giperjellipticheskih integralov i integrirovanie nelinejnyh differencial'nyh uravnenij. Vestn. Leningr. un-ta. Cer. Matem. Mehan. Astron., 7:2(1976), 39-46.

[6] A. O. Smirnov, Elliptic solutions of the nonlinear Schrödinger equation and the modified Korteweg–de Vries equation, Russian Acad. Sci. Sb. Math., 82:2 (1995), 461–470

[7] V. B. Matveev, A. O. Smirnov, Solutions of the Ablowitz–Kaup–Newell–Segur hierarchy equations of the "rogue wave" type: A unified approach, Theoret. and Math. Phys., 186:2 (2016), 156–182.

[8] V. B. Matveev, A. O. Smirnov, Two-phase periodic solutions to the AKNS hierarchy equations, J. Math. Sci. (N. Y.), 242:5 (2019), 722–741.

[9] Zaharov V.E., Manakov S.V., Novikov S.P., Pitaevskij L.P. Teorija solitonov: metod obratnoj zadachi. M., Nauka, 1980.

[10] Mitropol'skij Ju.A., Bogoljubov N.N (ml.), Prikarpatskij A.K., Samojlenko V.G. Integriruemye dinamicheskie sistemy: spektral'nye i differencial'no-geometricheskie aspekty. Kiev: Naukova dumka, 1987.

[11] Matveev V.B. 30 years of finite-gap integration theory. Phil. Trans. R Soc. A (2008) 366, p. 837-875.

[12] Babazhanov B.A., Khasanov A.B. Integration of equation of Toda periodic chain Kind. Ufa Mathematical Journal, 2017, 9(2), pp. 17-24.

[13] Khasanov A. B., Yakhshimuratov A. B. Inverse problem on the half-line for the Sturm-Liouville operator with periodic potential. Differential Equations, 2015, 51(1), pp. 23-32.

[14] Khasanov, A.B., Allanazarova, T.Z. On the Modified Korteweg-De-Vries Equation with Loaded Term. Ukrainian Mathematical Journal, 2022, 73(11), pp. 1783-1809.

[15] Zaharov V.E., Tahtadzhjan L.A., Faddeev L.D. Polnoe opisanie reshenij "Sin-Gordon" uravnenija. Dokl. ANSSSR. 1974. t. 219, vyp 6, s.1334-1337.

[16] Ablowitz M.J., Kaup D.J., Newell A.C. and Sugur H. Method for Solving the Sine-Gordon Equation. Phys. Rev.Lett. 1973. v.30, №25, p.1262-1264.

[17] Kozel V.A., Kotljarov V.P. Konechnozonnye reshenija uravnenija sine-Gordon. Preprint FTINT ANUSSR, №9-77, Har'kov, 1977.

[18] Kozel V.A., Kotljarov V.P. Javnye formuly dlja reshenij nelinejnogo uravnenija sinus-Gordon. Dokl. ANUSSR. ser.A. 1976, №11, str. 878-880.

[19] B. A. Dubrovin, S. M. Natanzon, "Real two-zone solutions of the sine-Gordon equation", Funct. Anal. Appl., 16:1 (1982), 21–33.

[20] Smirnov A.O. 3-Elliptik solutions of the sine-Gordon Equation. Mathematical Notes, vol. 62, No 3, 1997, p.368-376.
[21] Ince E.L. Ordinary Differential Equations. New York: Dover, 1956.
[22] P. B. Djakov, B. S. Mityagin, Instability zones of periodic 1-dimensional Schrödinger and Dirac operators, Uspekhi Mat. Nauk, 61:4(370) (2006), 77–182.
[23] A. B. Hasanov, M. M. Hasanov, Integration of the nonlinear Schrödinger equation with an additional term in the class of periodic functions, Theoret. and Math. Phys., 199:1 (2019), 525–532.
[24] A. B. Khasanov, M. M. Matjakubov, Integration of the nonlinear Korteweg–de Vries equation with an additional term, Theoret. and Math. Phys., 203:2 (2020), 596–607.
[25] Hasanov A. B., Hasanov T. G. Zadacha Koshi dlja uravnenija Kortevega-de Friza v klasse periodicheskih beskonechnozonnyh funkcij. Zapiski nauch. semin. POMI, t.506, 2021, s.258-279.
[26] Khasanov A.B., Khasanov, T.G. Integration of a nonlinear Korteweg-de Vries Equation with a loaded term and a source. Journal of Applied and Industrial Mathematics. 2022, 16(2), pp. 227-239
[27] Domrin A. V. Zamechanija o lokal'nom variante metoda obratnoj zadachi rassejanija. Tr. MIRAN, 2006, t.253, s.46-60.
[28] Khasanov A.B., Yaxshimuratov A.B. The Almost-Periodicity of infinite-Gap Potensials of the Dirac operator. Doklady Mathematics. 1996, 54(2), pp. 767-769.
[29] Mannonov G.A., Hasanov A.B. Zadacha Koshi dlja nelinejnogo uravnenija Hirota, v klasse periodicheskih beskonechnozonnyh funkcij. Algebra i analiz. Tom 34(2022), No5, s.139-172.
[30] U. B. Muminov, A. B. Khasanov, Integration of a defocusing nonlinear Schrödinger equation with additional terms, Theoret. and Math. Phys., 211:1 (2022), 514–531.
[31] Muminov U.B., Khasanov A.B. The Cauchy Problem for the Defocusing Nonlinear Schrödinger Equation with a Loaded Term. Siberian Advances in Mathematicsthis link is disabled, 2022, 32(4), 277–298.
[32] McKean H and Trubowitz E. Hill's operator and hyperelliptic function theory in the presence of infinitely many branchpoints. Commun.Pure Appl. Math., 1976, 29, 143-226.
[33] McKean H and Trubowitz E. Hill's surfaces and their theta functions. Bull. Amer. Math. Soc., 1978, 84, 1052-1085.
[34] Lax P. Almost periodic solutions of the KdV equation. SCAM Reveu. 1976.
[35] B. M. Levitan, "Almost periodicity of infinite-zone potentials", Math. USSR-Izv., 18:2 (1982), 249–273.
[36] A. A. Danielyan, B. M. Levitan, A. B. Khasanov, "Asymptotic behavior of Weyl–Titchmarsh m-function in the case of the Dirac system", Math. Notes, 50:2 (1991), 816–823.
[37] Misjura T.V. Harakteristika spektrov periodicheskoj kraevyh zadach, porozhdaemyh operaciej Diraka I. Teorija funkciej, funkcional'nyj analiz i ih prilozhenija. Har'kov: Vishha shkola, 1978. t.30 s.90-101.
[38] Khasanov A.B., Ibragimov A.M. Ob obratnoj zadache dlja operatora Diraka s periodicheskim potencialom. Uzb.matem.zhurn.,2001,No3-4.,s.48-55.
[39] Trubowtz E. The inverse problem for periodic potentials. Commun. Pure. Apple. Math. 1977. v. 30, p. 321-337.
[40] Khasanov A.B., Yahshimuratov A.B. Analog obratnoj teoremy G.Borga dlja operatora Diraka. Uzb.matem.zhurn., 2000, No3-4.,s.40-46.
[41] Currie S., Roth T., Watson B. Borg's periodicity theorems for first-order self-adjoint systems with complex potentials. Proc. Edinb. Math. Soc., 60:3(2017), 615-633.
[42] Battig D., Grebert B., Guillot J. C. and Kappeler T. Folation of phase space for the cubic non-linear Schrodinger equation. Composito Mathematica, 1993, 85(2), pp.163-199.
[43] Grebert B and Guillot J. C. .Gap of one dimensional periodic AKNS systems. Forum Math., 1993, 5(5), pp.459-504.
[44] Korotayev E. Inverse problem and estimates for periodic Zakharov-Shabat systems. J.Reine Angew. Math., 2005, 583(2005), pp. 87-115.
[45] Korotayev E and Mokeev D. Dubrovin equation for periodic Dirac operator on the half-line. Applicable Analysis, 2020, 101(1), pp.1-29.
[46] Stankevich I. V. "Ob odnoj obratnoj zadache spektral'nogo analiza dlja uravnenija Hilla", Dokl. AN SSSR, 192:1 (1970), 34–37.
[47] Ahiezer N. I. Kontinual'nye analogi ortogonal'nyh mnogochlenov na sisteme intervalov, Dokl. AN SSSR, 141:2 (1961), 263–266
[48] Flaschka H. On the inverse problem for Hill's operator// Arch. Rational Mech. Anal., 1975, v. 59, No4, p. 293-309.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

THE CRITERION OF INVERTIBILITY OF THE MIXED TYPE MINIMAL OPERATOR

ALEXANDER ROGOVOY [1,2], TYNYSBEK KALMENOV [1]

[1]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[2]Miras University, Shymkent, Kazakhstan
e-mail: rog2005@list.ru, kalmenov.t@mail.ru

Abstract. Minimal operators generated by overdetermined boundary conditions for differential equations are extremely important when describing regular boundary value problems for differential equations. In addition, for inverse problems of mathematical physics arising from applications, when determining unknown data, it is necessary to study problems with overdetermined boundary conditions, including the boundary conditions of minimal operators. Thus, the study of minimal differential operators is of both theoretical and applied interest. In this paper, a criterion for the regular solvability of the differential operator generated by the overdetermined Cauchy problem for the Gellerstedt equation is established. The proof is based on the Gellerstedt potential, the properties of solutions to the Goursat problem in the characteristic triangle, and the properties of special functions. It should be noted that differential operators of mixed type have numerous applications in transonic gas dynamics, the theory of infinitesimal surface bends, the instantaneous theory of shells with variable sign curvature, magnetodynamics and hydrodynamics.

Keywords: minimal differential operator, Gellerstedt equation, criterion, boundary condition, hypergeometric function.

AMS Subject Classification: 35L80, 35M10, 35N30, 33C05

Minimal operators generated by overdetermined boundary conditions for differential equations are used in the description of regular boundary value problems for differential equations.

The concept of a minimal operator is closely related to the theory of restrictions and extensions of operators, in particular, all regular boundary value problems for linear differential equations are boundary conditions of regular extensions of the corresponding minimal differential operators.

The theory of regular extensions for elliptic equations was first constructed by M.I.Vishik [1]. The development of M.I.Vishik's theory for arbitrary equations in Banach space, as well as the construction of invertible restrictions generated not necessarily by boundary conditions, was started with the work of M.O.Otelbaev and A.N.Shynybekov [2]. In the works of T.Sh.Kalmenov [3]-[4], using regular extensions method, a description of a wide class of regular boundary value problems for the Lavrent'ev-Bitsadze and the Tricomi equations of mixed type is given. In addition to the theoretical significance in the description of regular boundary value problems for differential equations, the study of overdetermined boundary value problems is also important from a practical point of view. For inverse problems of mathematical physics arising from applications, when determining unknown data, it is necessary to study problems with overdetermined boundary conditions, including the boundary conditions of minimal operators.

This paper is devoted to the study of the overdetermined Cauchy problem for the Gellerstedt equation. As is known, a deep study of mixed-type equations was started with the work of

F.Tricomi [5], and the problem he studied was called the Tricomi problem. S.Gellerstedt [6] solves the Tricomi problem for a more general equation, which was later called the Gellerstedt equation. It should be noted that differential equations of mixed type, in particular the Gellerstedt equation, have numerous applications in transonic gas dynamics, the theory of infinitesimal bending of surfaces, the instantaneous theory of shells with curvature of variable sign, magnetodynamics and hydrodynamics and are studied in the works of S.A.Chaplygin [7], F.I.Frankl [8], C.S.Moravetz [9]-[10].

In this paper, a criterion for the regular solvability of the differential operator generated by the overdetermined Cauchy problem for the Gellerstedt equation is established. The proof is based on the properties of the Gellerstedt potential, the properties of solutions to the Goursat problem in the characteristic triangle, and the properties of special functions.

Let $\Omega \subset R^2$ is a domain, bounded by segment $AB: y = 0,\ 0 < x < 1$, and by characteristics $AC:\ x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0$ and $BC:\ x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 1$ of the Gellerstedt equation

$$Lu \equiv -(-y)^m u_{xx} + u_{yy} = f(x,y), \tag{1}$$

The overdetermined Cauchy problem is considered: to find a regular solution of (1) in the Ω domain, satisfying the conditions:

$$u|_{y=0} = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \tag{2}$$

$$u\big|_{AC:\, x-\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=0} = 0, \quad u\big|_{BC:\, x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=1} = 0. \tag{3}$$

Let us first consider the Goursat problem ((1), (3) problem). In characteristic variables $\xi = x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}}$, $\eta = x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}}$ the Riemann function of the Goursat problem has the next form [11]:

$$R(\xi,\eta,\xi_1,\eta_1) = k \cdot \frac{(\eta_1-\xi_1)^{2\beta}}{(\eta-\xi_1)^{\beta}\cdot(\eta_1-\xi)^{\beta}} \cdot F(\beta,\beta;1;\sigma), \tag{4}$$

where

$$F(a,b;c;z) - hypergeometric function;$$

$$\sigma = \frac{(\xi_1-\xi)(\eta_1-\eta)}{(\xi_1-\eta)(\eta_1-\xi)}, \quad ,k = \frac{\Gamma(\beta)}{\Gamma(1-\beta)\cdot\Gamma(2\beta)}, \quad \beta = \frac{1}{2(m+2)}, \tag{5}$$

$$\Gamma(z) - gamma function.$$

In this case, the solution of the Goursat problem ((1), (3) problem) is represented as:

$$u(\xi,\eta) = \int_0^{\xi} d\xi_1 \int_1^{\eta} R(\xi,\eta,\xi_1,\eta_1)\cdot f_1(\xi_1,\eta_1)\, d\eta_1, \tag{6}$$

where

$$f_1(\xi_1,\eta_1) = \frac{1}{4}\left(\frac{4}{m+2}\right)^{4\beta}(\eta_1-\xi_1)^{-4\beta} f\left(\frac{\xi_1+\eta_1}{2}, -\left(\frac{m+2}{4}\right)^{1-2\beta}(\eta_1-\xi_1)^{1-2\beta}\right). \tag{7}$$

Thus, function (6) satisfies equation (1) and boundary conditions (3). Now we satisfy the initial Cauchy conditions (2) with function (6). We have

$$\lim_{\eta\to\xi} u(\xi,\eta) = \lim_{\eta\to\xi}\int_0^{\xi} d\xi_1 \int_1^{\eta} R(\xi,\eta,\xi_1,\eta_1)\, f_1(\xi_1,\eta_1)\, d\eta_1 = 0, \tag{8}$$

$$\lim_{\eta\to\xi}\left(\frac{m+2}{2}\right)^{2\beta}(\eta-\xi)^{2\beta}\left(\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\xi}\right)\int_0^{\xi} d\xi_1 \int_1^{\eta} R(\xi,\eta,\xi_1,\eta_1)\, f_1(\xi_1,\eta_1)\, d\eta_1 = 0. \tag{9}$$

From the ratio (8), taking into account the formulas [12, p. 556, p. 255]

$$F(a,b,c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\quad c\neq 0,-1,-2,\ldots,\quad Re(c-a-b)>0, \tag{10}$$

$$\Gamma(n+1)=n!, \tag{11}$$

we obtain, taking into account the relation (4), when $\eta\to\xi$:

$$\sigma|_{\xi=\eta}=\left.\frac{(\xi_1-\xi)(\eta_1-\eta)}{(\xi_1-\eta)(\eta_1-\xi)}\right|_{\xi=\eta}=\frac{(\xi_1-\xi)(\eta_1-\xi)}{(\xi_1-\xi)(\eta_1-\xi)}=1;$$

$$F(\beta,\beta;1;\sigma)|_{\xi=\eta}=F(\beta,\beta;1;1)=\frac{\Gamma(1)\Gamma(1-2\beta)}{\Gamma(1-\beta)\Gamma(1-\beta)}=\frac{\Gamma(1-2\beta)}{\Gamma(1-\beta)\Gamma(1-\beta)};$$

$$R(\xi,\eta,\xi_1,\eta_1)|_{\xi=\eta}=k\cdot\frac{(\eta_1-\xi_1)^{2\beta}}{(\eta-\xi_1)^{\beta}\cdot(\eta_1-\xi)^{\beta}}\cdot\frac{\Gamma(1-2\beta)}{\Gamma(1-\beta)\Gamma(1-\beta)},$$

that is , when $\eta\to\xi$:

$$\int_0^{\xi}d\xi_1\int_1^{\xi}k\cdot\frac{(\eta_1-\xi_1)^{2\beta}}{(\xi-\xi_1)^{\beta}\cdot(\eta_1-\xi)^{\beta}}\cdot\frac{\Gamma(1-2\beta)}{\Gamma(1-\beta)\Gamma(1-\beta)}f_1(\xi_1,\eta_1)\,d\eta_1=0,$$

that is, to fulfill the first of the initial Cauchy conditions (2), the function $f(x,y)$ must satisfy the condition

$$\int_0^{\xi}d\xi_1\int_1^{\xi}k\cdot\frac{(\eta_1-\xi_1)^{2\beta}}{(\xi-\xi_1)^{\beta}\cdot(\eta_1-\xi)^{\beta}}f_1(\xi_1,\eta_1)\,d\eta_1=0, \tag{12}$$

where the function $f_1(\xi_1,\eta_1)$ is defined by the formula (7).

To fulfill the second initial Cauchy condition (2), taking into account the formulas [12, p. 559, p. 557, p. 556]

$$F(a,b;c;z)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-z)+$$

$$+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b;c-a-b+1;1-z),\quad |arg(1-z)|<\pi, \tag{13}$$

$$\frac{d}{dz}F(a,b;c;z)=\frac{ab}{c}F(a+1,b+1;c+1;z), \tag{14}$$

$$F(a,b;c;0)=1, \tag{15}$$

and also considering the formula

$$\frac{d}{dy}\int_{\alpha(y)}^{\beta(y)}f(x,y)dx=\int_{\alpha(y)}^{\beta(y)}f(x,y)dx+\beta'(y)f(\beta(y),y)-\alpha'(y)f(\alpha(y),y), \tag{16}$$

taking into account the ratios (9), (4), (5), (10), (11), we get sequentially

$$1-\sigma=1-\frac{(\xi_1-\xi)(\eta_1-\eta)}{(\xi_1-\eta)(\eta_1-\xi)}=\frac{(\xi_1-\eta)(\eta_1-\xi)-(\xi_1-\xi)(\eta_1-\eta)}{(\xi_1-\eta)(\eta_1-\xi)}=$$

$$=\frac{\xi_1\eta_1-\eta\eta_1-\xi\xi_1+\xi\eta-\xi_1\eta_1+\xi\eta_1+\xi_1\eta-\xi\eta}{(\xi_1-\eta)(\eta_1-\xi)}=$$

$$=\frac{\xi\eta_1+\xi_1\eta-\eta\eta_1-\xi\xi_1}{(\xi_1-\eta)(\eta_1-\xi)}=\frac{\eta_1(\xi-\eta)-\xi_1(\xi-\eta)}{(\xi_1-\eta)(\eta_1-\xi)}=$$

$$=\frac{(\eta_1-\xi_1)(\xi-\eta)}{(\xi_1-\eta)(\eta_1-\xi)}=\frac{(\xi_1-\eta_1)(\eta-\xi)}{(\xi_1-\eta)(\eta_1-\xi)};$$

$$\frac{\partial}{\partial\eta}\sigma=\frac{\partial}{\partial\eta}\left(\frac{\xi_1-\xi}{\eta_1-\xi}\cdot\frac{\eta-\eta_1}{\eta-\xi_1}\right)=\frac{\xi_1-\xi}{\eta_1-\xi}\cdot\frac{\eta-\xi_1-(\eta-\eta_1)}{(\eta-\xi_1)^2}=$$

$$=\frac{(\xi_1-\xi)(\eta_1-\xi_1)}{(\eta_1-\xi)(\eta-\xi_1)^2};$$

$$F(\beta,\beta;1;\sigma)=\frac{\Gamma(1)\Gamma(1-2\beta)}{\Gamma(1-\beta)\Gamma(1-\beta)}F(\beta,\beta;2\beta;1-\sigma)+$$

$$+(1-\sigma)^{1-2\beta}\frac{\Gamma(1)\Gamma(2\beta-1)}{\Gamma(\beta)\Gamma(\beta)}F(1-\beta,1-\beta;2-2\beta;1-\sigma)=$$

$$=\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}F(\beta,\beta;2\beta;1-\sigma)+$$

$$+\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}(1-\sigma)^{1-2\beta}F(1-\beta,1-\beta;2-2\beta;1-\sigma)=Q_1(\beta,\sigma);$$

$$\frac{\partial}{\partial\eta}F(\beta,\beta;1;\sigma)=\frac{\partial}{\partial\eta}Q_1(\beta,\sigma)=$$

$$=\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}\frac{\beta\cdot\beta}{2\beta}F(\beta+1,\beta+1;2\beta+1;1-\sigma)\cdot\left(-\frac{(\xi_1-\xi)(\eta_1-\xi_1)}{(\eta_1-\xi)(\eta-\xi_1)^2}\right)+$$

$$+\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}(1-2\beta)(1-\sigma)^{-2\beta}\cdot\left(-\frac{(\xi_1-\xi)(\eta_1-\xi_1)}{(\eta_1-\xi)(\eta-\xi_1)^2}\right)\times$$

$$\times F(1-\beta,1-\beta;2-2\beta;1-\sigma)+$$

$$+\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}(1-\sigma)^{1-2\beta}\cdot\frac{(1-\beta)(1-\beta)}{2-2\beta}\times$$

$$\times F(2-\beta,2-\beta;3-2\beta;1-\sigma)\cdot\left(-\frac{(\xi_1-\xi)(\eta_1-\xi_1)}{(\eta_1-\xi)(\eta-\xi_1)^2}\right)=Q_2(\beta,\sigma);$$

$$\frac{\partial}{\partial\eta}R(\xi,\eta,\xi_1,\eta_1)=\frac{\partial}{\partial\eta}\left(k\cdot\frac{(\eta_1-\xi_1)^{2\beta}}{(\eta_1-\xi)^{\beta}}\cdot\frac{1}{(\eta-\xi_1)^{\beta}}F(\beta,\beta;1;\sigma)\right)=$$

$$=k\frac{(\eta_1-\xi_1)^{2\beta}}{(\eta_1-\xi)^{\beta}}\left(-\frac{\beta\cdot F(\beta,\beta;1;\sigma)}{(\eta-\xi_1)^{\beta+1}}+\frac{1}{(\eta-\xi_1)^{\beta}}\cdot Q_2(\beta,\sigma)\right);$$

$$\frac{\partial}{\partial\eta}\int_0^{\xi}d\xi_1\int_1^{\eta}R(\xi,\eta,\xi_1,\eta_1)\cdot f_1(\xi_1,\eta_1)\,d\eta_1=\int_0^{\xi}d\xi_1\int_1^{\eta}k\frac{(\eta_1-\xi_1)^{2\beta}}{(\eta_1-\xi)^{\beta}}\times$$

$$\times\left(-\frac{\beta\cdot F(\beta,\beta;1;\sigma)}{(\eta-\xi_1)^{\beta+1}}+\frac{1}{(\eta-\xi_1)^{\beta}}\cdot Q_2(\beta,\sigma)\right)f_1(\xi_1,\eta_1)\,d\eta_1+$$

$$+\int_0^{\xi}k\frac{(\eta-\xi_1)^{2\beta}}{(\eta-\xi_1)^{\beta}(\eta-\xi)^{\beta}}F\left(\beta,\beta;1;\frac{(\xi_1-\xi)(\eta-\eta)}{(\xi_1-\eta)(\eta-\xi)}\right)f_1(\xi_1,\eta)\,d\xi_1-0=$$

$$=-k\beta\int_0^{\xi}d\xi_1\int_1^{\eta}\frac{(\eta_1-\xi_1)^{2\beta}}{(\eta_1-\xi)^{\beta}(\eta-\xi_1)^{\beta+1}}F(\beta,\beta;1;\sigma)f_1(\xi_1,\eta_1)\,d\eta_1+$$

$$+k\int_0^{\xi}d\xi_1\int_1^{\eta}\frac{(\eta_1-\xi_1)^{2\beta}}{(\eta_1-\xi)^{\beta}(\eta-\xi_1)^{\beta}}Q_2(\beta,\sigma)f_1(\xi_1,\eta_1)\,d\eta_1+$$

$$+\frac{k}{(\eta-\xi)^{\beta}}\int\limits_{0}^{\xi}(\eta-\xi_1)^{\beta}f_1(\xi_1,\eta)\,d\xi_1;$$

$$\frac{\partial}{\partial\xi}\sigma=\frac{\partial}{\partial\xi}\left(\frac{\eta-\eta_1}{\eta-\xi_1}\cdot\frac{\xi-\xi_1}{\xi-\eta_1}\right)=\frac{\eta-\eta_1}{\eta-\xi_1}\cdot\frac{\xi-\eta_1-(\xi-\xi_1)}{(\xi-\eta_1)^2}=$$

$$=\frac{(\eta-\eta_1)(\xi_1-\eta_1)}{(\eta-\xi_1)(\xi-\eta_1)^2};$$

$$\frac{\partial}{\partial\xi}F(\beta,\beta;1;\sigma)=\frac{\partial}{\partial\xi}Q_1(\beta,\sigma)=$$

$$=\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}\frac{\beta\cdot\beta}{2\beta}F(\beta+1,\beta+1;2\beta+1;1-\sigma)\cdot\left(-\frac{(\eta-\eta_1)(\xi_1-\eta_1)}{(\eta-\xi_1)(\xi-\eta_1)^2}\right)+$$

$$+\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}(1-2\beta)(1-\sigma)^{-2\beta}\cdot\left(-\frac{(\eta-\eta_1)(\xi_1-\eta_1)}{(\eta-\xi_1)(\xi-\eta_1)^2}\right)\times$$

$$\times F(1-\beta,1-\beta;2-2\beta;1-\sigma)+$$

$$+\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}(1-\sigma)^{1-2\beta}\cdot\frac{(1-\beta)(1-\beta)}{2-2\beta}\times$$

$$\times F(2-\beta,2-\beta;3-2\beta;1-\sigma)\cdot\left(-\frac{(\eta-\eta_1)(\xi_1-\eta_1)}{(\eta-\xi_1)(\xi-\eta_1)^2}\right)=Q_3(\beta,\sigma);$$

$$\frac{\partial}{\partial\xi}R(\xi,\eta,\xi_1,\eta_1)=\frac{\partial}{\partial\xi}\left(k\cdot\frac{(\eta_1-\xi_1)^{2\beta}}{(\eta-\xi_1)^{\beta}}\cdot\frac{1}{(\eta_1-\xi)^{\beta}}F(\beta,\beta;1;\sigma)\right)=$$

$$=k\frac{(\eta_1-\xi_1)^{2\beta}}{(\eta-\xi_1)^{\beta}}\left(\frac{\beta\cdot F(\beta,\beta;1;\sigma)}{(\eta_1-\xi)^{\beta+1}}+\frac{1}{(\eta_1-\xi)^{\beta}}\cdot Q_3(\beta,\sigma)\right);$$

$$\frac{\partial}{\partial\xi}\int\limits_{0}^{\xi}d\xi_1\int\limits_{1}^{\eta}R(\xi,\eta,\xi_1,\eta_1)\cdot f_1(\xi_1,\eta_1)\,d\eta_1=\int\limits_{0}^{\xi}d\xi_1\int\limits_{1}^{\eta}k\frac{(\eta_1-\xi_1)^{2\beta}}{(\eta-\xi_1)^{\beta}}\times$$

$$\times\left(\frac{\beta\cdot F(\beta,\beta;1;\sigma)}{(\eta_1-\xi)^{\beta+1}}+\frac{1}{(\eta_1-\xi)^{\beta}}\cdot Q_3(\beta,\sigma)\right)f_1(\xi_1,\eta_1)\,d\eta_1+$$

$$+\int\limits_{1}^{\eta}k\frac{(\eta_1-\xi)^{2\beta}}{(\eta_1-\xi)^{\beta}(\eta-\xi)^{\beta}}F\left(\beta,\beta;1;\frac{(\xi-\xi)(\eta_1-\eta)}{(\xi-\eta)(\eta_1-\xi)}\right)f_1(\xi,\eta_1)\,d\eta_1-0=$$

$$=k\beta\int\limits_{0}^{\xi}d\xi_1\int\limits_{1}^{\eta}\frac{(\eta_1-\xi_1)^{2\beta}}{(\eta-\xi_1)^{\beta}(\eta_1-\xi)^{\beta+1}}F(\beta,\beta;1;\sigma)f_1(\xi_1,\eta_1)\,d\eta_1+$$

$$+k\int\limits_{0}^{\xi}d\xi_1\int\limits_{1}^{\eta}\frac{(\eta_1-\xi_1)^{2\beta}}{(\eta-\xi_1)^{\beta}(\eta_1-\xi)^{\beta}}Q_3(\beta,\sigma)f_1(\xi_1,\eta_1)\,d\eta_1+$$

$$+\frac{k}{(\eta-\xi)^{\beta}}\int\limits_{1}^{\eta}(\eta_1-\xi)^{\beta}f_1(\xi,\eta_1)\,d\eta_1;$$

$$\left(\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\xi}\right)\int\limits_{0}^{\xi}d\xi_1\int\limits_{1}^{\eta}R(\xi,\eta,\xi_1,\eta_1)\,f_1(\xi_1,\eta_1)\,d\eta_1=$$

$$= -k\beta \int_0^{\xi} d\xi_1 \int_1^{\eta} \frac{(\eta_1 - \xi_1)^{2\beta}}{(\eta_1 - \xi)^{\beta} (\eta - \xi_1)^{\beta+1}} F(\beta, \beta; 1; \sigma) f_1(\xi_1, \eta_1)\, d\eta_1 +$$

$$+k \int_0^{\xi} d\xi_1 \int_1^{\eta} \frac{(\eta_1 - \xi_1)^{2\beta}}{(\eta_1 - \xi)^{\beta} (\eta - \xi_1)^{\beta}} Q_2(\beta, \sigma) f_1(\xi_1, \eta_1)\, d\eta_1 +$$

$$+\frac{k}{(\eta - \xi)^{\beta}} \int_0^{\xi} (\eta - \xi_1)^{\beta} f_1(\xi_1, \eta)\, d\xi_1 -$$

$$-k\beta \int_0^{\xi} d\xi_1 \int_1^{\eta} \frac{(\eta_1 - \xi_1)^{2\beta}}{(\eta - \xi_1)^{\beta} (\eta_1 - \xi)^{\beta+1}} F(\beta, \beta; 1; \sigma) f_1(\xi_1, \eta_1)\, d\eta_1 +$$

$$-k \int_0^{\xi} d\xi_1 \int_1^{\eta} \frac{(\eta_1 - \xi_1)^{2\beta}}{(\eta - \xi_1)^{\beta} (\eta_1 - \xi)^{\beta}} Q_3(\beta, \sigma) f_1(\xi_1, \eta_1)\, d\eta_1 +$$

$$-\frac{k}{(\eta - \xi)^{\beta}} \int_1^{\eta} (\eta_1 - \xi)^{\beta} f_1(\xi, \eta_1)\, d\eta_1 = Q_4(\beta, \xi, \eta);$$

$$\lim_{\eta \to \xi} \sigma = \lim_{\eta \to \xi} \frac{(\xi_1 - \xi)(\eta_1 - \eta)}{(\xi_1 - \eta)(\eta_1 - \xi)} = \frac{(\xi_1 - \xi)(\eta_1 - \xi)}{(\xi_1 - \xi)(\eta_1 - \xi)} = 1;$$

$$\lim_{\eta \to \xi} (1 - \sigma) = \lim_{\eta \to \xi} \frac{(\xi_1 - \eta_1)(\eta - \xi)}{(\xi_1 - \eta)(\eta_1 - \xi)} =$$

$$= \frac{(\xi_1 - \eta_1)}{(\xi_1 - \eta)(\eta_1 - \xi)} \lim_{\eta \to \xi} (\eta - \xi);$$

$$\lim_{\eta \to \xi} F(\beta, \beta, 1; \sigma) = F(\beta, \beta, 1; 1) = \frac{\Gamma(1)\Gamma(1 - 2\beta)}{\Gamma(1 - \beta)\Gamma(1 - \beta)} = \frac{\Gamma(1 - 2\beta)}{\Gamma^2(1 - \beta)};$$

$$\lim_{\eta \to \xi} F(\beta + 1, \beta + 1, 2\beta + 1; 1 - \sigma) = F(\beta + 1, \beta + 1, 2\beta + 1; 0) = 1;$$

$$\lim_{\eta \to \xi} F(1 - \beta, 1 - \beta, 2 - 2\beta; 1 - \sigma) = F(1 - \beta, 1 - \beta, 2 - 2\beta; 0) = 1;$$

$$\lim_{\eta \to \xi} F(2 - \beta, 2 - \beta, 3 - 2\beta; 1 - \sigma) = F(2 - \beta, 2 - \beta, 3 - 2\beta; 0) = 1;$$

$$\lim_{\eta \to \xi} Q_2(\beta, \sigma) =$$

$$= \lim_{\eta \to \xi} \frac{\Gamma(1 - 2\beta)}{\Gamma^2(1 - \beta)} \frac{\beta}{2} F(\beta + 1, \beta + 1; 2\beta + 1; 0) \cdot \left(-\frac{(\xi_1 - \xi)(\eta_1 - \xi_1)}{(\eta_1 - \xi)(\xi - \xi_1)^2} \right) +$$

$$+ \lim_{\eta \to \xi} \frac{\Gamma(2\beta - 1)}{\Gamma^2(\beta)} (1 - 2\beta) \left(\frac{(\xi_1 - \eta_1)}{(\xi_1 - \eta)(\eta_1 - \xi)} \right)^{-2\beta} (\eta - \xi)^{-2\beta} \times$$

$$\times \left(-\frac{(\xi_1 - \xi)(\eta_1 - \xi_1)}{(\eta_1 - \xi)(\xi - \xi_1)^2} \right) \cdot F(1 - \beta, 1 - \beta; 2 - 2\beta; 0) +$$

$$+ \lim_{\eta \to \xi} \frac{\Gamma(2\beta - 1)}{\Gamma^2(\beta)} \left(\frac{(\xi_1 - \eta_1)}{(\xi_1 - \eta)(\eta_1 - \xi)} \right)^{1-2\beta} (\eta - \xi)^{1-2\beta} \cdot \frac{1 - \beta}{2} \times$$

$$\times F(2 - \beta, 2 - \beta; 3 - 2\beta; 0) \cdot \left(-\frac{(\xi_1 - \xi)(\eta_1 - \xi_1)}{(\eta_1 - \xi)(\xi - \xi_1)^2} \right) =$$

$$= \frac{\beta}{2} \cdot \frac{\Gamma(1 - 2\beta)}{\Gamma^2(1 - \beta)} \cdot \frac{\eta_1 - \xi_1}{(\eta_1 - \xi)(\xi - \xi_1)} +$$

$$+(1-2\beta)\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}\left(\frac{\xi_1-\eta_1}{(\xi_1-\xi)(\eta_1-\xi)}\right)^{1-2\beta}\lim_{\eta\to\xi}(\eta-\xi)^{-2\beta}+$$

$$+\frac{1-\beta}{2}\cdot\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}\left(\frac{\xi_1-\eta_1}{(\xi_1-\xi)(\eta_1-\xi)}\right)^{2-2\beta}\lim_{\eta\to\xi}(\eta-\xi)^{1-2\beta};$$

$$\lim_{\eta\to\xi}Q_3(\beta,\sigma)=$$

$$\lim_{\eta\to\xi}\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}\frac{\beta}{2}F(\beta+1,\beta+1;2\beta+1;0)\cdot\left(-\frac{(\xi-\eta_1)(\xi_1-\eta_1)}{(\xi-\xi_1)(\xi-\eta_1)^2}\right)+$$

$$+\lim_{\eta\to\xi}\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}(1-2\beta)\left(\frac{(\xi_1-\eta_1)}{(\xi_1-\xi)(\eta_1-\xi)}\right)^{-2\beta}(\eta-\xi)^{-2\beta}\times$$

$$\times\left(-\frac{(\xi-\eta_1)(\xi_1-\eta_1)}{(\xi-\xi_1)(\xi-\eta_1)^2}\right)\cdot F(1-\beta,1-\beta;2-2\beta;0)+$$

$$+\lim_{\eta\to\xi}\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}\left(\frac{(\xi_1-\eta_1)}{(\xi_1-\xi)(\eta_1-\xi)}\right)^{1-2\beta}(\eta-\xi)^{1-2\beta}\cdot\frac{1-\beta}{2}\times$$

$$\times F(2-\beta,2-\beta;3-2\beta;0)\cdot\left(-\frac{(\xi-\eta_1)(\xi_1-\eta_1)}{(\xi-\xi_1)(\xi-\eta_1)^2}\right)=$$

$$=\frac{\beta}{2}\cdot\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}\cdot\frac{\xi_1-\eta_1}{(\eta_1-\xi)(\xi-\xi_1)}-$$

$$-(1-2\beta)\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}\left(\frac{\xi_1-\eta_1}{(\xi_1-\xi)(\eta_1-\xi)}\right)^{1-2\beta}\lim_{\eta\to\xi}(\eta-\xi)^{-2\beta}-$$

$$-\frac{1-\beta}{2}\cdot\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}\left(\frac{\xi_1-\eta_1}{(\xi_1-\xi)(\eta_1-\xi)}\right)^{2-2\beta}\lim_{\eta\to\xi}(\eta-\xi)^{1-2\beta};$$

$$\lim_{\eta\to\xi}\left(\frac{m+2}{2}\right)^{2\beta}(\eta-\xi)^{2\beta}\left(\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\xi}\right)\int_0^{\xi}d\xi_1\int_1^{\eta}R(\xi,\eta,\xi_1,\eta_1)f_1(\xi_1,\eta_1)\,d\eta_1=$$

$$=\lim_{\eta\to\xi}\left(\frac{m+2}{2}\right)^{2\beta}(\eta-\xi)^{2\beta}Q_4(\beta,\xi,\eta)=\lim_{\eta\to\xi}\left(\frac{m+2}{2}\right)^{2\beta}(\eta-\xi)^{2\beta}\times$$

$$\times\left(-k\beta\int_0^{\xi}d\xi_1\int_1^{\xi}\frac{(\eta_1-\xi_1)^{2\beta}}{(\eta_1-\xi)^{\beta}(\xi-\xi_1)^{\beta+1}}\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}f_1(\xi_1,\eta_1)\,d\eta_1\right)+$$

$$+\lim_{\eta\to\xi}\left(\frac{m+2}{2}\right)^{2\beta}(\eta-\xi)^{2\beta}\cdot k\int_0^{\xi}d\xi_1\int_1^{\xi}\frac{(\eta_1-\xi_1)^{2\beta}}{(\eta_1-\xi)^{\beta}(\xi-\xi_1)^{\beta}}\times$$

$$\times\left(=\frac{\beta}{2}\cdot\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}\cdot\frac{\eta_1-\xi_1}{(\eta_1-\xi)(\xi-\xi_1)}+\right.$$

$$+(1-2\beta)\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}\left(\frac{\xi_1-\eta_1}{(\xi_1-\xi)(\eta_1-\xi)}\right)^{1-2\beta}(\eta-\xi)^{-2\beta}+$$

$$\left.+\frac{1-\beta}{2}\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}\left(\frac{\xi_1-\eta_1}{(\xi_1-\xi)(\eta_1-\xi)}\right)^{2-2\beta}(\eta-\xi)^{1-2\beta}\right)f_1(\xi_1,\eta_1)\,d\eta_1+$$

$$+\lim_{\eta\to\xi}\left(\frac{m+2}{2}\right)^{2\beta}(\eta-\xi)^{2\beta}\cdot\frac{k}{(\eta-\xi)^{\beta}}\int_0^{\xi}(\xi-\xi_1)^{\beta}f_1(\xi_1,\xi)\,d\xi_1-$$

$$-\lim_{\eta\to\xi}\left(\frac{m+2}{2}\right)^{2\beta}(\eta-\xi)^{2\beta}k\beta\times$$

$$\times\int_0^{\xi}d\xi_1\int_1^{\xi}\frac{(\eta_1-\xi_1)^{2\beta}}{(\eta-\xi_1)^{\beta}(\eta_1-\xi)^{\beta+1}}\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}f_1(\xi_1,\eta_1)\,d\eta_1-$$

$$-\lim_{\eta\to\xi}\left(\frac{m+2}{2}\right)^{2\beta}(\eta-\xi)^{2\beta}k\int_0^{\xi}d\xi_1\int_1^{\xi}\frac{(\eta_1-\xi_1)^{2\beta}}{(\xi-\xi_1)^{\beta}(\eta_1-\xi)^{\beta}}\times$$

$$\times\left(\frac{\beta}{2}\cdot\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}\cdot\frac{\xi_1-\eta_1}{(\eta_1-\xi)(\xi-\xi_1)}-\right.$$

$$-(1-2\beta)\cdot\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}\left(\frac{\xi_1-\eta_1}{(\xi_1-\xi)(\eta_1-\xi)}\right)^{1-2\beta}(\eta-\xi)^{-2\beta}-$$

$$\left.-\frac{1-\beta}{2}\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}\left(\frac{\xi_1-\eta_1}{(\xi_1-\xi)(\eta_1-\xi)}\right)^{2-2\beta}(\eta-\xi)^{1-2\beta}\right)f_1(\xi_1,\eta_1)\,d\eta_1-$$

$$-\lim_{\eta\to\xi}\left(\frac{m+2}{2}\right)^{2\beta}(\eta-\xi)^{2\beta}\frac{k}{(\eta-\xi)^{\beta}}\int_1^{\eta}(\eta_1-\xi)^{\beta}f_1(\xi,\eta_1)\,d\eta_1=$$

$$=2\cdot\left(\frac{m+2}{2}\right)^{2\beta}\cdot k\int_0^{\xi}d\xi_1\int_1^{\xi}\frac{(\eta_1-\xi_1)^{2\beta}}{(\eta_1-\xi)^{\beta}(\xi-\xi_1)^{\beta}}\times$$

$$\times(1-2\beta)\cdot\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}\cdot\left(\frac{\xi_1-\eta_1}{(\xi_1-\xi)(\eta_1-\xi)}\right)^{1-2\beta}f_1(\xi,\eta_1)\,d\eta_1=$$

$$=2k(1-2\beta)\left(\frac{m+2}{2}\right)^{2\beta}\frac{\Gamma(2\beta-1)}{\Gamma^2(\beta)}\times$$

$$\times\int_0^{\xi}d\xi_1\int_1^{\xi}\frac{\eta_1-\xi_1}{(\eta_1-\xi)^{1-\beta}(\xi-\xi_1)^{1-\beta}}f_1(\xi,\eta_1)\,d\eta_1;$$

As a result, to satisfy the second of the boundary conditions (2), the function $f(x,y)$ must satisfy condition (9), that is, by virtue of the last relation, the condition

$$\int_0^{\xi}d\xi_1\int_1^{\xi}\frac{\eta_1-\xi_1}{(\eta_1-\xi)^{1-\beta}(\xi-\xi_1)^{1-\beta}}f_1(\xi,\eta_1)\,d\eta_1=0. \tag{17}$$

Thus, the following is proved **Theorem 1.** The minimal operator ((1), (2), (3) problem) is invertible in $L_2(\Omega)$ if and only if (12) and (17) conditions are met, where the function f_1 is determined by a given function $f(x,y)$ from (1). In this case, the solution of overdetermined Cauchy problem is representable by the formula (6), where $R(\xi,\eta,\xi_1,\eta_1)$ is the Riemann function of the Goursat problem, defined by formula (4).

This work was supported by CS MSHE RK, grant No AP14871460

References

[1] Vishik M.I., On general boundary value problems for elliptic differential equations, *Trudy Matem. Islands* , No.1, 1952, pp.187-246.

[2] Otelbaev M.O., Shynybekov A.N., Well-posed problems of the Bitsadze-Samarsky type, *Reports of the Academy of Sciences of the USSR* , Vol.265, No.4, 1982, pp.815-819.

[3] Kalmenov T.Sh., On the spectrum of the Tricomi problem for the Lavrent'ev-Bitsadze equation, *Differential Equations* , Vol.13, No.8, 1977, pp.1418-1425.

[4] Kalmenov T.Sh., On self-adjoint boundary value problems for the Tricomi equation, *Differential Equations* , Vol.19, No.1, 1983, pp.66-75.

[5] Tricomi F., *Sulle equazioni lineari alle derivate parziali di* 2^0 *ordine di tipo misto* , Mem. Lincei, Ser. V, XIV, fasc. VII, 1923.

[6] Gellerstedt S., *Sur un probleme aux limites pour une equation lineaire aux derivees partielles du second ordre de type mixte* , Ph.D. Thesis, Almqvist & Wiksells, Uppsala, 1935.

[7] Chaplygin S., On gas jets, *Sci. Mem. Moscow Univ. Math. Phys.* , Vol.21, No.1063, 1902, pp.1-121.

[8] Frankl F.I., On the problems of Claplygin for mixed subsonic and supersonic flows, *Izv. Akad. Nauk. Ser. Mat.* , Vol.9, No.2, 1945, pp.121-143.

[9] Moravetz C.S., A uniqueness theorem for Frankl's problem, *Comm. Pure Appl. Math.* , Vol.7, No.4, 1954, pp.697-703.

[10] Moravetz C.S., The mathematical approach to the sonic barrier, *Bull. Am. Math. Soc. (N.S.)* , Vol.6, No.2, 1982, pp.127-145.

[11] Darboux G., *Lecons la theorie des surfaces* , 2-e ed. (Paris, Gauthier-Villars) p.2^n, livr. IV, Chap. III.

[12] Abramowitz M., Stegun I.A. (eds.), *Handbook of mathematical functions with formulas, graphs and mathematical tables* , National Bureau of Standards Applied Mathematics Series, vol. 55, U.S. Government Printing Office, Washington, D.C., 1964.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ONE PROBLEM FOR INTEGRO-DIFFERENTIAL EQUATION

ASAL YULDASHEVA[1]

[1] Lomonosov Moscow State University, Tashkent Branch, Tashkent, Uzbekistan
e-mail: a_v_yuldasheva@mail.ru

Abstract. In this paper we consider a singular integro-differential equation. The integral operator is a fractional power of the Laplace operator. The existence and uniqueness of the solution of the problem is proved, the class of the solution is found.

Keywords: fractional power of the Laplace operator, peridynamics, integro-differential equation, initial problem, Sobolev space, Fourier series.

AMS Subject Classification: MSC2010 45K05, 47G20

1. Statement of the problem

In the last years, the theory of fractional powers of operators on Banach spaces aroused great interest due to the increasing need for modeling in Peridynamics,Biology, Finance, etc. (See [1], [2], [3] and the references therein.)The fractional Laplacian refers to anomalous diffusion that uses fractional derivatives, to peridynamical equation [4], [5] and a fractional heat equation [7].

It turned out that the evolution equation with a fractional power of the Laplace operator is directly related to the peridynamic equation.

Peridynamic model, proposed by Silling [9], is described by integro-differential equations without derivatives with respect to spatial variables, so when a break occurs, the mathematical structure does not collapse. Fracture is seen as a natural result of deformation occurring in accordance with the equation of motion and the defining model. Peridynamics can be viewed as a continuum version of molecular dynamics, making this new approach a suitable candidate for large-scale material analysis. In addition, the peridynamic formulation can also be extended to other areas such as thermal, moisture, etc. And this makes it possible to use it as a single platform for multiphysics analysis of materials. We refer to [10]-[14] for reviews of the recent applications and theoretical development of the peridynamic framework.

The main problem that arises when solving such equations lies in the high singularity of the integral operator. For its regularization, the ideas of the works [4], [5], [6] were used. Note that the regularization method depends on the number of spatial variables.

We consider equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} + (-\Delta)^s u = f(x,t), \quad x \in \Omega,\ t > 0, \tag{1}$$

with initial data

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in \Omega, \tag{2}$$

where $s \in (0, 1/2)$, $\Omega \subset \mathbb{R}^n$ – domain with piecewise smooth boundary and $n \geq 3$.

We suppose, that $u : \Omega \times [0,T] \to \mathbb{R}$ is unknown function, kernel $K : \Omega \times \Omega \to \mathbb{R}$ and function $f : \Omega \times [0,T] \to \mathbb{R}$ are scalar functions.

According to definitions from [1], equation (1) can be rewrite as

$$\frac{\partial^2 u(x,t)}{\partial t^2} \; + \; \int\limits_{\Omega} \frac{[u(x,t) - u(y,t)]}{|x-y|^{n+2s}}\, dy \;=\; f(x,t), \quad x \in \Omega,\ t > 0. \tag{3}$$

Consider a self-adjoint extension of the Laplace operator $-\Delta$ generated by the Neumann boundary conditions. The spectrum of it extension consists of eigenvalues $\{\lambda_k\}$, and the eigenfunctions $\{v_k(x)\}$ satisfy the relations:

$$-\Delta v_k(x) = \lambda_k v(x),\ x \in \Omega, \qquad \frac{\partial v_k(s)}{\partial \nu} = 0,\ s \in \partial\Omega, \quad k = 0,1,2,... \tag{4}$$

We understand the solution of the spectral problem (4) in the sense of $W_2^1(\Omega)$.

We introduce the function $L(x,y)$, the expansion of which into a series by the eigenfunctions of problem (4) has the form

$$L(x,y) \;=\; \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{1-s}}\, v_k(x)\, v_k(y)\ . \tag{5}$$

Further we show that such a function exists, infinitely differentiable outside the diagonal $x = y$, but near the diagonal equal to

$$L(x,y) \;=\; \alpha\, \frac{1}{|x-y|^{n-2+2s}} \;+\; \eta(x,y), \tag{6}$$

where α is constant, depending only on the number of measurements n, and the function $\eta(x,y)$ is infinitely differentiable in the domain $\Omega \times \Omega$.

Laplace operator applied to a given function by one of the variables x or y, has on the diagonal non-integrable singularity:

$$\Delta L(x,y) \;= \frac{1}{|x-y|^{n+2s}} \;+\; \Delta\eta(x,y). \tag{7}$$

We define the kernel outside the diagonal $x = y$

$$K(x,y) \;=\; (-\Delta_y)\, L(x,y), \quad x \in \Omega,\ y \in \Omega. \tag{8}$$

Formally, we can assume that the eigenfunctions expansion of kernel $K(x,y)$ has the form

$$K(x,y) \;=\; \sum_{k=1}^{\infty} (\lambda_k^s)\, v_k(x)\, v_k(y). \tag{9}$$

For any $\beta \geq 0$ we introduce $H^\beta(\Omega) \;=\; D\left((I-\Delta)^{\beta/2}\right)$ with norm

$$\|u\|_\beta^2 \;=\; \sum_{k=1}^{\infty} (1+\lambda_k)^\beta |(u, v_k)|^2. \tag{10}$$

$W_2^\alpha(\Omega)$, where $\alpha > 0$, is symbol of classical Sobolev spaces of (fractional) order α.

For any $T > 0$ and $m = 0,1,...$ and arbitrary Banach space B by $C^m\big\{[0,T] \to B\big\}$ we mark m times continuously differentiable mappings of $[0,T]$ to B.

The solution of problem (1)-(2) from the class $H^\beta(\Omega)$ is called function $u \in C^2\big\{[0,T] \to H^\beta(\Omega)\big\}$, satisfying equation (1) and initial conditions (2).

The main result of this work is concluded in the following.

Theorem 1. *Let $0 < \beta < 2s/n$. Then for any $T > 0$ and $\varphi \in W_2^{2s}(\Omega)$, $\psi \in W_2^{2s}(\Omega)$ $f \in C\big\{[0,T] \to W_2^{2s}(\Omega)\big\}$ exists, and moreover, the only solution of the problem* (1)-(2) *from the class $C^2\big\{[0,T] \to H^{\beta}(\Omega)\big\}$.*

2. Special type kernel

In this section, we transform the hypersingular kernel of the integral operator of the equation (1) to a regular one. For this, we need to introduce a new function and consider its expansion in the Fourier series.

We introduce function $\chi \in C^{\infty}(\mathbb{R})$, non-growth and so that

$$\chi(r) \;=\; \begin{cases} 1, & \text{if} \quad r \le 1/2, \\ 0, & \text{if} \quad r \ge 1. \end{cases}$$

For fixed $\delta > 0$ and arbitrary point $x \in \Omega$

$$R = R(x) = \min\{\delta, \operatorname{dis}(x, \partial\Omega)\},$$

where$\operatorname{dis}(x, \partial\Omega)$ men as distance from x to $\partial\Omega$.

Let consider following function:

$$L_0(x,y) \;=\; \alpha\,|x-y|^{2-n-2s} \cdot \chi\left(\frac{|x-y|}{R}\right), \quad x \in \Omega, \quad y \in \Omega, \tag{11}$$

where

$$\alpha \;=\; \frac{\Gamma\left(\dfrac{n}{2}+s-1\right)}{2^{3-2s}\pi^{n/2}s(n-2+2s)\Gamma\,(1-s)}.$$

Taking x as a parameter, we find the Fourier coefficient of the function $L_0(x,y)$:

$$a_k(x) \;=\; \int\limits_{\Omega} L_0(x,y)\,v_k(y)\,dy \;=\; \alpha \int\limits_{|x-y|\le R} |x-y|^{2-n} \frac{1}{|x-y|^{n-2+2s}} \chi\left(\frac{|x-y|}{R}\right) \cdot v_k(y)\,dy.$$

Passing to the spherical coordinates with the center at the point x, we get

$$a_k(x) \;=\; \alpha \int\limits_0^R r^{n-1} dr \int_{S^{n-1}} r^{2-n-2s} \chi(r/R)\,v_k(x+r\theta)\,d\theta \;=$$

$$= \;\alpha \int\limits_0^R \chi(r/R)\,r^{1-2s}\,dr \int\limits_{S^{n-1}} v_k(x+r\theta)\,d\theta. \tag{12}$$

Next, we apply the mean value formula [8]

$$\int\limits_{S^{n-1}} v(x+r\theta)\,d\theta \;=\; v_k(x)(2\pi)^{n/2}(r\sqrt{\lambda_k})^{1-n/2} J_{n/2-1}(r\sqrt{\lambda_k}).$$

Substituting this formula into (12), we get

$$a_k(x) \;=\; v_k(x) \cdot (2\pi)^{n/2} \alpha \sqrt{\lambda_k}^{\,1-n/2} \int\limits_0^R J_{n/2-1}(r\sqrt{\lambda_k})\,\chi(r/R)\,r^{2-2s-n/2} dr.$$

We make the change $t = r\sqrt{\lambda_k}$. Then

$$a_k(x) \;=\; v_k(x)\cdot\alpha\frac{(2\pi)^{n/2}}{\lambda_k^{2-2s}}\int\limits_0^{R\sqrt{\lambda_k}}\chi\left(\frac{t}{R\sqrt{\lambda_k}}\right)J_{n/2-1}(t)\,t^{2-2s-n/2}dt. \tag{13}$$

Next, we use that for the integral

$$\int\limits_0^{\infty}J_\nu(t)\,t^\mu dt = 2^\mu\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)},$$

if

$$-Re\nu-1<Re\mu<\frac{1}{2}.$$

The choice of the parameter $s\in(0,1/2)$ follows from the last inequality.

Substituting this representation into (13), we get the following equality:

$$a_k(x) \;=\; v_k(x)\left[\frac{1}{\lambda_k^{1-s}} \;+\; O_x(\lambda_k^{-N})\right].$$

From above and the equality (13) follows the validity of the following approval.

Lemma 1. *The expansion of function* (11) *in a Fourier series in terms of eigenfunction* (4) *has the form*

$$L_0(x,y) \;=\; \sum_{\lambda_k>0}\frac{v_k(x)\,v_k(y)}{\lambda_k^{1-s}} \;+\sum_{k=1}^{\infty}c_k(x)v_k(x)v_k(y), \tag{14}$$

where the coefficients $c_k(x)$ for any number N satisfies the conditions

$$|c_k(x)| \;\le\; \frac{C_N(x)}{(1+\lambda_k)^N}, \tag{15}$$

and the value of $C_N(x)$ is bounded in x uniformly on each compact subdomain Ω.

Next, set the

$$R(x,y) \;=\; \sum_{k=1}^{\infty}c_k(x)v_k(x)v_k(y).$$

From estimate (15) it follows that the function $R(x,y)$ is infinite differentiable in $\Omega\times\Omega$. Let

$$L(x,y) \;=\; L_0(x,y) \;-\; R(x,y). \tag{16}$$

Lemma 2. *Fourier series of the function $L(x,y)$, defined by equality* (16) *has a form*

$$L(x,y) \;=\; \sum_{\lambda_k>0}\frac{v_k(x)\,v_k(y)}{\lambda_k^{1-s}}.$$

Proof. The validity of the last equality follows directly from formulas (14) and (16).

Note. Lemma 2 means that the function $L(x,y)$ satisfies the following ratios:

$$\int\limits_\Omega L(x,y)\,v_k(x,y)\,dy \;=\; \frac{1}{\lambda_k^{1-s}}\;v_k(x), \quad k=1,2,...$$

3. Transformation of the equation (1)

In this section, using the results obtained earlier, we represent the integral operator of equation (1) as a Fourier series by eigenvalues function of the boundary value problem (5).

Let

$$B(\varepsilon) \;=\; \{y \in \mathbb{R}^n \;:\; |x-y| < \varepsilon\}.$$

For any function $u \in C_0^\infty(\Omega)$ and any point $x \in \Omega$, for sufficiently small $\varepsilon > 0$ by Green's formula, we obtain

$$\int\limits_{\Omega\setminus B(\varepsilon)} \Delta_y L(x,y)\,[u(x)-u(y)]\,dy \;=\; \int\limits_{\Omega\setminus B(\varepsilon)} L(x,y)\,\Delta_y[u(x)-u(y)]\,dy \;+$$
$$+\int\limits_{\partial\Omega}\left([u(x)-u(y)]\,\frac{\partial L(x,y)}{\partial\nu_y} - L(x,y)\,\frac{\partial}{\partial\nu_y}[u(x)-u(y)]\right)d\sigma(y) \;+$$
$$+\int\limits_{\partial B(\varepsilon)}\left([u(x)-u(y)]\,\frac{\partial L(x,y)}{\partial\nu_y} - L(x,y)\,\frac{\partial}{\partial\nu_y}[u(x)-u(y)]\right)d\sigma(y).$$

Since the function u is finite and condition (3) implies that the integral over the surface $\partial\Omega$ is equal to zero.

According to (7), for any smooth function u, the surface the integral over the boundary of the ball $B(\varepsilon)$ is $O(\varepsilon \ln\frac{1}{\varepsilon})$. Therefore, in the limit at $\varepsilon \to 0$ we get

$$\int\limits_{\Omega} K(x,y)\,[u(x)-u(y)]\,dy \;=\; \int\limits_{\Omega} L(x,y)\,(-\Delta u(y))\,dy. \tag{17}$$

Further we use that for any function $u \in C_0^\infty(\Omega)$ following series converge in the $L_2(\Omega)$ metric

$$u(y) \;=\; \sum_{k=1}^{\infty}(u,v_k)v_k(y),$$

$$-\Delta\,u(y) \;=\; \sum_{k=1}^{\infty}\lambda_k\,(u,v_k)v_k(y).$$

In this case, the equality

$$L(x,y)(-\Delta u(y)) \;=\; \sum_{k=1}^{\infty}\lambda_k L(x,y)(u,v_k)v_k(y), \quad x\in\Omega, \quad y\in\Omega,$$

can be integrated. As a result, we get:

$$\int\limits_{\Omega} L(x,y)(-\Delta u(y))\,dy \;=\; \sum_{k=1}^{\infty}\lambda_k \int\limits_{\Omega} L(x,y)(u,v_k)v_k(y)\,dy \;=$$

$$=\; \sum_{k=1}^{\infty}\lambda_k^s(u,v_k)\,v_k(x), \quad x\in\Omega.$$

From this equality and from (17) the required ratio follows.

We have obtained an important relation that allows us to find a solution to the considering equation.

4. Solvability of the main equation

Let's move on to solving the problem (1) – (2).

We find the solution of the equation (1) in the form of series by eigenvalues function of the boundary value problem (5):

$$u(x,t) \;=\; \sum_{k=0}^{\infty} c_k(t)v_k(x), \quad x \in \Omega, \quad t \geq 0. \tag{18}$$

In this case, from equation (1) we get:

$$c_0''(t) \;=\; f_0(t), \tag{19}$$

$$\sum_{k=1}^{\infty} c_k''(t)v_k(x) \;+\; \sum_{k=1}^{\infty} \lambda_k^s \, c_k(t)v_k(x) \;=\; \sum_{k=1}^{\infty} f_k(t)v_k(x).$$

Where $f_k(t) = (f, v_k)$.

Consequently, for $k \geq 1$ we get

$$c_k''(t) \;+\; \lambda_k^s \, c_k(t) \;=\; f_k(t). \tag{20}$$

From (19) and (20) we get

$$c_0(t) \;=\; a_0 + b_0 t + \int_0^t (t-s) f_0(s)\, ds,$$

and

$$c_k(t) \;=\; a_k \cos(\lambda_k^{s/2} t) + b_k \frac{\sin(\lambda_k^{s/2} t)}{\lambda_k^{s/2}} + \frac{1}{\lambda_k^{s/2}} \int_0^t \sin(\lambda_k^{s/2}(t-s)) \; f_k(s)\, ds.$$

Taking into account the initial conditions (2), we come to the following ratios for the Fourier coefficients of the solution:

$$c_0(t) \;=\; (\varphi, v_0) \;+\; (\psi, v_0)t \;+\; \int_0^t (t-s)(f, v_0)(s)\, ds \;, \tag{21}$$

$$c_k(t) \;=\; (\varphi, v_k)\cos(\lambda_k^{s/2} t) + \; (\psi, v_k)\frac{\sin(\lambda_k^{s/2} t)}{\lambda_k^{s/2}} + \; \frac{1}{\lambda_k^{s/2}} \int_0^t \sin(\lambda_k^{s/2}(t-s)) \; f_k(s)\, ds \;, \quad k = 1, 2, ... \tag{22}$$

From above we find formal solution of the problem (1)-(2)

$$u(x,t) \;=\; ((\varphi, v_0) \;+\; (\psi, v_0)t \;+\; \int_0^t (t-s)(f, v_0)(s)\, ds) v_0(x) \;+$$

$$+ \sum_{k=1}^{\infty} (\lambda_k^s \, ((\varphi, v_k)\cos(\lambda_k^{s/2} t) + \; (\psi, v_k)\frac{\sin(\lambda_k^{s/2} t)}{\lambda_k^{s/2}} + \; \frac{1}{\lambda_k^{s/2}} \int_0^t \sin(\lambda_k^{s/2}(t-s)) \; f_k(s)\, ds)) \, v_k(x)$$

From (19) and (21) it follows that the second derivative by t is formally as follow:

$$\frac{\partial^2 u(x,t)}{\partial t^2} \;=\; (f, v_0)(t) v_0(x) \;+\; \sum_{k=1}^{\infty} \lambda_k^s \, c_k(t) \, v_k(x) \;+\; \sum_{k=1}^{\infty} (f, v_k)(t) \, v_k(x). \tag{23}$$

To prove the existence of a solution, it is sufficient to verify in convergence in the metric of the space $H^{\beta}(\Omega)$ of series (23).

5. Proof of Theorem 1

From the conditions of the theorem it follows that for any positive $\beta_1 < 2s/n$:

$$\sum_{k=1}^{\infty} \left(|(\varphi, v_k)|^2 \ + |(\psi, v_k)|^2 \ + \ |f_k(t)|^2 \right) \lambda_k^{\beta_1} \ < \ \infty. \tag{24}$$

Now we show that in this case for a positive $\beta \ < \ 2s/n$ series (23) converges in $H^{\beta}(\Omega)$. According to the definition of the norm in space $H^{\beta}(\Omega)$, for this we need to prove that

$$\sum_{k=1}^{\infty} |c_k(t)|^2 \, \lambda_k^{2s} \, \lambda_k^{\beta} \ < \ \infty \tag{25}$$

and

$$\sum_{k=1}^{\infty} |(f, v_k)(t)|^2 \, \lambda_k^{\beta} \ < \ \infty. \tag{26}$$

Note that condition (24) ensures the convergence of the series (26), moreover, the membership of the function f in the class $C\{[0,T] \to W_2^{2s}(\Omega)\}$ guarantees uniformity in $t \in [0,T]$ convergence of the indicated series.

It follows from the above that the solution of the problem (1)-(2) exists, is unique, and belongs to the class $C^2\big\{[0,T] \to H^{\beta}(\Omega)\big\}$.

References

[1] Maha Daoud, El Haj Laamri. Fractional Laplacians : a short survey, Discrete and Continuous Dynamical Systems Series S.,V.15 (1) , 2022, 95–116.

[2] Balakrishnan A. V. Fractional powers of closed operators and the semigroups generated by them. Pacific J. Math., V.10, 1960, 419–437.

[3] A. De Pablo , F. Quiros , A. Rodriguez , J. L. Vazquez . A fractional porous medium equation., Adv. Math., V.226, 2011, 1378–1409.

[4] Alimov S., Yuldasheva A. Solvability of Singular Equations of Peridynamics on Two-Dimensional Periodic Structures, Journal of Peridynamics and Nonlocal Modeling, 2021, 1–19.

[5] Yuldasheva A.V. On Solvability of One Singular Equation of Peridynamics. Lobachevskii J Math, V.41, 2020, 1131–1136.

[6] Alimov S. A., Sheraliev S. N. On the solvability of the singular equation of peridynamics,Complex Var. and El. Eq., V.64 (5), 2019, 873-887.

[7] Valdinoci E. From the long jump random walk to the fractional Laplacian, Bol. Soc. Esp. Mat. Apl., V.49, 2009, 33–44.

[8] Titchmarsh E. C., *Eigenfunction expansions associated with second-order differential equations. Vol. 2.*, Oxford, at the Clarendon Press, 1958.

[9] S. A. Silling, Reformulation of elasticity theory for discontinuities and long-range forces, J. Mech. Phys. Solids 48, 2000, 175-209.

[10] E. Askari, F. Bobaru, R. B. Lehoucq, M. L. Parks, S. A. Silling, and O. Weckner, Peridynamics for multiscale materials modeling, J. Phys.: Conf. Ser. 125, 2008.

[11] S. A. Silling and R. B. Lehoucq, Convergence of peridynamics to classical elasticity theory, J. Elasticity 93, 2008, 13-37.

[12] S. A. Silling and R. B. Lehoucq, Peridynamic theory of solid mechanics, Adv. Appl. Mech. 44, 2010, 73-168.

[13] O. Weckner and E. Emmrich, The peridynamics equation and its spatial discretisation, Math. Model. Anal. 12, 2007, 17-27.

[14] K. Zhou and Q. Du, Mathematical and numerical analysis of linear peridynamic models with nonlocal boundary conditions, SIAM J. Numer. Anal. 48,2010, 1759-1780.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

A NEW FUNCTIONAL RELATION FOR THE RIEMANN ZETA FUNCTIONS

ASSET DURMAGAMBETOV[1]

[1]L.Gumilyov Eurasian National University, Astana, Kazakhstan
e-mail: aset.durmagambet@gmail.com

Abstract. In the paper the Riemann's functional equation reduced to a Riemann–Hilbert boundary value problem, and the integral Hilbert transforms arising in its solution allow the calculation of an exact lower bounds for the zeta function.

Keywords: Riemann's functional equation , Riemann–Hilbert boundary value problem, and the integral Hilbert transforms, the zeta function.

1. Introduction

This study is concerned with the properties of modified zeta functions. Riemann's zeta function is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + it, \tag{1}$$

which is absolutely and uniformly convergent in any finite region of the complex s-plane for which $\sigma \geq 1 + \epsilon, \epsilon > 0$. If $\sigma > 1$, then ζ is represented by the following Euler product formula

$$\zeta(s) = \prod_{j \in N} \left[1 - \frac{1}{p_j^s}\right]^{-1}, \tag{2}$$

where p_j runs over all prime numbers. $\zeta(s)$ was first introduced by Euler in 1737 [1], who also obtained formula (2). Dirichlet and Chebyshev considered this function in their study on the distribution of prime numbers [2]. However, the most profound properties of $\zeta(z)$ were only discovered later, when it was extended to the complex plane. $\zeta(s)$ is a regular function for all values of s, except $s = 1$, where it has a simple pole with residue 1; it satisfies the following functional equation:

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s) \tag{3}$$

it equation is called Riemann's functional equation.

2. Results

As mentioned in Introduction, certain simple intermediate estimates are first obtained. To obtain the Riemann-Hilbert boundary value problem , the following lemma is required.

Lemma 2.1. *Let*

$$R(k) = \frac{e^{i2k}}{k + i\alpha} - 1$$
$$\alpha > 2$$

then

$$ind(R) = 0$$

,

Proof. By definition

$$ind(R) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{R'(k)}{R(k)}dk$$

As

$$Im(k) > 0, |e^{i2k}| \leq 1 \; and \; |k + i\alpha| > 2 \; yield \; \frac{R'(k)}{R(k))}$$

have nothing pole. Latest statement and Lemma of Jordan yield

$$ind(R) = 0$$

. □

To obtain the necessary asymptotics, the following lemma is required.
For

$$f \in W_2^1(R) = \{f \in L_2(R) : (1 + |\omega|^2)^{1/2}\hat{f}(\omega) \in L_2)\}.$$

, the operators $T_\pm$ and T are defined as follows:

$$T_+f = \frac{1}{2\pi i}\lim_{Imz\downarrow 0}\int_{-\infty}^{\infty}\frac{f(s)}{s-z}ds, \; Im\; z > 0, T_-f = \frac{1}{2\pi i}\lim_{Imz\uparrow 0}\int_{-\infty}^{\infty}\frac{f(s)}{s-z}ds, \; Im\; z < 0, \; Tf = \frac{1}{2}(T_+ + T_-)f.$$

These operators are closely related to the Hilbert transform, whose isometric properties were studied by Poincaré. The following result is from [3].

Lemma 2.2.

$$TT = \frac{1}{4}I, \; TT_+ = \frac{1}{2}T_+, \; TT_- = -\frac{1}{2}T_-, \; T_+ = T + \frac{1}{2}I, \; T_- = T - \frac{1}{2}I,$$

where I is the identit y operator $If = f$.

The reduction to a Riemann–Hilbert boundary value problem can now be formulated as follows.

Lemma 2.3. *Let*

$$\Psi_+(k) = R(k)\Psi_-(k) + G(k), \tag{4}$$

$$\lim_{Re(k)\to\infty}\Psi_+(k) = 0 \; as \; Im(k) \geq 0, \quad \lim_{Re(k)\to-\infty}\Psi_-(k) = 0 \; as\, Im(k) \leq 0 \tag{5}$$

$$\Gamma_+(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{ln(R(t))dt}{t-k-i0}, \; \Gamma_-(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{ln(R(t))dt}{t-k+i0}$$

$$X_+(k) = e^{\Gamma_+(k)}, \; X_-(k) = e^{\Gamma_-(k)}, \; R(k) = X_-(k)/X_+(k),$$

Then

$$\Psi_+(k) = \frac{X_+(k)}{2\pi i}\int_{-\infty}^{\infty}\frac{G(t)}{X_-(t)}\frac{dt}{t-k-i0}, \; \Psi_-(k) = \frac{X_-(k)}{2\pi i}\int_{-\infty}^{\infty}\frac{G(t)}{X_-(t)}\frac{dt}{t-k+i0} \tag{6}$$

Proof. Hilbert's formula and Lemma 2 gives the solution to the Riemann-Hilbert boundary value problem (1),(2) □

Applying Lemma to Riemans ζ function we get

Theorem 2.4. *Let*

$$...s_{-n}, s_{-n}....s_{-n+1}....s_{-1}, s_1, s_2....s_n.. \ is \ \ zeros \ \ \zeta \ \ function$$
$$Im(s) = const$$
$$Im(s_i) < Im(s) < Im(s_{i+1})$$

$$P(s) = \sum_{j\geq 1} \frac{1}{p_j^s}, \ Re(s) > 1+\delta, \delta > 0,$$

$$P(s) = ln(\zeta(s) - Q(s) \, Re(s) > 1/2 + \delta,$$

$$Q(s) = \sum_{n=2}^{\infty} P(ns)/n, \ \ Re(s) > 1/2 + \delta,$$

$$\psi_+(k) = \frac{1}{2\pi i} \frac{\int_0^1 (\ln(\zeta(s)))e^{i2kRe(s)}\theta(1/2-\delta-Re(s))dRe(s)}{k+i\alpha}$$

$$\psi_-(k) = \frac{1}{2\pi i} \int_0^1 ln(\zeta(s^*) - Q(s^*))e^{-i2kRe(s)}\theta(Re(s)-1/2-\delta)dRe(s)$$

$$\widetilde{\Phi}(k) = \int_0^1 \frac{s}{2}ln(\pi) - ln(\Gamma(s/2)) - \frac{1-s}{2}ln(\pi) + ln(\Gamma(1-s)/2) + Q(1-s))e^{i2kRe(s)}\theta(1/2-\delta-Re(s))dRe(s)$$

$$\widetilde{F}(k) = \frac{\widetilde{\Phi}(k)}{k+i\alpha}$$
$$G(k) = \psi_-(k) + F(k)$$

Then

$$\sup_s |ln(|\zeta(s)|)\theta(Re(s)-1/2-\delta)|\Big| < \frac{5CC_{Im(s)}}{\delta} \tag{7}$$

Proof. Taking the logarithm from (3) and then multiplying it by $e^{i2kRe(s)}$, and after integrating by $\{Re(s), \, Re(s) - 1/2 > \delta\}$ we get

$$\psi_+ = \frac{e^{i2k}}{k-i\alpha}\psi_- + \widetilde{F}(k) = \big(\frac{e^{i2k}}{k+i\alpha} - 1\big)\psi_- + \psi_- + \widetilde{F}(k) = R(k)\psi_- + G(k)$$

By Lemma 3 we get

$$\psi_+(k) = \frac{X_+(k)}{2\pi i} \int_{-\infty}^{\infty} \frac{G(t)}{X_-(t)} \frac{dt}{t-k-i0} = X_+(k)T_+\frac{G}{X_-} \tag{8}$$

$$\psi_-(k) = \frac{X_-(k)}{2\pi i} \int_{-\infty}^{\infty} \frac{G(t)}{X_-(t)} \frac{dt}{t-k+i0} = X_-(k)T_-\frac{G}{X_-} \tag{9}$$

$$\psi_+(k) = X_+(k)T_+\frac{G}{X_-} \tag{10}$$

$$\psi_-(k) = X_-(k)T_-\frac{G}{X_-} \tag{11}$$

$(10, 11) \Rightarrow$

$$\frac{\psi_+}{X_+} - \frac{\psi_-}{X_-} = T_+\frac{G}{X_-} - T_-\frac{G}{X_-} \Rightarrow$$

$$\frac{\psi_+}{X_+} - \frac{\psi_-}{X_-} = \frac{G}{X_-} = \frac{\psi_-}{X_-} + \frac{\widetilde{F}}{X_-}$$

$$2\frac{\psi_-}{X_-}=\frac{\psi_+}{X_+}-\frac{\widetilde{F}}{X_-}$$

$$\Big|\frac{\psi_-}{X_-}\Big|\Big|_{k=\pi n}<\Big|\frac{\widetilde{F}}{X_-}\Big|\Big|_{k=\pi n}+\Big|\frac{\int_0^1(\ln(\zeta(s)))e^{i2kRe(s)}\theta(1/2-\delta-Re(s))ds}{k-i\alpha}\Big|\Big|_{k=\pi n}$$

$$\phi_-=\int_0^1(\ln(\zeta(s)))e^{i2kRe(s)}\theta(1/2-\delta-Re(s))ds|_{k=\pi n}$$

$$\Big|\frac{\psi_-}{X_-}|_{k=\pi n}<\Big|\frac{\widetilde{F}}{X_-}\Big|\Big|_{k=\pi n}+\Big|\frac{\phi_-}{k-i\alpha}\Big|\Big|_{k=\pi n}$$

Lemma (2,3) $\Rightarrow$
$X_+(k)=1+O(1/k),\ \ X_-(k)=1+O(1/k)\ \ ,$
Theorem of Baclund $\Rightarrow |\phi_-|\in L_2$

$$|\psi_-|\big|_{k=\pi n}<|\widetilde{F}|\big|_{k=\pi n}+\frac{|\phi_-|}{k}|_{k=\pi n}$$

$$\sum_{n=1}^{\infty}|\psi_-|\big|_{k=\pi n}<\sum_{n=1}^{\infty}|\widetilde{F}|\big|_{k=\pi n}+\sum_{n=1}^{\infty}\frac{|\phi_-|}{k}|_{k=\pi n}$$

$$\sum_{n=1}^{\infty}|\psi_-|\big|_{k=\pi n}<\frac{C}{\delta}+\sqrt{\sum_{n=1}^{\infty}|\phi_-|^2|_{k=\pi n}\sum_{n=1}^{\infty}\frac{1}{k^2}|_{k=\pi n}}$$

$$\sum_{n=1}^{\infty}|\psi_-|\big|_{k=\pi n}<\frac{C}{\delta}+\sqrt{C_{Im(s)}\sum_{n=1}^{\infty}\frac{1}{k^2}|_{k=\pi n}}<\frac{3CC_{Im(s)}}{\delta}$$

Behaviour of the argument of the Riemann zeta function on the critical line by[6] $\Rightarrow$

$$\big|ln||\zeta(s^*)-Q(s^*)|\theta(Re(s)-1/2-\delta)\big|<\frac{3CC_{Im(s)}}{\delta}+\big|Q(s))\big|<\frac{4CC_{Im(s)}}{\delta}$$

$$\Rightarrow\sup_s|ln(|\zeta(s^*)|)\theta(Re(s)-1/2-\delta)|\big|<\frac{5CC_{Im(s)}}{\delta}$$

□

Analog of Theorem Davenport-Heilbronn

Theorem 2.5. *Let*

$$H(s)$$

is entire function

$$U(s)=H(s(1-s))$$

then

$$\zeta(s)U(s)$$

is solution Riemann's functional equation.

Proof. □

$$lnH(t)\Big|_{t=1-s}=ln(H(1-s)s)$$

$$ln(\zeta(s)H(s))=ln(\zeta(s))+lnH(s)=ln(\zeta(s))+lnH(1-s)$$

Theorem 2.6. *Riemann conjecture is true*

Proof. By Theorem Landau [5]

$$P(s) = ln(\zeta(s) - Q(s), \;\; Re(s) > 1/2 + \delta,$$

let

$$ln\mu(s) = \nu(s) + Q(s)$$

and $\mu(s)$ another solution (3) and $\mu(s)$ analitical in $(s_i < Im(s) < s_{i+1})$

$$\nu(s)\big|_{Re(s)>1, s_i<Im(s)<s_{i+1}} = (ln(\zeta(s)) - Q(s))\big|_{Re(s)>1, s_i<Im(s)<s_{i+1}}$$

then

$$\nu(s)\big|_{Re(s)>1/2, s_i<Im(s)<s_{i+1}} = (ln(\zeta(s)) - Q(s))\big|_{Re(s)>1/2, s_i<Im(s)<s_{i+1}} = P(s)$$

from last statement and the estimate Theorem (1-2) and its symmetry between ψ_+ and ψ_- relative to the critical line leads to the fact that there can be zero zeta-functions only on the critical line that completes the proof.

□

The results of this paper, together with the results of Voronin's theorem [7] on the universality of the Riemann zeta function, lead to the following applications

3. Applications

- Information compression and information transfer,
- Seismic exploration ,
- Optimal control of oil and gas production ,
- Optimal control of oil and gas transfer,
- Control of heat and mass transfer in thermonuclear processes,
- Digitalization of technological processes,
- Earthquake predictions,
- Stock market forecast,
- Stock market forecast,
- Generations of quantum computers and quantum computing
- Seismic exploration
- Tomography
- Early diagnosis of diseases

References

[1] Leonhard Euler. Introduction to Analysis of the Infinite by John Blanton (Book I, ISBN 0-387-96824-5, Springer-Verlag 1988;)

[2] Chebyshev P.L, Fav. mathematiical works, M,-L,-1946;

[3] Poincaré H., *Lecons de mecanique celeste, t. 3, P., 1910.*

[4] Backlund R., *Sur les zeros de la function $\zeta(s)$ de Riemann,C.R. Acad.Sci.,(1914)* 1979-1981 N3

[5] Landau E., Walfisz A. Ober die Nichtfortsetzbarkeit einiger durch Dirichletsrhe Reihen defi-nierter Funktionen, Rend, di Palermo 44 A919), 82—86. Congress Cambridge 1912, 1,

[6] Behaviour of the argument of the Riemann zeta function on the critical line A A Karatsuba1 and M A Korolev2 © 2006 Russian Academy of Sciences, (DoM) and London Mathematical Society, Turpion Ltd

[7] Voronin S. M. The "universality" theorem for the Riemann zeta function

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ABOUT SYSTEM OF MOMENT EQUATIONS AND MOMENT BOUNDARY CONDITIONS DEPENDING ON THE SPEED AND SURFACE TEMPERATURE OF THE AIRCRAFT

AUZHAN SAKABEKOV [1], SALTANAT MADALIEVA[1], RYSKUL YERGAZINA[1]

[1]Satbayev University, Almaty, Kazakhstan
e-mail: *auzhani@gmail.com, madalieva_s@mail.ru, ergazina.ryskul@gmail.com*

Abstract. We present a new one-dimensional non-stationary nonlinear moment system of equations and an approximation of the microscopic boundary condition when some of the molecules reflect from the surface specularly and others diffusively, following the Maxwell distribution. The system of moment equations and the macroscopic boundary conditions depend on the flight speed and the surface temperature of the aircraft. The new system of equations constitutes a complex nonlinear hyperbolic system with respect to the moments of the particle distribution function. The number of macroscopic boundary conditions for the moment equations system depends on the parity of the approximation number of moment equations in the system. We formulate the initial and boundary value problem for the system of moment equations in the third approximation, considering the macroscopic boundary conditions, and prove the existence and uniqueness of the solution for the aforementioned problem in the space of functions that are continuous in time and square summable by spatial variable. We have developed an iterative numerical method to solve direct and inverse problems for a non-stationary nonlinear system of moment equations in the third approximation. Additionally, we have implemented software algorithms and applied them to solve inverse problems in order to determine the speed and surface temperature of an aircraft. Finally, we present the results of numerical experiments.

Keywords: system of moment equations; Maxwell's microscopic condition; macroscopic boundary conditions; nonlinear hyperbolic system of equation.

1. INTRODUCTION

Rarefied gas dynamics deals with the study of gases in which the average distance traveled by molecules between collisions is comparable to or greater than the typical size of the region in which the gas flows. In this regime, the Boltzmann equation, which describes the time evolution of the gas molecule's distribution function, becomes relevant. However, obtaining analytical solutions for the highly nonlinear integro-differential Boltzmann equation under realistic assumptions remains a challenging task, primarily due to the evaluation of the collision term. Therefore, it becomes necessary to employ various approximate methods to solve the Boltzmann equation. One commonly used approach is the moment method, which aims to construct an approximate solution for the initial or initial-boundary value problem of the Boltzmann equation. When studying rarefied gas flows, particular attention must be given to the role of boundary conditions that describe the interaction between gas molecules and solid surfaces. This interaction governs the drag and lift experienced by bodies moving through the gas, as well as the heat transfer between the gas and the boundaries. The prediction of aerodynamic characteristics for aircraft operating at very high speeds and altitudes represents a significant challenge in aerospace

engineering. The methods of rarefied gas theory [1] can be employed to determine these characteristics. Describing a rarefied gas using the particle distribution function lies in the transitional regime between continuous flow and free-molecular flow, making it a complex problem. When calculating the aerodynamic characteristics of an aircraft in a high-speed rarefied gas flow, the Boltzmann equation needs to be augmented with a term dependent on the aircraft's velocity, while the boundary condition on the moving surface must incorporate a parameter related to the surface temperature of the aircraft. To analyze the aerodynamic characteristics of aircraft in the transitional regime, the complete integro-differential Boltzmann equation is utilized:

$$\frac{\partial f}{\partial t} + (c, \frac{\partial f}{\partial x}) + (U, \frac{\partial f}{\partial x}) = J(f, f)$$

where f=f(t,x,v)is the particle distribution function in space of velocity and time, $c = v - Uis$ the relative velocity, $U = (U_1, U_2, U_3)$ is the flying speed of aircraft,J(f,f) is the collision integral. The Boltzmann equation was examined under the corresponding boundary conditions, which must be satisfied by the particle distribution function on the mobile surface of a solid. Determining the boundary conditions on surfaces streamlined with rarefied gas is a crucial aspect of kinetic theory in gases. In high-altitude aerodynamics, the interaction between gas and the surface of a streamlined body plays a significant role [2]. The aerothermodynamic characteristics of bodies in gas flow are determined by the transfer of momentum and energy to the body's surface. This connection involves the velocities and energies of the incident and reflected gas molecules, which forms the basis of kinetic boundary conditions on the surface. In order to solve specific problems more accurately, Maxwell's boundary condition is used to describe the interaction of gas molecules with the surface. One of the approximate methods for solving the initial-boundary value problem for the Boltzmann equation is the moment method. This method allows for the determination of aerodynamic characteristics of aircraft, including atmospheric parameters, flight speed, geometric parameters, and more. It is worth mentioning that in [3], two new models of boundary conditions were proposed: diffusive-moment and mirror-moment, which generalize the known boundary conditions introduced by Cercignani. In [4], the aerodynamic characteristics of space vehicles were studied using the method of direct static modeling (Monte Carlo method) and various models of gas molecule interaction with the surface, examining their effects on aerodynamic characteristics. Different moment methods use various sets of basis functions. For instance, in works [5] and [6], Grad derived a moment system by decomposing the particle distribution function using Hermite polynomials near local Maxwell distributions. Grad utilized Cartesian coordinates of velocities, and his moment system included unknown hydrodynamic characteristics such as density, temperature, average speed, and so on. In works [7]-[8], we obtained a moment system that differs from Grad's system of equations. In this case, we employed spherical coordinates of velocity and decomposed the distribution function into a series using eigenfunctions of the linearized collision operator [1], [9], which is the product of Sonin polynomials and spherical functions. The expansion coefficients, the moments of the distribution function, were defined differently from Grad's approach. The resulting system of equations, corresponding to a partial sum of the series, was termed the Boltzmann's moment system equations. It is a nonlinear hyperbolic system with respect to the moments of the particle distribution function. The differential part of the resulting system is linear, and the quadratic nonlinearity takes the form of moments of the distribution function. The quadratic forms, representing the moments of the nonlinear collision integrals, were calculated in [10] and expressed in terms of Talmi coefficients [11] and Klebsh-Gordon coefficients [12]. In [5–8], it is assumed that the gas moves within a limited region with a fixed boundary. In works [13] - [14], moment systems for the spatially homogeneous Boltzmann equation were derived, along with conditions for representing the solution of the spatially homogeneous Boltzmann equation in the form of Poincaré series. The method proposed by Bobylev A.V. in [13], which applies Fourier transform with respect to the velocity variable in the isotropic case, greatly simplifies the collision integral and the calculation of moments. Vedeniapin V.V. in [14] generalized the results of [13] for the case of anisotropic scattering. C.D. Levermore presented a systematic nonperturbative derivation of

a hierarchy of closed systems of moment equations corresponding to any classical theory in [15]. The first member of this hierarchy is the Euler system, based on Maxwellian velocity distributions, while the second closure is based on non-isotropic Gaussian velocity distributions. The closure procedure involves two steps: the first step ensures that every member of the hierarchy is hyperbolic, has an entropy, and possesses realizability of its predicted moments. The second step involves modifying the collisional terms, which is a nonlinear generalization of Grad's "diagonal approximation" and ensures that the higher-order members of the hierarchy recover the correct Navier-Stokes behavior. C.D. Levermore's paper is a fundamental work for cases where closed systems of moment equations describe a transition regime. A new computational algorithm proposed in [16] serves as an essential component of the moment method for solving the Boltzmann equation. The approach is based on the invariance principle of the collision integral with respect to the choice of the basic system of functions used for expanding the distribution function. The relationships between the matrix elements of the interaction matrix are systematically studied, with recursive relationships deduced for the axially symmetric case. In [17], an approximation of the Boltzmann equation based on the method of moments is presented. The authors propose a generalization of the moment-closure problem from relative entropy to ϕ-divergences and introduce a corresponding closure based on the minimization of ϕ-divergences. This proposed description encompasses Grad's classical closure, which is based on an expansion in Hermite polynomials, and Levermore's entropy-based closure. The generalization to divergence-based closures enables the construction of extended thermodynamic theories that overcome limitations of standard moment-closure formulations, such as inadmissibility of the approximate phase-space distribution, potential loss of hyperbolicity, and singularity of flux functions at local equilibrium. The divergence-based closure leads to a hierarchy of tractable symmetric hyperbolic systems that retain the fundamental structural properties of the Boltzmann equation. In [18], the development of continuum models for describing processes in gases where particle collisions cannot maintain thermal equilibrium is discussed. These situations typically occur in rarefied or diluted gases, microscopic flows, or whenever the Knudsen number becomes significant. The continuum models are based on the stochastic description of the gas using Boltzmann's equation in kinetic gas theory. By using moment approximations, extended fluid dynamic equations such as the regularized 13-moment equations can be derived. The moment equations are explained in detail, and typical results are reviewed for channel flow, cavity flow, and flow past a sphere in the low-Mach number setting, where both evolution equations and boundary conditions are well established. However, nonlinear, high-speed processes require special closures that are still under development. The current approaches are examined, along with the challenge of computing shock wave profiles based on continuum equations. In [19], a globally hyperbolic regularization for the general Grad's moment system in multidimensional spaces is proposed. Systems with moments up to an arbitrary order are studied, and the characteristic speeds of the regularized moment system are analytically given, depending only on the macroscopic velocity and temperature. The structure of the eigenvalues and eigenvectors of the coefficient matrix is fully clarified. Additionally, all characteristic waves are proven to be genuinely nonlinear or linearly degenerate, and properties of rarefaction waves, contact discontinuities, and shock waves are included. In the past two decades, the Maximum Entropy Principle (MEP) has been successfully applied to construct macroscopic models that describe charge and heat transport in semiconductor devices [20]. These models are derived from the Boltzmann transport equations for the charge and phonon distribution functions by considering suitable moments of the distributions as macroscopic variables. MEP is then employed to close the evolution equations for the chosen moments. Significant results have been obtained in the description of charge transport in devices made of both elemental and compound semiconductors, particularly in cases involving charge confinement and one- or two-dimensional carrier flow. In [7]-[8], an approximation of the homogeneous boundary condition for the particle distribution function is presented, and the correctness of the initial-boundary value problem for the nonstationary nonlinear Boltzmann's moment system of equations in a

three-dimensional region with a fixed boundary is proven. Specifically, the existence of a unique generalized solution to the initial-boundary value problem for the Boltzmann's moment system of equations in a space of functions continuous in time and square summable in space variables is established. The paper is organized as follows. In Section 1, we present a new one-dimensional non-stationary nonlinear moment system of equations and an approximation of the microscopic boundary condition where some molecules are reflected from the surface specularly and others diffuse according to the Maxwell distribution. The moment system of equations and macroscopic boundary conditions depend on the flight speed and the surface temperature of the aircraft. Note that the moment system of equations lies between the kinetic theory described by Boltzmann's equation and the hydrodynamic level of gas state description, forming a class of nonlinear partial differential equations. In Section 2, we state the initial and boundary value problem for the system of moment equations in the third approximation under the derived macroscopic boundary conditions of Maxwell-Auzhan and prove the existence of a unique solution to the aforementioned problem in the space of functions that are continuous in time and square summable in spatial variables. We have developed an iterative numerical method for solving direct and inverse problems for the non-stationary nonlinear system of moment equations in the third approximation, implemented software algorithms, and applied them to solve inverse problems for determining the speed and surface temperature of an aircraft.

2. MOMENT SYSTEM OF EQUATIONS DEPENDING ON THE SPEED AND SURFACE TEMPERATURE OF AIRCRAFT AND MACROSCOPIC BOUNDARY CONDITIONS

In the case of a gas flow near a moving solid, the boundary conditions are specified as a ratio between the particles incident on the boundary and the particles reflected from the boundary. The interaction of gases with surfaces in aerodynamics is of great importance. The laws governing the interaction of molecular particles with surfaces become more significant as the gas becomes more rarefied. The boundary conditions for the Boltzmann equation establish the connection between the distribution function of incident and reflected molecules. If the initial distribution of gas molecules is known, the subsequent evolution of the gas is described by the Boltzmann integro-differential equation. Therefore, the problem reduces to solving the initial and boundary value problem for the Boltzmann equation in a region with a moving boundary. We will approximate this problem by considering the initial and boundary value problem for a system of moment equations that takes into account the flight speed of the aircraft under the microscopic conditions described by Maxwell's distribution on the moving boundary. We present a new system of one-dimensional nonstationary nonlinear moment equations that depend on the flight speed and the surface temperature of the aircraft, along with an approximation of Maxwell's microscopic conditions on the moving boundary. The problem of approximating Maxwell's microscopic conditions on a fixed boundary for the one-dimensional nonstationary nonlinear Boltzmann equation will be addressed in [21]. The existence of a global solution to the initial-boundary value problem for the three-dimensional nonlinear Boltzmann equation under the Maxwell boundary conditions was proven in [22]. Problem statement: Find a solution to the initial and boundary value problem for the one-dimensional nonlinear nonstationary Boltzmann equation [1].

$$\frac{\partial f}{\partial t} + |c|cos\theta\frac{\partial f}{\partial x}) + (U_3\frac{\partial f}{\partial x}) = J(f,f), t \in (0,T], x \in (-a,a), c \in R_3^c, \tag{1}$$

$$f|_{t=0} = f^0(x,c), (x,c) \in (-a,a)XR_3^c \tag{2}$$

$$f^+(t,x,|c|cos\theta) = \beta f^-(t,x,-|c|cos\theta)+(1-\beta)exp(-\frac{|c|^2}{2R\theta}, n, c = (n,|c|cos\theta) > 0, x = -aorx = a, \tag{3}$$

where f [[≡]]$f(t,x,c)$ is the particle distribution function in the space of velocity and time;
f^0[[≡]]$^0(x,c)$ is the particle distribution at initial moment of time (given function);

J(f,f)[[≡]] $\int$ [⌈] $f(c^{'})f(c_1^{'}) - f(c)f(c_1)$ [⌉]$\sigma(cosx)dc_1d\varepsilon$ is the
nonlinear collision operator, recorded for Maxwell molecules, n is the unit external normal vector of the boundary, $c = v - U$.

Condition (3) is a natural boundary condition for the Boltzmann equation, which makes it possible to determine the reflected half of the distribution function f if we know the half corresponding to the falling particles.
According to (3), some part of falling particles reflected specularly, and other particles are absorbed into the wall and emitted with the Maxwell distribution with corresponding wall temperature θ. α^2=1/((Rθ)) is also function of time and coordinate. Formula (3) is written under assumption that the boundary (wall or surface) moves with speed $|c|$ cos θ is the speed of the particles falling on the
boundary, $-|c|$ cos θ is the speed of the particles reflected from the boundary. Problem (1)-(3) is written in coordinate system associated with a moving wall, and the speed of movement is a function of time and coordinates, i.e.,$U_3 = U_3(t,x)$.

For one-dimensional problems the eigenfunctions of linearized operator are [1], [9]:
$g_{n1}(\alpha c) = \gamma_{n1}(\frac{\alpha|c|}{\sqrt{2}})^l S_n^{l+\frac{1}{2}}(\frac{\alpha^2|c|^2}{2})P_i(cos\theta), 2n+l = 0,1,2...$, where
$\gamma_{n1} = \sqrt{\frac{\sqrt{\pi}n!(2l+1)}{2\gamma(n+l+\frac{3}{2}}}$ is the normalize coefficient, $S_n^{l+\frac{1}{2}}(\frac{\alpha^2|c|^2}{2})$ is Sonin polynomials, P_l (cos θ) is the Legendre polynomials, Γ is the Gamma function.
To find an approximate solution of the problem (1)-(3) we apply the Galerkin method. We define the approximate solution of one-dimensional problem (1)-(3) as follows:

$$f_k(t,x,c) = f_0(\alpha|c|)\sum_{2n+l=0}^{k} f_{n1}(t,x)g_{n1}(\alpha c) \tag{4}$$

$$\int_R 3^c(\frac{\partial f_k}{\partial t} + |c|cos\theta\frac{\partial f_k}{\partial x} + U_3\frac{\partial f_k}{\partial x} - J(f_k,f_k))g_{n1}(\alpha c)dc = 02n+l = 0,1,2,...,k, x \in (-a,a), \tag{5}$$

$$\int_{(n,c)>0}(n,c)f_{2N+1}^{+}(t,x,c,)g_{n,2l}(\alpha c)dc - \beta\int_{(n,c,)>0}(n,c)f_{2N+1}^{-}(t,x,c)g_{n,2l}(\alpha c)dc - (1-\beta)\int_{(n,c)>0}(n,c) \tag{6}$$

$exp(-\frac{|c|^2}{2R\theta})g_{n,2l}(\alpha c)dc = 02(n+l) = 0,2,...,2N, x = -aorx = a$, with k=2N=1

$$\int_{(n,c)>0}(n,c)f_{2N}^{+}(t,x,c)g_{n,2l+1}(\alpha c)dc - \beta\int_{n,c)>0}(n,c)f_{2N}^{-}(t,x,c)g_{n,2l+1}(-\alpha c)dc - -(1-\beta)\int_{(n,c)<0}(n,c) \tag{7}$$

$exp(-\frac{|c|^2}{2R\theta})g_{b,2l+1}(-\alpha c)dc = 02(n+l)+1 = 1,3,...,2N-1, x = -aorx = a,$
with k=2N, where n=(0,0,1)with x=a and n=(0,0,-1)with x=-a;
$f_0(\alpha|c|) = (\alpha^2/2\pi)^(3/2)exp((-\alpha^2c^2)/2)$ is the local Maxwell distribution;

$$f_{nl}(t,x) = \int_{(R_3^c)} f_k(t,x,c)g_{nl}(\alpha c)dc \tag{8}$$

$f_k^0(x,c) = f_0(\alpha|c|)\sum_{(} 2n+l=0)^k f_n l^0(x)g_n l(\alpha c)dc,$
$f_n l^0(x) = \int_{(} R_3^c)f_k^0(x,c)g_n l(\alpha c)dc.$

The approximation of the boundary condition (3) using equalities (7) and (8) is similar to the approximation of the microscopic Maxwell boundary condition at a constant value of α. Note that the approximation of the boundary condition depends on the evenness or oddness of the approximation k. The quantity of macroscopic boundary conditions and obtaining them for moment equations system depends on the parity of the approximation number of moment equations system. More exactly, if k=2N+1 (k=2N) then both parts of the Maxwell boundary condition multiplies by even (odd) over eigenfunctions of linearized collision operator and integrates on velocity semispace . In case of k=2N+1 a quantity of moment equations system (5) equal to and quantity of macroscopic boundary conditions (7) equal to on the left endpoint and same boundary conditions on the right endpoint of the interval (-a,a). For example, if then a quantity of moment equations system equal to 6 and a quantity of macroscopic boundary conditions on the left endpoint and on the right endpoint of the interval (-a,a) equal to 3. When approximating the microscopic boundary condition we took into account the approximation of Boltzmann equation by the moment equations. Thus, the approximation orders for the expansion of the boundary condition and the expansion of the Boltzmann equation are consistent. The macroscopic conditions (7) and (8) we called the Maxwell-Auzhan boundary conditions [21]. We get the following moment system of equations with respect to the coefficients that will equivalent to the system of equations (5)(here we omit cumbersome calculations)[23]:

$$\frac{df_{nl}}{dt}+\frac{\partial}{\partial x}[\frac{l}{\alpha}(\sqrt{\frac{2(n+l+\frac{1}{2})}{(2l-1)(2l+1)}}f_{n,l-1}-\sqrt{\frac{2(n+1)}{(2l-1)(2l+1)}}f_{n+1,l-1})+]$$
$$+\frac{l+1}{\alpha}(\sqrt{\frac{2(n+l+\frac{3}{2}}{(2l+1)(2l+3)}}f_{n,l+1}-\sqrt{\frac{2n}{(2l+1)(2l+3)}}f_{n-1,l+1})]$$
$$+\frac{dln\alpha}{dt}b_1(f_{nl}+\alpha\frac{dU_3}{dt}b_2(f_{nl})+\frac{1}{\alpha}\frac{1}{\alpha}\frac{\partial ln\alpha}{\partial x}[l+2n)b_3(f_nl)-\ -2\sqrt{n(n+l+\frac{1}{2}b_3(f_{n-1,l}]}+\frac{\partial U_3}{\partial x}b_4(f_bl)=J_{nl},$$

$$2n+l=0,1,2,...,k. \tag{9}$$

where $\frac{d}{dt}\equiv\frac{\partial}{\partial t}+U_3\frac{\partial}{\partial x}$

The coefficients $b_1(f_nl),b_2(f_nl),b_3(f_nl)andb_4(f_nl)$ linear depend on the particle distribution function's moments and expressed in following form:

$b_1(f_nl)=(2n+l)f_nl-2\sqrt{(n(n+l+1/(2)))}f_(n-1,l);\ b_2(f_nl)=\frac{(l+1)\sqrt{(2)})}{(2l+1)(l+2)}\sqrt{\frac{(2l+1)}{(2l+3)}}[-(2n+l)\sqrt{n}f_(n-1,l+1)+2\sqrt{(n(n-1)(n+l+1/2))}f_(n-2,l+1)]-$

$-\frac{(2(2n+l)\gamma_nl)}{((2l+1)(l+2)\alpha^3)}[2l((n+l+1/2)I_(n-1,l-1)-nI_(n,l-1))-D_nl]-\frac{(4(n+l+1/2)\gamma_nl)}{((2l+1)(l+2)\alpha^3)}$

$[2l((n+l-1/2)I_(n-2,l-1)-(n-1)I_(n-1,l-1))-D_(1,nl)]$, where

$I_(n,l-1)\equiv\int_{(R_3^c)}f_0(\frac{\alpha|c|}{\sqrt{(2)}})^{(l-1)}\ S_n^{(l+1/2)}\frac{(\alpha^2c^2)}{2}P_(l-1)(\mu)f_k\ (t,x,c)d(\frac{\alpha|c|}{\sqrt{2}}/)d\mu d\phi$. For this integral following recurrent formula takes place $I_(n,l-1)=\frac{(2(n+l+1/(2)))}{(2n+l+2)}I_(n-1,l-1)+\frac{1}{(2n+l+2)}D_(1,nl)$;

$D_nl=\sum_(2N+L=0)^k f_NL(t,x)\int_(R_3^c)f_0\frac{(\alpha|c|}{\sqrt{2}})^(l+1)S_n^(l+1/2)\frac{\alpha^2c^2}{2}([(l+1)P_(l+1)+lP_(l-1)]$

$[-(\alpha|c|)^2g_NL+(L+2N)g_NL-2(N+L+1/2)\frac{\gamma_{NL}}{\gamma_(N-1,L)}g_(N-1,L)]d(\frac{\alpha|c|}{\sqrt{2}})d\mu d\phi,$

$D_{(1,nl)}=\sum_(2N+L=0)^k f_{NL}(t,x)\int_(R_3^c)f_0(\frac{\alpha|c|}{\sqrt{2}})^(l+1)S_n^(l+1/2)((\frac{\alpha^2c^2}{2}))P_(l-1)(cos\theta)$

$[-(\alpha|c|)^2g_{NL}+(L+2N)g_{NL}-2(N+L+1/2)\frac{\gamma_{NL}}{\gamma_(N-1,L)}g_{(N-1,L)}]d(\frac{\alpha|c|}{\sqrt{2}})d\mu d\phi;$

$b_3(f_nl)=l(\sqrt{\frac{(2(n+l+1/2)}{(2l-1)(2l+1)}})f_(n,l-1)-\sqrt{\frac{2(n+1)}{((2l-1)(2l+1))}}f_(n+1,l-1))+$

$+(l+1)\left(\sqrt{\frac{2(n+l+\frac{3}{2})}{(2l+1)(2l+3)}}f_{n,l+1}-\sqrt{\frac{2n}{(2l+1)(2l+3)}}f_{n-1,l+1}\right)$

$b_4(f_{n,l})=(2n+l+1)\left[\frac{2(l+1)(l+2)}{(2l+1)(2l+3)(l+3)}\sqrt{\frac{2l+1}{2l+5}}\sqrt{n(n+l+\frac{3}{2})}f_{n-1,l+2}-\ldots\right]$

$$- \sqrt{n(n-1)} f_{n-2,l+2}) + \frac{1}{2l+1}\left(\frac{(l+1)^2}{2l+3} + \frac{l^2}{2l-1}\right) f_{nl} + \frac{l(l-1)\gamma_{nl}}{(2l-1)(2l+1)} I_{n,l-2} -$$

$$- \left(\frac{(l+1)(l+2)\gamma_{n,l}}{(2l+1)(2l+3)}\partial_{n,l+2}\right) - 2(n+l+\tfrac{1}{2})\left[\frac{2(l+1)(l+2)}{(2l+1)(2l+3)(l+3)}\sqrt{\frac{2l+1}{2l+5}}\right] \times \dots$$

$$\left(\sqrt{n(n-1)} f_{n-2,l+2} - \sqrt{\frac{n(n-1)(n-2)}{n+l+\frac{1}{2}}} f_{n-3,l+2}\right) + \frac{1}{2l+1}\left(\frac{(l+1)^2}{2l+3} + \frac{l^2}{2l-1}\right) \times \dots$$

$$\sqrt{\frac{n}{n+l+\frac{1}{2}}} f_{n-1,l} + \frac{l(l-1)\gamma_{n,l}}{(2l-1)(2l+1)} I_{n-1,l-2} - \frac{(l+1)(l+2)\gamma_{n,l}}{(2l+1)(2l+3)}\partial_{n-1,l+2}$$

$$I_{n,l-2} \equiv \int_{R_3^c} \left(\frac{\alpha|c|}{\sqrt{2}}\right)^l S_n^{\left(l+\frac{1}{2}\right)}\left(\frac{\alpha^2 c^2}{2}\right) P_{l-2}(\cos\theta) f_k(t,x,c)\, dc$$

$$I_{n,l-2} = \frac{2(n+l+\frac{1}{2})}{2n+l+3} I_{n-1,l-2} + \frac{1}{2n+l+3}\partial_{n,l-2}$$

$$\partial_{n,l\pm 2} = \frac{2\sqrt{2}}{(l+3)\alpha^3} \sum_{2N+L=0}^{k} f_{NL}(t,x) \int_{R_3^c} f_0 \left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l+2} S_n^{\left(l+\frac{1}{2}\right)}\left(\frac{\alpha^2 c^2}{2}\right) P_{l\pm 2}(\mu)\, dc$$

$$\times \left[-(\alpha|c|)^2 g_{NL} + (L+2N) g_{NL} - \frac{2(N+L+\frac{1}{2})\gamma_{NL}}{\gamma_{N-1,L}} g_{N-1,L}\right] d\left(\frac{\alpha|c|}{\sqrt{2}}\right) d\mu\, d\phi.$$

The moments $J_n lof$ the collision integral can be expressed in terms of coefficients Talmi and Klebsh-Gordon[10]:

$J_{nl} = \sum \langle N_3 L_3 n_3 l_3 : l \,|\, nl00 : l\rangle \langle N_3 L_3 n_3 l_3 : l \,|\, n_1 l_1 n_2 l_2 : l\rangle \binom{l_1}{0}\binom{l_2}{0}(\sigma_{l_3} - \sigma_0) f_{n_1 l_1} f_{n_2 l_2}$, where $\langle N_3 L_3 n_3 l_3 : l \,|\, n_1 l_1 n_2 l_2 : l\rangle$

are generalized Talmi coefficients, $\binom{l_1}{0}\binom{l_2}{0}/\binom{l}{0}$ are Klebsh-Gordon coefficients. System of equations (12) is a nonlinear hyperbolic system of equations with respect to the moments $f_{nl}(t,x)$, since the moments of the collision integral are quadratic forms containing ${}_{nl}(t,x)$. In addition, the differential part of this system of equations contains U_3 as coefficients, the flight speed of the aircraft and $\alpha = \sqrt{\frac{1}{R\Theta}}$ where Θ is the surface temperature of the aircraft. The derivatives with respect to time and spatial variable of the flight speed and the surface temperature of the aircraft are also included in the system of equations (12) as coefficients for the lower terms. System (12) differs from existing Grad's system of equations since the moments of the distribution function are determined otherwise than Grad's and structures of the Grad's system and system (12) are different. During derivation of the system of Boltzmann moment equations [7]-[8] the particles distribution function was expanded into eigenfunctions of the linearized collision operator near the global Maxwell distribution, i.e. at a constant value of $\alpha = \sqrt{\frac{1}{R\Theta}}$ so the system of Boltzmann moment equations depended on only one parameter. In case of system (12) both quantities α and U_3 are functions of time and coordinates, so the structures of systems (12) and Boltzmann's moment equations are also different. If in system (12) parameter α is constant and $U_3 = 0$, then we obtain Boltzmann's moment system of equations [24]

$$\frac{\partial f_{nl}}{\partial t} + \frac{1}{\alpha}\frac{\partial}{\partial x}\left[l\left(\sqrt{\frac{2(n+l+\frac{1}{2})}{(2l-1)(2l+1)}} f_{n,l-1} - \sqrt{\frac{2(n+1)}{(2l-1)(2l+1)}} f_{n+1,l-1}\right) + \dots\right]$$

$$\left((l+1)\left(\sqrt{\frac{2(n+l+\frac{1}{2})}{(2l+1)(2l+3)}} f_{n,l+1} - \sqrt{\frac{2n}{(2l+1)(2l+3)}} f_{n-1,l+1}\right)\right) = I_{nl} \;\; 2n+l = 0,1,\dots,k.$$

Boltzmann's moment system of equations is the special case of system (12).

3. INITIAL AND BOUNDARY VALUE PROBLEM FOR THIRD APPROXIMATION OF THE MOMENT SYSTEM OF EQUATIONS WITH MAXWELL-AUZHAN BOUNDARY CONDITIONS

We will present the formulation of the initial and boundary value problem for the moment system of equations in third approximation. We write the initial and boundary value problem for the system of moment equations (12) in third approximation in vector-matrix form. If in (12) 2n+l takes values from 0 to 3, then we obtain the third approximation of the system of moment equations. We introduce the following vectors and matrices(we omit cumbersome calculations of elements of the matrixes B,$C_1, \ldots, C_8$ and vector F):

$$u(t,x) = \begin{pmatrix} f_{00}(t,x) \\ f_{02}(t,x) \\ f_{10}(t,x) \end{pmatrix}, \quad w(t,x) = \begin{pmatrix} f_{01}(t,x) \\ f_{03}(t,x) \\ f_{11}(t,x) \end{pmatrix}$$

$$J_1(t,x) = \begin{pmatrix} 0 \\ J_{02}(t,x) \\ 0 \end{pmatrix}, \quad J_2(t,x) = \begin{pmatrix} 0 \\ J_{03}(t,x) \\ J_{11}(t,x) \end{pmatrix}$$

$$F = \begin{pmatrix} 2 \\ \frac{2}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}'$$

$$\mathrm{A} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{\sqrt{3}} & \frac{3}{\sqrt{5}} & -\frac{2\sqrt{2}}{\sqrt{15}} \\ -\sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{5}{3}} \end{bmatrix}$$

$$B = \frac{1}{\sqrt{\pi}} \begin{bmatrix} \sqrt{2} & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & 2\sqrt{2} & -1 \\ -\frac{1}{\sqrt{3}} & -1 & 3\sqrt{2} \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ -\sqrt{6} & 0 & 2 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ -\sqrt{10} & 0 & 3 \end{bmatrix} \quad C_3 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{4}{5\sqrt{3}} & -\frac{6\sqrt{5}}{35} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{15}} \end{bmatrix}$$

$$C_4 = \begin{bmatrix} -\frac{1}{3} & -\frac{2}{5\sqrt{3}} & 0 \\ 0 & -\frac{9}{7\sqrt{5}} & 0 \\ -\frac{30-2\sqrt{2}}{3\sqrt{15}} & -\sqrt{\frac{3}{2}}\left(1+\frac{4}{3\sqrt{5}}\right) & \left(\frac{5\sqrt{2}-2}{\sqrt{15}}\right)+\frac{9+2\sqrt{6}}{4\sqrt{3}} \end{bmatrix}$$

$$C_5 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{4}{\sqrt{3}} & \frac{6}{\sqrt{7}} & -4\sqrt{\frac{2}{15}} \\ -\frac{4+2\sqrt{3}}{\sqrt{6}} & 0 & 2 \end{bmatrix}$$

$$C_6 = \begin{bmatrix} 1 & \frac{2}{\sqrt{5}} & -\sqrt{\frac{2}{3}} \\ 0 & \frac{9}{\sqrt{5}} & 0 \\ -2\sqrt{\frac{5}{2}} & -2(\sqrt{2}+\sqrt{\frac{6}{5}}) & 5\sqrt{\frac{5}{3}} \end{bmatrix}$$

$$C_7 = \begin{bmatrix} \frac{1}{3} & \frac{2}{5\sqrt{3}} & 0 \\ \frac{2\sqrt{3}}{5} & \frac{11}{7} & -\frac{2\sqrt{2}}{5} \\ -\sqrt{\frac{2}{3}} & -\frac{8}{5\sqrt{2}} & 1 \end{bmatrix}$$

$$C_8 = \begin{bmatrix} \frac{6}{5} & \frac{4\sqrt{15}}{35} & 0 \\ \frac{24}{7}\sqrt{\frac{2}{15}} & \frac{644}{315} & -\frac{24}{35} \\ -\frac{3\sqrt{2}}{5} & -\frac{8\sqrt{6}+10\sqrt{3}}{35} & \frac{12}{5}\sqrt{\frac{2}{3}} \end{bmatrix}$$

Then the moment system of equations (12) in third approximation we write in form

$\frac{du}{dt} + \frac{\partial}{\partial x}\left(\frac{1}{\alpha}Aw\right) + \frac{d\ln\alpha}{dt}C_1u + \alpha\frac{dU_3}{dt}C_3w + \frac{1}{\alpha}\frac{\partial\ln\alpha}{\partial x}C_5w + \frac{\partial U_3}{\partial x}C_7u = J_1(u,w)$
$\frac{dw}{dt} + \frac{\partial}{\partial x}\left(\frac{1}{\alpha}A'u\right) + \frac{d\ln\alpha}{dt}C_2w + \alpha\frac{dU_3}{dt}C_4u + \frac{1}{\alpha}\frac{\partial\ln\alpha}{\partial x}C_6u + \dots$

$$\frac{\partial U_3}{\partial x}C_8w = J_2(u,w), \quad (t,x) \in (0,T] \times (-a,a) \tag{10}$$

where

$J_{02} = (\sigma_2 - \sigma_0)\frac{f_{00}f_{02} - \frac{f_{01}^2}{\sqrt{3}}}{2}$
$J_{03} = \frac{1}{4}(\sigma_3 + 3\sigma_1 - 4\sigma_0)f_{00}f_{03} + \frac{1}{4\sqrt{5}}(2\sigma_1 + \sigma_0 - 3\sigma_3)f_{01}f_{02}$
$J_{11} = (\sigma_1 - \sigma_0)\left(f_{00}f_{11} + \frac{1}{2}\sqrt{\frac{5}{3}}f_{10}f_{01} - \frac{\sqrt{2}}{\sqrt{15}}f_{01}f_{02}\right)$

are the moments of collision integral, A' is transpose matrix. For system of equations (13) we set following initial conditions

where$u_0(x)$ and $w_0(x)$ are given vector functions. For third approximation of the moment system of equations we use boundary conditions (7). If in equality (7) expression 2(n+l) takes values 0 and 2 (2(n+l)=0→n=0,l=0;2(n+l)=2→n=0,l=1;n=1,l=0), then we obtain three boundary conditions on the left endpoint and same boundary conditions on the right endpoint of the interval (-a,a)

$\frac{1}{\alpha}(Aw^+ - Bu^+)(t,-a) = \beta\frac{1}{\alpha}(Aw^- + Bu^-)(t,-a) + \frac{\pi}{\alpha^4}(1-\beta)F, \quad t \in [0,T]$
$\frac{1}{\alpha}(Aw^+ + Bu^+)(t,a) = \beta\frac{1}{\alpha}(Aw^- - Bu^-)(t,a) - \frac{\pi}{\alpha^4}(1-\beta)F, \quad t \in [0,T]$

These boundary conditions match with boundary conditions for six-moment system of Boltzmann equations with constant value of parameter α [21], i.e. when the border was motionless. In problem (13)-(16) the boundary moves with velocity U_3, and parameter α depends from time and coordinates. Thus, it is required to find a solution of system (13), satisfying initial conditions (14) and boundary conditions (15)-(16), and to determine values of movement speed and surface temperature of the aircraft. The moments f_{00}, f_{01}, f_{10} expressed by the macroscopic caracteristics of the gas such that density, average speed and temperature. More exactly, we have following equalities $f_{00} = \rho, f_{01} = \alpha\rho V, f_{10} = \sqrt{\frac{3}{2}}\rho - \sqrt{\frac{2}{3}}\alpha^2\rho(\frac{3}{2}k\theta + \frac{1}{2}V^2)$

where is the density of gas, is the average speed of gas, is the temperature of gas and $\alpha = \sqrt{\frac{1}{R\Theta}}$. Moreover, we have following equality

$f_{00} + \frac{2}{\sqrt{3}} - \frac{2}{3}f_{10} = \alpha(P_{33} + \rho V^2)$, where P_33 is the component of stress tensor. It is possible to check through the direct calculations that $\det\begin{pmatrix} 0 & A \\ A' & 0 \end{pmatrix} \neq 0$

and eigenvalues of the matrix $\begin{pmatrix} 0 & A \\ A' & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 2/\sqrt{3} & 3/\sqrt{5} & -2\sqrt{2/15} \\ 0 & 0 & 0 & 0 & -\sqrt{2/3} & 0 & \sqrt{5/3} \\ 1 & 2/\sqrt{3} & -\sqrt{2/3} & 0 & 0 & 0 & \\ 0 & 3/\sqrt{5} & 0 & 0 & 0 & 0 & \\ 0 & -2\sqrt{2/15} & \sqrt{5/3} & 0 & 0 & 0 & \end{pmatrix}$

Let $\lambda_1 = -\sqrt{3+\sqrt{6}}, \lambda_2 = -1, \lambda_3 = -\sqrt{3-\sqrt{6}}, \lambda_4 = \sqrt{3-\sqrt{6}}, \lambda_5 = 1$, and $\lambda_6 = \sqrt{3+\sqrt{6}}$. In other words, the matrix $\begin{pmatrix} 0 & A \\ A' & 0 \end{pmatrix}$ has three positive and the same number of negative nonzero eigenvalues.

Therefore, the system of equations (13) with α constant and $U_3 = 0$ has three departure characteristics and three arrival characteristics.

The quantity of boundary conditions on the left and right ends of the interval $(-a, a)$ for a system of equations (13) is equal to the number of positive and negative eigenvalues of the matrix $\begin{pmatrix} 0 & A \\ A' & 0 \end{pmatrix}$.

First, we prove the correctness of the initial and boundary value problem (13)-(16) provided that certain functions are given. We show that a system of equations (13) with initial conditions (14) and boundary conditions (15)-(16) has a unique solution in the space of functions that are continuous in time and square summable by spatial variables.

For the problem (13)-(16), the following theorem is valid [26].

Theorem 3.1. *If the initial vector-function $U_0 = (u_0(x), w_0(x))'$ belongs to the space $L^2[-a,a]$ and $u(t,x)$, $w(t,x)$ are twice continuously differentiable functions on $[-a,a] \times [0,T]$, then the problem (13)-(16) has a unique solution $U = (u(t,x), w(t,x))'$ in the domain $[-a,a] \times [0,T]$, belonging to the space $C([0,T]; L^2[-a,a])$. Moreover,*

$$\|\|\mathbf{U}\|\|_{L^2[-a,a]} - y_1\|\|_{C[0,T]} \le C_1(\|\mathbf{U}_0\|\|_{L^2[-a,a]} - y_1(0)),$$

where C_1 is constant independent from U and $T \sim \mathcal{O}\left(\|\mathbf{U}_0\|_{L^2[-a,a]} - y_1(0)\right)^{-1}$, $y_1(t)$ is the particular solution of the Riccati equation, which was built in an explicit form.

The theorem was proved by the method of a priori estimation, the Galerkin method, and Tartar's compensated compactness method.

Let us bring the system of equations (13) to the canonical form. Denote by $\mathbf{U} = L\mathbf{\Psi}$, where $\mathbf{\Psi} = (\psi_1, \dots, \psi_6)'$. Here, $\mathbf{U} = (u, w)'$, and L is the matrix of eigenvectors of the matrix $\begin{pmatrix} 0 & A \\ A' & 0 \end{pmatrix}$. We multiply both parts of the system of equations (13) on the left by L', where L' is the transpose matrix. Then we obtain the following problem:

$$\frac{d}{dt}\begin{pmatrix} \psi_1 & \psi_2 & \psi_3 \\ \psi_4 & \psi_5 & \psi_6 \end{pmatrix} + \frac{\partial}{\partial x}\frac{1}{\alpha}\begin{pmatrix} -\sqrt{3+\sqrt{6}} & \psi_1 & -\psi_2 \\ -\sqrt{3-\sqrt{6}} & \psi_3 & \psi_4 \\ \sqrt{3-\sqrt{6}} & \psi_5 & \sqrt{3+\sqrt{6}} \end{pmatrix} + (L'G(U_3, \alpha, L\mathbf{\Psi})) = (L'J(L\mathbf{\Psi}, L\mathbf{\Psi})), \quad t \in (0,T],\ x \in$$

(11)

$$\mathbf{\Psi}(0,x) = L'\mathbf{U}_0(x), \quad x \in [-a,a], \tag{12}$$

$$\frac{1}{\alpha}\begin{pmatrix} \pm B \\ A \end{pmatrix}\mathbf{L\Psi}^+(t, \pm a) = \frac{\beta}{\alpha}\begin{pmatrix} \mp B \\ A \end{pmatrix}\mathbf{L\Psi}^-(t, \pm a) \pm \frac{\pi}{\alpha^4}(1-\beta)F, \quad t \in (0,T]. \tag{13}$$

Let us construct a finite-difference scheme for the numerical solution of problems (17)-(19). We divide the segment $[0,T]$ into N equal parts and divide the segment $[-a,a]$ into M equal parts. Let us introduce grid functions $\mathbf{u}_{i,j} = (t_i, x_j)$. We approximate the differential problem

(17)-(19) by the first-order finite difference scheme. We now present the statement of the inverse coefficient problem for the system of equations (17). Find the values of the flight speed and surface temperature of the aircraft with the following additional information about the solution of the direct problem. Additional information will be specified on the left boundary of the interval $(-a, a)$ according to the arrival characteristic and on the right border of the interval $(-a, a)$ according to the departure characteristic [26-27], i.e. set the value on the left border, and the value on the right border of the interval $(-a, a)$. Let's solve the problem (17)-(19) for the given values of U_3 and α by the numerical method and determine the values $\psi_k(t, -a, U_3, \alpha)$, $k = 1, \ldots, 3$ and $\psi_k(t, a, U_3, \alpha)$, $k = 4, \ldots, 6$. Let us choose these values as measured ones, i.e. $\psi_k(t, -a, U_3, \alpha) = (\psi_k)_{\text{measure}}, k = 1, \ldots, 3$ and $\psi_k(t, a, U_3, \alpha) = (\psi_k)_{\text{measure}}, k = 4, \ldots, 6$ and carry out an iterative process according to the definition of U_3 and $\alpha = \sqrt{\frac{1}{R\Theta}}$. We will minimize the functional

$J(U_3, \alpha) = \int_0^T \left[\sum_{k=1}^3 \left((\psi_k)_{\text{measure}} - (\psi_k)_{\text{calc}}\right)^2 + \sum_{k=3}^6 \left((\psi_k)_{\text{measure}} - (\psi_k)_{\text{calc}}\right)^2\right] dt,$

where $(\psi_k)_{\text{calc}} = \psi_k(t, -a, U_3^n, \alpha^n)$ for $k = 1, \ldots, 3$ and $(\psi_k)_{\text{calc}} = \psi_k(t, a, U_3^n, \alpha^n)$ for $k = 4, \ldots, 6$, values resulting from the calculation. We continue the iterative process until we obtain the required accuracy.

The numerical experiment was carried out with the following data:

$$J(U_3, \alpha) = \int_0^T \left[\sum_{k=1}^3 \left((\psi_k)_{\text{measure}} - (\psi_k)_{\text{calc}}\right)^2 + \sum_{k=3}^6 \left((\psi_k)_{\text{measure}} - (\psi_k)_{\text{calc}}\right)^2\right] dt,$$

where $(\psi_k)_{\text{calc}} = \psi_k(t, -a, U_3^n, \alpha^n)$ for $k = 1, \ldots, 3$ and $(\psi_k)_{\text{calc}} = \psi_k(t, a, U_3^n, \alpha^n)$ for $k = 4, \ldots, 6$, are values resulting from the calculation. We continue the iterative process until we obtain the required accuracy.

The numerical experiment was carried out with the following data: $[-a, a] \approx [0, 1]$, $\sigma = 120$, $\sigma = -24$. Segment $[0, 1]$ is divided into 10 equal parts, $h = 0.1$ step in space variable x. Denote by τ the time step and choose $\tau = 0.02$. The ratio τ/h satisfies the stability condition.
$R = 8.617333 \times 10^{-5}$ Ev/K;

$\beta = 0.5$, $b = 0.1$, $c = 0.8$, $d = 0.63$;

$F = \frac{1}{4\sqrt{2}} = 0.176776695296637$;

The initial functions are chosen as follows: $\psi_1^{(0)} = \gamma_1 \cdot (c + \sin x)$, $\psi_2^{(0)} = \gamma_2 \cdot (b + e^{-x})$, $\psi_3^{(0)} = \gamma_3 \cdot (d + \cos x)$, $\psi_4^{(0)} = \gamma_4 \cdot (d - \cos x)$, $\psi_5^{(0)} = \gamma_5 \cdot (b - e^{-x})$, $\psi_6^{(0)} = \gamma_6 \cdot (c - \sin x)$.

Coefficients $\gamma_1, \ldots, \gamma_6$ are chosen such that initial functions $\psi_1^{(0)}(x), \ldots, \psi_6^{(0)}(x)$ satisfy boundary conditions: $\gamma_1 = 0.2674$, $\gamma_2 = 0.4080$, $\gamma_3 = 0.2458$, $\gamma_4 = 7.6236$, $\gamma_5 = -0.4916$, $\gamma_6 = 3.4559$.
$\psi_{1_{\text{izm}}}(t) = 0.8925$;
$\psi_{2_{\text{izm}}}(t) = 0.3546$;
$\psi_{3_{\text{izm}}}(t) = 0.3593$;
$\psi_{4_{\text{izm}}}(t) = 0.0017$;
$\psi_{5_{\text{izm}}}(t) = 0.1565$;
$\psi_{6_{\text{izm}}}(t) = 0.2260$.

Using an iterative process, we determine the flight speed $U_3 = 0.5\,\text{km/s}$ and the value of $\alpha = 1/\sqrt{R\Theta} = 5.41334827830624$, which corresponds to the surface temperature $\Theta = 396\,\text{K}$.

4. Conclusion

The system of moment equations (12) depend on the flight speed and the surface temperature of the aircraft and the boundary conditions (7) and (8) depend on the surface temperature of the aircraft. The system of equations (12) is a complex nonlinear hyperbolic system of equations with respect to the moments of the particle distribution function. The left side of the system of

equations (12) depends on unknown parameters α and U_3, and the right side is a quadratic form - the moments of the collision integral. In addition, the boundary conditions (7) and (8) also depend on the unknown parameter α. The characteristics of the system of moment equations depend on such unknown parameters as flight speed and surface temperature of aircraft, and in the general case, it is a very difficult task to formulate boundary conditions for such systems. The approximation of the microscopic Maxwell boundary condition and the formulation of the initial and boundary value problem for the system of moment equations at a constant value of the parameter α allowed us to approximate the microscopic Maxwell boundary condition and formulate the initial and boundary value problem for the system of moment equations with variable values of the parameter α. We managed to formulate the boundary conditions for the system of moment equations depending on the evenness and oddness of the approximation order in the form (7) and (8).

The proof of the correctness of the initial and boundary value problems for the system of equations (12) in various approximations under boundary conditions (7) or (8) is very important from a mathematical point of view, and the determination of flight speeds, and surface temperatures of aircraft is very important for aerospace engineering. We proved the correctness of the initial and boundary value problem for a system of moment equations (13) with boundary conditions (15)-(16), more precisely, we proved the theorem on the existence and uniqueness of a local-in-time solution of an initial and boundary value problem for a system of moment equations in the space of functions that are continuous in time and square summable in spatial variables. Determination of the unknown parameters α and U_3 is the inverse problem for the nonlinear hyperbolic system of equations (13) with initial and boundary conditions (14)-(16). The system of moment equations in the third approximation is reduced to the canonical form and the initial-boundary value problem for the transformed system of equations is formulated. The initial-boundary value problem for the system of equations reduced to the canonical form is approximated by a finite-difference scheme. Determining the flight speed and surface temperature of an aircraft is an inverse problem for a system of equations reduced to a canonical form. This nonlinear system of six equations of hyperbolic type has three departure characteristics and three arrival characteristics. To determine the value of the flight speed and surface temperature of the aircraft, additional information about the solution of the direct problem is required. Additional information is given on the left border of the interval $(-a, a)$ for the arrival characteristic and on the right border of the interval $(-a, a)$ for the departure characteristic. By minimizing the functional...

$J(U_3, \alpha) = \int_0^T \left[\sum_{k=1}^{3} \left((\psi_k)_{\text{measure}} - (\psi_k)_{\text{calc}} \right)^2 + \sum_{k=3}^{6} \left((\psi_k)_{\text{measure}} - (\psi_k)_{\text{calc}} \right)^2 \right] dt$

The flight speed and surface temperature of the aircraft are determined, where

$$(\psi_k)_{\text{calc}} = \psi_k(t, -a, U_3^n, \alpha^n), \quad k = 1, \ldots, 3,$$

and

$$(\psi_k)_{\text{calc}} = \psi_k(t, a, U_3^n, \alpha^n), \quad k = 4, \ldots, 6,$$

are values resulting from the calculation.

Supports: This research was funded by the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan, grant number AP08856926.

References

[1] M.N. Kogan. *Dynamic of rarefied gas.* Moscow, Nauka, 1967, 440p.

[2] R.G. Barantsev. *Interaction of rarefied gases with streamlined surfaces.* Moscow, Nauka, 1975, 343p.

[3] A. Latyshev, A. Yushkanov. *Moment Boundary Conditions in Rarefied Gas Slip-Flow Problems.* Fluid Dynamics, March, 2004, No2, pp. 193-208.

[4] Y.I. Khlopkov, M.M. Zeia, A.Y. Khlopkov. *Techniques for solving high-altitude tasks in a rarefied gas.* International Journal of Applied and Fundamental Research, 2014, No1, pp. 156-162.

[5] G. Grad. *Kinetic theory of rarefied gases.* Comm. Pure Appl. Math, 2, 331, 1949.

[6] G. Grad. *Principle of the kinetic theory of gases.* Handuch der Physik, Volume 12, Springer, Berlin, pp. 205-294.
[7] A. Sakabekov. *Initial-boundary value problems for the Boltzmann's moment system equations in an arbitrary approximation.* Sb. Russ. Acad. Sci. Math, 77(1), 57-76 (1994).
[8] A. Sakabekov. *Initial-boundary value problems for the Boltzmann's moment system equations.* Gylym: Almaty, Kazakhstan, 2002; 276p.
[9] C. Cercignani. *Theory and application of the Boltzmann equation.* Milano, Italy, 1975.
[10] K. Kumar. *Polynomial expansions in Kinetic theory of gases.* Annals of physics, 57, 115-141 (1966).
[11] V.G. Neudachin, U.F. Smirnov. *Nucleon association of easy kernel.* Moscow, Nauka, 1969.
[12] M. Moshinsky. *The harmonic oscillator in modern physics: from atoms to quarks.* New York – London - Paris, 1960, 152p.
[13] A.V. Bobylev. *The Fourier transform method in the theory of the Boltzmann equation for Maxwellian molecules.* Dokl.Akad.Nauk USSR225, 1041-1044 (1975).
[14] V.V. Vedeniapin. *Anisotropic solutions of the nonlinear Boltzmann equation for Maxwellian molecule.* Dokl.Akad.Nauk USSR256, 338-342 (1981).
[15] C.D. Levermore. *Moment closure Hierarchies for Kinetic Theories.* Journal of Statistical Physics, 1996, Vol.83, Nos. 5/6, pp. 1021-1065.
[16] A.I. Endera, I.A. Enderb, M.B. Lutenkoa. *Development of the nonlinear moment method for solving the Boltzmann equation in axially symmetric case.* Matem. Mod., 2002, Volume 14, Number 12, Pages 98–104.
[17] Abdel Malik, M.R.A., van Brummelen E.H. *Moment Closure Approximations of the Boltzmann Equation Based on ϕ-Divergences.* Journal of Statistical Physics volume 164, pages77–104(2016).
[18] Manuel Torrilhon. *Modeling Nonequilibrium Gas Flow Based on Moment Equations.* Annual Review of Fluid Mechanics, 2016, vol.48, 429-458.
[19] Zhenning Cai, Yuwei Fan, Ruo Li. *Globally Hyperbolic Regularization of Grad's Moment System.* Communications on Pure and Applied Mathematics, volume 67, Issue 3, March 2014, pp 464-518.
[20] Giovanni Mascali, Vittorio Romano. *Exploitation of the Maximum Entropy Principle in Mathematical Modeling of Charge Transport in Semiconductors.* January 2017, Entropy 19 (1): 36, DOI: 10.3390/e19010036.
[21] A. Sakabekov, A. Auzhani. *Boundary conditions for the one-dimensional nonlinear nonstationary Boltzmann's moment system equations.* Journal of Mathematical Physics, 55, 123507, (2014).
[22] S. Mischler. *Kinetic equations with Maxwell boundary conditions.* Annales Scientifique de l'ENS 43, fascicule 5, 719-760 (2010).
[23] A. Sakabekov, Y. Auzhani. *Application of the Boltzmann equation for determining the aerodynamic characteristics of aircrafts.* Vestnik KazNITU, vol. 143, no. 1, pp. 54–67, 2021.
[24] A. Sakabekov. *Mixed value problem for one-dimensional Boltzmann's moment system of equations in odd approximation.* Differential Equations, 1992, V.28, No5, pp. 892-900 (in Russian).
[25] A. Sakabekov, S. Madaliyeva, R. Yergazina. *Investigation of aerodynamic characteristics of aircrafts in a rarefied gas flow using the moment method.* Hindawi International Journal of Mathematics and Mathematical Sciences, Volume 2022, Article ID 6943602, 20 pages, https://doi.org/10.1155/2022/6943602.
[26] V. Romanov. *Inverse Problems of Mathematical Physics.* Moscow, Nauka, 265 pages, 1984.
[27] S. Kabanikhin and K. Iskakov. *Inverse and Ill-Posed Problems for Hyperbolic Equations.* Almaty, 330 pages, 2007.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

CONTROLLED EQUATIONS TO PRESENT TRANSFORMING VERBS WITH FUNCTIONAL RELATIONS

ELAMAN KENENBAEV[1]

[1]Institute of Mathematics of NAS of KR, Bishkek, Kyrgyzstan
e-mail: Elaman0527@gmail.com

Abstract. Independent interactive computer presentation of natural languages (without other languages as media) is being developed in Kyrgyzstan. Verbs are main in this approach. They are presented as the user's actions on screen. Some verbs (put, take, push) were implemented as parallel shifts of solids. The aim of the paper is developing presentation of transforming verbs and corresponding transformable objects in various media. Functional relations between points of an object proposed by the authors are used for this purpose. Controlled motion of an object with fixed distances between marked points for presentation of verbs (pull, straighten) is considered. Systems of differential equations (two vector ones or four scalar ones) and corresponding difference ones are built for two marked points. A procedure of concatenation of such equations for several marked points is proposed. The results are new because nobody presented such verbs independently earlier. The results can be used for objective estimation of complexity of notions in the new grammar.

Keywords: controlled equation, mathematical model, language, algorithm, computer model, independent presentation, functional relation

AMS Subject Classification: 34H05

1. Introduction

In 1878 M.Berlitz proposed "immersive teaching method" to teach languages without other languages as media (now its on-line implementation includes a native-speaker instructor).

Combining this idea with ideas [1] and [2] and computers' capacities yielded the notion of independent interactive computer presentation of natural languages (from zero, without other languages as media) which is being developed in Kyrgyzstan [3-6].

Verbs are main in this approach and they are presented as the user's actions on the display. First versions of such soft, with verbs BER (give), KOY(put), AL (take), an avatar verb TÜRT (push) were implemented by parallel shifts of solids. The task to implement transforming verbs arises.

Remark 1.1. *Existing softs such as Rosetta Stone, FluentU, Rocket Languages, Transparent Language, Mango Languages ... mean any primary knowledge of the language, contain instructions written in the language.*

Remark 1.2. *Existing computer and printed vocabularies do not give definitions of notions. They explain words by other words which leads to a circular reasoning. Vocabularies cannot be used for learning a language from zero.*

The improved method of independent presentation of notions is proposed in Section 2. We proposed [7] to use functional relations between points of an object for investigation and constructing objects in various branches of mathematics. Exposition of the definitions and operations of functional relations is in Section 3.

The purpose of this paper is construction of mathematical models (controlled differential and difference equations) for some transforming verbs and corresponding transformable objects within the frames of the method of independent presentation. Obtained results are in Section 4.

As well as other authors, we proceed from the fact that many people (children) in the world know how to use a computer mouse. We will consider intertialess (very viscous) physical media. H-media is meant as horizontal position of a display: an object after ungrasping remains on the spot. V-media (gravitational) is meant as vertical position of a display: an object after ungrasping falls down. Animated and Avatar are stirring slightly.

For convenience of programming (imitation of 3D-space), 2D-media is made multi-layer: the plane itself is (0), all objects (2D-sets) are assigned natural numbers (layers). If $k = n$ then $Object1(k)$ and $Object2(n)$ cannot overlap; if $k > n$ then $Object1(k)$ is shown over $Object2(n)$.

2. "Drag-and-drop" method of independent interactive computer presentation of natural languages

There are some pieces of software to learn nouns and adjectives of natural languages (independently by our terminology) by "Point-and-lick" method as follows. Firstly, some nouns are shown and named. Further, a command of type "show / point name " sounds. The user (child) does not understand the word "show" or "point" but s/he hears a familiar word (after a verb in English, Russian or before a verb in Turkic languages) and guesses what to do. The computer confirms the correctness of the answer. But by such a way only concrete nouns (for instance, animals, countries on a map) and, with some complications, some concrete adjectives (colors) can be taught. By our opinion, as well as the [3-6] authors' one, the verb is the main part of speech and the constituent of a language and it can be presented as the user's actual action. Let any "notion" (word W of a natural language L) be given.

Definition 1. A "Drag-and-Drop" algorithm checks fulfilling of actions "grasp" and "ungrasp" in prescribed domains of a display (and makes transformations due to media) successfully.

If the algorithm generates (randomly) a different situation when restarted, gives a command in L involving W in each situation and such situations cover all essential aspects of the "notion" and the user is able to guess what to do then such algorithm is said to be an independent computer interactive presentation of W in L (commands may contain other words, learned earlier. But these words must not give any definitions or explanations of W).

Definition 2. A description of domains, conditions, properties of media, corresponding command containing the word W and written in a formalized subset of the language L such that these actions correspond to the common sense of the notion is said to be a mathematical model of W.

The environment consists of (generated randomly) constant objects, moving objects and controlled movable objects. (By default, any declared object is fixed).

Some techniques are proposed for the user's guessing:

G1) uniqueness of the action (or the sequence of actions) which subdues the command and the situation naturally;

G2) similarity (some objects have the same property and this property is mentioned twice);

G3) alternation (new object and new word appear together as an alternative to preceding objects);

G4) distinguishability (one of objects differs sufficiently and attracts the user's attention).

Grammar of the method:

Definition 3. If all words being used in Definition 2 are unknown to the user nevertheless s/he is be able to fulfil the meant action (because it is the only natural one in this situation) then the notion (word of a language) is said to be primary. If the user has to know supplementary words to complete the action then the notion is said to be secondary. Thus, there is a natural

hierarchy of notions. Complexity of such model is the objective measure of complexity of the notion.

We extended this definition, see Section 3 below. Preceding implemented examples of mathematical models (parallel shifts only).

Example 2.1. *Verb KOY-PUT (G1). Primary.*
H-Environment: Thing(2) (to be shifted); Place(1).
Command "PUT"+Name(Thing)+"into"+ Name(Place)+"!"
Sequence of conditions: 1) Grasp in Thing. 2) Ungrasp in Place.

Example 2.2. *Avatar verbs KIR-ENTER, CHYK-EXIT (G1). Primary.*
H-Environment: Outer-domain(1), Inner-domain(1), Avatar (2) (to be shifted) on Outer-domain; Yard-with-narrow-entrance(2).
Command "ENTER"+"the yard"+"!"
Sequence of conditions: 1) Grasp in Avatar. 2) Ungrasp in Inner-domain.
Command "EXIT"+"the yard"+"!"
Sequence of conditions: 1) Grasp in Avatar. 2) Ungrasp in Outer-domain.

Example 2.3. *Adjective GREEN (G2). Secondary, after the word PUT.*
Environment: green Thing (to be shifted); blue Thing (to be shifted), green Place (fixed).
Command "Put"+"GREEN"+"Name(Thing)+"into"+"GREEN"+ Name(Place)+"!"
Sequence of conditions: 1) Grasp in green Thing. 2) Ungrasp in green Place.

3. Functional relations

Consider any mathematical object F with marked points $z_1, z_2, z_k, k \geq 2$ of a topological space S.

If F is a function then
$z_1 = (x_1, F(x_1)), z_2 = (x_2, F(x_2)), ..., z_k = (x_k, F(x_k))$.

Definition 4. If values $z_1, z_2, ..., z_k$ meet a non-trivial predicate:

$$P(z_1, z_2, ..., z_k) = yes \tag{1}$$

for all $F \in S$ then S is said to be $(P, \leq k)$ - functional-related.

If the number k is the minimal then S is said to be $(P - k)$ - functional-related.

We propose: if all solutions of a differential equation or other mathematical problem satisfy a functional relation, then the classification of the functional relation is also the classification of the given mathematical problem.

Definition 5. If for given points $x_1, x_2, ..., x_{k-1}$ the point x_k is unique then the functional relation for a function is said to be hard otherwise it is said to be soft.

We found the following well-known examples:

Example 3.1. *For F being a polynomial of $(k-2)$-power, $k \geq 3$, its k values are connected, the predicate P can be written by Lagrange polynomial. (Soft relation). In general, for a 2D-object of $(k-2)$-order, its k points are connected; for a 3D-object of $(k-3)$-order, its k points are connected.*

Example 3.2. *Ch.J.Vallee-Poussin, 1929:*
Multipoint boundary value problem for a linear ordinary differential equation

$$y^{(k-2)}(x) + p_1(x)y^{(k-3)}(x) + ... + p_{k-2}(x)y(x) = 0 (a \leq x \leq b), \tag{2}$$

$p_1(x), ..., p_{k-2}(x) \in C[a, b]; y(x[j]) = c[j], j = 1..k-1$
has a unique solution when sum of norms of coefficients is small (soft relation):
$\Sigma_{j=1}^{k-2} \|p_j\|_{[a,b]}(b-a)^j/j! < 1$.

Example 3.3. *A solution of the hyperbolic equation* $\partial^2 u(x_1, x_2)/\partial x_1/\partial x_2 = 0$ *fulfills the Asgeirsson's identity* $(k = 4)$ *(hard relation)*

$$u(w_1, v_1) + u(w_2, v_2) - u(w_1, v_2) - u(w_2, v_1) = 0. \tag{3}$$

A solution of the wave equation $\partial^2 u(x_1, x_2)/\partial x_1^2 = \partial^2 u(x_1, x_2)/\partial x_2^2$ *fulfills the similar identity* $(k = 4)$*: for four vertices of a rectangle obtained by means of rotation of the rectangle on* 45^0*.*

Example 3.4. *A solution of the equation* $\partial^3 u(x_1, x_2, x_3)/\partial x_1 \partial x_2 \partial x_3 = 0$ *fulfills the generalized Asgeirsson's identity* $(k = 8)$ *(hard relation):*

$$u(v_1, v_2, v_3) - u(v_1, v_2, v_6) - u(v_1, v_5, v_3) + u(v_1, v_5, v_6) - \\ - u(v_4, v_2, v_3) + u(v_4, v_2, v_6) + u(v_4, v_5, v_3) - u(v_4, v_5, v_6) = 0. \tag{4}$$

The following definition extends Definition 2.

Definition 6. A transformable object is a (varying) set or union of sets on display, some points of it are connected with functional relations. Also, conditions on these points may be included in mathematical models of notions.

Definition 6. A transformable object is a (varying) set or union of sets on display, some points of it are connected with functional relations. Also, conditions on these points may be included in mathematical models of notions.

4. Equations for horizontal media

Consider the verb TART-DRAW for a transformable object with two marked points in H-environment. The functional relation is: the distance between these points is constant. The user is to draw it by the first point to a spot (in screen).

We have the system (in vector form)

$$z_1'(t) = w(t), z_2'(t) = f(z_1(t), z_2(t), w(t)) \tag{5}$$

where $z_k(t) = (x_k(t), y_k(t)), k = 1..2$ are vector-functions of marked points;

$w(t) = (u(t), v(t))$ is the user's control;

the vector-function $f(z_1, z_2, w) = (g(z_1, z_2, w), h(z_1, z_2, w))$ is defined from the following conditions:

$$|z_1(t) - z_2(t)| = c = const; \tag{6}$$

motion of object is close to natural one.

Rewrite the system (5) in scalar form:

$$x_1'(t) = u(t), y_1'(t) = v(t), \\ x_2'(t) = g(x_1(t), y_1(t), x_2(t), y_2(t), u(t), v(t)), \\ y_2'(t) = h(x_1(t), y_1(t), x_2(t), y_2(t), u(t), v(t)). \tag{7}$$

Rewrite the condition (6) in scalar form:

$$(x_1(t) - x_2(t))^2 + (y_1(t) - y_2(t))^2 = c^2. \tag{8}$$

Suppose that an infinitesimal shift of the point $z_2(t)$ is along the vector

$$z_1(t) - z_2(t) = (x_1(t) - x_2(t), y_1(t) - y_2(t)). \tag{9}$$

Let $p_k, k = 1..2$ be infinitesimals. We obtain the equation

$$|(z_1(t) + p_1 w) - (z_2(t) + p_2(z_1(t) - z_2(t)))| = |z_1(t) - z_2(t)|. \tag{10}$$

In scalar form

$$((x_1(t)+p_1u)-(x_2(t)+p_2(x_1(t)-x_2(t))))^2+$$
$$+((y_1(t)+p_1v)-(y_2(t)+p_2(y_1(t)-y_2(t))))^2=$$
$$=(x_1(t)-x_2(t))^2+(y_1(t)-y_2(t))^2;$$
$$((x_1(t)+p_1u)-(x_2(t)+p_2(x_1(t)-x_2(t))))^2+$$
$$+((y_1(t)+p_1v)-(y_2(t)+p_2(y_1(t)-y_2(t))))^2=$$
$$=(x_1(t)-x_2(t))^2+(y_1(t)-y_2(t))^2;$$
$$((x_1(t)-x_2(t))-(-p_1u+p_2(x_1(t)-x_2(t))))^2+$$
$$+((y_1(t)-y_2(t))-(-p_1v+p_2(y_1(t)-y_2(t))))^2=$$
$$=(x_1(t)-x_2(t))^2+(y_1(t)-y_2(t))^2. \quad (11)$$

Omitting squared infinitesimals we obtain

$$2(x_1(t)-x_2(t))(-p_1u+p_2(x_1(t)-x_2(t)))++2(y_1(t)-y_2(t))(-p_1v+p_2(y_1(t)-y_2(t)))=0;$$
$$p_2(x_1(t)-x_2(t))^2-p_1u(x_1(t)-x_2(t))+$$
$$+p_2(y_1(t)-y_2(t))^2-p_1v(y_1(t)-y_2(t))=0;$$
$$p_2c^2=p_1u(x_1(t)-x_2(t))+p_1v(y_1(t)-y_2(t)). \quad (12)$$

Hence,

$$p_2=p_1(u(x_1(t)-x_2(t))+v(y_1(t)-y_2(t)))/c^2;$$
$$g(x_1,y_1,x_2,y_2,u,v)=(u(x_1-x_2)+v(y_1-y_2))(x_1-x_2)/c^2;$$
$$h(x_1,y_1,x_2,y_2,u,v)=(u(x_1-x_2)+v(y_1-y_2))(y_1-y_2)/c^2. \quad (13)$$

In vector form

$$f(z_1,z_2,w)=\langle w,z_1-z_2\rangle(z_1-z_2)/c^2. \quad (14)$$

For computer implementation, choose a number p_1 such that $p_1|w|$ 10 pixels, we have the following system of controlled difference equations:

$$X_1[m+1]=X_1[m]+p_1u; Y_1[m+1]=Y_1[m]+p_1w; \quad (15)$$

$$P_2:=p_1(u(X_1[m]-X_2[m])+v(Y_1[m]-Y_2[m]))/c^2; \quad (16)$$

$$X_2[m+1]=X_2[m]+P_2(X_1[m]-X_2[m]); \quad (17)$$

$$Y_2[m+1]=Y_1[m]+P_2(Y_1[m]-Y_2[m]), m=0,1,2,... \quad (18)$$

Example 4.1. *Verb DRAW, preposition BY, noun ARROW (G1). Secondary. Environment: Arrow-with-red-point-in-head(2) (to be transformed); Place(1). Command "DRAW"+"the ARROW"+"BY"+"the red point"+"into"+ Name(Place)+"!" Sequence of conditions: 1) Grasp in Red-point. 2) Ungrasp in Place.*

Example 4.2. *Noun PASS, preposition THROUGH (G3). Secondary. Environment: Arrow-with-red-point-in-head(2) (to be transformed); colored Pass(2), Place(1). Command "DRAW"+"the arrow"+"THROUGH"+"the green PASS"+"into"+ Name(Place)+"!" Sequence of conditions: 1) Grasp in Red point. 2) Red point is in Pass. 3) Second marked point is in Pass. 4) Ungrasp in Place.*

Consider a transformable object with three marked points. The functional relations are: the distances between the first and the second points and between the second and the third points are constant. Then the shift of the third point is defined by the shift of the second point. Such object can be used for mathematical models of verbs STRAIGHTEN, BEND. Also, a fourth point, a fifth point can be added. Then the noun CHAIN arises.

5. Equations for vertical media

Consider the object drawn stick as above. At each step the second point should move down due to the force of gravity (defined by the constant $g > 0$ which should be chosen to make motion more "natural"). If z_1 is fixed then the component of the force of gravity perpendicular to the vector $(z_1 - z_2)$ is proportional to $cos(x - z_1 - z_2)$. Introduce the sequence $Y_2'[m]$, denoting the y-coordinate of the second point after moving down and before "drawing" (15). Then we obtain:

$$Y_2'[m] = Y_2[m] - ((1 - (Y_1[m] - Y_2[m])^2/c^2)^{1/2} g. \tag{19}$$

$$P_2' := p_1(u(X_1[m] - X_2[m]) + v(Y_1[m] - Y_2'[m]))/c^2; \tag{20}$$

$$X_2[m+1] = X_2[m] + P_2'(X_1[m] - X_2[m]); \tag{21}$$

$$Y_2[m+1] = Y_1[m] + P_2'(Y_1[m] - Y_2'[m]), m = 0, 1, 2, ... \tag{22}$$

Changing (15) to (20)-(21)-(22) we obtain the system of controlled difference equations.

Example 5.1. *Verb DROP, preposition IN, noun BOTTLE.*

Environment: Tree as the sign of V-media; Arrow-with-red-point-in-head(2) (to be transformed); Bottle(2); Domain-near-bottom-of-bottle(1) (Arrowhead cannot pass the neck of Bottle).

Command "DROP"+"the arrow"+"IN"+"the BOOTLE"+"!"

Sequence of conditions: 1) Grasp in Red point. 2) Second marked point is in Domain.

Corresponding systems of equations can be constructed for the verbs TÜZDÖ (straighten the chain $z_1 - z_2 - z_3 - ...$), IIY (bend a stick $z_1 - z_2 - z_3$) and other transforming verbs.

Difference approximation of such systems would be used for computer presentations.

6. Conclusion

We hope that our results are new because none soft with transforming verbs was found in Internet. We introduced definitions enlarging the notion of mathematical model for independent computer presentation of notions of natural languages, and constructed systems of differential equations and corresponding difference ones for transformable object with two marked points connected by a functional relation, proposed a procedure of concatenation of such equations for several marked points. The purpose of the paper is achieved. These results can be used for developing virtual spaces with controlled motion and transformation in them. A number of connected points being necessary for independent presentation of a notion yields the objective estimation of complexity of notion in the grammar of the method. Investigations can be continued for adequate presentations of various verbs and other notions of natural languages.

References

[1] Asher J. The strategy of the total physical response: an application to learning Russian. *International Review of Applied Linguistics*, 1965, 3, pp. 291-300.

[2] Winograd T. *Understanding Natural Language.* Massachusetts Institute of Technology, New York, 1972.

[3] Pankov P.S., Alimbay E. Virtual Environment for Interactive Learning Languages. *Human Language Technologies as a Challenge for Computer Science and Linguistics: Proceedings of 2nd Language and Technology Conference*, Poznan, 2005, pp. 357-360.

[4] Bayachorova B.J., Pankov P.S. Independent Computer Presentation of a Natural Language. *Varia Informatica.* Polish Information Processing Society, Lublin, 2009, pp.73-84.

[5] Pankov P.S., Bayachorova B.J., Juraev M. Mathematical Models for Independent Computer Presentation of Turkic Languages. *TWMS Journal of Pure and Applied Mathematics*, Volume 3, No.1, 2012, pp. 92-102.
[6] Bayachorova B., Pankov P. Independent Computer Presentation of Abstract Notions of Natural Languages. *Actual Problems of Computer Science*, ECCC Foundation, Lublin, 2013, No. 1(3), pp. 64-69.
[7] Kenenbaeva G.M., Kenenbaev E. Category of objects with functional relations for differential equations. *Herald of Institute of Mathematics of National Academy of Sciences of Kyrgyz Republic*, 2020, No. 2, pp.80-86.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

MIXED PROBLEM FOR A PARABOLIC INTEGRO-DIFFERENTIAL EQUATION WITH INVOLUTION

FARHOD RAKHMANOV

National University of Uzbekistan named after Mirzo Ulugbek, 100174, University street 4, Olmazor district, Tashkent
e-mail: farxod_frd@bk.ru

Abstract. In this paper, we consider an inhomogeneous parabolic type partial integro-differential equation with degenerate kernel and involution. With respect to spatial variable x is used Dirichlet boundary value conditions and spectral problem is obtained. Eigenvalues and eigenfunctions of the spectral problems are found. The Fourier method of separation of variables is applied. The countable system of functional-integral equations is obtained. Theorem on a unique solvability of countable system of integral equations is proved. The method of successive approximations is used in combination with the method of contraction mapping. The unique solution of the mixed problem is obtained in the form of Fourier series. Absolutely and uniformly convergence of Fourier series is proved.

Keywords: Mixed problem, parabolic equation, degenerate kernel, involution, unique solvability.

1. Formulation of the problem statement

Investigations on ordinary and partial integro-differential equations of the Fredholm type are great interest from the point of theoretical research and applications in the different fields of physics, mechanics, engineering and chemistry [1-7]. Today some new problems are posed for ordinary and partial integro-differential equations, and a large number of papers devoted to study of boundary value and mixed problems for the parabolic integro-differential equations are published [8-28]. In [29-37], integro-differential equations with a degenerate kernel were considered.

In this paper, we study the solvability of the mixed problem for a parabolic integro-differential equation with a degenerate kernel and involution.

In the rectangular domain, $\Omega = \{0 < t < T,\ -1 < x < 1\}$ we consider the following partial integro-differential equation

$$U_t(t,x) - \omega^2 \left[U_{xx}(t,x) + \varepsilon\, U_{xx}(t,-x)\right] = \nu \int_0^T K(t,\, s)\, U(s,x)\, d\, s + \alpha(t)\, U(t,x), \qquad (1)$$

where $0 < \alpha(t) \in C[0,T]$, T is given positive number, ε is parameter, $|\varepsilon| < 1$, ω is positive parameter, ν is nonzero real parameter, $K(t,\, s) = \sum_{r=1}^{k} a_r(t)\, b_r(s)$, $a_r(t)\,, b_r(s) \in C\,[0;\, T]$. It is assumed that the systems of functions $\{a_r(t)\}_{r=1}^{k}$ and $\{b_r(s)\}_{r=1}^{k}$ are linear independent.

In solving partial integro-differential equation (1), with respect to spatial variable x we use Dirichlet boundary value conditions

$$U(t,-1) = U(t,1) = 0, \qquad 0 \le t \le T. \qquad (2)$$

This work was partially supported by the Foundation for Basic Research of the Republic of Uzbekistan (Grant No. FZ–20200929224).

With respect to time variable t we use initial value condition

$$U(0,x)=\varphi(x), \quad -1\le x\le 1, \tag{3}$$

where $\varphi(x)$ is enough smooth on the segment $[-1,1]$. For this function the following conditions are fulfilled

$$\varphi(-1)=\varphi(1)=0.$$

Problem statement. To find a function $U(t,x)\in C\left(\overline{\Omega}\right)\cap C^{1,2}_{t,x}(\Omega)$, which satisfies the parabolic integro-differential equation (1) and the specified conditions (2) and (3), where $\overline{\Omega}=\{0\le t\le T,\ -1\le x\le 1\}$.

2. Formal solution of the problem (1)-(3)

First, consider the homogeneous differential equation

$$U_t(t,x)-\omega^2\left[U_{xx}(t,x)+\varepsilon\, U_{xx}(t,-x)\right]=0 \tag{4}$$

with boundary value conditions of the Dirichlet type

$$U(t,-1)=U(t,1)=0, \quad 0\le t\le T. \tag{5}$$

Problem (4), (5) will be solved by the method of separation of variables:$U(t,x)=u(t)\vartheta(x)$. Then from this problem we arrive at the following spectral problem for an ordinary differential equation

$$\vartheta''(x)+\varepsilon\vartheta''(-x)+\lambda\vartheta(x)=0 \tag{6}$$

with boundary value conditions

$$\vartheta(-1)=0, \quad \vartheta(1)=0. \tag{7}$$

For the case of even eigenfunctions, equation (6) takes the form

$$(1+\varepsilon)\vartheta_1''(x)+\lambda_1\vartheta_1(x)=0. \tag{8}$$

Solving the differential equation (8) with conditions (7), we find the eigenvalues

$$\lambda_{1n}=(1+\varepsilon)\,\pi^2(n+0.5)^2 \tag{9}$$

and eigenfunctions

$$\vartheta_{1n}(x)=\cos\,\pi(n+0.5)x, \quad n\in N. \tag{10}$$

In the case of odd eigenfunctions, equation (6) takes the form

$$(1-\varepsilon)\vartheta_2''(x)+\lambda_2\vartheta_2(x)=0. \tag{11}$$

Solving the differential equation (11) with conditions (7), we find the eigenvalues and the corresponding eigenfunctions of this problem

$$\lambda_{2n}=(1-\varepsilon)\,\pi^2n^2, \quad |\,\varepsilon\,|<1, \tag{12}$$

$$\vartheta_{2n}(x)=\sin\,\pi n\,x, \quad n\in N\,. \tag{13}$$

Note that the eigenfunctions $\vartheta_{i\,n}(x)\,(i=1,2)$ determined by (10) and (13) form a complete system of orthonormal eigenfunctions in the space $L_2[-1,1]$. Therefore, we seek nontrivial solutions to the inhomogeneous partial integro-differential equation (1) in the forms $U(t,x)=U_1(t,x)+U_2(t,x)$ of the Fourier series

$$U_i(t,x)=\sum_{n=1}^{\infty}u_{i\,n}(t)\,\vartheta_{i\,n}(x), \quad i=1,2, \tag{14}$$

where $U_1(t,x)$ and $U_2(t,x)$ satisfy the given integro-differential equation (1)

$$U_{i\,t}(t,x)-\omega^2\left[U_{i\,xx}(t,x)+\varepsilon\, U_{i\,xx}(t,-x)\right]=\alpha(t)\,U_i(t,x)+\nu\int_0^T K\,(t\,,\,s)\,U_i\,(s,x)\,d\,s,$$

$$u_{i\,n}(t)=\int_{-1}^{1}U_i(t,x)\,\vartheta_{i\,n}(x)\,dx, \quad i=1,2. \tag{15}$$

Substituting the Fourier series (14) into the given integro-differential equation (1), we obtain a countable system of first order ordinary integro-differential equations

$$u'_{in}(t)+\omega^2\lambda_{in}\,u_{in}(t)=\,\nu\,\sum_{r=1}^{k}a_r\,(t\,)\,\tau_{inr}\,+\,\alpha(t)\,u_{in}(t),\;\;i=1,2, \tag{16}$$

where λ_{in} are eigenvalues determined by (9) and (12),

$$\tau_{inr}=\int_0^T\,b_r\,(s)\,u_{in}(s)\,d\,s. \tag{17}$$

Solving the countable system of inhomogeneous differential equation (16) by the method of variation of arbitrary constants, we obtain the representation

$$u_{in}(t)=A_{in}\;\exp\,\{-\lambda_{in}\,\omega\,t\}+\nu\;\sum_{r=1}^{k}\tau_{inr}\;\int_0^t\exp\,\{-\lambda_{in}\,\omega\,(t-s)\}\;a_r\,(s)\,d\,s+$$

$$+\int_0^t\exp\,\{-\lambda_{in}\,\omega\,(t-s)\}\;\alpha(s)\,u_{in}(s)\,d\,s, \tag{18}$$

where A_{in} are yet arbitrary coefficients, which will be determined.
For given function $\varphi(x)$ we set $\varphi(x)\,=\,\varphi_1(x)+\varphi_2(x)$. Now, supposing that the functions $\varphi_1(x)$ and $\varphi_2(x)$ are expanding into a Fourier series and using the Fourier coefficients (15), from conditions (3) we obtain

$$u_{in}\,(0)=\,\int_{-1}^{1}U_i(0,x)\,\vartheta_{in}(x)\,dx=\,\int_{-1}^{1}\varphi_i(x)\,\vartheta_{in}(x)\,dx=\varphi_{in},\;\;i=1,2. \tag{19}$$

To find the unknown (arbitrary) coefficients A_{in} in the presentations (18), we use the boundary conditions (19). Then we have

$$A_{in}=\varphi_{in},\;\;i=1,2. \tag{20}$$

Substituting these values of (20) into representation (18), we obtain

$$u_{in}(t)=\varphi_{in}\;\exp\,\{-\lambda_{in}\,\omega\,t\}+\nu\;\sum_{r=1}^{k}\tau_{inr}\;\int_0^t\exp\,\{-\lambda_{in}\,\omega\,(t-s)\}\;a_r\,(s)\,d\,s+$$

$$+\int_0^t\exp\,\{-\lambda_{in}\,\omega\,(t-s)\}\;\alpha(s)\,u_{in}(s)\,d\,s. \tag{21}$$

Although functions (21) are Fourier coefficients of solution to the problem (1)-(3), it contains extra quantities τ_{inr} that are still unknown. To find these quantities, we substitute representation (21) into designation (17) and arrive at a SAE:

$$\tau_{inr}\;-\nu\;\sum_{j=1}^{k}\tau_{ijnr}\,\sigma_{i1rjn}\,(t)=\varphi_{in}\,\sigma_{i2rn}\,+\sigma_{i3rn}\,(u_{in}),\;\;i=1,2,\;\;r=1,...,k, \tag{22}$$

where

$$\sigma_{i1rjn}\,=\int_0^T\,b_r(s)\int_0^s\exp\,\{-\lambda_{in}\,\omega\,(s-\theta)\}\;a_j\,(\theta)\,d\theta\,ds,$$

$$\sigma_{i2rn}\,=\int_0^T\,b_r(s)\exp\,\{-\lambda_{in}\,\omega\,s\}\;d\,s,\;\;\sigma_{i3rn}(u_{in})=\int_0^T\,b_r(s)\int_0^s\exp\,\{-\lambda_{in}\,\omega\,(s-\theta)\}\;\alpha(\theta)\,u_{in}(\theta)\,d\theta\,ds.$$

To establish the unique solvability of SAE (22), we introduce the following matrix

$$\Theta_{i0\,n}\,(\nu\,,\,\omega)=\left(\begin{array}{cccc}1-\nu\,\sigma_{i11\,1n}(\omega) & \nu\,\sigma_{i11\,2n}(\omega) & \ldots & \nu\,\sigma_{i11kn}(\omega)\\ \nu\,\sigma_{i12\,1n}(\omega) & 1-\nu\,\sigma_{i12\,2n}(\omega) & \ldots & \nu\,\sigma_{i12kn}(\omega)\\ \ldots & \ldots & \ldots & \ldots\\ \nu\,\sigma_{i1k\,1n}(\omega) & \nu\,\sigma_{i1k\,2n}(\omega) & \ldots & 1-\nu\,\sigma_{i1\,kkn}(\omega)\end{array}\right)$$

and consider the values of the parameter ν, for which the Fredholm determinant differs from zero:

$$\Delta_{i0n}(\nu,\omega) = \det \Theta_{i0n}(\nu,\omega) \neq 0. \tag{23}$$

Determinant $\Delta_{i0n}(\nu,\omega)$ in (23) is a polynomial with respect to ν of the degree not higher than k. The countable system of algebraic equations $\Delta_{i0n}(\nu,\omega) = 0$ has no more than k different real roots for every value of n. We denote them by μ_{il} $(l = \overline{1,\,p}\ ,\ 1 \leq p \leq k)$. Then $\nu_i = \nu_{il} = \mu_{il}$ are the characteristic (irregular) values of the kernel of the integro-differential equation (1). So, we introduce the following two designations

$$\Lambda_{i\,1} = \{\,(\nu_i,\,\omega)\ :\ \nu_i = \mu_{il}\,,\ i = 1,2,\ \ \omega \in (0,\infty)\}\,,$$

$$\Lambda_{i\,2} = \{\,(\nu_i,\,\omega)\ :\ |\,\Delta_{i\,0}(\nu_i,\omega)\,| > 0,\ \ \nu_i \neq \mu_{il}\,,\ \ i = 1,2,\ \ \omega \in (0,\infty)\}\ .$$

On the number set Λ_{i2} we consider a matrix $\Theta_{i\,r\,m-1}(\nu,\omega) =$

$$= \begin{pmatrix} 1-\nu_i\,\sigma_{i11\,1} & \dots & \nu_i\,\sigma_{i11\,(j-1)} & \sigma_{i\,m\,1} & \nu_i\,\sigma_{i11\,(j+1)} & \dots & \nu_i\,\sigma_{i11\,k} \\ \nu_i\,\sigma_{i12\,1} & \dots & \nu_i\,\sigma_{i12\,(j-1)} & \sigma_{i\,m\,2} & \nu_i\,\sigma_{i12\,(j+1)} & \dots & \nu_i\,\sigma_{i12k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \nu_i\,\sigma_{i1k1} & \dots & \nu_i\,\sigma_{i1k\,(j-1)} & \sigma_{i\,m\,k} & \nu_i\,\sigma_{i\,1k\,(j+1)} & \dots & 1-\nu_i\,\sigma_{i1kk} \end{pmatrix},$$

$m = 2,3$. Taking into account the known properties of the matrix $\Theta_{i\,r\,m}(\nu,\omega)$, we modified the Cramer method on the set Λ_{i2} and obtain solutions of SAE (22) in the form

$$\tau_{ir} = \varphi_{in}\frac{\Delta_{i1\,r}(\nu,\omega)}{\Delta_{\,i0}(\nu,\omega)} + \frac{\Delta_{i2\,r}(\nu,\omega,u_{\,in})}{\Delta_{\,i\,0}\,(\nu,\omega)}\ ,\quad i = 1,2,\ \ r = \overline{1\,,\,k}\ ,(\nu,\omega) \in \Lambda_{i2}\,, \tag{24}$$

where $\Delta_{i\,r\,m-1\,n}(\nu,\omega) = \det\,\Theta_{i\,r\,m-1\,n}\,(\nu,\omega)$, $m = 2,3$. Substituting solutions (24) into function (21), we obtain

$$u_{\,i\,n}(t,\,\nu,\omega) = \Im(t;u_{\,in}) \equiv \varphi_{in}\,h_{\,i\,1jn}\,(t,\,\nu,\omega) + h_{\,i\,2jn}\,(t,\,\nu,\omega,u_{\,in}) +$$

$$+\int_0^t \exp\,\{-\lambda_{\,i\,n}\,\omega\,(t-s)\}\ \alpha(s)\,u_{\,i\,n}(s)\,d\,s, \tag{25}$$

where

$$h_{\,i\,1n}\,(t,\,\nu,\omega) = \exp\,\{-\lambda_{\,i\,n}\,\omega\,t\} + \nu_i\ \sum_{r=1}^{k}\frac{\Delta_{i1\,r}\,(\nu,\omega)}{\Delta_{\,i0}\,(\nu,\omega)}\int_0^t \exp\,\{-\lambda_{\,i\,n}\,\omega\,(t-s)\}\ a_{\,r}\,(s)\,ds,$$

$$h_{\,i\,2n}\,(t,\,\nu,\omega,u_{\,in}) = \nu_i\sum_{r=1}^{k}\frac{\Delta_{i2\,r}\,(\nu,\omega,u_{\,in})}{\Delta_{\,i\,0}\,(\nu,\omega)}\int_0^t \exp\,\{-\lambda_{\,i\,n}\,\omega\,(t-s)\}\ a_{\,r}\,(s)\,d\,s.$$

The representation (25) is countable system of functional-integral equations. Substituting representation (25) in the Fourier series (14), we obtain formal solution of the direct problem (1)-(3) on the domain Ω

$$U_i(t,x,\,\nu,\omega) = \sum_{n=1}^{\infty}\vartheta_{\,i\,n}(x)\,[\varphi_{in}\,h_{\,i\,1n}\,(t,\,\nu,\omega) + h_{\,i\,2n}\,(t,\,\nu,\omega,u_{\,in}) +$$

$$+\int_0^t \exp\,\{-\lambda_{\,i\,n}\,\omega\,(t-s)\}\ \alpha(s)\,u_{\,i\,n}(s,\,\nu,\omega)\,ds\Big]\,,\quad (\nu,\omega) \in \Lambda_{i\,2},\quad i = 1,2. \tag{26}$$

Note that the functions (26) make sense only for values of parameters ν, ω from the set Λ_{i2}. In addition, in the countable system of functional-integral equations (25), the unknown functions $u_{\,i\,n}(t,\,\nu,\omega)$ are under the sign of the determinant and under the sign of integral.

3. Solvability of the of countable system of functional integral equations

Let us investigate the system of functional-integral equations (25) in the question of unique solvability. To this end, consider the following well-known Banach spaces, which we will use in our further actions:
The space B_2 of function sequences $\{u_n(t)\}_{n=1}^{\infty}$ on the segment $[0;\,T]$ with the norm

$$\|u(t)\|_{B_2} = \sqrt{\sum_{n=1}^{\infty}\left(\max_{t\in[0;T]}|u_n(t)|\right)^2} < \infty;$$

The Hilbert coordinate space ℓ_2 of number sequences $\{\varphi_n\}_{n=1}^{\infty}$ with the norm

$$\|\varphi\|_{\ell_2} = \sqrt{\sum_{n=1}^{\infty}|\varphi_n|^2} < \infty;$$

The space $L_2[-1,1]$ of square-integrable functions on an interval $[-1,1]$ with norm

$$\|\vartheta(x)\|_{L_2[-1,1]} = \sqrt{\int_{-1}^{1}|\vartheta(x)|^2\,dx} < \infty\,.$$

4. Smoothness conditions.

Let for the functions $\varphi_i(x) \in C^3[-1,1]$, $i=1,2$ on the segments $[-1,1]$ exist peace-wise continuous derivatives up fourth order on x. Then, after integration the integrand functions $\varphi_{in} = \int_{-1}^{1}\varphi_i(x)\,\vartheta_{in}(x)\,dx$, $i=1,2$ by part four times on the variable E, we obtain the following relation

$$|\varphi_{in}| \le \left(\frac{1}{\pi}\right)^4 \frac{\left|\varphi_{in}^{(4)}\right|}{n^4}, \quad i=1,2, \tag{27}$$

where

$$\varphi_{in}^{(4)} = \int_{-1}^{1}\frac{\partial^4\varphi_i(x)}{\partial x^4}\,\vartheta_{in}(x)\,dx, \quad i=1,2.$$

Here the Bessel inequality is valid

$$\sum_{n=1}^{\infty}\left[\varphi_{in}^{(4)}\right]^2 \le \int_{-1}^{1}\left[\frac{\partial^4\varphi_i(x)}{\partial x^4}\right]^2 dx, \quad i=1,2. \tag{28}$$

Theorem 1. Let the smoothness conditions and the following conditions be fulfilled:

$$\max_{t\in[0,T]}|h_{i1n}(t,\nu,\omega)| \le \delta_{i1} < \infty, \max_{t\in[0,T]}|h_{i2n}(t,\omega,\nu,u_{in}(t))| \le \delta_{i2}\,|u_{in}(t)|\,.$$

Then the countable system of functional integral equations (25) is uniquely solvable in the space B_2. In this case, the desired solution can be founded by the following iterative process:

$$\begin{cases} u_{in}^0(t,\nu,\omega) = \varphi_{in}\,h_{i1n}(t,\nu,\omega), \\ u_{in}^{m+1}(t,\nu,\omega) = \Im(t,\nu,\omega;u_{in}^m), \quad i=1,2, \quad m=0,1,2,... \end{cases} \tag{29}$$

Proof. We use the method of contraction maps in combination with the method of successive approximations in the space B_2. Then, by virtue of smoothness condition (27) and conditions of the theorem, applying the Cauchy-Schwartz inequality and Bessel inequality (28), we obtain from the approximations (29) that the following estimate is valid:

$$\sum_{n=1}^{\infty}\max_{t\in[0,T]}|u_{in}^0(t)| \le \sum_{n=1}^{\infty}|\varphi_{i1n}|\max_{t\in[0,T]}|h_{i1n}(t,\nu,\omega)| \le \delta_{i1}\left(\frac{1}{\pi}\right)^4\sum_{n=1}^{\infty}\frac{\left|\varphi_{in}^{(4)}\right|}{n^4} \le$$

$$\leq \delta_{i\,1} \left(\frac{1}{\pi}\right)^4 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^8}} \left\| \frac{\partial^4 \varphi_i(x)}{\partial x^4} \right\|_{L_2[-1,1]} = \delta_{i\,0} < \infty. \tag{30}$$

Taking into account estimate (30), applying the Cauchy–Schwartz inequality, for the first difference of approximations (29) we obtain:

$$\sum_{n=1}^{\infty} \max_{t\in[0,T]} \left| u_{in}^{1}(t,\omega,\nu) - u_{in}^{0}(t,\omega,\nu) \right| \leq \sum_{n=1}^{\infty} \max_{t\in[0,T]} \left| h_{i2n}\left(t,\omega,\nu,u_{in}^{0}(t)\right) \right| +$$

$$+ \sum_{n=1}^{\infty} \max_{t\in[0,T]} \left| \int_0^t \exp\left\{-\lambda_{i\,n}\,\omega\,(t-s)\right\} \alpha(s)\, u_{in}^{0}(s)\, ds \right| \leq$$

$$\leq \delta_{i2} \sum_{n=1}^{\infty} \left| u_{in}^{0}(t) \right| + \delta_3 \sum_{n=1}^{\infty} \left| u_{in}^{0}(t) \right| \leq (\delta_{i2} + \delta_3)\, \delta_{i\,0} < \infty, \tag{31}$$

where

$$\delta_3 = \max_{t\in[0,T]} \int_0^t |\,\alpha(s)\,|\, ds.$$

Continuing this process, similarly to the estimate (31) we obtain

$$\sum_{n=1}^{\infty} \max_{t\in[0,T]} \left| u_{in}^{m+1}(t) - u_{in}^{m}(t) \right| \leq$$

$$\leq \sum_{n=1}^{\infty} \max_{t\in[0,T]} \left| h_{i2n}\,(t,\omega,\nu,u_{in}^{m}(t)) - h_{i2n}\left(t,\omega,\nu,u_{in}^{m-1}(t)\right) \right| +$$

$$+ \sum_{n=1}^{\infty} \max_{t\in[0,T]} \left| \int_0^t \exp\left\{-\lambda_{i\,n}\,\omega\,(t-s)\right\} \alpha(s) \left| u_{in}^{m}(s) - u_{in}^{m-1}(s) \right| ds \right| \leq$$

$$\leq (\delta_{i2} + \delta_3)^{m+1}\, \delta_{i\,0}, \tag{32}$$

According to the last condition of the theorem, $\delta_{i2} + \delta_3 < 1$. So, from (32) we obtain

$$\lim_{m\to\infty} \left\| u_i^{m+1}(t) - u_i^{m}(t) \right\| \leq \lim_{m\to\infty} (\delta_{i2} + \delta_3)^{m+1} = 0. \tag{33}$$

Consequently, it follows from estimate (33) that the operator on the right-hand sides of countable system of functional integral equations (25) is contracting. It follows from estimates (30) and (33) that there is a unique fixed point, which is a solution to the countable system of functional integral equations (25) in space B_2. Theorem 1 is proved.

5. Uniform convergence of series (26)

Theorem 2. Let the conditions of the theorem 1 are fulfilled. Then the series (26) are convergence in the segment $[-1, 1]$.

Proof. Let $u_{\,in}(t,\,\nu,\omega) \in B_2$ are solution of the system (25). As in the case of estimates (30) and (32), for $(\nu,\omega) \in \Lambda_{i\,2}$ we obtain

$$|\,U_i(t,x,\,\nu,\omega)\,| \leq \delta_{i\,1} \left(\frac{1}{\pi}\right)^4 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^8}} \left\| \frac{\partial^4 \varphi_i(x)}{\partial x^4} \right\|_{L_2[-1,1]} +$$

$$+ (\delta_{i2} + \delta_3) \,\|\, u_{\,i}\,(t,\,\nu,\,\omega)\,\|_{B_2} < \infty, \;\; i = 1, 2. \tag{34}$$

From the estimate (34) implies absolutely and uniformly convergence of the series (26). The theorem 2 is proved.

Theorem 3. Let the conditions of the Theorem 1 are fulfilled. Then the main unknown function $U(t,x) = U_1(t,x) + U_2(t,x)$ of the mixed problem (1)–(3) is defined by the Fourier series (26) and belongs to the class $C(\overline{\Omega}) \cap C^{1,2}_{t,x}(\Omega)$.

The **proof** of the Theorem 3 is similar to the proof of the Theorem 2.

Conclusion

In the rectangular domain $\Omega = \{0 < t < T,\ -1 < x < 1\}$ we consider an inhomogeneous parabolic integro-differential equation (1) with degenerate kernel and involution. With respect to spatial variable x is used Dirichlet boundary value conditions (2). First, we considered homogeneous problem (4), (5). This problem we replace with spectral problem (6), (7). Eigenvalues (9), (12) and eigenfunctions (10), (13) of the spectral problem (6), (7) are found. The Fourier series method of separation of variables is applied. The countable system of functional integral equations (25) is obtained. Theorem 1 on a unique solvability of countable system of functional integral equations (25) is proved. The method of successive approximations is used in combination with the method of contraction mappings. The solution of the mixed problem (1)–(3) is obtained in the form of Fourier series (26). The absolute convergence of Fourier series is proved (Theorem 2 and 3).

References

[1] Cavalcanti M. M., Domingos Cavalcanti V. N., Ferreira J. Existence and uniform decay for a nonlinear viscoelastic equation with strong damping. Math. Methods in the Appl. Sciences. 2001. 24. 1043-1053.

[2] Bykov Ya. V. On some problems in the theory of integro-differential equations. Izdatelstvo Kirg. Gos. Univ-ta, Frunze, 1957. 327 p. (in Russian).

[3] Dzhumabaev D. S., Bakirova E. A. Criteria for the well-posedness of a linear two-point boundary value problem for systems of integro-differential equations. Differential Equations. 2010. 46 (4). 553-567.

[4] Dzhumabaev D. S., Mynbayeva S. T. New general solution to a nonlinear Fredholm integro-differential equation. Eurasian Math. Journal. 2019. 10 (4). 24-33.

[5] Vainberg M. M. Integro-differential equations. Itogi Nauki. 1962. VINITI, Moscow, 1964. 5{37} (in Russian).

[6] Sidorov N. A. Solution of the Cauchy problem for a class of integro-differential equations with analytic nonlinearities. Differ. Uravneniya. 1968. 4 (7). 1309-1316 (in Russian).

[7] Ushakov E. I. Static stability of electrical circuits. Nauka, Novosibirsk, 1988. 273 p. (in Russian).

[8] Abildayeva A. T., Kaparova R. M., Assanova A. T. To a unique solvability of a problem with integral condition for integro-differential equation. Lobachevskii Journal of Mathematics. 2021. 42 (12). 2697-2706.

[9] Asanova A. T., Dzhumabaev D. S. Correct solvability of a nonlocal boundary value problem for systems of hyperbolic equations. Doklady Mathematics. 2003. 68 (1). 46-49.

[10] Assanova A. T., On the solvability of nonlocal problem for the system of Sobolev-type differential equations with integral condition. Georgian Mathematical Journal. 2021. 28 (1). 49-57.

[11] Asanova A. T., Dzhumabaev D. S. Well-posedness of nonlocal boundary value problems with integral condition for the system of hyperbolic equations. Journal of Mathematical Analysis and Applications. 2013. 402 (1). 167-178.

[12] Assanova A. T., Imanchiyev A. E., Kadirbayeva Zh. M. A nonlocal problem for loaded partial differential equations of fourth order. Bulletin of the Karaganda university-Mathematics. 2020. 97 (1). 6-16.

[13] Assanova A. T., Tokmurzin Z. S. A nonlocal multipoint problem for a system of fourth-order partial differential equations. Eurasian Math. Journal. 2020. 11 (3). 8-20.

[14] Gordeziani D. G., Avalishvili G. A. Gordeziani solutions of nonlocal problems for one-dimensional vibrations of a medium. Matematicheskoye Modelirovaniye. 2000. 12 (1). 94-103 (in Russian).

[15] Dzhumabaev D. S. Well-posedness of nonlocal boundary value problem for a system of loaded hyperbolic equations and an algorithm for finding its solution. Journal of Mathematical Analysis and Applications. 2018. 461 (1). 1439-1462.

[16] Isgenderov N. Sh., Allahverdiyeva S. I. On solvability of an inverse boundary value problem for the Boussinesq-Love equation with periodic and integral condition. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics. 2021. 41 (1). 118-132.

[17] Ivanchov N. I. Boundary value problems for a parabolic equation with integral conditions. Differential Equations. 2004. 40 (4). 591-609.

[18] Mammedzadeh G. S. On a boundary value problem with spectral parameter quadratically contained in the boundary condition. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics. 2022. 42 (1), 141-150.
[19] Ochilova N. K., Yuldashev T. K. On a nonlocal boundary value problem for a degenerate parabolichyperbolic equation with fractional derivative. Lobachevskii Journal of Mathematics 2022. 43 (1). 229-236.
[20] Tikhonov I. V. Theorems on uniqueness in linear nonlocal problems for abstract differential equations. Izvestiya: Mathematics. 2003. 67 (2). 333-363.
[21] Yurko V. A. Inverse problems for first-order integro-differential operators. Math Notes. 2016. 100 (6). 876-882.
[22] Yuldashev T. K. Determination of the coefficient and boundary regime in boundary value problem for integro-differential equation with degenerate kernel. Lobachevskii Journal of Mathematics. 2017. 38 (3). 547-553.
[23] Yuldashev T. K., Odinaev R. N., Zarifzoda S. K. On exact solutions of a class of singular partial integrodifferential equations. Lobachevskii Journal of Mathematics. 2021. 42 (3). 676-684.
[24] Yuldashev T. K., Rakhmonov F. D. Nonlocal problem for a nonlinear fractional mixed type integrodifferential equation with spectral parameters. AIP Conference Proceedings. 2021. 2365 (060003). 1-20.
[25] Yuldashev T. K., Zarifzoda S. K. On a new class of singular integro-differential equations. Bulletin of the Karaganda University. Mathematics series. 2021. 101 (1). 138-148.
[26] Zaripov S. K. Construction of an analog of the Fredholm theorem for a class of model first order integrodifferential equations with a singular point in the kernel. Vestnik Tomsk. gos. universiteta. Matematika i mekhanika. 46. 2017. 24-35.
[27] Zaripov S. K. A construction of analog of Fredgolm theorems for one class of first order model integrodifferential equation with logarithmic singularity in the kernel. Vestnik Samar. gos. tekh. univ. Fiz.-mat. nauki. 21 (2). 2017. 236-248.
[28] Zaripov S. K. On a new method of solving of one class of model first-order integro-differential equations with singularity in the kernel. Matematicheskaya fizika i compyuternoe modelirovanie. 20 (4). 2017. 68-75.
[29] Yuldashev T. K. On Fredholm partial integro-differential equation of the third order. Russian Mathematics. 2015. 59 (9). 62-66.
[30] Yuldashev T. K. On a boundary-value problem for a fourth-order partial integro-differential equation with degenerate kernel. Journal of Mathematical Sciences. 2020. 245 (4). 508-523.
[31] Yuldashev T. K. Determining of coefficients and the classical solvability of a nonlocal boundary-value problem for the Benney-Luke integro-differential equation with degenerate kernel. Journal of Mathematical Sciences. 254 (6). 2021. 793-807.
[32] Yuldashev T. K. Inverse boundary-value problem for an integro-differential Boussinesq-type equation with degenerate kernel. Journal of Mathematical Sciences. 2020. 250 (5). 847-858.
[33] Yuldashev T. K. On a boundary value problem for a fifth orderpartial integro-differential equation. Azerbaijan Journal of Mathematics. 2022. 12 (2). 154-172.
[34] Yuldashev T. K. On a nonlocal boundary value problem for a partial integro-differential equations with degenerate kernel. Vladikavkaz Mathematical Journal. 2022. 24 (2). 130-141.
[35] Yuldashev T. K., Apakov Yu. P., Zhuraev A. Kh. Boundary value problem for third order partial integro-differential equation with a degenerate kernel. Lobachevskii Journal of Mathematics. 2021. 42 (6). 1317-1327.
[36] Yuldashev T. K., Rakhmonov F. D. On a boundary value problem for Benney–Luke type differential equation with nonlinear function of redefinition and integral conditions. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics. 2021. 41 (1). 172-183.
[37] Yuldashev T.K. Rakhmonov F.D. On a Benney–Luke type Differential Equation with Nonlinear Boundary Value Conditions. Lobachevski Journal of Math. 2020. 41 (1). 111-123.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

INVESTIGATION OF SYSTEM OF NONLINEAR PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS OF THE FIRST ORDER BY MEANS OF THE METHOD OF ADDITIONAL ARGUMENT

GULKHAN K. SADYKOVA[1]

[1]Osh State University, Osh, Kyrgyzstan
e-mail: gsadykova.osh@gmail.com

Abstract. At present, the application of the method of an additional argument to a system of nonlinear partial differential equations of a new class is one of the topical issues. In this article, we consider the solution of a system of nonlinear integro-differential equations in partial derivatives, consisting of two unknown functions in three-dimensional space. The initial problem for the system under consideration is reduced to a system of integral problem equations, to which the principle of compressive reflections has been applied.

Keywords: method of an additional argument, system of equations, partial derivatives, integro-differential equation, non-linear equation, unique solution.

We use results of articles [1-3].

Let $\overline{C}^{(k)}$ be the class of functions being continuous and bounded together with their derivatives up to the k-th order.

$Lip(N|u, M|v,)$ is the class of functions satisfying the Lipschitz condition with the variable with coefficient N, in variable v with coefficient $M, ;$ for a function of one variable, the index will be omitted.

Denote the differential operator $D[\omega_1, \omega_2] = \frac{\partial}{\partial t} + \omega_1 \frac{\partial}{\partial x} + \omega_2 \frac{\partial}{\partial y}$.

The system of nonlinear partial integro-differential equations of the first order of the form is considered:

$$\begin{cases} D[a_{11}(t,x,y,u_1,u_2), a_{12}(t,x,y,u_1,u_2)]u_1(t,x,y) = \sum\limits_{k=1}^{2} a_{1k}(t,x,y,u_1,u_2) + \int\limits_0^1 \int\limits_0^1 u(t,\xi,\eta)d\xi d\eta \\ D[a_{21}(t,x,y,u_1,u_2), a_{22}(t,x,y,u_1,u_2)]u_2(t,x,y) = b(t,x,y,u_1,u_2), \end{cases} \tag{1}$$

$(t,x,y) \in G_3(T) = [0,T] \times R_+^2,$

There is considering system (1) with the initial conditions:

$$\begin{array}{l} u_1(0,x,y)=x+y, \\ u_2(0,x,y)=\varphi(x,y), (x,y)\in R_+^2. \end{array} \tag{2}$$

The given functions are sufficiently smooth:

$\varphi(x,y) \in \overline{C}^1(R_+^2), a_{ij}(t,x,y,u_1,u_2) \in \overline{C}^1(G_3(T) \times R^2),$

$b(t,x,y,u_1,u_2) \in \overline{C}^1(G_3(T) \times R^2), i,j = 1,2.$

From the first equation of system (1), using the method of an additional argument, we have:

$$u_1(t,x) = x - \int\limits_0^t a_{11}(v, p_1(v,t,x,y), p_2(v,t,x,y), u_1(v,p_1,p_2), u_2(v,p_1,p_2))dv +$$

$$+y-\int_0^t a_{12}(v,p_1(v,t,x,y),p_2(v,t,x,y),u_1(v,p_1,p_2),u_2(v,p_1,p_2))dv+$$

$$+\int_0^t a_{11}(v,p_1(v,t,x,y),p_2(v,t,x,y),u_1(v,p_1,p_2),u_n(v,p_1,p_2))dv+$$

$$+\int_0^t a_{12}(v,p_1(v,t,x,y),p_2(v,t,x,y),u_1(v,p_1,p_2),u_n(v,p_1,p_2))dv+$$

$$+\int_0^t\int_0^1\int_0^1 u_1(s,\xi,\eta)d\eta d\xi ds, \tag{3}$$

Here
$p_1(s,t,x,y), p_2(s,t,x,y)$-are determined from the following system of integral equations:

$$\begin{cases} p_1(s,t,x,y)=x-\int_0^t a_{11}(v,p_1(v,t,x,y),p_2(v,t,x,y),u_1(v,p_1,p_2),u_2(v,p_1,p_2))dv \\ p_2(s,t,x,y)=y-\int_0^t a_{12}(v,p_1(v,t,x,y),p_2(v,t,x,y),u_1(v,p_1,p_2),u_2(v,p_1,p_2))dv, \end{cases} \tag{4}$$

$(s,t,x,y)\in G_4(T)=\{(s,t,x,y)\mid 0\le s\le t\le T,(x,y)\in R^2_+\}$.
From (3) we get:

$$u_1(t,x,y)=x+y+\int_0^t\int_0^1\int_0^1 u_1(s,\xi,\eta)d\eta d\xi ds. \tag{5}$$

From (4), respectively, follow the relations

$$D[a_{11}(t,x,y,u_1,u_2),a_{12}(t,x,y,u_1,u_2)]p_1(s,t,x,y)=0, p_1(s,s,x,y)=x,$$

$$D[a_{11}(t,x,y,u_1,u_2),a_{12}(t,x,y,u_1,u_2)]p_2(s,t,x,y)=0, p_2(s,s,x,y)=y,$$

For (5), the method of successive approximations is applied, assuming $u_1^0(t,x,y)=0:\ u_1^N(t,x,y)=x+y+\int_0^t\int_0^1\int_0^1 u_1^{N-1}(s,\xi,\eta)d\eta d\xi ds,\quad N=1,2,3,...,$

$u_1^1(t,x,y)=x+y,$

$u_1^2(t,x,y)=x+y+\int_0^t\int_0^1\int_0^1(\xi+\eta)d\eta d\xi ds=x+y+t,$

$u_1^3(t,x,y)=x+y+\int_0^t\int_0^1\int_0^1(\xi+\eta+s)d\eta d\xi ds=x+y+t+\frac{t^2}{2},$

$u_1^4(t,x,y)=x+y+\int_0^t\int_0^1\int_0^1(\xi+\eta+s+\frac{s^2}{2})d\eta d\xi ds=x+y+t+t+\frac{t^2}{2}+\frac{t^3}{3!},$

$- - - -

Where

$$\widetilde{a}_{2i}(t,x,y,u_2) = a_{2i}(t,x,y,x+y+e^t-1,u_2)$$
$$\widetilde{b}(t,x,y,u_2) = b_1(t,x,y,x+y+e^t-1,u_2), i=1,2.$$

Problem (6), (2), in turn, using the method of an additional argument, can be reduced to a system of integral equations:

$$\begin{cases} u_2(t,x,y) = \varphi(q_1(0,t,x,y), q_2(0,t,x,y)) + \\ + \int\limits_0^t \widetilde{b}(v, q_1(v,t,x,y), q_2(v,t,x,y), u_2(v,t,x,y), q_2(v,t,x,y))dv, \\ q_1(s,t,x,y) = x - \int\limits_s^t \widetilde{a}_{21}(v, q_1(v,t,x,y), q_2(v,t,x,y), u_1(v, q_1(v,t,x,y), q_2(v,t,x,y)))dv \\ q_2(s,t,x,y) = y - \int\limits_s^t \widetilde{a}_{22}(v, q_1(v,t,x,y), q_2(v,t,x,y), u_1(v, q_1(v,t,x,y), q_2(v,t,x,y)))dv \end{cases} \tag{7}$$

$(s,t,x,y) \in G_4(T)$.
$q_1(s,t,x,y), q_2(s,t,x,y)$-for them are true:

$$D[a_{11}(t,x,y,u_1,u_2), a_{22}(t,x,y,u_1,u_2)]q_1(s,t,x,y) = 0, q_1(s,s,x,y) = x,$$
$$D[a_{21}(t,x,y,u_1,u_2), a_{22}(t,x,y,u_1,u_2)]q_2(s,t,x,y) = 0, q_2(s,s,x,y) = y.$$

For

$$\begin{gathered} a_{21}(t,x,y,u_1,u_2), a_{22}(t,x,y,u_1,u_2) \in \overline{C}^(1)(G_3(T) \times R_+^2) : \\ q_1(s,\tau,q_1(\tau,t,x,y), q_2(\tau,t,x,y)) = q_1(s,\tau,x,y), \\ q_2(s,\tau,q_1(\tau,t,x,y), q_2(\tau,t,x,y)) = q_2(s,\tau,x,y), \\ (s,t,x,y) \in G_4(T). \end{gathered} \tag{8}$$

Using these equalities (8), from system (7) we have:

$$\begin{cases} \omega(s,t,x,y) = \varphi(q_1(0,t,x,y), q_2(0,t,x,y)) + \\ + \int\limits_0^t \widetilde{b}_1(v, q_1(v,t,x,y), q_2(v,t,x,y), \omega(v,t,x,y))dv, \\ q_1(s,t,x,y) = x - \int\limits_s^t \widetilde{a}_{21}(v, q_1(v,t,x,y), q_2(v,t,x,y), \omega(v,t,x,y))dv, \\ q_2(s,t,x,y) = y - \int\limits_s^t \widetilde{a}_{22}(v, q_1(v,t,x,y), q_2(v,t,x,y), \omega(v,t,x,y))dv, \end{cases} \tag{9}$$

The following notation is introduced here:

$$\omega(v,t,x,y) = u_2(v, q_1(s,t,x,y), q_2(s,t,x,y)) \tag{10}$$

From (10) at $s = t$, we get $\omega(t,t,x,y) = u_2(t,x,y)$.

We write (9) in the form of a vector equation: $\theta = A\theta$,

Where $\theta = (\theta_1, \theta_2, \theta_3)$- a function $\theta_1 = \omega(s,t,x,y), \theta_2 = q_1(s,t,x,y), \theta_3 = q_2(s,t,x,y)$, $A = (A_1, A_2, A_3)$:

$$\begin{cases} A_1\theta = \varphi(\theta_1(0,t,x,y), \theta_3(0,t,x,y)) + \\ + \int\limits_0^t \widetilde{b}_1(v, \theta_2(v,t,x,y), \theta_3(v,t,x,y), \theta_1(v,t,x,y)dv, \\ A_2\theta = x - \int\limits_s^t \widetilde{a}_{21}(v, \theta_2(v,t,x,y), \theta_3(v,t,x,y), \theta_1(v,t,x,y))dv, \\ A_3\theta = y - \int\limits_s^t \widetilde{a}_{22}(v, \theta_2(v,t,x,y), \theta_3(v,t,x,y), \theta_1(v,t,x,y))dv, \end{cases} \tag{11}$$

System (11) in the area $G_4(T)$ at $T < T_*$ we have a solution satisfying the inequality
$\| \theta - \theta_0 \| \leq M$.

We use the norm: $\| \theta \| = \max\limits_{0 \leq i \leq 3} \max\limits_{(t,x,y) \in G_3(T)} \{ | \theta_i |, i = 1, 2, 3 \}$

$| A_1\theta - \varphi | \leq M_0 T,$
$| A_2\theta - x | \leq M_1 T,$
$| A_3\theta - y | \leq M_2 T,$
Where
$| \widetilde{b}_1(t, x, y, u_2) | \leq M_0 = const, \quad | a_{2ii}(t, x, y, u_2) | \leq M_0 = const, \quad i = 1, 2.$
Fair:
$\| A_1\theta^1 - A_1\theta^2 \| \leq \Omega_1(T), \| \theta^1 - \theta^2 \|,$
$\| A_2\theta^1 - A_2\theta^2 \| \leq \Omega_2(T), \| \theta^1 - \theta^2 \|,$
$\| A_3\theta^1 - A_3\theta^2 \| \leq \Omega_3(T), \| \theta^1 - \theta^2 \|,$
Here:
$\Omega_1(T) = L_1 + L_2 + (K_1 + K_2 + K_3)T,$
$\Omega_2(T) = (N_1 + N_2 + N_3)T,$
$\Omega_3(T) = (H_1 + H_2 + H_3)T,$
Where
$\varphi(x, y) \in Lip(L_1 |_x, L_2 |_y), L_1, L_2 \geq 0, L_1, L_2 - const,$
$\widetilde{b}_1(t, x, y, u_2) \in Lip(K_1 |_x, K_2 |_y, K_3 |_{u_2}), K_1, K_2, K_3 \geq 0, K_1, K_2, K_3 - const,$
$\widetilde{a}_{21}(t, x, y, u_2) \in Lip(N_1 |_x, N_2 |_y, N_3 |_{u_2}), N_1, N_2, N_3 \geq 0, N_1, N_2, N_3 - const,$
$\widetilde{a}_{22}(t, x, y, u_2) \in Lip(H_1 |_x, H_2 |_y, H_3 |_{u_2}), H_1, H_2, H_3 \geq 0, H_1, H_2, H_3 - const,$
Solution equation $\Omega_i(T) = 1, i = 1, 2, 3$ denote T_1, T_2, T_3.
Then at $T < T^* = min\{T_1, T_2, T_3\}$ equation (11), therefore, problem (6), (2) has a unique solution.

In this work, the developed method of an additional argument is applied to a system of nonlinear integro-differential equations in partial derivatives with three independent variables. The results can be generalized to the multivariate case.

References

[1] Ashirbaeva A.J., Sadykova G.K. Developing of the method of additional argument for system of nonlinear partial differential equations (in Russian). *International Scientific-Research journal*, Ekaterinburg, 2019, No. 4-1(82), pp. 6-10.

[2] Ashirbaeva A.J., Sadykova G.K. Developing of the method of additional argument for system of nonlinear differential equations (in Russian). *Science, new technologies and innovations of Kyrgyzstan*, Ekaterinburg, 2019, No. 4-1(82), pp. 6-10.

[3] Sadykova G.K. Investigation of solution of one system of nonlinear partial differential equations or the first order (in Russian). *Izvestia vuzov Kyrgyzstana*, Bishkek, 2019, No. 11, pp. 15-19.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

OPTIMIZATION OF ADAMS-TYPE DIFFERENCE FORMULAS IN THE HILBERT SPACE $W_2^{(2,1)}(0,1)$

KHOLMAT SHADIMETOV[1,2], ROZIK KARIMOV[1,3]

[1]V.I. Romanovskiy Institute of Mathematics, Tashkent, Uzbekistan
[2]Tashkent State Transport University, Tashkent, Uzbekistan
[3]Bukhara Institute of Natural Resources Management, Bukhara, Uzbekistan
e-mail: kholmatshadimetov@mail.ru (Kh.M. Shadimetov), roziq.s.karimov@gmail.com (R.S. Karimov)

Abstract. In this paper, we consider the problem of constructing new optimal explicit and implicit Adams-type difference formulas for finding an approximate solution to the Cauchy problem for an ordinary differential equation in a Hilbert space. In this work, I minimize the norm of the error functional of the difference formula with respect to the coefficients, we obtain a system of linear algebraic equations for the coefficients of the difference formulas. This system of equations is reduced to a system of equations in convolution and the system of equations is completely solved using a discrete analog of a differential operator d^2/dx^2-1. Here we present an algorithm for constructing optimal explicit and implicit difference formulas in a specific Hilbert space. In addition, comparing the Euler method with optimal explicit and implicit difference formulas, numerical experiments are given. Experiments show that the optimal formulas give a good approximation compared to the Euler method.

Keywords: Hilbert space, initial-value problem, multistep method, the error functional, optimal difference formula.

AMS Subject Classification: 39A06, 65D25, 65L05, 65L70.

1. Introduction

It is known that the solutions of many practical problems lead to solutions of differential equations or their systems. Although differential equations have so many applications and only a small number of them can be solved exactly using elementary functions and their combinations. Even in the analytical analysis of differential equations, their application can be inconvenient due to the complexity of the obtained solution. If it is very difficult to obtain or impossible to find an analytic solution to a differential equation, one can find an approximate solution.

In the present paper we consider the problem of approximate solution to the first order linear ordinary differential equation

$$y' = f(x,y),\ x \in [0,1] \tag{1}$$

with the initial condition

$$y(0) = y_0. \tag{2}$$

We assume that $f(x,y)$ is a suitable function and the differential equation (1) with the initial condition (2) has a unique solution on the interval $[0,1]$.

For approximate solution of problem (1)-(2) we divide the interval $[0,1]$ into N pieces of the length $h = \frac{1}{N}$ and find approximate values y_n of the function $y(x)$ for $n = 0,1,...,N$ at nodes $x_n = nh$.

A classic method of approximate solution of the initial-value problem (1)-(2) is the Euler method. Using this method, the approximate solution of the differential equation is calculated

as follows: to find an approximate value y_{n+1} of the function at the node x_{n+1}, it is used the approximate value y_n at the node x_n:

$$y_{n+1} = y_n + hy'_n, \tag{3}$$

where $y'_n = f(x_n, y_n)$, so that y_{n+1} is a linear combination of the values of the unknown function $y(x)$ and its first-order derivative at the node x_n.

Everyone are known that there are many methods for solving the initial-value problem for ordinary differential equation (1). For example, the initial-value problem can be solved using the Euler, Runge-Kutta, Adams-Bashforth and Adams-Moulton formulas of varying degrees [1]. In [2] by Ahmad Fadly Nurullah Rasedee, et al., research they discussed the order and stepsize strategies of the variable order stepsize algorithm. The stability and convergence estimations of the method are also established. In the work [3] by Adekoya Odunayo M. and Z.O.Ogunwobi, it was shown that the Adam-Bashforth-Moulton method is better than the Milne Simpson method in solving a second-order differential equation. Some studies have raised the question of whether Nordsieck's technique for changing the step size in the Adams-Bashforth method is equivalent to the explicit continuous Adams-Bashforth method. And in N.S.Hoang and R.B.Sidje's work [4] they provided a complete proof that the two approaches are indeed equivalent. In the works [5] and [6] there were shown the potential superiority of semi-explicit and semi-implicit methods over conventional linear multi-step algorithms.

However, it is very important to choose the right one among these formulas to solve the Initial-value problem and it is not always possible to do this. Also, in this work, in contrast to the above-mentioned works, exact estimates of the error of the formula is obtained.

Our aim, in this paper, is to construct new difference formulas that are exact for e^{-x} and optimal in the Hilbert space $W_2^{(2,1)}(0,1)$. Also these formulas can be used to solve certain classes of problems with great accuracy.

The rest of the work is organized as follows. In the first paragraph, an algorithm for constructing an explicit difference formula in the space is given. The above algorithm is used to obtain an analytical formula for the optimal coefficients of an explicit difference formula. In the second section, the same algorithm is used to obtain an analytical formula for the optimal coefficients of the implicit difference formula. In the third and fourth sections, respectively, exact formulas are given for the square of the norm of the error functionals of explicit and implicit difference formulas. Numerical experiments are presented at the end of the work.

2. Optimal explicit difference formulas of Adams-Bashforth type in the Hilbert space $W_2^{(2,1)}(0,1)$

We consider a difference formula of the following form for the approximate solution of the problem (1)-(2) [7, 8]

$$\sum_{\beta=0}^{k} C\left[\beta\right]\varphi\left[\beta\right] - h\sum_{\beta=0}^{k-1} C_1\left[\beta\right]\varphi'\left[\beta\right] \cong 0, \tag{1}$$

where $h = \frac{1}{N}$, N is a natural number, $C[\beta]$ and $C_1[\beta]$ are the coefficients, functions φ belong to the Hilbert space $W_2^{(2,1)}(0,1)$. The space $W_2^{(2,1)}(0,1)$ is defined as follows

$$W_2^{(2,1)}(0,1) = \{\varphi : [0,1] \to \mathbf{R} | \varphi' \text{ is abs.contunuous, } \varphi'' \in L_2(0,1)\}$$

equipped with the norm [9, 10]

$$\|\varphi|W_2^{(2,1)}\| = \Big\{ \int\limits_0^1 \Big(\varphi''(x) + \varphi'(x)\Big)^2 dx \Big\}^{1/2}. \tag{2}$$

The following difference between the sums given in the formula (1) is called the error of the formula (1) [11]

$$(\ell, \varphi) = \sum_{\beta=0}^{k} C[\beta]\varphi(h\beta) - h\sum_{\beta=0}^{k-1} C_1[\beta]\varphi'(h\beta).$$

To this error corresponds the error functional [12]

$$\ell(x) = \sum_{\beta=0}^{k} C[\beta]\delta(x-h\beta) + h\sum_{\beta=0}^{k-1} C_1[\beta]\delta'(x-h\beta), \tag{3}$$

where $\delta(x)$ is Dirac's delta-function. We note that (ℓ, φ) is the value of the error functional ℓ at a function φ and it is defined as [13, 14]

$$(\ell, \varphi) = \int_{-\infty}^{\infty} \ell(x)\varphi(x)dx.$$

It should be also noted that since the error functional ℓ is defined on the space $W_2^{(2,1)}(0,1)$ it satisfies the following conditions

$$(\ell, 1) = 0, \quad (\ell, e^{-x}) = 0.$$

These give us the following equations with respect to coefficients $C[\beta]$ and $C_1[\beta]$:

$$\sum_{\beta=0}^{k} C[\beta] = 0, \tag{4}$$

$$\sum_{\beta=0}^{k} C[\beta]e^{-h\beta} + h\sum_{\beta=0}^{k-1} C_1[\beta]e^{-h\beta} = 0. \tag{5}$$

Based on the Cauchy-Schwartz inequality for the absolute value of the error of the formula (1) we have the estimation

$$|(\ell, \varphi)| \le \|\varphi|W_2^{(2,1)}\| \cdot \|\ell|W_2^{(2,1)*}\|.$$

Hence, the absolute error of the difference formula (1) in the space $W_2^{(2,1)}$ is estimated by the norm of the error functional ℓ on the conjugate space $W_2^{(2,1)*}$. From this we get the following[15].

Problem 1. Calculate the norm $\|\ell|W_2^{(2,1)*}\|$ of the error functional ℓ.

From the formula (3) one can see that the norm $\|\ell|W_2^{(2,1)*}\|$ depends on the coefficients $C[\beta]$ and $C_1[\beta]$.

Problem 2. Find such coefficients $C_1[\beta] = \mathring{C}_1[\beta]$ that satisfy the equality

$$\|\mathring{\ell}|W_2^{(2,1)*}\| = \inf_{\mathring{C}_1[\beta]} \sup_{\|\varphi|W_2^{(2,1)}\|\neq 0} \frac{|(\ell, \varphi)|}{\|\varphi|W_2^{(2,1)}\|}.$$

In this case $\mathring{C}_1[\beta]$ are called the optimal coefficients and the corresponding difference formula (1) is called the *optimal difference formula*.

A function ψ_ℓ satisfying the following equation is called the extremal function of the difference formula (1) [13]

$$(\ell, \psi_\ell) = \|\ell|W_2^{(2,1)*}\| \cdot \|\psi_\ell|W_2^{(2,1)}\|. \tag{6}$$

Since the space $W_2^{(2,1)}(0,1)$ is a Hilbert space, then from the Riesz theorem on the general form of a linear continuous functional on a Hilbert space there is a function ψ_ℓ (which is the extremal function) that satisfies the following equation [16, 17]

$$(\ell, \varphi) = \langle \varphi, \psi_\ell \rangle_{W_2^{(2,1)}} \tag{7}$$

and the equality $\|\ell|W_2^{(2,1)*}\| = \|\psi_\ell|W_2^{(2,1)}\|$ holds, here $\langle\varphi,\psi_\ell\rangle_{W_2^{(2,1)}}$ is the inner product in the space $W_2^{(2,1)}(0,1)$ and is defined as follows [18]

Theorem 2.1. *The solution of equation (7) has the form*

$$\psi_\ell(x) = \ell(x) * G_2(x) + de^{-x} + p_0 \tag{8}$$

and it is an extremal function for the difference formula (1), where $G_2(x) = \frac{\operatorname{sgn}(x)}{2}\left(\frac{e^x - e^{-x}}{2} - x\right)$, *$d$ and p_0 are real numbers.*

According to the above mentioned Riesz's theorem, the following equalities is fulfilled

$$\|\ell|W_2^{(2,1)*}\|^2 = (\ell,\psi_\ell) = \|\ell|W_2^{(2,1)*}\| \cdot \|\psi_\ell|W_2^{(2,1)}\|.$$

By direct calculation from the last equality for the norm of the error functional for the difference formula (1) we have the following result [18].

Theorem 2.2. *For the norm of the error functional of the difference formula (1) we have the following expression*

$$\|\ell|W_2^{(2,1)*}\|^2 = \sum_{\gamma=0}^{k}\sum_{\beta=0}^{k} C[\gamma]C[\beta]G_2(h\gamma - h\beta) - 2h\sum_{\gamma=0}^{k-1} C_1[\gamma]\sum_{\beta=0}^{k} C[\beta]G_2'(h\gamma - h\beta) -$$

$$-h^2\sum_{\gamma=0}^{k-1}\sum_{\beta=0}^{k-1} C_1[\gamma]C_1[\beta]G_2''(h\gamma - h\beta), \tag{9}$$

where $G_2'(x) = \frac{\operatorname{sgn}(x)}{2}\left(\frac{e^x + e^{-x}}{2} - 1\right)$ *and* $G_2''(x) = \frac{\operatorname{sgn}(x)}{2}\left(\frac{e^x - e^{-x}}{2}\right)$.

It is known that stability in the Dahlquist sense, just like strong stability, is determined only by the coefficients $C[\beta]$, $\beta = \overline{0,k}$. For this reason, our search for the optimal formula is only related to finding $C_1[\beta]$. Therefore, in this subsection we consider difference formulas of the Adams-Bashforth type, i.e. $C[k] = -C[k-1] = 1$ and $C[k-i] = 0$, $i = \overline{2,k}$, [19, 20]. Then is easy to check, that the coefficients satisfy the condition (4).

In this work, we find the minimum of the norm (9) by the coefficients $C_1[\beta]$ under the condition (5) in the space $W_2^{(2,1)}(0,1)$ [21]. Then using Lagrange method of undetermined multipliers we get the following system of linear equations with respect to the coefficients $C_1[\beta]$:

$$h\sum_{\gamma=0}^{k-1} C_1[\gamma]\, G_2''(h\beta - h\gamma) + de^{-h\beta} = -\sum_{\gamma=0}^{k} C[\gamma]\, G_2'(h\beta - h\gamma), \tag{10}$$

$\beta = \overline{0,k-1}$,

$$h\sum_{\gamma=0}^{k-1} C_1[\gamma]\, e^{-h\gamma} = -\sum_{\gamma=0}^{k} C[\gamma]\, e^{-h\gamma}. \tag{11}$$

It is easy to prove that the solution of this system gives the minimum value to the expression (9) under the condition (5). Here d is an unknown constant, $\overset{\circ}{C}_1[\beta]$ are optimal coefficient. Given that $C[k] = 1$, $C[k-1] = -1$, $C[k-i] = 0$, $i = \overline{2,k}$ the system (10),(11) is reduced to the form,

$$h\sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma]\, G_2''(h\beta - h\gamma) + de^{-h\beta} = f[\beta], \qquad \beta = \overline{0,k-1} \tag{12}$$

$$h\sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma]\, e^{-h\gamma} = g, \tag{13}$$

where

$$f[\beta] = \frac{1-e^h}{4}\left(e^{h\beta-hk} - e^{-h\beta+hk-h}\right), \tag{14}$$

$$g = e^{-hk+h} - e^{-hk}. \tag{15}$$

Assuming that $C_1[\beta] = 0$, for $\beta < 0$ and $\beta > k-1$, we rewrite the system (12), (13) in the convolution form

$$\begin{cases} h\overset{\circ}{C}_1[\beta] * G_2''(h\beta) + de^{-h\beta} = f[\beta] \ \ for\ \beta = \overline{0,k-1}, \\ h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}_1[\gamma]\, e^{-h\gamma} = g. \end{cases} \tag{16}$$

We denote first equation of the system (16) by U_{exp}

$$U_{exp}[\beta] = h\overset{\circ}{C}_1[\beta] * G_2''(h\beta) + de^{-h\beta}. \tag{17}$$

(12) implies that

$$U_{exp}[\beta] = f[\beta] \ for\, \beta = \overline{0,k-1}. \tag{18}$$

Now calculating the convolution we have

$$U_{exp}[\beta] = \overset{\circ}{C}_1[\beta] * G_2''(h\beta) + de^{-h\beta} = h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}_1[\gamma]\, G_2''(h\beta - h\gamma) + de^{-h\beta}.$$

For $\beta < 0$ we get

$$U_{exp}[\beta] = h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}_1\frac{sgn(h\beta - h\gamma)}{2}\left(\frac{e^{h\beta-h\gamma} - e^{-h\beta+h\gamma}}{2}\right) + de^{-h\beta}$$

$$= -\frac{e^{h\beta}}{4}h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}_1[\gamma]\, e^{-h\gamma} + \frac{e^{-h\beta}}{4}h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}_1[\gamma]\, e^{h\gamma} + de^{-h\beta} = -\frac{e^{h\beta}}{4}g + e^{-h\beta}(d+b).$$

For $\beta > k-1$

$$U_{exp}[\beta] = \frac{e^{h\beta}}{4}g + e^{-h\beta}(d-b).$$

Then $d^+ = d+b$ and $d^- = d-b$ the function $U_{exp}[\beta]$ becomes

$$U_{exp}[\beta] = \begin{cases} -\frac{e^{h\beta}}{4}g + e^{-h\beta}d^+ & for\ \beta > k-1, \\ f[\beta] & for\ \beta = \overline{0,k-1}, \\ \frac{e^{h\beta}}{4}g + e^{-h\beta}d^- & for\ \ \beta < 0. \end{cases} \tag{19}$$

We use to find the unknowns d^+ and d^- from the discrete analogue of the differential operator $\frac{d^2}{dx^2} - \frac{d}{dx}$ which is given below [22]

$$D_1[\beta] = \frac{1}{1-e^{2h}}\begin{cases} -2e^h & for\ \ |\beta| = 1, \\ 2(1+e^{2h}) & for\ \beta = 0, \\ 0 & for\ \ |\beta| \geq 2. \end{cases} \tag{20}$$

The unknowns d^+ and d^- are determined from the conditions

$$\overset{\circ}{C}_1[\beta] = h^{-1}D_1[\beta] * U_{exp}[\beta] = 0 \ for\ \beta < 0 \ and\ \beta > k-1. \tag{21}$$

Calculate the convolution

$$h^{-1}D_1[\beta] * U_{exp}[\beta]$$

$$= h^{-1}\sum_{\gamma=1}^{\infty} D_1[\beta+\gamma]\, U_{exp}[-\gamma] + h^{-1}\sum_{\gamma=0}^{k-1} D_1[\beta-\gamma]\, U_{exp}[\gamma]$$

$$+h^{-1}\sum_{\gamma=1}^{\infty} D_1\left[\beta-k-\gamma+1\right]U_{exp}\left[k+\gamma-1\right].$$

From (19) with $\beta=k$ and $\beta=-1$, we have

$$\begin{cases} h^{-1}D_1\left[0\right]U_{exp}\left[-1\right]+h^{-1}D_1\left[1\right]U_{exp}\left[-2\right]+h^{-1}D_1\left[-1\right]U_{exp}\left[0\right]=0, \\ h^{-1}D_1\left[0\right]U_{exp}\left[k\right]+h^{-1}D_1\left[1\right]U_{exp}\left[k-1\right]+h^{-1}D_1\left[-1\right]U_{exp}\left[k+1\right]=0. \end{cases}$$

Hence, due to (21), we get

$$\begin{cases} 2\left(1+e^{2h}\right)\left[-\frac{1}{4}e^{-h}g+e^{h}d^{+}\right]-2e^{h}\left[-\frac{1}{4}e^{-2h}g+e^{2h}d^{+}\right]-2e^{h}f[0]=0 \\ 2\left(1+e^{2h}\right)\left[\frac{1}{4}e^{hk}g+e^{-hk}d^{-}\right]-2e^{h}\left[\frac{1}{4}e^{hk+h}g+e^{-hk-h}d^{-}\right]-2e^{h}f[hk-h]=0. \end{cases}$$

From the first equation d^{+} is equal to the following

$$d^{+}=\frac{e^{hk}-e^{hk-h}}{4}.$$

From the second equation d^{-} is equal to the following

$$d^{-}=\frac{e^{hk}-3e^{hk-h}+2e^{hk-2h}}{4}.$$

so

$$d=\frac{d_0^{+}+d_0^{-}}{2}=\frac{e^{hk}-2e^{hk-h}+e^{hk-2h}}{4}\quad and\quad b=\frac{d_0^{+}-d_0^{-}}{2}=\frac{e^{hk-h}-e^{hk-2h}}{4}.$$

Now we calculate the optimal coefficients $\overset{\circ}{C}_1\left[\beta\right]$

$$\overset{\circ}{C}_1\left[\beta\right]=h^{-1}D_1\left[\beta\right]*U_{exp}\left[\beta\right]=h^{-1}\sum_{\gamma=-\infty}^{\infty}D_1\left[\beta-\gamma\right]U_{exp}\left[\gamma\right],\ \beta=\overline{0,k-1}.$$

Let $\beta=k-1$, then

$$\overset{\circ}{C}_1\left[k-1\right]=h^{-1}\sum_{\gamma=-\infty}^{\infty}D_1\left[k-1-\gamma\right]U_{exp}\left[\gamma\right]$$

$$=h^{-1}\left\{D_1\left[0\right]U_{exp}\left[k-1\right]+D_1\left[1\right]U_{exp}\left[k-2\right]+D_1\left[-1\right]U_{exp}\left[k\right]\right\}$$

$$=\frac{h^{-1}}{(1-e^{2h})}\cdot\left\{1-e^{-h}+e^{h}-e^{2h}\right\}=\frac{e^{h}-1}{he^{h}},$$

thus, $\overset{\circ}{C}_1\left[k-1\right]=\frac{e^{h}-1}{he^{h}}$ for $\beta=k-1$.

Compute $\overset{\circ}{C}_1\left[0\right]$

$$\overset{\circ}{C}_1\left[0\right]=h^{-1}\sum_{\gamma=-\infty}^{\infty}D_1\left[-\gamma\right]U_{exp}\left[\gamma\right]$$

$$=h^{-1}\left\{D_1\left[0\right]U_{exp}\left[0\right]+D_1\left[1\right]U_{exp}\left[-1\right]+D_1\left[-1\right]U_{exp}\left[1\right]\right\}$$

$$=\frac{h^{-1}}{2(1-e^{2h})}\cdot\left\{(1+e^{2h})(e^{-hk}-e^{hk-h}-e^{-hk+h}+e^{hk})\right.$$

$$\left.-e^{-h}(-e^{-hk}+e^{-hk-h}-e^{hk+h}-e^{hk})\right\}$$

$$-\frac{h^{-1}}{2(1-e^{2h})}\cdot\left\{e^{-h}(e^{-hk+h}-e^{hk-2h}-e^{-hk+2h}+e^{hk-h})\right\}=\frac{h^{-1}}{2(1-e^{2h})}\cdot 0=0,$$

hence, $\overset{\circ}{C}_1\left[0\right]=0$ for $\beta=0$.

Now calculate $\overset{\circ}{C}_1\left[\beta\right]$ for $\beta=\overline{1,k-2}$

$$\overset{\circ}{C}_1\left[\beta\right]=h^{-1}\sum_{\gamma=-\infty}^{\infty}D_1\left[-\gamma\right]U_{exp}\left[\gamma\right]$$

$$= h^{-1}\left\{D_1[0]U_{exp}[\beta] + D_1[1]U_{exp}[\beta-1] + D_1[-1]U_{exp}[\beta+1]\right\}$$
$$= \frac{h^{-1}}{2(1-e^{2h})}\cdot\left\{(1+e^{2h})(1-e^h)(e^{-hk+h\beta} - e^{hk-h\beta-h})\right\}$$
$$-\frac{h^{-1}}{2(1-e^{2h})}\cdot\left\{e^{-h}(1-e^h)(e^{-hk+h\beta-h} - e^{hk-h\beta})\right\}$$
$$-\frac{h^{-1}}{2(1-e^{2h})}\cdot\left\{e^{-h}(1-e^h)(e^{-hk+h\beta+h} - e^{hk-h\beta-2h})\right\} = \frac{h^{-1}}{2(1-e^{2h})}\cdot 0 = 0,$$

thereby, $\overset{\circ}{C}_1[\beta] = 0$ for $\beta = \overline{1, k-2}$.

Finally, we have proved the following theorem.

Theorem 2.3. *In the Hilbert space $W_2^{(2,1)}(0,1)$ there is a unique optimal explicit difference formula of the Adams-Bashforth type whose coefficients are determined by following expressions*

$$C[\beta] = \begin{cases} 1 & for \ \ \beta = k, \\ -1 & for \ \ \beta = k-1, \\ 0 & for \ \ \beta = \overline{0, k-2}, \end{cases} \tag{22}$$

$$\overset{\circ}{C}_1[\beta] = \begin{cases} \frac{e^h-1}{he^h} & for \ \ \beta = k-1, \\ 0 & for \ \ \beta = \overline{0, k-2}. \end{cases} \tag{23}$$

Thus, the optimal explicit difference formula in $W_2^{(2,1)}(0,1)$ has the form

$$\varphi_{n+k} = \varphi_{n+k-1} + \frac{e^h-1}{e^h}\varphi'_{n+k-1}, \tag{24}$$

where $n = 0, 1, ..., N-k, \ k \geq 1$.

3. Optimal implicit difference formulas of Adams-Moulton type in the Hilbert space $W_2^{(2,1)}(0,1)$

Consider an implicit difference formula of the form

$$\sum_{\beta=0}^{k} C[\beta]\varphi[\beta] - h\sum_{\beta=0}^{k} C_1[\beta]\varphi'[\beta] \cong 0 \tag{1}$$

with the error function

$$\ell(x) = \sum_{\beta=0}^{k} C[\beta]\delta(x-h\beta) + h\sum_{\beta=0}^{k} C_1[\beta]\delta'(x-h\beta) \tag{2}$$

in the space $W_2^{(2,1)}(0,1)$.

In this section, we also consider the case $C[k] = -C[k-1] = 1$, and $C[k-i] = 0$, $i = \overline{2,k}$, i.e. Adams-Moulton type formula. Minimizing the norm of the error functional (2) of an implicit difference formula of the form (1) with respect to the coefficients $C_1[\beta]$, $\beta = \overline{0,k}$ in the space $W_2^{(2,1)}(0,1)$ we obtain a system of linear algebraic equations

$$\begin{cases} h\sum_{\gamma=0}^{k} \overset{\circ}{C}_1[\gamma]\, G_2''(h\beta - h\gamma) + de^{-h\beta} = f[\beta], \qquad \beta = \overline{0,k} \\ h\sum_{\gamma=0}^{k} \overset{\circ}{C}_1[\gamma]\, e^{-h\gamma} = g. \end{cases}$$

Here $\overset{\circ}{C}_1[\beta]$ are unknowns coefficients of the implicit difference formulas (1), $\beta = \overline{0,k}$ and d is an unknown constant,

$$f[\beta] = G_2'(h\beta - hk + h) - G_2'(h\beta - hk)$$

$$= \begin{cases} \frac{1}{4}\left(1-e^{h}\right)\left(e^{-hk+h\beta}-e^{hk-h\beta-h}\right), & \beta=\overline{0,k-1}, \\ \frac{1}{4}\left(e^{h}+e^{-h}-2\right), & \beta=k, \end{cases} \tag{3}$$

$$g = e^{-hk+h} - e^{-hk}. \tag{4}$$

Assuming, in general, that

$$\overset{\circ}{C}_1[\beta] = 0 \ , for\ \beta < 0\ and\ \beta > k, \tag{5}$$

rewrite the system in the convolution form

$$\begin{cases} h\overset{\circ}{C}_1[\beta] * G_2''(h\beta) + de^{-h\beta} = f[\beta], & \beta=\overline{0,k}, \\ h\sum_{\gamma=0}^{k}\overset{\circ}{C}_1[\gamma]\, e^{-h\gamma} = g. \end{cases}$$

Denote by $U_{imp}[\beta] = h\overset{\circ}{C}_1[\beta] * G_2''(h\beta) + de^{-h\beta}$. Shows that

$$U_{imp}[\beta] = f[\beta]\ for\ \beta=\overline{0,k} \tag{6}$$

Now we find $U_{imp}[\beta]$ for $\beta < 0$ and $\beta > k$. Let $\beta < 0$, then

$$U_{imp}[\beta] = h\sum_{\gamma=0}^{k}\overset{\circ}{C}_1[\gamma]\,\frac{sgn(h\beta-h\gamma)}{2}\left(\frac{e^{h\beta-h\gamma}-e^{-h\beta+h\gamma}}{2}\right)+de^{-h\beta}$$

$$= -\frac{e^{h\beta}}{4}h\sum_{\gamma=0}^{k}\overset{\circ}{C}_1[\gamma]\,e^{-h\gamma} + \frac{e^{-h\beta}}{4}h\sum_{\gamma=0}^{k}\overset{\circ}{C}_1[\gamma]\,e^{h\gamma} + de^{-h\beta}.$$

Here d^+ is defined by the equality

$$d^+ = \frac{e^{-h\beta}}{4}h\sum_{\gamma=0}^{k}\overset{\circ}{C}_1[\gamma]\,e^{h\gamma} + de^{-h\beta}. \tag{7}$$

Similarly, for $\beta > k$ we have

$$U_{imp}[\beta] = h\sum_{\gamma=0}^{k}\overset{\circ}{C}_1[\gamma]\,\frac{sgn(h\beta-h\gamma)}{2}\left(\frac{e^{h\beta-h\gamma}-e^{-h\beta+h\gamma}}{2}\right)+de^{-h\beta}$$

$$= \frac{e^{h\beta}}{4}h\sum_{\gamma=0}^{k}\overset{\circ}{C}_1[\gamma]\,e^{-h\gamma} - \frac{e^{-h\beta}}{4}h\sum_{\gamma=0}^{k}\overset{\circ}{C}_1[\gamma]\,e^{h\gamma} + de^{-h\beta}.$$

Here d^- is defined by the equality

$$d^- = -\frac{e^{-h\beta}}{4}h\sum_{\gamma=0}^{k}\overset{\circ}{C}_1[\gamma]\,e^{h\gamma} + de^{-h\beta}. \tag{8}$$

(7) and (8) immediately imply that

$$d = \frac{d^+ + d^-}{2}. \tag{9}$$

So $U_{imp}[\beta]$ for any $\beta \in Z$ is defined by the formula

$$U_{imp}[\beta] = \begin{cases} -\frac{e^{h\beta}}{4}g + e^{-h\beta}d^+ & for\ \beta > k, \\ f[\beta] & for\ \beta=\overline{0,k}, \\ \frac{e^{h\beta}}{4}g + e^{-h\beta}d^- & for\ \beta < 0. \end{cases} \tag{10}$$

If we operate operator (20) on expression $U_{imp}[\beta]$, we get

$$\overset{\circ}{C}_1[\beta] = h^{-1}D_1[\beta] * U_{imp}[\beta]\ ,\ \beta \in Z. \tag{11}$$

Assuming that $\overset{\circ}{C}_1[\beta] = 0$ for $\beta < 0$ and $\beta > k$, we get a system of linear equations for finding the unknowns d^+ and d^- in the formula (10). Indeed, calculating the convolution, we have

$$h^{-1}D_1[\beta] * U_{imp}[\beta] = h^{-1}\sum_{\gamma=-\infty}^{\infty} D_1[\beta-\gamma]\,U_{imp}[\gamma]$$

$$= h^{-1}\sum_{\gamma=-\infty}^{-1} D_1[\beta-\gamma]\,U_{imp}[\gamma] + h^{-1}\sum_{\gamma=0}^{k} D_1[\beta-\gamma]\,U_{imp}[\gamma]$$

$$+h^{-1}\sum_{\gamma=k+1}^{\infty} D_1[\beta-\gamma]\,U_{imp}[\gamma]$$

$$= h^{-1}\sum_{\gamma=1}^{\infty} D_1[\beta+\gamma]\,U_{imp}[-\gamma] + h^{-1}\sum_{\gamma=0}^{k} D_1[\beta-\gamma]\,U_{imp}[\gamma]$$

$$+h^{-1}\sum_{\gamma=1}^{\infty} D_1[\beta-k-\gamma]\,U_{imp}[k+\gamma]. \tag{12}$$

Equating the expression (12) to zero with $\beta = -1$, $\beta = k+1$ and using the formulas (10), (20) we get

$$\begin{cases} h^{-1}D_1[0]\,U_{imp}[-1] + h^{-1}D_1[1]\,U_{imp}[-2] + h^{-1}D_1[-1]\,U_{imp}[0] = 0, \\ h^{-1}D_1[0]\,U_{imp}[k+1] + h^{-1}D_1[1]\,U_{imp}[k] + h^{-1}D_1[-1]\,U_{imp}[k+2] = 0 \end{cases}$$

or

$$\begin{cases} 2\left(1+e^{2h}\right)\left[-\frac{1}{4}e^{-h}g + e^h d^+\right] - 2e^h\left[-\frac{1}{4}e^{-2h}g + e^{2h}d^+\right] - 2e^h f[0] = 0, \\ 2\left(1+e^{2h}\right)\left[\frac{1}{4}e^{hk+h}g + e^{-hk-h}d^-\right] - 2e^h\left[\frac{1}{4}e^{hk+2h}g + e^{-hk-2h}d^-\right] - 2e^h f[hk] = 0. \end{cases}$$

By virtue of the formulas (3) and (4), finally, we find

$$d^+ = \frac{1}{4}\left(e^{hk} - e^{hk-h}\right), \tag{13}$$

$$d^- = \frac{1}{4}\left(e^{hk-h} - e^{hk}\right). \tag{14}$$

Then from (9) we find that $d = 0$.

As a result, we rewrite $U_{imp}[\beta]$ through the (13) and (14) as follows

$$U_{imp}[\beta] = \begin{cases} -\frac{e^{h\beta}}{4}g + \frac{e^{-h\beta}}{4}\left(e^{hk} - e^{hk-h}\right) & for\ \beta > k, \\ f[\beta] & for\ \beta = \overline{0,k}, \\ \frac{e^{h\beta}}{4}g + \frac{e^{-h\beta}}{4}\left(e^{hk-h} - e^{hk}\right) & for\ \beta < 0. \end{cases} \tag{15}$$

Now we turn to calculating the optimal coefficients of implicit difference formulas $\overset{\circ}{C}_1[\beta]$, $\beta = \overline{0,k}$ according to the formula (11)

$$\overset{\circ}{C}_1[k] = h^{-1}\sum_{\gamma=-\infty}^{\infty} D_1[k-\gamma]U_{imp}[\gamma] =$$

$$= h^{-1}\left\{D_1[0]U_{imp}[k] + D_1[1]U_{imp}[k-1] + D_1[-1]U_{imp}[k+1]\right\} =$$

$$= \frac{h^{-1}}{2\left(1-e^{2h}\right)}\left(-2e^{2h} + 4e^h - 2\right) = \frac{e^h - 1}{h\left(e^h+1\right)}.$$

So $\overset{\circ}{C}_1[k] = \frac{e^h-1}{h\left(e^h+1\right)}$.

Calculate the next optimal coefficient

$$\overset{\circ}{C}_1[k-1] = h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[k-\gamma-1] U_{imp}[\gamma] =$$

$$= h^{-1} \left\{ D_1[0] U_{imp}[k-1] + D_1[1] U_{imp}[k-2] + D_1[-1] U_{imp}[k] \right\} =$$

$$= \frac{h^{-1}}{2\left(1-e^{2h}\right)} \left(-2e^{2h} + 4e^{h} - 2 \right) = \frac{e^h - 1}{h\left(e^h + 1\right)}.$$

Thus $\overset{\circ}{C}_1[k-1] = \frac{e^h-1}{h\left(e^h+1\right)}$.

Go to computed $\overset{\circ}{C}_1[\beta]$ when $\beta = \overline{1, k-2}$

$$\overset{\circ}{C}_1[\beta] = h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[\beta-\gamma] U_{imp}[\gamma]$$

$$= h^{-1} \left\{ D_1[0] U_{imp}[\beta] + D_1[1] U_{imp}[\beta-1] + D_1[-1] U_{imp}[\beta+1] \right\}$$

$$= \frac{h^{-1}}{2\left(1-e^{2h}\right)} \left\{ \left(1+e^{2h}\right) \left(e^{-hk+h\beta} - e^{hk-h\beta-h} - e^{-hk+h\beta+h} + e^{hk-h\beta} \right) \right\}$$

$$- \frac{h^{-1}}{2\left(1-e^{2h}\right)} \left\{ e^h \left(e^{-hk+h\beta-h} - e^{hk-h\beta} - e^{-hk+h\beta} + e^{hk-h\beta+h} \right) \right\}$$

$$- \frac{h^{-1}}{2\left(1-e^{2h}\right)} \left\{ e^h \left(e^{-hk+h\beta+h} - e^{hk-h\beta-2h} - e^{-hk+h\beta+2h} + e^{hk-h\beta-h} \right) \right\}$$

$$= \frac{h^{-1}}{2\left(1-e^{2h}\right)} \cdot 0 = 0\,.$$

Thereby, $\overset{\circ}{C}_1[\beta] = 0$, for $\beta = \overline{1, k-2}$.
Then calculate $\overset{\circ}{C}_1[0]$

$$\overset{\circ}{C}_1[0] = h^{-1} \sum_{\gamma=-\infty}^{\infty} D_1[-\gamma] U_{imp}[\gamma]$$

$$= h^{-1} \left\{ D_1[0] U_{imp}[0] + D_1[1] U_{imp}[-1] + D_1[-1] U_{imp}[1] \right\}$$

$$= \frac{h^{-1}}{2\left(1-e^{2h}\right)} \left\{ \left(1+e^{2h}\right) \left(e^{-hk} - e^{hk-h} - e^{-hk+h} + e^{hk} \right) \right\}$$

$$- \frac{h^{-1}}{2\left(1-e^{2h}\right)} \left\{ e^h \left(-e^{-hk} + e^{-hk-h} + e^{hk+h} - e^{hk} \right) \right\}$$

$$- \frac{h^{-1}}{2\left(1-e^{2h}\right)} \left\{ e^h \left(e^{-hk+h} - e^{hk-2h} - e^{-hk+2h} + e^{hk-h} \right) \right\} = \frac{h^{-1}}{2\left(1-e^{2h}\right)} \cdot 0 = 0.$$

hence $\overset{\circ}{C}_1[0] = 0$.

Finally, we have proved the following.

Theorem 3.1. *In the Hilbert space $W_2^{(2,1)}(0,1)$, there exists a unique optimal implicit difference formula, of Adams-Moulton type, whose coefficients are determined by formulas*

$$C[\beta] = \begin{cases} 1 & for \quad \beta = k, \\ -1 & for \quad \beta = k-1, \\ 0 & for \quad \beta = \overline{0, k-2}, \end{cases} \tag{16}$$

$$\overset{\circ}{C}_1[\beta] = \begin{cases} \frac{e^h-1}{h\left(e^h+1\right)} & for \quad \beta = k, \\ \frac{e^h-1}{h\left(e^h+1\right)} & for \quad \beta = k-1, \\ 0 & for \quad \beta = \overline{0, k-2}\,. \end{cases} \tag{17}$$

Consequently, the optimal implicit difference formula in $W_2^{(2,1)}(0,1)$ has the form

$$\varphi_{n+k} = \varphi_{n+k-1} + \frac{e^h-1}{e^h+1}\left(\varphi'_{n+k} + \varphi'_{n+k-1}\right), \tag{18}$$

where $n = 0, 1, ..., N-k,\ k \geq 1$.

4. Norm of the error functional of the optimal explicit difference formula

The square of the norm of an explicit Adams-Bashforth type difference formula is expressed by the equality

$$\left\|\ell \left| W_2^{(2,1)*}(0,1) \right.\right\|^2 = \sum_{\gamma=0}^{k}\sum_{\beta=0}^{k} C[\gamma]C[\beta]G_2\left[\gamma-\beta\right] -$$

$$-2h\sum_{\gamma=0}^{k-1} C_1[\gamma]\sum_{\beta=0}^{k} C[\beta]G_2'\left[\gamma-\beta\right] - h^2\sum_{\gamma=0}^{k-1}\sum_{\beta=0}^{k-1} C_1[\gamma]C_1[\beta]G_2''\left[\gamma-\beta\right]. \tag{1}$$

In this section, we deal with the calculation of the squared norm (1) in the space $W_2^{(2,1)}(0,1)$. For this we use the coefficients $C[\beta]$ and $\overset{\circ}{C}_1[\beta]$, which is detected in the formulas (22) and (23).

Then we calculate (1) in sequence as follows.

$$\left\|\overset{\circ}{\ell} \left| W_2^{(2,1)*}(0,1) \right.\right\|^2 = \sum_{\gamma=0}^{k} C[\gamma]\left\{G_2\left[\gamma-k\right] - G_2\left[\gamma-k+1\right]\right\}$$

$$-2h\sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma]\left\{G_2'\left[\gamma-k\right] - G_2'\left[\gamma-k+1\right]\right\}$$

$$-h^2\frac{e^h-1}{he^h}\sum_{\gamma=0}^{k-1} \overset{\circ}{C}_1[\gamma]\left\{G_2''\left[\gamma-k+1\right]\right\}$$

$$= G_2\left[0\right] - G_2\left[1\right] - G_2\left[-1\right] + G_2\left[0\right] - \frac{2(e^h-1)}{e^h}\left\{G_2'\left[-1\right] - G_2'\left[0\right]\right\} - \frac{(e^h-1)^2}{e^{2h}}G_2''\left[0\right]$$

$$= -2G_2\left[1\right] + \frac{2(e^h-1)}{e^h}G_2'\left[1\right] = \frac{2(e^h-1)}{e^h}\cdot\frac{sgn(h)}{2}\left(\frac{e^h+e^h}{2}-1\right)$$

$$-2\cdot\frac{sgn(h)}{2}\left(\frac{e^h-e^h}{2}-h\right) = \frac{e^h-1}{e^h}\cdot\frac{e^{2h}-2e^h+1}{2e^h} - \frac{e^{2h}-1}{2e^h} + h$$

$$= h - \frac{\left(e^h-1\right)\left(3e^h-1\right)}{2e^{2h}}.$$

As a result, we get the following outcome.

Theorem 4.1. *The square of the norm of the optimal error functional of an explicit difference formula of the form (1) in the quotient space $W_2^{(2,1)}(0,1)$ is expressed as formula*

$$\left\|\overset{\circ}{\ell} \left| W_2^{(2,1)*}(0,1) \right.\right\|^2 = h - \frac{\left(e^h-1\right)\left(3e^h-1\right)}{2e^{2h}}\ .$$

5. Norm of the error functional of the implicit optimal difference formula

In this case, the square of the norm of the error functional of an implicit Adams-Moulton type difference formula of the form (1) is expressed by the equality

$$\left\|\ell\left|W_2^{(2,1)*}(0,1)\right.\right\|^2=\sum_{\gamma=0}^{k}\sum_{\beta=0}^{k}C[\gamma]C[\beta]G_2\left[\gamma-\beta\right]-$$

$$-2h\sum_{\gamma=0}^{k}C_1[\gamma]\sum_{\beta=0}^{k}C[\beta]G_2'\left[\gamma-\beta\right]-h^2\sum_{\gamma=0}^{k}\sum_{\beta=0}^{k}C_1[\gamma]C_1[\beta]G_2''\left[\gamma-\beta\right]. \tag{1}$$

Here we use the optimal coefficients of an implicit difference formula of the form (1), which is detected in the formulas (16) and (17).
Then, we calculate (1) as follows

$$\left\|\overset{\circ}{\ell}\left|W_2^{(2,1)*}(0,1)\right.\right\|^2=\sum_{\gamma=0}^{k}C[\gamma]\left\{G_2\left[\gamma-k\right]-G_2\left[\gamma-k+1\right]\right\}$$

$$-2h\sum_{\gamma=0}^{k}\overset{\circ}{C}_1[\gamma]\left\{G_2'\left[\gamma-k\right]-G_2'\left[\gamma-k+1\right]\right\}$$

$$-h^2\frac{e^h-1}{h\left(e^h+1\right)}\sum_{\gamma=0}^{k}\overset{\circ}{C}_1[\gamma]\left\{G_2''\left[\gamma-k\right]+G_2''\left[\gamma-k+1\right]\right\}$$

$$=G_2\left[0\right]-G_2\left[1\right]-G_2\left[-1\right]+G_2\left[0\right]-\frac{2h(e^h-1)}{h\left(e^h+1\right)}\left\{G_2'\left[0\right]-G_2'\left[1\right]+G_2'\left[-1\right]-G_2'\left[0\right]\right\}$$

$$-\frac{h^2\left(e^h-1\right)^2}{h^2\left(e^h+1\right)}\left\{G_2''\left[0\right]+G_2''\left[1\right]+G_2''\left[-1\right]+G_2''\left[0\right]\right\}$$

$$=-2G_2\left[1\right]+\frac{4(e^h-1)}{e^h+1}G_2'\left[1\right]-2\frac{\left(e^h-1\right)^2}{\left(e^h+1\right)}G_2''\left[1\right]$$

$$=\frac{4(e^h-1)}{e^h+1}\cdot\frac{sgn(h)}{2}\left(\frac{e^h+e^h}{2}-1\right)-2\cdot\frac{sgn(h)}{2}\left(\frac{e^h-e^h}{2}-h\right)$$

$$-2\frac{\left(e^h-1\right)^2}{\left(e^h+1\right)}\cdot\frac{sgn(h)}{2}\left(\frac{e^h-e^h}{2}\right)$$

$$=h-\frac{e^{2h}-1}{2e^h}+\frac{2\left(e^h-1\right)^2\left(e^h-1\right)}{2e^h\left(e^h+1\right)}-\frac{\left(e^h-1\right)^2\left(e^h-1\right)}{2e^h\left(e^h+1\right)}$$

$$=h+\frac{\left(e^h-1\right)}{2e^h}\left(\frac{\left(e^h-1\right)^2}{e^h+1}-e^h-1\right)=h-\frac{2\left(e^h-1\right)}{e^h+1}\ .$$

Consequently, we get the following result.

Theorem 5.1. *Among all implicit difference formulas of the form (1) in the Hilbert space* $W_2^{(2,1)}(0,1)$*, there is a unique implicit optimal difference formula square the norms of the error functional of which is determined by the equality*

$$\left\|\overset{\circ}{\ell}\left|W_2^{(2,1)*}(0,1)\right.\right\|^2=h-\frac{2\left(e^h-1\right)}{e^h+1}.$$

6. Numerical results

In this section, we give some numerical results in order to show tables and graphs of solutions and errors of our optimal explicit difference formulas (24) and optimal implicit difference formulas (18), with coefficients given correspondingly in Theorem 2.3 and Theorem 3.1. We show the results of the created formulas in some examples in the form of tables and graphs. Here, of course, the results presented in the table are then shown in the graph.

Example: y'=xsin(-3x)-2, N = 10, h = 0.1, y_0=1; exact solution: 1/3xcos(-3x)-1/3cos(-3x)-2x+4/3
N=10
y_0=1

t	Exact solution	Solution of the Euler Method	Error of the Euler Method	Solution of the Optimal exp.diff .formula	Error of the Optimal exp.dif f.formula
0.0	1.0	1.0	0.0	1.0	0.0
0.1	0.846732386595652	0.8	0.046732386595651	0.809674836071919	0.037057550523732
0.2	0.713243836024086	0.597044797933387	0.116199038090699	0.616537425554994	0.096706410469092
0.3	0.588291007403512	0.385751948465486	0.202539058938026	0.415465694495946	0.172825312907565
0.4	0.460861782437999	0.162252141176661	0.298609641261337	0.202777507195246	0.258084275242752
0.5	0.321543799722049	-0.075029422262028	0.396573221984077	-0.023025755097653	0.344569554819702
0.6	0.163626945959078	-0.32490417159223	0.488531117551308	-0.260813018236448	0.424439964195526
0.7	-0.016182056206681	-0.583335029444922	0.567152973238241	-0.506742495160889	0.490560438954208
0.8	-0.217507085630584	-0.843759685110343	0.62625259947976	-0.754569321563065	0.537062235932482
0.9	-0.436530928599431	-1.097796739554436	0.661265810955004	-0.996317541717459	0.559786613118027
1.0	-0.666666666666667	-1.33626092877548	0.669594262108813	-1.22324622123982	0.556579554573153

Figure 1

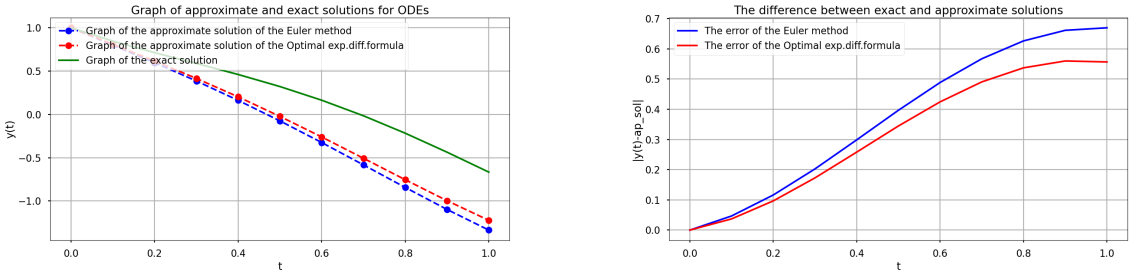


Figure 2

The tables in Figures 1 and 3 show the exact and approximate solutions and the differences between the exact and approximate solutions.

According to the tables in Figures 1 and 3 on the left side of these Figures 2 and 4 are graphs of approximate and exact solutions, and on the right side of these Figures 2 and 4 graphs of the difference between the actual and approximate solutions are shown. As can be seen from the results presented above, in a certain sense, the optimal explicit formula gives better results than the classical Euler formula. Also, with the help of these methods, it is possible to solve problems in various fields of mechanics. The cited numerical results were obtained using the Python programming language.

```
Example: y'=xsin(-3x)-2, N = 10, h = 0.1, y_0=0; exact solution: 1/3xcos(-3x)-1/3cos(-3x)-2x+4/3
N=10
y_0=0
```

t	Exact solution	Solution of the Euler Method	Error of the Euler Method	Solution of the Optimal exp.diff. formula	Error of the Optimal exp.dif f.formula
0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.177582570455131	0.2	0.022417429544869	0.190325163928081	0.01274259347295
0.2	0.318937005892713	0.35751574640716	0.038578740514447	0.340221215209065	0.021284209316352
0.3	0.433883226340232	0.48432171812668	0.050438491786449	0.460893051981951	0.02700982564172
0.4	0.529205824465697	0.588522961386072	0.059317136920375	0.560053645506219	0.030847821040522
0.5	0.609635519478213	0.675744617789102	0.066109098310888	0.643056025771146	0.033420506292933
0.6	0.67850887558362	0.749931212084026	0.071422336500406	0.713653904373384	0.035145028789763
0.7	0.738212180552286	0.81388417102237	0.075671990470084	0.774513191341514	0.036301010789229
0.8	0.790480484888187	0.869623707664033	0.079143222775846	0.827556373584513	0.037075888696326
0.9	0.836599409647578	0.918632824473592	0.082033414826014	0.874194714538262	0.037595304890684
1.0	0.877541649106374	0.962022162692381	0.084480513586007	0.915485129084372	0.037943479977998

FIGURE 3

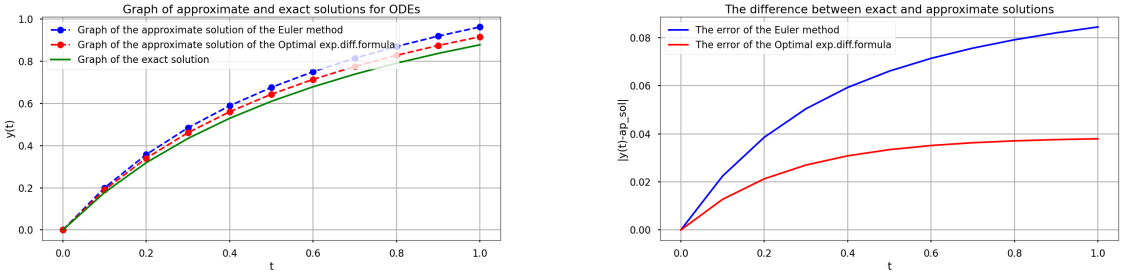


FIGURE 4

7. Conclusion

In conclusion, In this paper, new Adams-type optimal difference formulas are constructed and exact expressions for the exact estimation of their error are obtained. Moreover, we have shown that the results obtained by the optimal explicit difference formulas constructed in the $W_2^{(2,1)}(0,1)$ Hilbert space are better than the results obtained by the Euler formula. In addition, the optimal implicit formula is more accurate than the optimal explicit formula and the effectiveness of the new optimal difference formulas was shown in the numerical results.

References

[1] R.L.Burden, D.J.Faires, A.M.Burden *Numerical Analysis.* - Boston, MA : Cengage Learning, 2016, 896 p.

[2] Ahmad Fadly Nurullah Rasedee, Mohammad Hasan Abdul Sathar, Siti Raihana Hamzah, Norizarina Ishak, Tze Jin Wong, Lee Feng Koo and Siti Nur Iqmal Ibrahim. Two-point block variable order step size multistep method for solving higher order ordinary differential equations directly. *Journal of King Saud University - Science*, vol.33, 2021, 101376, https://doi.org/10.1016/j.jksus.2021.101376

[3] M. Adekoya Odunayo and Z.O.Ogunwobi. Comparison of Adams-Bashforth-Moulton Method and Milne-Simpson Method on Second Order Ordinary Differential Equation. *Turkish Journal of Analysis and Number Theory*, vol.9, no.1, 2021: 1-8., https://doi:10.12691/tjant-9-1-1.

[4] N.S.Hoang, R.B.Sidje. On the equivalence of the continuous Adams-Bashforth method and Nordsiecks technique for changing the step size. *Applied Mathematics Letters*, 2013, 26, pp. 725-728.

[5] Loïc Beuken, Olivier Cheffert, Aleksandra Tutueva, Denis Butusov and Vincent Legat. Numerical Stability and Performance of Semi-Explicit and Semi-Implicit Predictor-Corrector Methods. *Mathematics*, 2022, 10(12), https://doi.org/10.3390/math10122015

[6] Aleksandra Tutueva and Denis Butusov. Stability Analysis and Optimization of Semi-Explicit Predictor-Corrector Methods. *Mathematics*, 2021, 9, 2463. https://doi.org/10.3390/math9192463

[7] I.Babuška, S.L.Sobolev *Optimization of numerical methods.* - Apl. Mat., 1965, 10, 9-170.

[8] I.Babuška, E.Vitasek, M.Prager *Numerical processes for solution of differential equations.* - Mir, Moscow, 1969, 369 p.

[9] Kh.M.Shadimetov, A.R.Hayotov Optimal quadrature formulas in the sense of Sard in $W_2^{(m,m-1)}$ space. *Calcolo,* Springer, 2014, V.51, pp. 211-243.

[10] Kh.M.Shadimetov, A.R.Hayotov Construction of interpolation splines minimizing semi-norm in $W_2^{(m,m-1)}$ space. *BIT Numer Math,* Springer, 2013, V.53, pp. 545-563.

[11] Kh.M.Shadimetov Functional statement of the problem of optimal difference formulas. *Uzbek mathematical Journal,* Tashkent, 2015, no.4, pp.179-183.

[12] Kh.M.Shadimetov, R.N.Mirzakabilov The problem on construction of difference formulas. *Problems of Computational and Applied Mathematics,* - Tashkent, 2018, no.5(17). pp. 95-101.

[13] S.L.Sobolev *Introduction to the theory of cubature formulas.* - Nauka, Moscow, 1974, 808 p.

[14] S.L.Sobolev, V.L.Vaskevich *Cubature fromulas.* - Novosibirsk, 1996, 484 p.

[15] Kh.M.Shadimetov, R.N.Mirzakabilov On a construction method of optimal difference formulas. *AIP Conference Proceedings*, 2365, 020032, 2021.

[16] D.M.Akhmedov, A.R.Hayotov, Kh.M.Shadimetov Optimal quadrature formulas with derivatives for Cauchy type singular integrals. *Applied Mathematics and Computation,* Elsevier, 2018, V.317, pp. 150-159.

[17] N.D.Boltaev, A.R.Hayotov, Kh.M.Shadimetov Construction of Optimal Quadrature Formula for Numerical Calculation of Fourier Coefficients in Sobolev space $L_2^{(1)}$. *American Journal of Numerical Analysis,* 2016, v.4, no.1, pp. 1-7.

[18] Kh.M.Shadimetov, A.R.Hayotov, R.S.Karimov Optimization of Explicit Difference Methods in the Hilbert Space $W_2^{(2,1)}$, *AIP Conference Proceedings*, 2023, 2781, 00054.

[19] G.Dahlquits Convergence and stability in the numerical integration of ordinary differential equations. -*Math. Scand.*, 1956, v.4, pp. 33-52.

[20] G.Dahlquits *Stability and error bounds in the numerical integration of ordinary differential equations.* - Trans. Roy. Inst. Technol. Stockholm, 1959.

[21] Kh.M.Shadimetov, R.N.Mirzakabilov Optimal Difference Formulas in the Sobolev Space. - *Contemporary Mathematics. Fundamental Directions*, 2022, Vol.68, No.1, 167-177.

[22] Kh.M.Shadimetov, A.R.Hayotov Construction of a discrete analogue of a differential operator *Uzbek Matematical Journal,* - Tashkent, 2004, no. 2, pp. 85-95.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON THE CONTINUUM OF EXACT SOLUTIONS TO THE SYSTEM OF MAXWELL'S EQUATIONS

MAKSUT ABENOV[1], TURAR ABZHALIULY[1]

[1]Kazakh National University, Almaty, Kazakhstan
e-mail: abenov60@gmail.com, turar.abzh.02@mail.ru

As is known [1], the basic equations of electrodynamics in theoretical physics are given in the form of a system of Maxwell's equatins of the following form

$$\begin{cases} div(\varepsilon\vec{E}) = 4\pi\rho, \\ rot\vec{E} = -\dfrac{1}{c}\dfrac{\partial(\mu\vec{H})}{\partial t}, \\ div(\mu\vec{H}) = 0, \\ rot\vec{H} = \dfrac{1}{c}\dfrac{\partial(\varepsilon\vec{E})}{\partial t} + \dfrac{4\pi\vec{J}}{c}, \end{cases} \tag{1}$$

where c is the speed of light, $\varepsilon > 0, \mu > 0$ - are known, dimensionless constants, $\vec{E}(t,x,y,z), \vec{H}, \vec{J}$ - are unknown vectors of electric (magnetic) field strengths and saturation current density of electric charges. The system of equations (1) is considered key in theoretical physics. To date, only a few exact solutions of this system are known. For example, the motion of a plane electromagnetic wave it is described in various textbooks on theoretical physics.

Definition 1. *The four - dimensional potential of the electromagnetic field is the vector function* $\Pi_{4d} = (\theta, u, v, \omega)$, *and it satisfies the following continuity equation for the electromagnetic field:*

$$\frac{\varepsilon\mu}{c}\frac{\partial\theta}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial\omega}{\partial z} = 0 \tag{2}$$

Theorem 1. *(2) The continuity equation has a continuum of exact solutions of the following form:*

$$\begin{cases} \varepsilon\mu\theta(t,x,y,z) = \beta_1 u_1(\alpha_1 ct, \alpha_2 x, \alpha_3 y, \alpha_4 z) \\ u(t,x,y,z) = \beta_2 u_2(\alpha_1 ct, \alpha_2 x, \alpha_3 y, \alpha_4 z) \\ v(t,x,y,z) = \beta_3 u_3(\alpha_1 ct, \alpha_2 x, \alpha_3 y, \alpha_4 z) \\ \omega(t,x,y,z) = \beta_4 u_4(\alpha_1 ct, \alpha_2 x, \alpha_3 y, \alpha_4 z) \end{cases} \tag{3}$$

here: $\alpha_k, \beta_k, k = 1,2,3,4$ - is an arbitrary set of scalars satisfying condition $\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4 = 0$, and $U(X) = (u_1(x_1,x_2,x_3,x_4), u_2, u_3, u_4)$ is four - dimensional arbitraty regular function in the domain $G \subset R^4$.

Theorem 2. *For each four - dimensional potential of the electromagnetic field* $\Pi_{4d} = (\theta, u, v, \omega)$ *an exact solution of Maxwell's equations of the following form corresponds:*

$$\begin{cases} \overrightarrow{E} = \nabla\theta + \dfrac{1}{c}\dfrac{\partial\overrightarrow{\varphi}}{\partial t} \\ \overrightarrow{H} = -\dfrac{1}{\mu} rot\overrightarrow{\varphi} \\ \rho = \dfrac{\varepsilon}{4\pi}\left(\Delta\theta - \dfrac{\varepsilon\mu}{c^2}\dfrac{\partial^2\theta}{\partial t^2}\right) \\ \overrightarrow{j} = \dfrac{c}{4\pi\mu}\left(\Delta\varphi - \dfrac{\varepsilon\mu}{c^2}\dfrac{\partial^2\overrightarrow{\varphi}}{\partial t^2}\right) \end{cases} \tag{4}$$

formulas (3) - (4) give a continuum of exact solutions to Maxwell's equations.

Let the four-dimensional function $u(X) = [u_1(x_1, x_2, x_3, x_4), u_2, u_3, u_4]$ is defined and continuous in some neighborhood of $O_\varepsilon(X)$.

Let's increment the ΔX argument so that $X + \Delta X \in O_\varepsilon(X)$

Definition 2. A four - dimensional function is called a differentiable at point X if there is a limit of the form:

$$u'(X) = \lim_{\Delta X \to \theta} \frac{u(X + \Delta X) - u(X)}{\Delta X}, \tag{5}$$

and this limit is called the derivative of the function at X.

Naturally this definition assumes that

$$\Delta X = (\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4) \to \theta = (0, 0, 0, 0)$$

by any path consisting only of non - degenerate numbers. Otherwise, the definition of differentability loses its meaning, since the inverse quantity exists only for non - degenerate numbers.

This is always achievable. For example, there is a non - degenerate path aspirations to zero, of the following kind:

$$\Delta X^{(n)} = \left(\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \frac{4}{n}\right) \to (0, 0, 0, 0); n = \overline{1, \infty}$$

Further, the differentability conditions of a four - dimensional function are given by the following theorem generalizing a well - known theorem from complex analysis.

Theorem 3. *Let all components of a four - dimensional function $u(X) = [u_1(x_1, x_2, x_3, x_4), u_2, u_3, u_4]$ have continuous derivatives, in the neighborhood of $O_\varepsilon(X)$, as ordinary scalar functions of four independent variables. Then, for the differentiability of the function $u(X)$ at the point X, it is necessary and sufficient to fulfill the generalized Cauchy - Riemann (Euler - Dalembert) conditions:*

$$\begin{cases} \dfrac{\partial u_1}{\partial x_1} = \dfrac{\partial u_2}{\partial x_2} = \dfrac{\partial u_3}{\partial x_3} = \dfrac{\partial u_4}{\partial x_4}, \\ \dfrac{\partial u_1}{\partial x_2} = -\dfrac{\partial u_2}{\partial x_1} = \dfrac{\partial u_3}{\partial x_4} = -\dfrac{\partial u_4}{\partial x_3}, \\ \dfrac{\partial u_1}{\partial x_3} = \dfrac{\partial u_2}{\partial x_4} = \dfrac{\partial u_3}{\partial x_1} = \dfrac{\partial u_4}{\partial x_2}, \\ \dfrac{\partial u_1}{\partial x_4} = -\dfrac{\partial u_2}{\partial x_3} = \dfrac{\partial u_3}{\partial x_2} = -\dfrac{\partial u_4}{\partial x_1} \end{cases} \tag{6}$$

Proof. Necessity: Let the limit (5) exists. Then it should not depend on the method of aspiration $\Delta X \to \theta$ as long as the path consists of non - degenerate points. We will consider the following four ways that are convenient for us when proximity $(X + \Delta X)$ to the X point.

1. $\Delta X = (\Delta x_1, 0, 0, 0) \to \theta$, that is, $\Delta X = J_1 \Delta x_1 \to \theta \Leftrightarrow x_1 \to 0$.

In this case, using the base number decomposition, we obtain an equivalent (5) entry of the form:

$$u'(X)=\lim_{\Delta x_1\to 0}\frac{\sum_{k=1}^{4}[u(x_1+\Delta x_1,x_2,x_3,x_4)-u(x_1,x_2,x_3,x_4)]J_k}{J_1\Delta x_1}=\frac{\partial u_1}{\partial x_1}J_1+\frac{\partial u_2}{\partial x_1}J_2+\frac{\partial u_3}{\partial x_1}J_3+\frac{\partial u_4}{\partial x_1}J_4.$$

2.$\Delta X=(0,\Delta x_2,0,0)\to\theta$, that is, $\Delta X=J_2\Delta x_2\to\theta\Leftrightarrow x_2\to 0$.

In this case, the equivalent (5) entry would be:

$$u'(X)=\lim_{\Delta x_2\to 0}\frac{\sum_{k=1}^{4}[u(x_1,x_2+\Delta x_2,x_3,x_4)-u(x_1,x_2,x_3,x_4)]J_k}{J_2\Delta x_2}=$$

$$=\lim_{\Delta x_2\to 0}\frac{\sum_{k=1}^{4}[u(x_1,x_2+\Delta x_2,x_3,x_4)-u(x_1,x_2,x_3,x_4)]J_k(-J_2)}{\Delta x_2}=\frac{\partial u_2}{\partial x_2}J_1-\frac{\partial u_1}{\partial x_2}J_2+\frac{\partial u_4}{\partial x_2}J_3-\frac{\partial u_3}{\partial x_2}J_4.$$

Here the formulas [2] of multiplication of basic numbers are used.

3.$\Delta X=(0,0,\Delta x_3,0)\to\theta$, that is, $\Delta X=J_3\Delta x_3\to\theta\Leftrightarrow x_3\to 0$.

$$u'(X)=\lim_{\Delta x_3\to 0}\frac{\sum_{k=1}^{4}[u(x_1,x_2,x_3+\Delta x_3,x_4)-u(x_1,x_2,x_3,x_4)]J_k}{J_3\Delta x_3}=$$

$$=\lim_{\Delta x_3\to 0}\frac{\sum_{k=1}^{4}[u(x_1,x_2,x_3+\Delta x_3,x_4)-u(x_1,x_2,x_3,x_4)]J_kJ_3}{\Delta x_3}=\frac{\partial u_3}{\partial x_3}J_1+\frac{\partial u_4}{\partial x_3}J_2+\frac{\partial u_1}{\partial x_3}J_3+\frac{\partial u_2}{\partial x_3}J_4.$$

4.$\Delta X=(0,0,0,\Delta x_4)\to\theta$, that is, $\Delta X=J_4\Delta x_4\to\theta\Leftrightarrow x_4\to 0$.

$$u'(X)=\lim_{\Delta x_4\to 0}\frac{\sum_{k=1}^{4}[u(x_1,x_2,x_3,x_4+\Delta x_4)-u(x_1,x_2,x_3,x_4)]J_k}{J_4\Delta x_4}=$$

$$=\lim_{\Delta x_4\to 0}\frac{\sum_{k=1}^{4}[u(x_1,x_2,x_3,x_4+\Delta x_4)-u(x_1,x_2,x_3,x_4)]J_k(-J_4)}{\Delta x_4}=\frac{\partial u_4}{\partial x_4}J_1-\frac{\partial u_3}{\partial x_4}J_2+\frac{\partial u_2}{\partial x_4}J_3-\frac{\partial u_1}{\partial x_4}J_4.$$

Now, comparing the right - hand sides of the last formulas in pairs, we obtain the conditions (6).

Sufficiency. Let (6) be fulfilled. By definition of the differential of a real function of four variables, for each components of a four - dimensional function can be written:

$$u_k(x_1+\Delta x_1,x_2+\Delta x_2,x_3+\Delta x_3,x_4+\Delta x_4)-u(x_1,x_2,x_3,x_4)=$$

$$=\frac{\partial u_k}{\partial x_1}\Delta x_1+\frac{\partial u_k}{\partial x_2}\Delta x_2+\frac{\partial u_k}{\partial x_3}\Delta x_3+\frac{\partial u_k}{\partial x_4}\Delta x_4+\alpha_k|h|;k=\overline{1,4}\qquad(7)$$

Here: α_k tend to zero together with $|h|=\sqrt{\sum(\Delta x_k)^2}$.

Then from (7), simultaneously using (6) we easily get:

$$u(X+\Delta X)-u(X)=\left(\frac{\partial u_1}{\partial x_1}J_1+\frac{\partial u_2}{\partial x_2}J_2+\frac{\partial u_3}{\partial x_3}J_3+\frac{\partial u_4}{\partial x_4}J_4\right)\Delta X+\eta(\Delta X).\qquad(8)$$

where: $\eta(\Delta X)=o(\Delta X),\Delta X\to\theta$. Now (8) implies the existense of a limit (5). **The theorem is proved.**

The proved theorem is one of the main and key theorems of four dimensional analysis.

Definition 3. A four - dimensional function $u(X)$ having a derivative at all points of some domain $G\subset R^4$ is called regular in this domain. A function that is regular in the whole space is called an integer function.

In the future, to denote the derivative of a regular function, we will use one of two forms of notation

$$u'(X) = \frac{du}{dX}.$$

It follows from the main theorem that it is possible to use four equivalent formulas for determining the derivative:

$$\frac{du}{dX} = \left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_3}{\partial x_1}, \frac{\partial u_4}{\partial x_1}\right) = \left(\frac{\partial u_2}{\partial x_2}, -\frac{\partial u_1}{\partial x_2}, \frac{\partial u_4}{\partial x_2}, -\frac{\partial u_3}{\partial x_2}\right) =$$

$$= \left(\frac{\partial u_3}{\partial x_3}, \frac{\partial u_4}{\partial x_3}, \frac{\partial u_1}{\partial x_3}, \frac{\partial u_2}{\partial x_3}\right) = \left(\frac{\partial u_4}{\partial x_4}, -\frac{\partial u_3}{\partial x_4}, \frac{\partial u_2}{\partial x_4}, -\frac{\partial u_1}{\partial x_4}\right). \tag{9}$$

Now let's give an example.

Example. Take the four - dimensional exponental function $u(X) = e^X$.

$$u_1(x_1, x_2, x_3, x_4) = \frac{1}{2}[e^{x_1+x_3}\cos(x_2 + x_4) + e^{x_1-x_3}\cos(x_2 - x_4)]$$

$$u_2(x_1, x_2, x_3, x_4) = \frac{1}{2}[e^{x_1+x_3}\sin(x_2 + x_4) + e^{x_1-x_3}\sin(x_2 - x_4)]$$

$$u_3(x_1, x_2, x_3, x_4) = \frac{1}{2}[e^{x_1+x_3}\cos(x_2 + x_4) - e^{x_1-x_3}\cos(x_2 - x_4)]$$

$$u_4(x_1, x_2, x_3, x_4) = \frac{1}{2}[e^{x_1+x_3}\sin(x_2 + x_4) - e^{x_1-x_3}\sin(x_2 - x_4)]$$

Let's make sure that the above generalized Cauchy - Riemann conditions are met. In fact, all components of the function satisfy the generalized Cauchy - Riemann conditions.

Now let the solvability parameters have the following values: $\alpha_1 = -1, \alpha_2 = 1, \alpha_3 = i, \alpha_4 = 1, \beta_1 = 1, \beta_2 = -1, \beta_3 = -i, \beta_4 = 1$.

We see that the condition

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4 = 0$$

is met.

It means that

$$\begin{cases} \theta(t, x, y, z) = \dfrac{1}{\varepsilon\mu}\beta_1 Reu_1(\alpha_1 ct, \alpha_2 x, \alpha_3 y, \alpha_4 z) \\ u(t, x, y, z) = \beta_2 Reu_2(\alpha_1 ct, \alpha_2 x, \alpha_3 y, \alpha_4 z) \\ v(t, x, y, z) = \beta_3 Reu_3(\alpha_1 ct, \alpha_2 x, \alpha_3 y, \alpha_4 z) \\ \omega(t, x, y, z) = \beta_4 Reu_4(\alpha_1 ct, \alpha_2 x, \alpha_3 y, \alpha_4 z) \end{cases}$$

$$\begin{cases} \theta(t, x, y, z) = \dfrac{1}{\varepsilon\mu}e^{-\frac{ct}{L}}\cos\dfrac{x}{L}\cos\dfrac{y}{L}\cos\dfrac{z}{L} \\ u(t, x, y, z) = -e^{-\frac{ct}{L}}\sin\dfrac{x}{L}\cos\dfrac{y}{L}\cos\dfrac{z}{L} \\ v(t, x, y, z) = e^{-\frac{ct}{L}}\cos\dfrac{x}{L}\sin\dfrac{y}{L}\cos\dfrac{z}{L} \\ \omega(t, x, y, z) = e^{-\frac{ct}{L}}\cos\dfrac{x}{L}\cos\dfrac{y}{L}\sin\dfrac{z}{L} \end{cases}$$

$$\begin{cases} E_1(t,x,y,z) = \dfrac{(1-\frac{1}{\varepsilon\mu})}{L} e^{-\frac{ct}{L}} \sin\dfrac{x}{L}\cos\dfrac{y}{L}\cos\dfrac{z}{L} \\ E_2(t,x,y,z) = -\dfrac{(1+\frac{1}{\varepsilon\mu})}{L} e^{-\frac{ct}{L}} \cos\dfrac{x}{L}\sin\dfrac{y}{L}\cos\dfrac{z}{L} \\ E_3(t,x,y,z) = -\dfrac{(1+\frac{1}{\varepsilon\mu})}{L} e^{-\frac{ct}{L}} \cos\dfrac{x}{L}\cos\dfrac{y}{L}\sin\dfrac{z}{L} \\ H_1(t,x,y,z) = 0 \\ H_2(t,x,y,z) = -\dfrac{2}{\mu L} e^{-\frac{ct}{L}} \sin\dfrac{x}{L}\cos\dfrac{y}{L}\sin\dfrac{z}{L} \\ H_3(t,x,y,z) = \dfrac{2}{\mu L} e^{-\frac{ct}{L}} \sin\dfrac{x}{L}\sin\dfrac{y}{L}\sin\dfrac{z}{L} \\ \rho(t,x,y,z) = -\dfrac{(\varepsilon+\frac{3}{\mu})}{4\pi L^2} e^{-\frac{ct}{L}} \cos\dfrac{x}{L}\cos\dfrac{y}{L}\cos\dfrac{z}{L} \\ j_1(t,x,y,z) = \dfrac{c(\varepsilon+\frac{3}{\mu})}{4\pi L^2} e^{-\frac{ct}{L}} \sin\dfrac{x}{L}\cos\dfrac{y}{L}\cos\dfrac{z}{L} \\ j_2(t,x,y,z) = -\dfrac{c(\varepsilon+\frac{3}{\mu})}{4\pi L^2} e^{-\frac{ct}{L}} \cos\dfrac{x}{L}\sin\dfrac{y}{L}\cos\dfrac{z}{L} \\ j_3(t,x,y,z) = -\dfrac{c(\varepsilon+\frac{3}{\mu})}{4\pi L^2} e^{-\frac{ct}{L}} \cos\dfrac{x}{L}\cos\dfrac{y}{L}\sin\dfrac{z}{L} \end{cases}$$

If we check the results obtained by placing them in the system of Maxwell's equations, we will see that the solutions satisfy the equations. That is, we have obtained a continuum of real solutions to the system of Maxwell's equations. In addition, these solutions are limited and periodic.

Conclusion. In this paper, we investigate the long-known continuum of real solutions of Maxwell's equations using a new four-dimensional analysis and confirm these formulas by our own example. So, we took one autonomous solution that has a physical meaning and fully studied its differential properties. To do this, we first investigated the history of the origin of Maxwell's equation, its role in modern physics, its physical meaning, and their practical application. Then we outlined the basic concepts of four-dimensional analysis necessary to solve this problem. The most fundamental thing in this work was to introduce a four-dimensional analogue of the continuity equation and find its solution. The continuity equation is derived from Maxwell's equation by simple methods, and its solution leads to the search for a solution to Maxwell's equations. Along with this, Maxwell's equation gives the equation of vector and scalar potentials. We implement the solution of Maxwell's equation by introducing these potentials. The main purpose of the work is also to study the continuum of real solutions of the system of Maxwell's equations found using these potentials. In conclusion, I would like to express my gratitude to the scientific supervisor Abenov Mahsut Mnaidarovich, who provided great help and advice when writing the work.

Keywords: Maxwell's equations, electromagnetic field, four - dimensional potential, continuity equation, regular function.

References

[1] Vladimirov V.S., *Equations of Mathematical Physics*, Nauka, Moscow, 1981.
[2] Abenov M.M., *Four - dimensional mathematics methods and applications*, KazNU, Almaty, 2019.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

SOLVING AN INITIAL-BOUNDARY VALUE PROBLEM FOR A LOADED FRACTIONAL-ORDER DIFFUSION EQUATION

MINZILYA KOSMAKOVA[1], DANNA AKHMANOVA[1], KAMILA IZHANOVA[1]

[1]Karaganda Buketov University, Karaganda, Kazakhstan
e-mail: svetlanamir578@gmail.com

1. Problem statement

In a domain

$$G = \{(x, y; t) | -\infty < x, y < +\infty,\ t > 0\}$$

find a solution to the equation

$$_{RL}D^{\alpha}_{0t}u = u_{xx} + u_{yy} + \lambda\{_{RL}D^{\beta}_{0t}\}\Big|_{x-1=t,y-1=t,t>0} + f(x, y, t), \tag{1}$$

where $_{RL}D^{\sigma}_{0t}f(t)$ is a Riemann-Liouville derivative of the order $\sigma, 0 < \sigma < 1$, and $0 < \beta \le \alpha < 1$.

The solution of equation (1) must satisfy the initial condition:

$$\lim_{t\to 0} {}_{RL}D^{\alpha-1}_{0t}u = \varphi(x, y). \tag{2}$$

2. Reduction to an integral equation

Let us denote

$$\mu(t) = \{_{RL}D^{\beta}_{0t}u\}\Big|_{x-1=t,y-1=t,t>0}. \tag{3}$$

Then a solution of Problem (1) – (2) can be represented as [1; p.174]:

$$\begin{aligned} u(x, y, t) = &\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varphi(\xi, \eta)\Gamma_{\alpha,2}(x-\xi, y-\eta, t)d\xi d\eta + \\ &+ \lambda\int_0^t\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mu(\tau)\Gamma_{\alpha,2}(x-\xi, y-\eta, t-\tau)d\xi d\eta d\tau + \\ &+ \int_0^t\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\xi, \eta, \tau)\Gamma_{\alpha,2}(x-\xi, y-\eta, t-\tau)d\xi d\eta d\tau \end{aligned} \tag{4}$$

where [1; p.152]

$$\Gamma_{\alpha,2}(x, y, t) = \frac{1}{4\sqrt{\pi}t} f_{\frac{\alpha}{2}}\left(\frac{\sqrt{x^2+y^2}}{t^{\alpha/2}}, 1; 0\right) \tag{5}$$

[1; p.145]

$$f_{\frac{\alpha}{2}}(z, \mu, \sigma) = \begin{cases} \frac{2}{\Gamma(\frac{\mu}{2})}\int_1^{+\infty} \phi(-\alpha/2; \sigma; -z\tau)(\tau^2-1)^{\mu/2-1}d\tau, & \mu > 0 \\ \phi(-\alpha/2; \sigma; -z), & \mu = 0 \end{cases} \tag{6}$$

$$\phi(-a, b; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(b-ak)}$$

This work was partially supported by the Science Committee of the Ministry of Higher Education and Science of the Republic of Kazakhstan (Grants no. AP09259780) .

is a Wright function.

In our case

$$\mu = 1 \neq 0; z = \frac{\sqrt{x^2+y^2}}{t^{\frac{\alpha}{2}}}; \sigma = 0 => f_{\frac{\alpha}{2}}(z;1;0) = \frac{2}{\sqrt{\pi}} \int_1^{+\infty} \phi\left(-\frac{\alpha}{2}; 0; -zt\right)(t^2-1)^{-\frac{1}{2}} dt.$$

Let us denote:

$$\varphi_1(x,y,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varphi(\xi,\eta)\Gamma_{\alpha,2}(x-\xi, y-\eta, t)d\xi d\eta, \tag{7}$$

$$f_1(x,y,t) = \int_0^t \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\xi,\eta,\tau)\Gamma_{\alpha,2}(x-\xi, y-\eta, t-\tau)d\xi d\eta d\tau \tag{8}$$

Then we rewrite representation (4) in the form:

$$u(x,y,t) = \lambda \int_0^t \mu(\tau) \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Gamma(\alpha,2)(x-\xi, y-\eta, t-\tau)d\xi d\eta d\tau + \\ + \varphi_1(x,y,t) + f_1(x,y,t)$$

or

$$u(x,y,t) = \lambda(\mu(t) * \gamma_{\alpha,2}(x,y,t))(t) + \varphi_1(x,y,t) + f_1(x,y,t), \tag{9}$$

where

$$\gamma_{\alpha,2}(x,y,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Gamma_{\alpha,2}(x-\xi, y-\eta, t-\tau)d\xi d\eta. \tag{10}$$

Let us apply to equality (9) the Riemann-Liouville fractional differentiation operator of an order β. There is an equality [1, p.14]

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} {}_{RL}D_{0t}^{\alpha-\zeta}\Gamma_{\alpha,n}(x,y,t)dxdy = \frac{t^{\zeta-1}}{\Gamma(\zeta)}. \tag{11}$$

In our case $\beta = \alpha - \zeta => \zeta = \alpha - \beta \geq 0; n = 2$. We consider the case $\zeta = \alpha - \beta > 0$, i.e.$\alpha \neq \beta$.

According to formula (11), function (10) takes the form:

$$\gamma_{\alpha,2}(x,y,t) = \left.\frac{t^{\zeta-1}}{\Gamma(\zeta)}\right|_{\zeta=\alpha} = \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

The formula for fractional differentiation of the convolution of two functions is also valid:

$${}_{RL}D_{0t}^{\alpha}(f*g)(t) = \left({}_{RL}D_{0t}^{\alpha} g * f\right)(t) + f(t)\lim_{t\to 0} {}_{RL}D_{0t}^{\alpha-1} g(t), 0 < \alpha \leq 1.$$

Then, after the operation of fractional differentiation of an order β, equality (9) takes the form:

$$ {}_{RL}D_{0t}^{\beta}u(x,y,t) = \lambda\left(\mu(t) * {}_{RL}D_{0t}^{\beta}\gamma_{\alpha,2}(x,y,t)\right)(t) + \mu(t)\lim_{t\to 0} {}_{RL}D_{0t}^{\beta-1}\gamma_{\alpha,2}(x,y,t) + \\ + {}_{RL}D_{0t}^{\beta}\varphi_1(x,y,t) + {}_{RL}D_{0t}^{\beta} f_1(x,y,t). \tag{12}$$

Taking into account the notation (10) and equality (11), we have for the first term in (12)

$${}_{RL}D_{0t}^{\beta}\gamma_{\alpha,2}(x,y,t) = \frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}$$

we also have for the second term:

$$\lim_{t\to 0} {}_{RL}D_{0t}^{\beta-1}\gamma_{\alpha,2}(x,y,t) = \lim_{t\to 0}\frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} = 0, \quad \text{since} \quad \alpha > \beta$$

Then, substituting $x-1=t$ and $y-1=t$ into (12), taking into account notation (3), and introducing the notation

$$f_2(t) = {}_{RL}D_{0t}^{\beta} f_1(x,y,t)\Big|_{x-1=t, y-1=t, t>0}; \quad \varphi_2(t) = {}_{RL}D_{0t}^{\beta}\varphi_1(x,y,t)\Big|_{x-1=t, y-1=t, t>0}, \tag{13}$$

we get the integral equation:

$$\mu(t) - \frac{\lambda}{\Gamma(\alpha-\beta)} \int_0^t \frac{\mu(t)}{(t-\tau)^{\beta-\alpha+1}} d\tau = f_2(t) + \varphi_2(t) \tag{14}$$

Here $0 < \beta - \alpha + 1 < 1$, and the functions $f_2(t)$ and $\varphi_2(t)$ are defined by formulas (13), (5) – (8).

So using the representation formula for the solution of Problem (1) – (2), we obtain Volterra-type integral equation (14) with respect to the function (3).

3. Solving the integral equation

The solution of the integral equation is obtained by the Laplace transform method.

Let us denote $r = \beta - \alpha + 1$

$$\mu(t) - \frac{\lambda}{\Gamma(1-r)} \int_0^t \frac{\mu(t)}{(t-\tau)^{r}} d\tau = F(t). \tag{15}$$

Here $\varphi(t) = f_2(t) + \varphi_2(t)$.

Let $\Phi(s) = L[\mu(t)]$ be a Laplace image of the function $\mu(t)$. Applying the integral Laplace transform to equation (15), we obtain [2, p.235, formula 7]

$$\Phi(s) - \frac{\lambda \Phi(s)}{s^{1-r}} = \Psi(s),$$

where $\Psi(s) = L[\psi(t)]$, or

$$\Phi(s) = \Psi(s) + \lambda \frac{\Psi(s)}{s^{1-r} - \lambda}.$$

Applying the inverse Laplace transform, we obtain a solution of the integral equation (15):

$$\mu(t) = \psi(t) + \lambda \psi(t) * t^{-r} \mathcal{E}_{1-r;1-r}(\lambda t^{1-r}),$$

where $\mathcal{E}_{a,b}(z)$ is a Mittag-Leffler function.

4. Solution to Problem (1) – (2). Main result

Taking into account the representation (9), we obtain

$$u(x,y,t) = \lambda\big(\psi(t) * \gamma_{\alpha,2}(x,y,t)\big)(t) + \lambda^2\big((\psi(t) * t^{-r}\ \mathcal{E}_{1-r;1-r}(\lambda t^{1-r})) * \gamma_{\alpha,2}(x,y,t)\big)(t) + \varphi_1(x,y,t) + f_1(x,y,t).$$

Due to the associativity property of convolution, we have

$$u(x,y,t) = \lambda\big(\psi(t) * \gamma_{\alpha,2}(x,y,t)\big)(t) + \lambda^2 \psi(t) * \big(\big(\gamma_{\alpha,2}(x,y,t) * (t^{-r}\ \mathcal{E}_{1-r,1-r}(\lambda t^{1-r}))\big)(t)\big) + \varphi_1(x,y,t) + f_1(x,y,t), \tag{16}$$

where $r = \beta - \alpha + 1$.

Let's calculate

$$\left(\gamma_{\alpha,2}(x,y,t) * (t^{-r}\ \mathcal{E}_{1-r,1-r}(\lambda t^{1-r}))\right)(t).$$

Since

$$\gamma_{\alpha,2}(x,y,t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)},$$

then

$$\left(\gamma_{\alpha,2}(x,y,t) * (t^{-r}\ \mathcal{E}_{1-r,1-r}(\lambda t^{1-r}))\right)(t) = \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} * (t^{-r}\ \mathcal{E}_{1-r,1-r}(\lambda t^{1-r}))\right)(t)$$

is a fractional integral of order α of the function $g(t) = t^{-r}\ \mathcal{E}_{1-r,1-r}(\lambda t^{1-r})$ starting at 0 and ending at t. Then, taking into account the notation $r = \beta - \alpha + 1$ we have

$$\left(\gamma_{\alpha,2}(x,y,t) * (t^{-r}\ \mathcal{E}_{1-r,1-r}(\lambda t^{1-r}))\right)(t) = I_{0t}^{\alpha}\left(t^{\alpha-\beta-1}\ \mathcal{E}_{\alpha-\beta,\alpha-\beta}(\lambda t^{\alpha-\beta})\right) = t^{2\alpha-\beta-1}\ \mathcal{E}_{\alpha-\beta,2\alpha-\beta}(\lambda t^{\alpha-\beta}).$$

Solution (16) can be rewritten as:

$$u(x,y,t)=\lambda\left(\psi(t)*\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)(t)+\lambda^2\left(\psi(t)*\left(t^{2\alpha-\beta-1}\ \mathcal{E}_{\alpha-\beta,2\alpha-\beta}(\lambda t^{\alpha-\beta})\right)\right)(t)+ \\ \varphi_1(x,y,t)+f_1(x,y,t), \tag{17}$$

where the functions $\varphi_1(x,y,t)$ and $f_1(x,y,t)$ are defined by formulas (7) and (8), respectively.

The following theorem is proved.

Theorem 1. Function (17) is a solution to Problem (1) – (2), where the functions $\varphi_1(x,y,t)$ and $f_1(x,y,t)$ are defined by the initial function $\varphi(x,y)$ from condition (2) and the right side $f(x,y,t)$ from equation (1), respectively. $\mathcal{E}_{a,b}(z)$ is a Mittag-Leffler function and

$$\psi(t)={}_{RL}D^{\beta}_{0t}\left(\varphi_1(x,y,t)+f_1(x,y,t)\right)\Big|_{x-1=t,y-1=t,t>0}.$$

Keywords: loaded equation, fractional derivative, Wright function, Laplace transform.

AMS Subject Classification: 45D05, 35K20, 26A33.

References

[1] Pskhu A.V., The fundamental solution of a diffusion-wave equation of fractional order, *Izv. RAN. Ser. Mat.*, Vol.73, No.2, 2009, pp.141182. DOI: https://doi.org/10.4213/im2429

[2] Korn G.A., Korn T.M. Handbook of mathematics for scientists and engineers. M: Fizmatlit, 1973 (In Russian).

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

AN EXPLICIT POSITIVITY-PRESERVING NONSTANDARD FINITE DIFFERENCE SCHEME

MOHAMMAD MEHDIZADEH KHALSARAEI, ALI SHOKRI, REZA SHOKRI JAHANDIZI

Department of Mathematics, Faculty of Basic Science, University of Maragheh, Maragheh, Iran
e-mail: mehdizadeh@maragheh.ac.ir, Shokri@maragheh.ac.ir, Reza.shokri.j@gmail.com

Abstract. Solving problems in various fields of science and engineering, such as physical and biological systems, leads to solving parabolic equations of the advection-diffusion reaction (ADR) type. In these problems, usually the concentration of chemical compounds or the size of the population are our unknowns, which are positive in nature. In general, the use of common methods, such as the classical finite difference method, may produce numerical drawbacks such as spurious oscillations and negative values in the solutions due to truncation errors. By using the nonstandard finite difference (NSFD) method, a better finite difference model is constructed. The proposed NSFD scheme ensures that the solutions are positive and there are no spurious oscillations in the solutions.

Keywords: Advection-diffusion reaction equations, Nonstandard finite differences, Positivity preserving.

AMS Subject Classification: 65M06; 65M12.

1. Introduction

Analytical and numerical solution techniques are used to solve differential equations [3, 19, 20, 21, 22, 24, 25, 27, 29], especially ADR equation. The analytical solution of this equation may be obtained if the simple and idealized conditions are satisfied, but if the solution parameters are time-dependent, the use of numerical solution techniques to solve the ADR equation is necessary. It is desirable that the numerical solutions satisfy the same properties as the exact solution such as positivity, total variation dimensioning or monotonicity. The standard ADR model deals with the physical and biological phenomena such as reaction of chemical species, transport, heat transfer, adsorption of pollutants in soil and population density in biology [1, 4, 18, 23]. In these cases, the components of the unknown can denote concentrations of chemical species and population sizes which are positive quantities, so we need to develop numerical techniques that preserve the positivity of solution.

The standard finite difference method has this disadvantage that the above mentioned qualitative properties of the exact solution usually are not transferred to the numerical solution. In addition, many problems may affect the stability properties of the standard approach. One way of avoiding this defect is to use the NSFD strategy presented by Mickens [13].

The present work, inspired by many successful papers on the matter [2, 9, 10, 12, 14, 15, 16, 17], introduces a family of NSFD schemes to approximate consistent solutions of the standard ADR equations, which is an equation for which the existence of non-negative solutions is a well-known fact. The proposed methods satisfy property of positivity, and they are stable for appropriate choice of the model parameter.

The rest of the paper is organized as follows: In Section 2, we propose the new method and investigate the positivity and stability requirements. In Section 3, we apply the method to four problems and compared with the other schemes. Finally, we end the paper with some conclusions in Section 4.

2. Scheme construction

Consider the standard advection-diffusion reaction equation

$$\frac{\partial C(x,t)}{\partial t} + P\frac{\partial C(x,t)}{\partial x} - Q\frac{\partial^2 C(x,t)}{\partial x^2} = -RC(x,t), \quad (x,t) \in [0, x_{max}] \times [0,T], \tag{1}$$

where $C_j^n \simeq C(x_j, t_n)$, and the solution domain of the problem is covered by a mesh of grid-lines $x_j = j\Delta x$ and $t_n = n\Delta t$, such that $j = 0, 1, 2, \cdots, M$ and $n = 0, 1, 2, \cdots, N$, where x_j and t_n are parallel to the space and time coordinate axes. The constant spatial and temporal grid-spacing are $\Delta x = x_{max}/M$ and $\Delta t = T/N$.
In order to derive an accurate NSFD scheme for (1) which is positivity preserving, we replace the reaction term in (1) by

$$a(C_{j+1}^n + C_{j-1}^n) + (1-2a)C_j^{n+1}, \tag{2}$$

here a is arbitrary parameter to be determined below. Therefore, by using the central second order finite difference approximation schemes for the spatial operators (first and second order derivatives), the discretization of (1) can be considered as

$$\left(\frac{1}{\Delta t} + (1-2a)R\right)C^{n+1} = AC^n, \tag{3}$$

where C^n is the transpose of vector $[C_0^n, C_1^n, \cdots, C_M^n]$ and A is the following tridiagonal matrix

$$A = tridiag\left\{\frac{P}{2\Delta x} + \frac{Q}{\Delta x^2} - Ra; \frac{1}{\Delta t} - \frac{2Q}{\Delta x^2}; -\frac{P}{2\Delta x} + \frac{Q}{\Delta x^2} - Ra\right\}. \tag{4}$$

The parameter a is chosen according to the following theorem.

Theorem 1. Sufficient for scheme (3) to be positive is,

$$a \le \frac{\frac{Q}{\Delta x^2} - \frac{P}{2\Delta x}}{R} \quad , \quad \Delta t \le \frac{\Delta x^2}{2Q}. \tag{5}$$

Proof. From (3) it is enough to show that $A \ge 0$, then we have to put

$$\frac{P}{2\Delta x} + \frac{Q}{\Delta x^2} - Ra \ge 0, \tag{6}$$

$$-\frac{P}{2\Delta x} + \frac{Q}{\Delta x^2} - Ra \ge 0, \tag{7}$$

$$\frac{1}{\Delta t} - \frac{2Q}{\Delta x^2} \ge 0, \tag{8}$$

from (6) and (7), we have $a \le \frac{\frac{Q}{\Delta x^2} - \frac{P}{2\Delta x}}{R}$ and from (8), we can derive $\Delta t \le \frac{\Delta x^2}{2Q}$, with conditions (5), we have $\frac{1}{\Delta t} + (1-2a)R \ge 0$, then the solution is positive. □

Theorem 2. The new scheme is conditionally stable and convergent with local truncation error $O(\Delta t, \Delta x^2)$.

Proof. Under condition (5), we have $\frac{1}{\frac{1}{\Delta t}+(1-2a)R} \geq 0$. In the other hand we have $\|A\|_\infty = \frac{1}{\Delta t} - 2Ra$, and for spectral radius, ρ of the iteration matrix we derive

$$\rho\bigg(\bigg[\frac{1}{\frac{1}{\Delta t}+(1-2a)R}\bigg]A\bigg) \leq \bigg\|\bigg[\frac{1}{\frac{1}{\Delta t}+(1-2a)R}\bigg]A\bigg\|_\infty \leq \bigg|\frac{1}{\frac{1}{\Delta t}+(1-2a)R}\bigg|\bigg\|A\bigg\|_\infty$$

$$\leq \frac{\frac{1}{\Delta t}-2Ra}{\frac{1}{\Delta t}-2Ra+R} < 1, \tag{9}$$

therefore the scheme is stable and then via the Lax-theorem [26] convergent with local truncation error

$$\begin{aligned} T_j^n =& \frac{C(x_j,t_{n+1})-C(x_j,t_n)}{\Delta t} + P\frac{C(x_{j+1},t_n)-C(x_{j-1},t_n)}{2\Delta x} \\ & - Q\frac{C(x_{j-1},t_n)-2C(x_j,t_n)+C(x_{j+1},t_n)}{\Delta x^2} \\ & + R\bigg(a\big(C(x_{j+1},t_n)+C(x_{j-1},t_n)\big)+(1-2a)C(x_j,t_{n+1})\bigg), \end{aligned} \tag{10}$$

by Taylor's expansion

$$C(x_j,t_{n+1}) = C(x_j,t_n) + \Delta t\bigg(\frac{\partial C}{\partial t}\bigg)(x_j,t_n) + \frac{1}{2}\Delta t^2\bigg(\frac{\partial^2 C}{\partial t^2}\bigg)(x_j,t_n) + \frac{1}{6}\Delta t^3\bigg(\frac{\partial^3 C}{\partial t^3}\bigg)(x_j,t_n) + \cdots,$$

$$C(x_{j+1},t_n) = C(x_j,t_n) + \Delta x\bigg(\frac{\partial C}{\partial x}\bigg)(x_j,t_n) + \frac{1}{2}\Delta x^2\bigg(\frac{\partial^2 C}{\partial x^2}\bigg)(x_j,t_n) + \frac{1}{6}\Delta x^3\bigg(\frac{\partial^3 C}{\partial x^3}\bigg)(x_j,t_n) + \cdots,$$

$$C(x_{j+1},t_n) = C(x_j,t_n) - \Delta x\bigg(\frac{\partial C}{\partial x}\bigg)(x_j,t_n) + \frac{1}{2}\Delta x^2\bigg(\frac{\partial^2 C}{\partial x^2}\bigg)(x_j,t_n) - \frac{1}{6}\Delta x^3\bigg(\frac{\partial^3 C}{\partial x^3}\bigg)(x_j,t_n) + \cdots,$$

by substitution into (10) we have

$$\begin{aligned} T_j^n =& \bigg(\frac{\partial C}{\partial t} + Pe\frac{\partial C}{\partial x} - \frac{\partial^2 C}{\partial x^2} + RC\bigg)(x_j,t_n) \\ & + R(1-2a)\Delta t\bigg(\frac{\partial C}{\partial t}\bigg)(x_j,t_n) + \frac{1}{2}\Delta t\bigg(\frac{\partial^2 C}{\partial t^2}\bigg)(x_j,t_n) + Ra\Delta x^2\bigg(\frac{\partial^2 C}{\partial x^2}\bigg)(x_j,tn) + \cdots, \end{aligned} \tag{11}$$

hence the scheme is consistent with (1) and $T_j^n = O(\Delta t + \Delta x^2)$.□

3. Numerical simulations

To test the proposed method with respect to positivity and stability, developed in previous section, we have integrated several problems of different applications nature. We validate the method by comparing it to exact solutions and also with solutions obtained from the other methods.

3.1. **Example 1: Exponential traveling wave.** The first example consists of equation (1) for $P = 1$, $Q = 1$ and $R = 1$

$$\frac{\partial C(x,t)}{\partial t} + \frac{\partial C(x,t)}{\partial x} - \frac{\partial^2 C(x,t)}{\partial x^2} = -C(x,t), \quad (x,t) \in [0, x_{max}] \times [0, T], \tag{12}$$

with initial condition

$$C(x,0) = exp(-x), \quad x \in [0, x_{max}], \tag{13}$$

and boundary conditions

$$\begin{aligned} C(0,t) &= exp(t), \quad t \in [0,T], \\ \frac{\partial C(x_{max},t)}{\partial x} &= -C(x_{max},t), \quad t \in [0,T]. \end{aligned} \tag{14}$$

The exact solution is given by

$$C(x,t) = exp(t-x). \tag{15}$$

In order to show the advantages of the proposed new method, we numerically solve (12) for $x_{max} = 10$ and $T = 0.85$ using $\Delta x = 0.1$ and $\Delta t = 0.005$. In addition to comparing the solution of the new scheme with the exact solution, we also compare it to the numerical solution produced by a standard upwind forward Euler finite difference method (EE):

$$\frac{C_j^{n+1} - C_j^n}{\Delta t} + \frac{C_j^n - C_{j-1}^n}{\Delta x} - \frac{C_{j-1}^n - 2C_j^n + C_{j+1}^n}{\Delta x^2} = -C_j^n, \tag{16}$$

and the nonstandard finite-difference (NSFD) method, proposed by Mickens in [14]:

$$\frac{C_j^{n+1} - C_j^n}{\Delta t} + \frac{C_j^n - C_{j-1}^n}{\Delta x} - \frac{C_{j-1}^n - 2C_j^n + C_{j+1}^n}{\Delta x^2} = -C_j^{n+1}, \tag{17}$$

using the same values for the parameters. As can be seen from Figure 1, the proposed method is stable and produces a solution that is very close to the exact solution, but both EE and NSFD methods are unstable for this choice of a time step $\Delta t = 0.005$ and larger.The parameters used in this simulation are taken from [2].

3.2. **Example 2: Colonization of Europe by oaks.** In the second example, we deal with the model for the recolonization by oaks of Europe after the last glaciation. The model assumes Malthusian growth and a standard advection-diffusion reaction equation for the local density $C(x,t)$ of oaks at time t

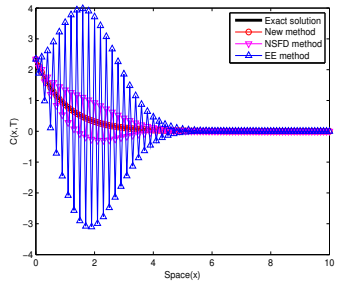
$$\frac{\partial C(x,t)}{\partial t} + u\frac{\partial C(x,t)}{\partial x} - D\frac{\partial^2 C(x,t)}{\partial x^2} = rC(x,t), \quad (x,t) \in [0, x_{max}] \times [0, T], \tag{18}$$

where r is the reproduction rate, u is an advection parameter taking into account the displacement of acorns by squirrels, and D is the diffusivity. If the population size at time 0 is M and is concentrated at the origin, the exact solution of this equation is

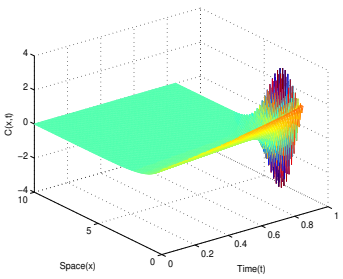
$$C(x,t) = \frac{M}{2\sqrt{\pi D t}} exp\left(rt - \frac{(x-ut)^2}{4Dt}\right), \tag{19}$$

for more details see [2, 6]. In Figure 2 numerical solutions for (18) are shown with the parameters $u = 1$, $D = 1$, $r = 0.1$, $x_{max} = 10$, $T = 2$, $\Delta x = 0.1$ and $\Delta t = 0.005$, which are taken from [2]. Comparing the proposed new method with the upwind EE method

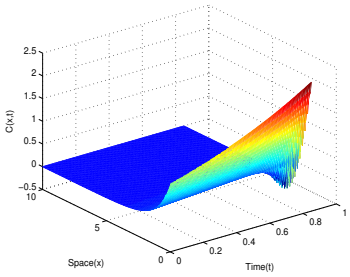
$$\frac{C_j^{n+1} - C_j^n}{\Delta t} + u\frac{C_j^n - C_{j-1}^n}{\Delta x} - D\frac{C_{j-1}^n - 2C_j^n + C_{j+1}^n}{\Delta x^2} = rC_j^n, \tag{20}$$



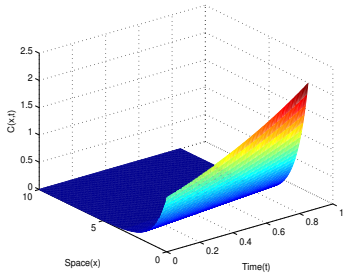
(a) Solutions at T=0.85



(b) Solution of the EE method



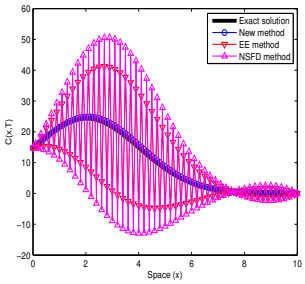
(c) Solution of the NSFD method



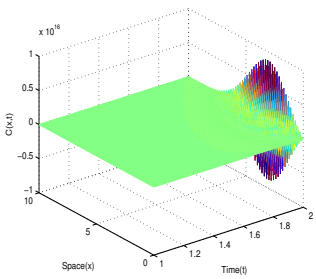
(d) Solution of the new method

FIGURE 1. Solutions for the exponential traveling wave model.

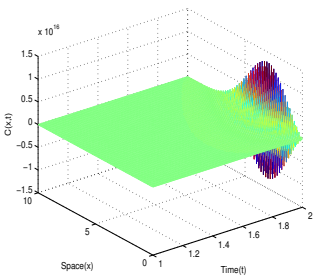
we observe that the new method performs very well. Furthermore, NSFD method for (18), proposed by Mickens in [14], is the same as the EE method.



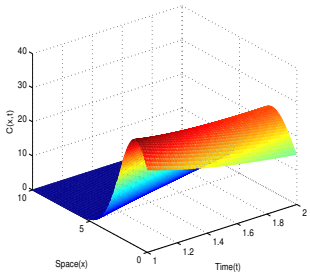
(a) Solutions at T=2



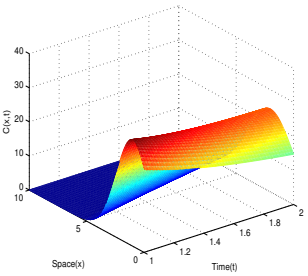
(b) Solution of the EE method



(c) Solution of the NSFD method



(d) Solution of the new method



(e) Exact solution

FIGURE 2. Solutions for the oak propagation model.

3.3. **Example 3: Catalytic particle.** As our third example we consider (1) with $P = Pe$, $Q = 1$ and $R = \phi^2$

$$\frac{\partial C(x,t)}{\partial t} + Pe\frac{\partial C(x,t)}{\partial x} - \frac{\partial^2 C(x,t)}{\partial x^2} = -\phi^2 C(x,t), \quad (x,t) \in [0, x_{max}] \times [0,T], \tag{21}$$

with initial and boundary conditions

$$C(x,0) = 0, \quad C(0,t) = 1, \quad C(1,t) = 1. \tag{22}$$

The unknown $C(x,t)$ corresponds to the normalized concentration and endowed, Pe is the *Peclet* number, which denotes the relationship between the advective and diffusive transport and ϕ is Thiele modulus, which relates chemical reaction rate and the diffusive transport; the dimensionless parameters $x \in [0,1]$ and $t > 0$ denote the spatial coordinate and time, respectively.

In traditional FD schemes, the spatial operators of (21) can be discretized in different ways. By *method of lines* (MOL) approach, we replace the spatial derivatives C_x and C_{xx} by a finite difference approximation to arrive at a semi-discrete system where $C_i(t) \simeq C(x_i, t)$. According to the MOL approach, *fully discrete* approximation $C_i^n \simeq C(x_i, t_n)$ are now obtained by applying some suitable ordinary differential equations (ODEs) solver. For instance, for an equidistant grid $X_{N+1} = \{x_a, x_1, \cdots, x_N, x_b\}$ where $x_a = 0$ and $x_b = 1$, with $x_i - x_{i-1} = \Delta x$ and For the advective operator, it is also possible to use backward or forward approximations for obtaining the following schemes

- Forward finite difference (FFD) scheme

$$\frac{dC_i(t)}{dt} = \frac{C_{i-1}(t) - (2 - Pe\Delta x)C_i(t) + (1 - Pe\Delta x)C_{i+1}(t)}{\Delta x^2} - \phi^2 C_i(t), \tag{23}$$

- Backward finite difference (BFD) scheme

$$\frac{dC_i(t)}{dt} = \frac{(1 + Pe\Delta x)C_{i-1}(t) - (2 + Pe\Delta x)C_i(t) + C_{i+1}(t)}{\Delta x^2} - \phi^2 C_i(t). \tag{24}$$

To obtain a reference solution of (21) the Laplace transform was applied and for the analytical solution we found

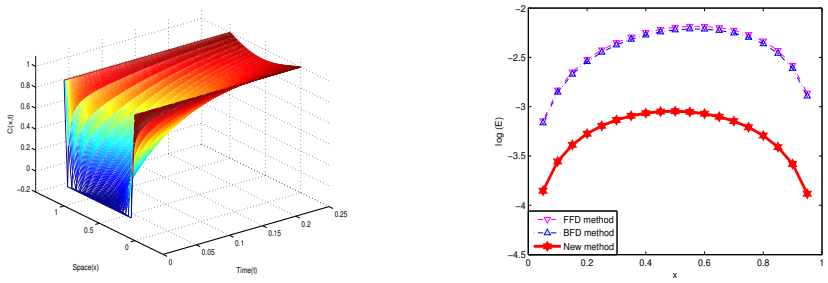
$$\hat{C}(x,s) = LC(x,t) = \frac{exp(m_2 x)[exp(m_1) - 1] + exp(m_1 x)[1 - exp(m_2)]}{exp(m_1) - exp(m_2)} \tag{25}$$

with

$$m_1 = \frac{Pe - \sqrt{Pe^2 + 4(s + \phi^2)}}{2} \quad , \quad m_2 = \frac{Pe + \sqrt{Pe^2 + 4(s + \phi^2)}}{2}, \tag{26}$$

where $\hat{C}(x,s)$ is the Laplace transform of $C(x,t)$. Unfortunately, the inverse Laplace transform for $\hat{C}(x,s)$ is not available. In order to determinate the solution in the time-domain, we have used the numerical inversion by Zakians algorithm [28, 30].
We apply new scheme to (21) with $Pe = 1$ and $\phi = 0.1$. Figure 3(a) shows the concentration profile for new scheme. A comparison with the FFD and BFD schemes one can find that new scheme performs well, see Figure 3(b).

3.4. **Example 4: Adsorption of contaminants.** Contaminants on the surface may seep into the ground with water and contaminate groundwater. These pollutants may come from industrial and mineral wastes, household wastes, pesticides and agricultural herbicides, chemical fertilizers, and so on. All or part of these contaminants may be adsorbed by the soil before reaching the aquifers. We assume that a non-volatile trace element is transported by the water phase. Let C be the concentration of the element in the water phase and let q be the concentration in the solid matrix. Following [1], the two balance equations involving C and q, for a



(a) (b)

FIGURE 3. Solution of the new method (a) and logarithm of absolute errors (b) with $Pe = 1$ and $\phi = 0.1$.

nonstationary Langmuir isotherm, are

$$\begin{cases} \frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} - D\frac{\partial^2 C}{\partial x^2} = \frac{\rho}{\theta}\frac{\partial q}{\partial t} \\ \\ \frac{\partial q}{\partial t} = k_f C(b-q) - k_b q, \end{cases} \tag{27}$$

where k_f is the sorption rate constant, b is the maximum solid phase concentration, and k_b is the desorption rate constant, $u = u_\omega/\theta_\omega$ is the pore water velocity, $\rho = \rho_s\theta_s$ is the bulk density of the porous medium, $\theta = \theta_\omega$, θ_ω is the volumetric water content, u_ω is the water-phase velocity, D is the dispersion coefficient, θ_s is the volumetric content of the solid matrix, and ρ_s is the density of the solid matrix. An exact solution for the system (27) given by [8] is the following

$$C(x,t) = \frac{-\frac{a_1}{a_2}}{1 + exp[-a_1(x - \alpha u t)]}, \tag{28}$$

where

$$a_1 = -0.8428 \quad , \quad a_2 = 0.1876 \quad , \quad \alpha = 0.1329.$$

The values of the parameters, $u = 40.1$, $D = 36.09$, $\rho = 1.64$, $\theta = 0.41$, $k_f = 2.0$, $k_b = 0.643$ and $b = 7.85$, used in our simulation are also taken from [7].

The proposed nonstandard explicit method for system (27) is

$$\begin{cases} \frac{C_j^{n+1}-C_j^n}{\Delta t} + u\frac{C_j^n - C_{j-1}^n}{\Delta x} - D\frac{C_{j-1}^n - 2C_j^n + C_{j+1}^n}{\Delta x^2} \\ \qquad + \frac{\rho}{\theta}\big(k_f b\big(a(C_{j+1}^n + C_{j-1}^n) + (1-2a)C_j^{n+1}\big) - k_f C_j^n q_j^n - k_b q_j^n\big) = 0 \\ \\ \frac{q_j^{n+1}-q_j^n}{\Delta t} = k_f b C_j^n - k_f C_j^n q_j^{n+1} - k_b q_j^n, \end{cases} \tag{29}$$

the system (29) can be written as

$$\begin{cases} \big(\frac{1}{\Delta t} + \frac{\rho}{\theta}bk_f(1-2a)\big)C_j^{n+1} = \big(\frac{u}{\Delta x} + \frac{D}{\Delta x^2} - \frac{\rho}{\theta}bk_f a\big)C_{j-1}^n + \big(\frac{1}{\Delta t} - \frac{u}{\Delta x} - \frac{2D}{\Delta x^2}\big)C_j^n \\ \qquad\qquad + \big(\frac{D}{\Delta x^2} - \frac{\rho}{\theta}bk_f a\big)C_{j+1}^n + \frac{\rho}{\theta}k_f C_j^n q_j^n + \frac{\rho}{\theta}k_b q_j^n \\ \\ \big(\frac{1}{\Delta t} + k_f C_j^n\big)q_j^{n+1} = \big(\frac{1}{\Delta t} + k_b\big)q_j^n + bk_f C_j^n. \end{cases} \tag{30}$$

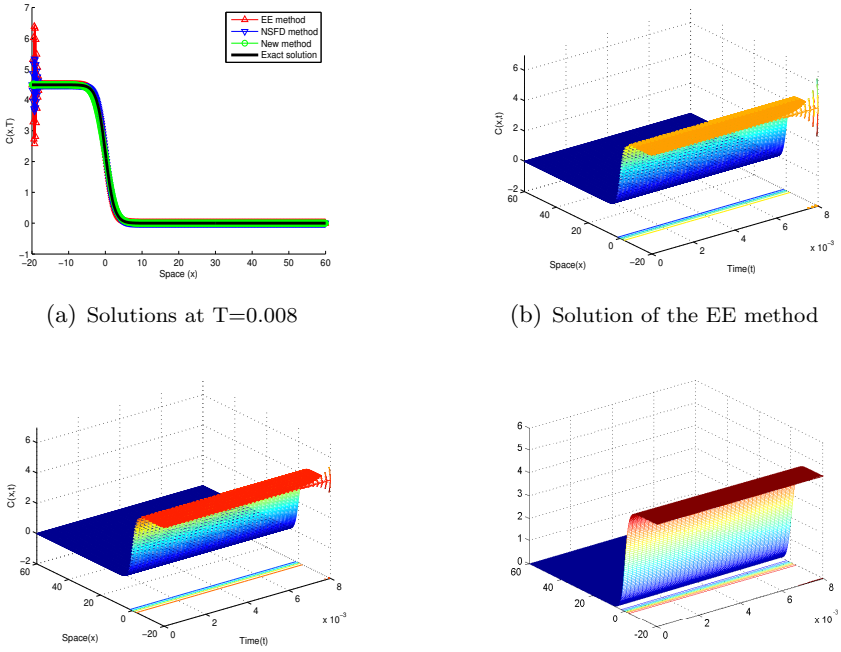
The upwind explicit Euler (EE) scheme is

$$\begin{cases} \frac{C_j^{n+1}-C_j^n}{\Delta t} + u\frac{C_j^n - C_{j-1}^n}{\Delta x} - D\frac{C_{j-1}^n - 2C_j^n + C_{j+1}^n}{\Delta x^2} + \frac{\rho}{\theta}\big(k_f b C_j^n - k_f C_j^n q_j^n - k_b q_j^n\big) = 0 \\ \\ \frac{q_j^{n+1}-q_j^n}{\Delta t} = k_f b C_j^n - k_f C_j^n q_j^{n+1} - k_b q_j^n, \end{cases} \tag{31}$$

and the NSFD scheme, proposed by Mickens in [7], is

$$\begin{cases} \frac{C_j^{n+1}-C_j^n}{\Delta t} + u\frac{C_j^n-C_{j-1}^n}{\Delta x} - D\frac{C_{j-1}^n-2C_j^n+C_{j+1}^n}{\Delta x^2} + \frac{\rho}{\theta}\left(k_f b C_j^{n+1} - k_f C_j^n q_j^n - k_b q_j^n\right) = 0 \\ \frac{q_j^{n+1}-q_j^n}{\Delta t} = k_f b C_j^{n+1} - k_f C_j^{n+1} q_j^{n+1} - k_b q_j^{n+1}. \end{cases} \tag{32}$$

The values used for these calculations are $\Delta x = 0.1$, $\Delta t = 0.0002$, the model was solved in $(x,t) \in [-20,60] \times [0,T]$ and $T = 0.008$. In Figure 4 numerical approximations of the (27) have been shown with the proposed scheme. The new scheme is stable and positive at time T, while the approximations obtained by the EE (31) and NSFD (32) methods are unstable.



(a) Solutions at T=0.008

(b) Solution of the EE method

(c) Solution of the NSFD method

(d) Solution of the new method

FIGURE 4. Solutions for the adsorption model.

4. Conclusions and discussion

Schemes preserving the positivity are great importance. Such schemes can be employed to prevent the occurrence of negative values where even very small negative values are unacceptable. Within the strategy suggested by Mickens, consisting of a combined use of different finite difference schemes in order to satisfy all the severe requirements of the problem, we have presented a family of NSFD scheme that is positivity preserving. We have presented the new method for an advection-diffusion reaction equation with constant velocity and diffusion and different reactions in one spatial dimension. Comparisons with a standard explicit upwind Euler (EE) method and with the Mickens' NSFD method, show that our NSFD method performs very well and it is stable under conditions for which the other methods are very unstable. We studied the suffusion conditions on positivity for the new method. A future work can be investigate the necessity of condition for positivity in Theorem 1.

References

[1] Bear J., *Hydraulics of Groundwater*, Dover, Minneola, 2007.

[2] Chen-Charpentier B. M., Kojouharov H. V., An unconditionally positivity preserving scheme for advection-diffusion reaction equations. *Mathematical and computer modelling*, 57(9), 2013, pp.2177-2185.

[3] F.A. Aliev, N.A. Aliyev, N.S. Hajiyeva, N.I. Mahmudov, Some Mathematical Problems and Their Solutions for the Oscillating Systems with Liquid Dampers: a Review, *Appl. Comput. Math.*, 20(3), 2021, pp. 339-365.

[4] Hetrick D. K., *Dynamics of Nuclear Reactors*, University of Chicago, Chicago, 1971.

[5] Hundsdorfer W., Verwer J. G., *Numerical Solution of Time-Dependent Advection-Diffusion-Reaction Equation*, Springer, 2003.

[6] Istas J., *Mathematical Modeling for the Life Sciences*, Springer-Verlag, Berlin, 2005.

[7] Liu L., Clemence D. P., Mickens R. E., A nonstandard finite difference scheme for contaminant transport with kinetic Langmuir sorption, *Numer. Methods Partial Differential Equations*, 2010, doi: 10. 1002 / num. 20551.

[8] Manoranjan V. S., Stauffer T. B., Exact solution for contaminant transport with kinetic Langmuir sorption, *Water Resour. Res.*, 32, 1996, pp.749-752.

[9] Mehdizadeh Khalsaraei M., Khodadoosti F., A new total variation diminishing implicit nonstandard finite difference scheme for conservation laws, *Computational Methods for Differential Equations*, 2, 2014, pp.85-92.

[10] Mehdizadeh Khalsaraei M., Khodadoosti F., Nonstandard finite difference schemes for differential equations, *Sahand Commun. Math. Anal.*, 1(2), 2014, pp.47-54.

[11] M. Mehdizadeh Khalsaraei, Positivity of an explicit Runge-Kutta method, *Ain Shams Eng. J.*, 6(4), 2015, pp.1217-1223.

[12] Mickens R. E., Exact solution to a finite difference model of a nonlinear reaction-advection equation: implications for numerical analysis, *Numer. Meth. Part. D. E.*, 5, 1989, pp.313-325.

[13] Mickens R. E., *Nonstandard Finite Difference Models of Differential Equations*, World Scientific, Singapore 1994.

[14] Mickens R. E., Nonstandard finite difference schemes for reaction-diffusion equations having linear advection, *Numer. Methods Partial Differential Equations*, 16(4), 2000, pp.361-364.

[15] Mickens R. E., A nonstandard finite difference scheme for a Fisher PDE having nonlinear diffusion, *Computer and Mathematics with Applications*, 45, 2003, pp.429-436.

[16] Mickens R. E., Jordan P. M., A positivity-preserving nonstandard finite difference scheme for the Damped Wave Equation, *Numer Methods Partial Differential Eq.*, 20, 2004, pp.639-649.

[17] Mickens R. E., Jordan P. M., A new positivity-preserving nonstandard finite difference scheme for the DWE, *Numer Methods Partial Differential Eq.*, 21, 2005, pp.976-985.

[18] Murray J. D., *Mathematical Biology I*, Springer-Verlag, Berlin, 2002.

[19] Musaev, H.K., The Cauchy problem for degenerate parabolic convolution equation, *TWMS J. Pure Appl. Math.*, 12(2), 2021, pp.278-288.

[20] Noor, M.A., Noor, K.I., Some new classes of strongly generalized preinvex functions, *TWMS J. Pure Appl. Math.*, 12(2), 2021, pp.181-192.

[21] Pankov, P.S., Zheentaeva, Z.K. and Shirinov, T., Asymptotic reduction of solution space dimension for dynamical systems, *TWMS J. Pure Appl. Math.*, 12(2), 2021, pp.243-253.

[22] R.S. Adıgüzel, Ü. Aksoy, E. Karapınar, İ.M. Erhan, On the Solutions of Fractional Differential Equations via Geraghty Type Hybrid Contractions, *Appl. Comput. Math.*, 20(3), 2021, pp.313-333.

[23] Shih T. M., *Numerical Heat Transfer*, Springer-Verlag, Berlin, 1984.

[24] Shokri, A., Saadat, H, P-stability, TF and VSDPL technique in Obrechkoff methods for the numerical solution of the Schrödinger equation, *Bull. Iran. Math. Soc.*, 42(3), 2016, pp.687–706.

[25] Shokri, A., The Symmetric P-Stable Hybrid Obrenchkoff Methods for the Numerical Solution of Second Order IVPs, *TWMS J. Pure Appl. Math.*, 5, 2012, pp.28-35.

[26] Smith G. D., *Numerical Solution of Partial Differential Equations: Finite Difference Methods*, Oxford University Press, Oxford, 1985.

[27] Sunday, J., Shokri, A. and Marian, D., Variable step hybrid block method for the approximation of Kepler problem, *Fractal and Fractional*, 6(6), 2022, pp.343.

[28] Taiwo O., Schultz J., Krebs V., A comparison of two methods for the numerical inversion of Laplace transform, *Comput. Chem. Eng.*, 19, 1995, pp.303-305.

[29] X.-Y. Gao, Y.-J. Guo, W.-R. Shan, Similarity Reductions for a (3+1)-Dimensional Generalized Kadomtsev-Petviashvili Equation in Nonlinear Optics, Fluid Mechanics and Plasma Physics, *Appl. Comput. Math.*, 20(3), 2021, pp.421-429.

[30] Zakian V., Properties of IMN and JMN approximates and applications to numerical inversion of Laplace transforms and initial value problems, *J. Math. Anal. Appl.*, 50, 1975, pp.191-222.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON A SPECTRAL PROBLEM FOR A BIHARMONIC OPERATOR IN A RECTANGULAR DOMAIN

MUVASHARKHAN JENALIYEV[1,A], MADI YERGALIYEV[1,B]

[1]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[A]e-mail: muvasharkhan@gmail.com, [B]e-mail: ergaliev.madi.g@gmail.com

1. Introduction

The theory of the Navier-Stokes equations attracts unrelenting interest and attention not only from theoreticians but also from applied scientists. As is known, these equations describe the motion of a fluid, taking into account its pressure, as well as many accompanying processes, such as thermal conductivity, electromagnetic phenomena, etc. To date, many qualitative issues of solutions to initial boundary value problems have been studied, but a number of open problems remain unresolved [1, 41, 3], including inverse problems. For the past three or four decades, inverse problems for the system of Navier-Stokes equations have been actively studied. These are related to the so-called problems of identification of mathematical models described by the Navier-Stokes equations: the determination of unknown data in initial boundary value problems. This includes coefficients in equations and boundary conditions, i.e. functions that determine the properties and nature of the interaction of individual parts of the medium both inside the domain and on its boundary; source functions that set external programmatic effects on the process both in equations and in boundary and initial conditions, etc. Analogs of the above problems formulated as optimal control problems are given, for example, in ([5], Chapter 4, 1.1), such as the problem of accelerating a stationary liquid to a given velocity vector field while acting on the liquid with the help of volumetric forces, and the problem of influencing the velocity field from the boundary of a domain filled with liquid ([5], Chapter 5, 1.1). Similar problems of optimal control are also considered in [6].

To determine unknown data, additional conditions are specified, which are measured or observed quantities and are referred to as overdetermination conditions.

The literature on inverse problems for the system of Navier-Stokes equations is quite extensive and numerous. Among them, we would like to mention works on inverse problems for stationary [7, 8, 9, 10] and non-stationary Navier-Stokes equations [11, 12, 13, 14, 15, 16]. The closest to this article can be called the works [12, 14, 19, 10, 17, 18] on defining the right-hand parts of the equation. In the paper [20], one can find the results of numerical experiments that confirm the theoretical results obtained. One should also point to the works [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32] devoted to inverse problems for parabolic and elliptic equations, which also, in a certain sense, ideologically adjoin the above studies.

We also note the papers [33, 34, 35, 36] devoted to the general theory of ill-posed problems and various questions about its connection with inverse problems.

This research is funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grants No. AP09258892, 2021–2023 .

Direct boundary value problems for linearized Navier-Stokes equations have been the subject of many studies (see, [1, 41, 3]), in which theorems on their unique solvability in the corresponding Sobolev spaces have been established.

In this paper, we consider the inverse problem for a linearized system of Navier-Stokes equations in a cylindrical domain with the condition of final overdetermination. The main difference between this work and works [43, 37] is the active use of the theory of spectral decomposition of self-adjoint operators [38, 39, 40] and the special operator equation introduced by us, the solvability of which allows us to propose new sufficient conditions for the solvability of the inverse problem.

The work consists of this Introduction, three sections, and a Conclusion. In Section 2 the formulation of the inverse problem is given, and the solvability theorems are given in Section 3. In Section 4 we also compare the results from work [43] ("Theorem 3") and ours. In Section 5, we study the spectral problem. A brief conclusion completes the work.

In the notation of spaces, we adhere to works [2, 3, 4, 41].

2. Statement of the inverse problem

Let $\Omega \subset R^n$ be an open bounded domain with boundary $\partial\Omega \in C^2$. We introduce the notation of spaces $\mathbf{V}$, $\mathbf{H}$, $\mathbf{L}^2(\Omega)$, $\mathbf{H}_0^1(\Omega)$ and $\mathbf{H}^2(\Omega)$:

$$\mathbf{V} = \{v:\ v \in \mathbf{H}_0^1(\Omega) = \left(H_0^1(\Omega)\right)^n,\ \operatorname{div} v = 0\},\ \ \mathbf{H} = \left\{v:\ v \in \mathbf{L}^2(\Omega),\ \operatorname{div} v = 0\right\},$$

$$\mathbf{L}^2(\Omega) = \left(L^2(\Omega)\right)^n,\ \ \mathbf{H}^2(\Omega) = \left(H^2(\Omega)\right)^n.$$

The following dense embeddings take place

$$\mathbf{V} \subset \mathbf{H} \equiv \mathbf{H}' \subset \mathbf{V}',\ \ \mathbf{H}_0^1(\Omega) \subset \mathbf{L}^2(\Omega) \equiv \left(\mathbf{L}^2(\Omega)\right)' \subset \mathbf{H}^{-1}(\Omega),$$

where $\mathbf{H}^{-1}(\Omega) = (\mathbf{H}_0^1(\Omega))'$ and $(\cdot,\cdot)$, $((\cdot,\cdot))$ are scalar products in spaces $\mathbf{H}$, $\mathbf{L}^2(\Omega)$ and $\mathbf{V}$, $\mathbf{H}_0^1(\Omega)$, respectively. The orthogonal decomposition of the space $\mathbf{L}^2(\Omega)$: $\mathbf{L}^2(\Omega) = \mathbf{H} \oplus \mathbf{H}^{\perp}$,

$$\mathbf{H}^{\perp} \text{ is an orthogonal complement to } \mathbf{H} \text{ in the space } \mathbf{L}^2(\Omega),$$

$$\mathbf{H}^{\perp} = \{v:\ v \in \mathbf{L}^2(\Omega),\ v = \nabla u,\ u \in H^1(\Omega)\},$$

$$\left(\mathbf{H} \oplus \mathbf{H}^{\perp}\right)' \equiv \left(\mathbf{L}^2(\Omega)\right)' \equiv \mathbf{L}^2(\Omega) \equiv \mathbf{H} \oplus \mathbf{H}^{\perp}.$$

Further, let $Q = \{(y,t) : y \in \Omega,\ 0 < t < T\}$, $\Sigma = \{(y,t) : y \in \partial\Omega,\ 0 < t < T\}$. The following inverse problem of determining functions $\{w(y,t), P(y,t), f(y)\}$ is considered:

$$\partial_t w - \nu\Delta w = g(t)f(y) - \nabla P, \quad (y,t) \in Q, \tag{1}$$

$$\operatorname{div} w = 0, \quad (y,t) \in Q, \tag{2}$$

$$w(y,t) = 0, \quad (y,t) \in \Sigma, \tag{3}$$

$$w(y,0) = 0, \quad y \in Q, \tag{4}$$

with overdetermination condition:

$$w(y,T) = w_T(y), \tag{5}$$

where $w_T(y)$ is a given function.

So, in problem (1)–(5) $g(t)$ is a given scalar function (although, under certain conditions, one could take a vector- or even a matrix-function), $F(y,t) = \{g(t)f_1(y), ..., g(t)f_n(y)\}$, and the vector function $w(y,t) = \{w_1(y,t), ..., w_n(y,t)\}$, scalar function $P(y,t)$ and the functions $f_1(y), ..., f_n(y)$ are unknown.

3. Solvability of the inverse problem

Let $\mathcal{P}$ be an orthogonal projector from $\mathbf{L}^2(\Omega)$ to $\mathbf{H}$.

According to [38] we have: $A = -\mathcal{P}\Delta$ is a self-adjoint operator in $\mathbf{H}$, $D(A) \equiv \mathbf{V} \cap \mathbf{H}^2(\Omega)$,

$$G(t) = \mathcal{P}F(\cdot, t) = g(t)\tilde{f}, \text{ where } \tilde{f} = \mathcal{P}f(\cdot) \in \mathbf{H}.$$

Following [38, 41], we write inverse problem (1)–(5) in the differential-operator form:

$$w'(t) + \nu A w(t) = g(t)\tilde{f}, \ \ w(0) = 0, \ \ w(T) = w_T, \tag{6}$$

where the description of the operator A will be given below.

Further, integrating the equation (6) with respect to t from 0 to T, and taking into account the overdetermination condition, we obtain

$$\tilde{f} = \frac{1}{B_T g}\left[w_T + \nu A B_T w\right], \tag{7}$$

$$w'(t) + \nu A w(t) - \nu M(t) A B_T w = M(t) w_T, \ \ w(0) = 0, \tag{8}$$

where

$$B_T v = \int_0^T v(t)dt, \ \ M(t) = \frac{g(t)}{B_T g}. \tag{9}$$

Our goal is to prove the solvability of the Cauchy problem (8) under certain given conditions, and then use formula (7) to find the unknown element $\tilde{f}$.

3.1. **Theorem on the solvability of the inverse problem.** We study the solvability of the inverse problem (1)–(5). We will consider a loaded (according to terminology from [42]) equation (8) with an unbounded self-adjoint, positive definite operator A [38]:

$$A = \ -\mathcal{P}\Delta : \mathbf{H} \to \mathbf{H}, \ \ D(A) \equiv \mathbf{V} \cap \mathbf{H}^2(\Omega), \ \ (Aw, w) \geq \bar{\lambda}\|w\|^2_{\mathbf{H}} \text{ for } w \in D(A), \tag{10}$$

where $(\cdot, \cdot)$ is the scalar product in $\mathbf{H}$, $\bar{\lambda}$ is a positive number (the smallest eigenvalue of the operator A).

The equation (8) is called loaded due to the presence in it of the term $B_T w$ given in formula (9).

3.2. **The following statement is true.**

Theorem 3.1. *Let $w_T \in \mathbf{V} \cap \mathbf{H}^2(\Omega)$, $g(t) \in C^1([0,T])$, $|B_T g| \geq \varepsilon > 0$, and one of the following conditions is met*

a^0. $M_0 = M(0) = 0, \ \exists \varepsilon_1 > 0$, *such that*

$$\left\| \int_0^T M'(t) \frac{I - e^{-\nu(T-t)A}}{\nu A} dt \right\| \geq \left(\int_0^T M'(t) \frac{I - e^{-\nu(T-t)A}}{\nu A} dt\, w, w \right) \geq \varepsilon_1 \|w\|^2_{\mathbf{H}}, \ \ w \in \mathbf{H}$$

or

$$\text{vrai} \max_{\lambda \in P_\sigma\{A\}} \left| \int_0^T M'(t) \frac{1 - e^{-\nu(T-t)\lambda}}{\nu\lambda} dt \right| \geq \varepsilon_1 > 0. \tag{11}$$

b^0. $M_0 = M(0) \neq 0, \ \exists \varepsilon_2 > 0$, *such that*

$$\left\| M_0 \frac{I - e^{-\nu T A}}{\nu A} + \int_0^T M'(t) \frac{I - e^{-\nu(T-t)A}}{\nu A} dt \right\|$$

$$\geq \left((M_0 \frac{I - e^{-\nu T A}}{\nu A} + \int_0^T M'(t) \frac{I - e^{-\nu (T-t) A}}{\nu A} dt)\, w, w \right) \geq \varepsilon_2 \|w\|^2_{\mathbf{H}}, \;\; w \in \mathbf{H}$$

or

$$\operatorname{vrai} \max_{\lambda \in P_\sigma\{A\}} \left| M_0 \frac{1 - e^{-\nu T \lambda}}{\nu \lambda} + \int_0^T M'(t) \frac{1 - e^{-\nu (T-t) \lambda}}{\nu \lambda} dt \right| \geq \varepsilon_2 > 0, \tag{12}$$

where $P_\sigma\{A\} \subset [\bar{\lambda}, +\infty)$, $P_\sigma\{A\}$ *is the spectrum of the operator* A, I *is the identity operator.*

Then inverse problem (1)–(5) *is uniquely solvable,*

$$w \in \mathbf{W} \equiv \left\{ v : \, v(t) \in L^2(0,T; \mathbf{V} \cap \mathbf{H}^2(\Omega)), \;\; v'(t) \in L^2(0,T; \mathbf{H}) \right\}, \;\; \nabla P \in L^2(0,T; \mathbf{H}^\perp), \;\; \tilde{f} \in \mathbf{H},$$

and the following estimates ($C_j = C_j(\nu, T, \bar{\lambda}, \varepsilon, \varepsilon_1, \varepsilon_2)$, $j = 1, 2$)*:*

$$\|w\|_{\mathbf{W}} + \|\nabla P\|_{L^2(0,T;\mathbf{H}^\perp)} \leq C_1 \|w_T\|_{\mathbf{V} \cap \mathbf{H}^2(\Omega)}, \tag{13}$$

$$\|\tilde{f}\|_{\mathbf{H}} \leq C_2 \|w_T\|_{\mathbf{V} \cap \mathbf{H}^2(\Omega)} \tag{14}$$

hold, where

$$\|w\|^2_{\mathbf{W}} = \int_0^T \left[\|w(t)\|^2_{\mathbf{V} \cap \mathbf{H}^2(\Omega)} + \|w'(t)\|^2_{\mathbf{H}} \right] dt.$$

4. Comparison of the statement of Theorem 3.1 with the known result

Let us briefly compare the statement of Theorem 3.1 with the result of work [43] in the case when the function $g(x,t)$ does not depend on x.

"Theorem 3" ([43], p. 111). *Let* $w_T \in \mathbf{V} \cap \mathbf{H}^2(\Omega)$, $g(t) \in C^1([0,T])$, $|g(T)| \geq g_T > 0$. *Let the domain* Ω, *the function* g, *the values* ν *and* T *be such that condition*

$$m < 1, \tag{a}$$

is satisfied, where

$$m = \frac{1}{|g(T)|} \left\{ |g(0)| e^{-\nu T / d^2} + \int_0^T |g'(t)| e^{-\nu (T-t)/d^2} \, d\, t \right\} \tag{b}$$

and d *is the diameter of the domain* Ω.

Then there is a unique generalized solution to inverse problem (1)–(5) *and the estimates*

$$\|f\|_{\mathbf{H}} \leq \tilde{m}, \tag{c}$$

$$\|w\|_{L^2(0,T;\mathbf{V} \cap \mathbf{H}^2(\Omega))} + \|\nabla P\|_{L^2(0,T;\mathbf{H}^\perp)} \leq c \|g\|_{L^2(0,T)} \tilde{m} \tag{d}$$

are valid, where

$$\tilde{m} = \frac{\nu}{(1-m)|g(T)|} \|\Delta w_T\|_{\mathbf{L}_2(\Omega)}. \tag{e}$$

Remark 4.1. *First, note that if the values of* d *and* $\bar{\lambda}$ *are given for the unit circle, then from* ([44], *Charter IX,* **211**) *we have* $d^2 = 4 \leq 4,201 \approx \bar{\lambda}^{-1}$, *i.e.* $d^{-2} = 0.25 \geq \bar{\lambda} \approx 0.2378$. *This shows that the values of* d^{-2} *and* $\bar{\lambda}$ *are quite similar.*

Remark 4.2. *In Theorem 4.2.2 from* [45]*, analogous to "Theorem 3", the value* d^2 *is replaced by the smallest constant* $c_1(\Omega)$ *from the Poincare-Friedrichs inequality. Note that the constant* $c_1(\Omega)$ *coincides with the smallest eigenvalue* $\bar{\lambda}$ *of the self-adjoint, positive-definite operator* A *defined in* (10)*.*

Next, a comparison of the conditions of Theorem 3.1 and "Theorem 3" (using the example of the function $g(t)$) shows:

1^0. Let the function $g(t)$ be chosen so that the following equality takes place

$$\left| \int_0^T M'(t) \frac{1 - e^{-\nu(T-t)\bar{\lambda}}}{\nu\bar{\lambda}} \, dt \right| = \varepsilon_1 = 0. \tag{15}$$

Then the requirements (a)–(b) of "Theorem 3" can be satisfied (for the chosen function $g(t)$), but not for all T. Indeed, let us set the function $g(t)$ in formula (a) in the form:

$$\begin{aligned} &g(t) = \int_0^t h(s)ds, \;\; h(s) = a + \cos\frac{2\pi s}{T}, \\ &g(T) = aT, \;\; g(0) = 0, \;\; B_T g = \frac{aT^2}{2}, \end{aligned} \tag{16}$$

where the unknown constant $a = a_1$ is determined from condition (15). After simple calculations, we get:

$$a_1 = \pm\frac{1}{T} \cdot \frac{\left(\nu T\bar{\lambda}/(2\pi)\right)^2}{1 + \left(\nu T\bar{\lambda}/(2\pi)\right)^2} \cdot \left[\frac{\nu T\bar{\lambda}}{1 - e^{-\nu T\bar{\lambda}}} - 1 \right]^{-1}. \tag{17}$$

When specifying the function $g(t)$ given in (16), where a_1 is defined according to (17), the condition (11) of Theorem 3.1 is either violated or not violated, depending on whether equality or strict inequality is fulfilled in the following relation, respectively:

$$\text{vrai} \max_{\lambda \in P_\sigma\{A\}} \left| \int_0^T M'(t) \frac{1 - e^{-\nu(T-t)\lambda}}{\nu\lambda} dt \right| \geq \left| \int_0^T M'(t) \frac{1 - e^{-\nu(T-t)\bar{\lambda}}}{\nu\bar{\lambda}} \, dt \right|.$$

Let us check the conditions (a)–(b) of "Theorem 3". By virtue of equality $1/c_1(\Omega) = \bar{\lambda}$ (see Remark 4.2) we have

$$\begin{aligned} &\frac{1}{|g(T)|} \int_0^T |g'(t)| e^{-\nu(T-t)/c_1(\Omega)} \, dt \geq \left| \frac{1}{g(T)} \int_0^T g'(t) e^{-\nu(T-t)\bar{\lambda}} \, dt \right| \\ &= \frac{1 - e^{-\nu T\bar{\lambda}}}{\nu T\bar{\lambda}} + T\left(1 - \frac{1 - e^{-\nu T\bar{\lambda}}}{\nu T\bar{\lambda}}\right) \\ &= T + (1 - T)\frac{1 - e^{-\nu T\bar{\lambda}}}{\nu T\bar{\lambda}} > 0, \;\; \forall\{\nu > 0, \, T > 0, \, \bar{\lambda} > 0\}. \end{aligned} \tag{18}$$

However, the right side of (18) becomes greater than or equal to one as soon as $T \geq 1$, i.e. conditions (a)–(b) of "Theorem 3" are violated. However, the inequality $0 < m < 1$ will only occur at $0 < T < 1 - \delta$ for some $0 \leq \delta < 1$.

2^0. Now let (a)–(b) of "Theorem 3" be violated. To do this, we choose in (16) the value of the number $a = a_2$ so that inequality

$$m = \frac{1}{|g(T)|} \int_0^T \left| a_2 + \cos\frac{2\pi t}{T} \right| e^{-\nu(T-t)/c_1(\Omega)} d\, t \geq \left| \frac{1}{g(T)} \int_0^T \left[a_2 + \cos\frac{2\pi t}{T} \right] e^{-\nu(T-t)/c_1(\Omega)} d\, t \right| = 1.1$$

is satisfied.

Then, replacing $1/c_1(\Omega)$ by $\bar{\lambda}$, we have:

$$\frac{1}{a_2 T}\int\limits_0^T \left[a_2+\cos\frac{2\pi t}{T}\right] e^{-\nu(T-t)\bar{\lambda}}dt = \frac{1-e^{-\nu T\bar{\lambda}}}{\nu T\bar{\lambda}} + \frac{1}{a_2}\cdot\frac{\left(\nu T\bar{\lambda}/(2\pi)\right)^2}{1+\left(\nu T\bar{\lambda}/(2\pi)\right)^2}\cdot\frac{1-e^{-\nu T\bar{\lambda}}}{\nu T\bar{\lambda}} = \pm 1.1.$$

Hence, for the value of the number a_2 we get:

$$a_2 = \frac{\left(\nu T\bar{\lambda}/(2\pi)\right)^2}{1+\left(\nu T\bar{\lambda}/(2\pi)\right)^2}\cdot\frac{1-e^{-\nu T\bar{\lambda}}}{\nu T\bar{\lambda}}\left[\pm 1.1 - \frac{1-e^{-\nu T\bar{\lambda}}}{\nu T\bar{\lambda}}\right]^{-1}. \tag{19}$$

Let us proceed to check condition (11) of our Theorem 3.1. Calculations show that the second inequality from relation

$$\operatorname{vrai}\max_{\lambda\in P_\sigma\{A\}}\left|\int\limits_0^T M'(t)\frac{1-e^{-\nu(T-t)\lambda}}{\nu\lambda}dt\right| \geq \left|\int\limits_0^T M'(t)\frac{1-e^{-\nu(T-t)\bar{\lambda}}}{\nu\bar{\lambda}}\,dt\right| \neq 0$$

is equivalent to inequality

$$a_2\cdot\left(1-\frac{1-e^{-\nu T\bar{\lambda}}}{\nu T\bar{\lambda}}\right) \neq \frac{\left(\nu T\bar{\lambda}/(2\pi)\right)^2}{1+\left(\nu T\bar{\lambda}/(2\pi)\right)^2}\cdot\frac{1-e^{-\nu T\bar{\lambda}}}{\nu T\bar{\lambda}},$$

or substituting the value a_2 from (19) into the last inequality, we get $1 \neq \pm 1.1$. Thus, conditions (11) of Theorem 3.1 are satisfied, and the inverse problem is uniquely solvable on any nonzero interval $(0, T)$, although conditions (a)–(b) of "Theorem 3" are violated.

3^0. Let $g(t) = \text{const}$. Then the conditions b^0 (12) of Theorem 3.1 and "Theorem 3" are fulfilled and are equivalent for any values of T.

So, we can see that the assertions of Theorem 3.1 are significant and are of independent interest. Therefore, Theorem 3.1 and "Theorem 3" complement each other.

5. Spectral Problem

In rectangular region $\Omega_a = \{x, y|\ -1 < x < 1,\ -a < y < a,\ 0 < a = \text{const} \leq 1\}$ with boundary $\partial\Omega_a$ we study the spectral problem:

$$\Delta^2 w = \lambda^2(-\Delta w),\ (x,y)\in\Omega_a, \tag{20}$$

$$w = 0,\ \partial_{\vec{n}} w = 0,\ (x,y)\in\partial\Omega_a, \tag{21}$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplace operator, $\vec{n}$ is the unit vector of the outward normal to $\partial\Omega_a$, excluding the vertices of the rectangle Ω_a.

First, we consider the case of a square domain Ω_1, i.e. $a = 1$. In this case, problem (20)–(21) was the subject of study in works [46, 47], as well as monographs [48, 49]. It is called the problem of studying bending vibrations of a clamped square plate, in contrast to the problem for a fixed plate, where the second condition in (21) is replaced by the following:

$$\Delta w = 0,\ (x,y)\in\partial\Omega_1, \tag{22}$$

and the latter is called the basic spectral problem. The spectrum of the basic problem gives a lower bound for the eigenvalues of the original problem, which, generally speaking, can be quite rough. To improve this estimate, a family of special intermediate problems is used, which are obtained by successive "strengthening" of conditions (22) and obtaining conditions approximating the second condition from (21). This approach is called the Aronshine-Weinstein method of intermediate problems [48, 49]. Note that in [48, 49] the importance of problem (20)–(21) for structural mechanics, shipbuilding mechanics, etc. is also noted.

We arrived at problem (20)–(21) (exactly, without any artificial transformations) by studying the numerical solution of direct and inverse problems for the linearized two-dimensional Navier-Stokes equation. In this case, we used the well-known (for the two-dimensional case) concept of the stream function.

Earlier, the spectral problem (20)–(21) was studied by us in [50] for the case when the region Ω is a circle with unit radius.

Note. In ([48], chapter VII, item 2) the following is stated: "If the clamped plate has the shape of a square or any other shape other than a circle, then" ... eigenvalues ... "cannot be expressed exactly in terms of elementary or tabulated special functions."

To formulate the basic problem, we will replace the second boundary conditions from (21) only partially. Namely, instead of conditions (21) we will have

$$\begin{cases} w(-1,y) = \partial_x w(-1,y) = w(1,y) = \partial_x w(1,y) = 0, \\ w(x,-1) = \partial_y^2 w(x,-1) = w(x,1) = \partial_y^2 w(x,1) = 0. \end{cases} \tag{23}$$

Secondly, the case $a \neq 1$, is considered separately, i.e. when the region Ω_a is a rectangle. Differences are noted in the lower estimates of the eigenvalues of problem (20)–(21) using intermediate spectral problems both in accordance with formulas (22) and – with formulas (23).

6. Conclusion

In the paper, we have studied the inverse problem for a linearized multidimensional system of Navier-Stokes equations in a cylindrical domain with a final overdetermination condition. Theorems on the solvability of the inverse problem under consideration have been proved.

Keywords: System of Navier-Stokes equations, Inverse problem, Final overdetermination condition, Biharmonic operator, Spectral problem, Rectangular domain.

AMS Subject Classification: 35Q30, 35R30, 65N21

References

[1] O. A. Ladyzhenskaya, *Mathematical Questions in the Dynamics of a Viscous Incompressible Fluid,* Nauka, Moscow, 1970. [in Russian]

[2] R. A. Adams, J. J. F. Fournier, *Sobolev Spaces,* 2nd ed., Elsevier, Amsterdam, 2003.

[3] J.-L. Lions, *Some Methods for Solving Nonlinear Boundary Problems,* Mir, Moscow, 1972. [in Russian]

[4] J.-L. Lions, E. Magenes, *Problemes aux Limites non Homogenes et Applications, vol. 1,* Dunod, Paris, 1968.

[5] A. V. Fursikov, *Optimal control of distributed systems. Theory and applications,* AMS, Rhode Island, 2000.

[6] R. Lattes, J.-L. Lions, *Methode de qusi-reversibility et applications,* Dunod, Paris, 1967.

[7] F. Caubet, Instability of an inverse problem for the stationary Navier-Stokes equations, *SIAM J. Control Optim.* **51** (2013), 2949–2975.

[8] A. Yu. Chebotarev, Inverse problems for stationary Navier-Stokes systems, *Comput. Math. Math. Phys.* **54** (2014), 537–545.

[9] W. Yan, Y. He, Y. Ma, Shape reconstruction of an inverse boundary value problem of two-dimensional Navier-Stokes equations, *Int. J. Numer. Meth. Fluids* **62** (2009), 632–646.

[10] A. Yu. Chebotarev, Determination of the right-hand side of the Navier-Stokes system of equations and inverse problems for the thermal convection equations, *Comput. Math. Math. Phys.* **51** (2011), 2146–2154.

[11] A. I. Prilepko, I. A. Vasin, On a non-linear non-stationary inverse problem of hydrodynamics, *Inverse Probl.* **7** (1991), L13.

[12] M. Choulli, O. Yu. Imanuvilov, J.-P. Puel, M. Yamamoto, Inverse source problem for linearized Navier-Stokes equations with data in arbitrary sub-domain, *Appl. Anal.* **92** (2013), 2127–2143.

[13] J. Fan, M. Di Cristo, Y. Jiang, G. Nakamura G, Inverse viscosity problem for the Navier-Stokes equation, *J. Math. Anal. Appl.* **365** (2010), 750–757.

[14] J. Fan, G. Nakamura, Local solvability of an inverse problem to the density-dependent Navier-Stokes equations, *Appl. Anal.* **87** (2008), 1255–1265.

[15] A. S. Kozelko, O. L. Krutyakova, V. V. Kurulin, D. Yu. Strelets, M. A. Shishlenin, The accuracy of numerical simulation of the acoustic wave propagations in a liquid medium based on Navier-Stokes equations, *Siberian Electron. Math. Rep.*, **18** (2021), 1238–1250.
[16] J. Satouri, Numerical resolution of optimal control problem for the in-stationary Navier-Stokes equations, *J. Inverse Ill-Posed Probl.* **27** (2019), 43–52.
[17] S. N. Antontsev, H. B. Oliveira, K. Khompysh, Kelvin-Voigt equations with anisotropic diffusion, relaxation and damping: Blow-up and large time behavior, *Asymptot. Anal.* **121** (2021), 125–157.
[18] S. N. Antontsev, H. B. Oliveira, K. Khompysh, The classical Kelvin-Voigt problem for incompressible fluids with unknown non-constant density: existence, uniqueness and regularity, *Nonlinearity.* **34** (2021), 3083–3111.
[19] J. Fan, G. Nakamura, Well-posedness of an inverse problem of Navier-Stokes equations with the final overdetermination, *J. Inv. Ill-Posed Problems.* **17** (2009), 565–584.
[20] G. Fourestey, M. Moubachir, Solving inverse problmes involving the Navier-Stokes equations discretized by a Lagrange-Galerkin method, *Comput. Methods Appl. Mech. Eng.* **194** (2005), 877–906.
[21] M. M. Amangaliyeva, M. T. Jenaliyev, M. I. Ramazanov, S. A. Iskakov, On a boundary value problem for the heat equation and a singular integral equation associated with it, *Appl. Math. Comput.* **399** (2021), 126009.
[22] A. Hasanov, M. Otelbev, B. Akpayev, Inverse heat conduction problems with boundary and final time measured output data, *Inverse Probl. Sci. Eng.* **19** (2011), 985–1006.
[23] M. Jenaliyev, M. Ramazanov, M. Yergaliyev, On the coefficient inverse problem of heat conduction in a degenerating domain, *Appl. Anal.* **99** (2020), 1026–1041.
[24] M. T. Jenaliyev, M. I. Ramazanov, M. G. Yergaliyev, On an inverse problem for a parabolic equation in a degenerating angular domain, *Eurasian Math. J.* **12** (2021), 25–38.
[25] M. Jenaliyev, M. Ramazanov, M. Yergaliyev, On the numerical solution of one inverse problem for a linearized two-dimensional system of Navier-Stokes equations, *Opusc. Math.* **42** (2022).
[26] A. I. Kozhanov, Inverse problems of finding the absorption parameter in the diffusion equation, *Math. Notes* **106** (2019), 378–389.
[27] A. I. Kozhanov , T. N. Shipina, Loaded differential equations and linear inverse problems for elliptic equations, *Complex Var. Elliptic Equ.* **66** (2021), 910–928.
[28] O. Krivorotko, S. Kabanikhin, S. Zhang, V. Kashtanova, Global and local optimization in identification of parabolic systems, *J. Inverse Ill-Posed Probl.* **28** (2020), 899–913.
[29] S. I. Kabanikhin, M. A. Bektemesov, D. V. Nechaev, Optimizational method for solving Ill-posed boundary value problems for elliptic and hyperbolic equations, in: *Inverse Problems in Engineering Mechanics IV*, Elsevier, Amsterdam (2003), 447–456.
[30] S. Kabanikhin, O. Krivorotko, M. Bektemesov, Z. Bektemesov, S. Zhang, Differential evolution algorithm of solving an inverse problem for the spatial Solow mathematical model, *J. Inverse Ill-Posed Probl.* **28** (2020), 761–774.
[31] D. N. Hao, L. T. T. Giang, S. Kabanikhin, M. Shishlenin, A finite difference method for the very weak solution to a Cauchy problem for an elliptic equation, *J. Inverse Ill-Posed Probl.* **26** (2018), 835–857.
[32] D. V. Lukyanenko, M. A. Shishlenin, V. T. Volkov, Asymptotic abalysis of solving an inverse boundary value problem for a nonlinear singularly perturbed time-periodic reaction-diffusion-advection equation, *J. Inverse Ill-Posed Probl.* **27** (2019), 745–758.
[33] V. K. Ivanov, V. V. Vasin, V. P. Tanana, *Theory of Linear Ill-Posed Problems and Its Applications*, Nauka, Moscow, 1978. [in Russian]
[34] M. M. Lavrentiev, V. G. Romanov, S. P. Shishatsky, *Ill-Posed Problems of Mathematical Physics and Analysis,* Nauka, Moscow, 1980. [in Russian]
[35] A. N. Tikhonov, V. Ya. Arsenin, *Methods of Solution of Ill-Posed Problems,* Nauka, Moscow, 1986. [in Russian]
[36] S. I. Kabanikhin, *Inverse and Ill-Posed Problems. Theory and Applications,* Inverse Ill-Posed Probl. Ser. 55, Walter de Gruyter, Berlin, 2012.
[37] A. I. Prilepko, I. A. Vasin, An inverse initial-boundary value problem for the nonlinear Navier-Stokes system in the case of final overdetermination, *Differ. Equ.* **25** (1989), 1547–1557. [in Russian]
[38] H. Fujita, T. Kato, On the Navier-Stokes Initial Value Problem. I, *Arch. Ration. Mech. Anal.* **16** (1964), 269–315.
[39] V. Hutson, J. S. Pym, M. J. Cloud, *Applications of Functional Analysis and Operator Theory*, 2nd ed., Elsevier, Amsterdam, 2005.
[40] L. V. Kantorovich, G. P. Akilov, *Functional Analysis*, 2nd ed., Elsevier, Amsterdam, 2014.
[41] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis,* Mir, Moscowm 1981. [in Russian]
[42] A. M. Nakhushev, *Loaded Equations and Their Applications,* Nauka, Moscow, 2012.
[43] A. I. Prilepko, I. A. Vasin, Some inverse initial-boundary value problems for non-stationary linearized Navier-Stokes equations, *Differ. Equ.* **25** (1989), 106–117. [in Russian]

[44] J. W. Strutt (Rayleigh Baron), *The Theory of Sound,* 2nd ed., Dover Publication, N.-Y., 1945.
[45] A. I. Prilepko, D. G. Orlovsky, I. A. Vasin, *Methods for Solving Inverse Problems in Mathematical Physics,* N.-Y.-Basel: Marcel Dekker Inc., 2000.
[46] Weinstein A., Étude des spectres des equations aux dérivées partielles de la théoriè des plaques élastiques, *Mém. des Sciences math., facs. Thèses de l´entre-deux-guerres, Gauthier-Villars, Paris*, Vol.88, 1937, pp.1-63.
[47] Aronszajn N., Rayleigh-Ritz and A. Weinstein methods for approximation of eigenvalues. I, II, *Proc. Nat. Acad. Sci. USA*, Vol.34, 1943, pp.474-480, 594-601.
[48] Gould S.H., *Variational Methods for Eigenvalue Problems. An Introduction to the Weinstein Method of Intermediate Problems*, London : Oxford University Press, 1966, XVI+275 p.
[49] Weinstein A., Stenger W., *Methods of Intermediate Problems for Eigenvalues. Theory and Ramifications*, New York : Academic Press, 1972, XII+236 p.
[50] Jenaliyev M.T., Ramazanov M.I., Yergaliyev M.G., On the numerical solution of one inverse problem for a linearized two-dimensional system of Navier-Stokes equations, *Opuscula Math.*, Vol.42, No.5, 2022, pp.711-727.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

THE CAUCHY PROBLEM FOR THE MODIFIED KORTEWEG-DE VRIES EQUATION WITH FINITE DENSITY IN THE CLASS OF PERIODIC FUNCTIONS

RAYKHONBEK ESHBEKOV[1], GAYRAT MANNONOV[2]

[1,2]Samarkand State University named after Sharof Rashidov, Samarkand, Uzbekistan
e-mail: [1]rayxonbek@mail.ru

Abstract. In this paper, the method of inverse spectral problems is used to integrate the nonlinear modified Korteweg-de Vries equation with finite density in the class of periodic functions. The solvability of the Cauchy problem for an infinite system of Dubrovin differential equations in the class of fivefold continuously differentiable periodic functions is proven. It is shown that the sum of a uniformly convergent functional series constructed by solving the system of Dubrovin equations and the first trace formula satisfies the mKdV equations with finite density and the stability of this solution.

Keywords: modified Korteweg-de Vries equation with finite density, Dirac operator, spectral data, system of Dubrovin differential equations, trace formulas.

AMS Subject Classification: 35J10,34A55.

1. Introduction

Inverse spectral problems play an important role in the integration of some important evolutionary equations of mathematical physics. An important breakthrough was made by Gardner, Green, Kruskal, and Miura [1] in 1967, where they discovered a deep connection between the well-known nonlinear Korteweg-de Vries (KdV) equation

$$q_t = 6qq_x - q_{xxx}, q(x,t)|_{t=0} = q_0(x), x \in \mathbb{R}, t > 0,$$

and the spectral theory of the Sturm-Liouville operator

$$L(t)y \equiv -y'' + q(x,t)y = \lambda y, x \in \mathbb{R}, t > 0.$$

The authors of [1] managed to find a global solution of the Cauchy problem for the KdV equation by reducing it to an inverse problem of scattering theory. The inverse scattering theory problem for the Sturm-Liouville operator on the whole line was studied in the works of Faddeev[2], Marchenko[3], Levitan[4] and others. In [5], Lax showed the universality of the inverse scattering problem method (ISPM) and generalized the equation KdV, introducing the concept of the higher KdV equation. In their original paper [6], Zakharov and Shabat showed that the following nonlinear Schrodinger equation (NSE)

$$iu_t \pm 2|u|^2u + u_{xx} = 0$$

can also be included in the ISPM formalism. Using the technique proposed by Lax, they found the NSE solution for given initial functions $u(x,0)$ that sufficiently rapidly decreased at $|x| \to \infty$.

This work was partially supported by prof. Aknazar Khasanov .

Soon Wadati [7], based on the ideas of Zakharov and Shabat, proposed a method for solving the Cauchy problem for the modified KdV (mKdV) equation

$$u_t \pm 6u^2u_x + u_{xxx} = 0.$$

It is well known that finding an explicit formula for solving the nonlinear evolution equation of KdV and the modified equation of KdV(mKdV) in the class of periodic functions essentially depends on the number of nontrivial gaps in the spectrum of the Sturm-Liouville and periodic Dirac operators. In this regard, the class of periodic functions is conveniently divided into two sets:

1. *The class of periodic finite-gap functions;*
2. *The class of periodic infinite-gap functions.*

Using the method of the inverse spectral problem for Sturm-Liouville and Dirac operators with periodic potentials, when the spectrum has only a finite number of nontrivial gaps, in the works of Its-Matveev[8], Dubrovin-Novikov[9], Its-Kotlyarov[10], Matveev-Smirnov [11], the complete integrability of the equation (KdV, NSE, mKdV) was established in the class of finite-gap periodic and quasiperiodic functions. In addition, for finite-gap solutions of equations (KdV, NSE, mKdV, etc.), an explicit formula was derived in terms of the Riemann theta functions.

Thus, in these papers [8]–[11], the solvability of the Cauchy problem for nonlinear equations (KdV, NSE, mKdV, etc.) was proven for any finite-gap initial data.

It was shown (see [12]) that if the initial conditions are in fourfold differentiable periodic infinite-gap zone functions for the KdV equation, then the Cauchy problem has a smooth solution at all times $t > 0$. This is proven by analysing the Dubrovin equations for a dvisor. In this paper, the solvability of problems (1) and (2) is established as periodic and infinite gap functions in the class $C^n(\mathbb{R}), n \geq 5$. From this, in particular $\rho = 0$, follows the result of the work [13].

2. Statement of the problem

Consider the Cauchy problem for the modified Korteweg-de Vries equation with finite density of the following form:

$$\begin{cases} q_t = q_{xxx} - 6(q^2 - \rho^2)q_x,\ 0 < \rho < \infty \\ q(x,t)|_{t=0} = q_0(x),\ q_0(x+\pi) = q_0(x) \in C^5(\mathbb{R}) \end{cases} \tag{1}$$

in the class of real infinite-gap π periodic with respect to x functions:

$$q(x+\pi,t) = q(x,t) \in C_x^3(t>0) \cap C_t^1(t>0) \cap C(t \geq 0),\ x \in \mathbb{R}, t > 0. \tag{2}$$

In this paper, we propose an algorithm for constructing the exact solution $q(x,t)$, $x \in \mathbb{R}, t > 0$ of problem (1)-(2) by reducing it to an inverse spectral problem for the following Dirac operator:

$$L(\tau,t)y \equiv B\frac{dy}{dx} + \Omega(x+\tau,t)y = \lambda y, x \in \mathbb{R}, \tau \in \mathbb{R}, t > 0, \tag{3}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \Omega(x+\tau,t) = \begin{pmatrix} 0 & q(x+\tau,t) \\ q(x+\tau,t) & 0 \end{pmatrix},\ y = \begin{pmatrix} y_1(x) \\ y_1(x) \end{pmatrix}.$$

Denote by $c(x,\lambda,\tau,t) = (c_1(x,\lambda,\tau,t), c_2(x,\lambda,\tau,t))^T$ and $s(x,\lambda,\tau,t) = (s_1(x,\lambda,\tau,t), s_2(x,\lambda,\tau,t))^T$ solutions of equation (3) with initial conditions $c(0,\lambda,\tau,t) = (1,0)^T$ and $s(0,\lambda,\tau,t) = (0,1)^T$. The function $\Delta(\lambda,\tau,t) = c_1(\pi,\lambda,\tau,t) + s_2(\pi,\lambda,\tau,t)$ is called the Lyapunov function for equation (3).

The spectrum of the Dirac operator $L(\tau,t)$ is purely continuous and consists of the following set:

$$\sigma(L) = \mathbb{R} \setminus \left(\bigcup_{n=-\infty}^{+\infty} (\lambda_{2n-1}, \lambda_{2n}) \right).$$

The intervals $(\lambda_{2n-1}, \lambda_{2n}), n \in \mathbb{Z}$ are called gaps, where λ_n are the roots of the equation $\Delta(\lambda) \mp 2 = 0$. They coincide with the eigenvalues of the periodic or antiperiodic $(y(0) = y(\pi))$ problem for equation (4). We denote the roots of equation $s_1(\pi, \lambda, \tau, t) = 0$ by $\xi_n(\tau, t), n \in \mathbb{Z}$, while $\xi_n(\tau, t) \in [\lambda_{2n-1}, \lambda_{2n}], n \in \mathbb{Z}$. The numbers $\xi_n(\tau, t), n \in \mathbb{Z}$ and the signs $\sigma_n = sgn\{s_2(\pi, \xi_n, \tau, t) - c_1(\pi, \xi_n, \tau, t)\}, n \in \mathbb{Z}$ are called spectral parameters of the operator $L(\tau, t)$. The spectral parameters $\xi_n(\tau, t), \sigma_n(\tau, t) = \pm 1, n \in \mathbb{Z}$ and the spectrum boundaries $\lambda_n(\tau, t), n \in \mathbb{Z}$ are called the spectral data of the Dirac operator $L(\tau, t)$.

The problem of recovering the coefficient $\Omega(x, t)$ of the operator $L(\tau, t)$ from spectral data is called the inverse problem. The coefficient $q(x, t)$ of the operator $L(\tau, t)$ is determined uniquely from the spectral data $\{\lambda_n(\tau, t), \xi_n(\tau, t), \sigma_n(\tau, t) = \pm 1, n \in \mathbb{Z}\}$. If using the initial function $q_0(x + \tau), \tau \in \mathbb{R}$, we construct the Dirac operator $L(\tau, 0)$ of the following form:

$$L(\tau, 0)y \equiv B\frac{dy}{dx} + \Omega_0(x + \tau)y = \lambda y, x \in \mathbb{R}, \tau \in \mathbb{R},$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \Omega_0(x + \tau) = \begin{pmatrix} 0 & q_0(x + \tau) \\ q_0(x + \tau) & 0 \end{pmatrix}, \ y = \begin{pmatrix} y_1(x) \\ y_1(x) \end{pmatrix},$$

then we will see that the boundaries of the spectrum $\lambda_n(\tau), n \in \mathbb{Z}$ do not depend on the parameter $\tau \in \mathbb{R}$, i.e., $\lambda_n(\tau) = \lambda_n, n \in \mathbb{Z}$, and the spectral parameters depend on the parameter τ: $\xi_n^0 = \xi_n^0(\tau), \sigma_n^0 = \sigma_n^0(\tau) = \pm 1$, which are also periodic functions: $\xi_n^0(\tau+\pi) = \xi_n^0(\tau), \sigma_n^0(\tau+\pi) = \sigma_n^0(\tau), \tau \in \mathbb{R}, n \in \mathbb{Z}$. Solving the direct problem, we find the spectral data$\{\lambda_n, \xi_n^0(\tau), \sigma_n^0(\tau), n \in \mathbb{Z}\}$ of the operator $L(\tau, 0)$.

3. Main result and discussion

The main result of this paper is contained in the following theorem:

Theorem 1. Let $q(x, t), x \in \mathbb{R}, t > 0$ is the solution of the Cauchy problem (1)-(2). Then, the boundaries of the spectrum $\lambda_n(\tau, t), n \in \mathbb{Z}$ of the operator $L(\tau, t)$ do not depend on τ and t, i.e., $\lambda_n(\tau, t) = \lambda_n, n \in \mathbb{Z}$ and the spectral parameters $\xi_n(\tau, t), \sigma_n(\tau, t) = \pm 1, n \in \mathbb{Z}$, satisfy the first and second equations of the system of Dubrovin differential equations, respectively:

$$1.\frac{\partial \xi_n(\tau, t)}{\partial \tau} = 2(-1)^{n-1}\sigma_n(\tau, t)h_n\left(\xi(\tau, t)\right)\xi_n(\tau, t), n \in \mathbb{Z}; \tag{4}$$

$$2.\frac{\partial \xi_n(\tau, t)}{\partial t} = 2(-1)^{n}\sigma_n(\tau, t)h_n\left(\xi(\tau, t)\right)g_n\left(\xi(\tau, t)\right), n \in \mathbb{Z}. \tag{5}$$

Here, the signs $\sigma_n(\tau, t) = \pm 1, n \in \mathbb{Z}$ changes to the opposite at each collision of the point $\xi_n(\tau, t), n \in \mathbb{Z}$ with the boundaries of its gap $[\lambda_{2n-1}, \lambda_{2n}]$. In addition, the following initial conditions are satisfied

$$\xi_n(\tau, t)|_{t=0} = \xi_n^0(\tau), \sigma_n(\tau, t)|_{t=0} = \sigma_n^0(\tau), n \in \mathbb{Z} \tag{6}$$

where $\xi_n^0(\tau), \sigma_n^0(\tau) = \pm 1, n \in \mathbb{Z}$ are spectral parameters of the Dirac operator $L(\tau, 0)$. Sequences $h_n(\xi)$ and $g_n(\xi), n \in \mathbb{Z}$ are determined by the following formulas:

$$h_n(\xi) = \sqrt{\left(\xi_n(\tau, t) - \lambda_{2n-1}\right)\left(\lambda_{2n} - \xi_n(\tau, t)\right)} \cdot f_n(\xi),$$

$$f_n(\xi) = \sqrt{\prod_{k=-\infty, k\neq n}^{+\infty} \frac{\left(\lambda_{2k-1} - \xi_n(\tau, t)\right)\left(\lambda_{2k} - \xi_n(\tau, t)\right)}{\left(\xi_k(\tau, t) - \xi_n(\tau, t)\right)^2}}, \tag{7}$$

$$g_n(\xi) = 4\xi_n^3(\tau, t) + 2\xi_n(\tau, t)\left(q^2(\tau, t) + q_\tau(\tau, t) - 3\rho^2\right), n \in \mathbb{Z}. \tag{8}$$

Proof. Let $q(x, t), x \in \mathbb{R}, t > 0$ is π periodic function with respect to x, satisfy equation (1). Denote by $y_n = (y_{n,1}(x, \tau, t), y_{n,2}(x, \tau, t))^T, n \in \mathbb{Z}$ the orthonormal eigenfunctions of the operator $L(\tau, t)$ considered on the interval $[0; \pi]$ with the Dirichlet boundary conditions:

$$y_1(0, \tau, t) = 0, y_1(\pi, \tau, t) = 0,$$

corresponding to the eigenvalues $xi_n = \xi_n(\tau,t), n \in \mathbb{Z}$. Differentiating with respect to t the following identity:

$$\xi_n(\tau,t) = \left(L(\tau,t)y_n, y_n\right), n \in \mathbb{Z},$$

and using the symmetry of the operator $L(\tau,t)$, we have

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = \left(\dot{\Omega}(x+\tau,t)y_n, y_n\right), n \in \mathbb{Z}. \tag{9}$$

Using the following explicit form of the scalar product

$$(y,z) = \int_0^\pi (y_1(x)\overline{z_1(x)} + y_2(x)\overline{z_2(x)})dx,\ y = \begin{pmatrix} y_1(x) \\ y_1(x) \end{pmatrix},\ z = \begin{pmatrix} z_1(x) \\ z_1(x) \end{pmatrix},$$

equality (9) can be written in the following form:

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = \int_0^\pi 2y_{n,1}(x,\tau,t)y_{n,2}(x,\tau,t)q_t(x+\tau,t)dx. \tag{10}$$

Putting expression (1) into (10), we obtain the following equality:

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = I_1(\tau,t) + I_2(\tau,t) + I_3(\tau,t)$$

where

$$I_1(\tau,t) = 2\int_0^\pi y_{n,1}(x,\tau,t)y_{n,2}(x,\tau,t)q_{xxx}(x+\tau,t)dx,$$

$$I_2(\tau,t) = -12\int_0^\pi y_{n,1}(x,\tau,t)y_{n,2}(x,\tau,t)q^2(x+\tau,t)q_x(x+\tau,t)dx,$$

$$I_3(\tau,t) = 12\rho^2\int_0^\pi y_{n,1}(x,\tau,t)y_{n,2}(x,\tau,t)q_x(x+\tau,t)dx.$$

Using the following identities

$$\begin{cases} y_{n,1}^{'} = qy_{n,1} - \xi_n y_{n,2} \\ y_{n,2}^{'} = \xi_n y_{n,1} - qy_{n,2} \end{cases},$$

it is easy to check that the following equalities are valid:

$$\begin{cases} (y_{n,1}\cdot y_{n,2})^{'} = \xi_n(y_{n,1}^2 - y_{n,2}^2) \\ (y_{n,1}^2 - y_{n,2}^2)^{'} = 2q\left(y_{n,1}^2 + y_{n,2}^2\right) - 4\xi_n y_{n,1}y_{n,2} \\ \left(y_{n,1}^2 + y_{n,2}^2\right)^{'} = 2q(y_{n,1}^2 - y_{n,2}^2) \end{cases}$$

where

$$y_{n,1} = y_{n,1}(x,\tau,t), y_{n,2} = y_{n,2}(x,\tau,t), q = q(x+\tau,t), \xi_n = \xi_n(\tau,t).$$

Using these equalities and boundary conditions $y_{n,1}(0,\tau,t) = y_{n,1}(\pi,\tau,t) = 0$, it is easy to calculate $I_1(\tau,t)$:

$$I_1(\tau,t) = 2\int_0^\pi y_{n,1}(x,\tau,t)y_{n,2}(x,\tau,t)q_{xxx}(x+\tau,t)dx = 2\int_0^\pi y_{n,1}(x,\tau,t)y_{n,2}(x,\tau,t)dq_{xx}(x+\tau,t) =$$

$$= -2\xi_n(\tau,t)\int_0^\pi \left(y_{n,1}^2(x,\tau,t) - y_{n,2}^2(x,\tau,t)\right)dq_x(x+\tau,t) = 2\xi_n(\tau,t)q_\tau(\tau,t)\left[y_{n,2}^2(\pi,\tau,t) - y_{n,2}^2(0,\tau,t)\right] +$$

$$+2\xi_n(\tau,t)\int_0^\pi \left(y_{n,1}^2(x,\tau,t) + y_{n,2}^2(x,\tau,t)\right)dq^2(x+\tau,t) - 8\xi_n^2(\tau,t)\int_0^\pi \left(y_{n,1}(x,\tau,t) - y_{n,2}(x,\tau,t)\right)dq(x+\tau,t) =$$

$$= 4\xi_n^3(\tau,t) + 2\xi_n(\tau,t)\{q_\tau(\tau,t) + q^2(\tau,t)\}\left[y_{n,2}^2(\pi,\tau,t) - y_{n,2}^2(0,\tau,t)\right] -$$

$$-4\xi_n(\tau,t)\int_0^\pi q^3(x+\tau,t)\left\{y_{n,1}^2(x,\tau,t) - y_{n,2}^2(x,\tau,t)\right\}dx.$$

Then, calculate $I_2(\tau,t)$ and $I_3(\tau,t)$:

$$I_2(\tau,t) = -12\int_0^\pi y_{n,1}(x,\tau,t)y_{n,2}(x,\tau,t)q^2(x+\tau,t)q_x(x+\tau,t)dx =$$

$$= -4\int_0^\pi y_{n,1}(x,\tau,t)y_{n,2}(x,\tau,t)dq^3(x+\tau,t) = 4\xi_n(\tau,t)\int_0^\pi q^3(x+\tau,t)\left\{y_{n,1}^2(x,\tau,t) - y_{n,2}^2(x,\tau,t)\right\}dx.$$

$$I_3(\tau,t) = 12\rho^2\int_0^\pi y_{n,1}(x,\tau,t)y_{n,2}(x,\tau,t)q_x(x+\tau,t)dx = 12\rho^2\int_0^\pi y_{n,1}(x,\tau,t)y_{n,2}(x,\tau,t)dq(x+\tau,t) =$$

$$= -6\rho^2\xi_n(\tau,t)\left[y_{n,2}^2(\pi,\tau,t) - y_{n,2}^2(0,\tau,t)\right].$$

Thus

$$\frac{\partial\xi_n(\tau,t)}{\partial t} = 4\xi_n^3(\tau,t) + 2\xi_n(\tau,t)\{q^2(\tau,t) + q_\tau(\tau,t) - 3\rho^2\}\left[y_{n,2}^2(\pi,\tau,t) - y_{n,2}^2(0,\tau,t)\right]. \qquad (11)$$

Since the eigenvalues $\xi_n(\tau,t)$ of the Dirichlet problem for equation (3) are simple, then the following equality holds:

$$y_n(x,\tau,t) = \frac{1}{c_n(\tau,t)}s(x,\xi_n(t),\tau,t),$$

where $c_n^2(\tau,t) = \int_0^\pi[s_1^2(x,\xi_n(t),\tau,t) + s_2^2(x,\xi_n(t),\tau,t)]dx = -\frac{\partial s_1(\pi,\xi_n(t),\tau,t)}{\partial\lambda}\cdot s_2(\pi,\xi_n(t),\tau,t)$. Using these equalities, we have

$$y_{n,2}^2(\pi,\tau,t) - y_{n,2}^2(0,\tau,t) = -\frac{s_2(\pi,\xi_n(t),\tau,t) - \frac{1}{s_2(\pi,\xi_n(t),\tau,t)}}{\frac{\partial s_1(\pi,\xi_n(t),\tau,t)}{\partial\lambda}}.$$

Putting into this relation the following equality

$$s_2(\pi,\xi_n(t),\tau,t) - \frac{1}{s_2(\pi,\xi_n(t),\tau,t)} = \sigma_n(\tau,t)\sqrt{\Delta^2(\xi_n) - 4},$$

we obtain

$$y_{n,2}^2(\pi,\tau,t) - y_{n,2}^2(0,\tau,t) = -\frac{\sigma_n(\tau,t)\sqrt{\Delta^2(\xi_n) - 4}}{\frac{\partial s_1(\pi,\xi_n(t),\tau,t)}{\partial\lambda}}. \qquad (12)$$

Given the decomposition

$$\Delta^2(\lambda) - 4 = -4\pi^2\prod_{k=-\infty}^{\infty}\frac{(\lambda-\lambda_{2k-1})(\lambda-\lambda_{2k})}{\alpha_k^2}, s_1(\pi,\lambda,t) = \pi\prod_{k=-\infty}^{\infty}\frac{\xi_k-\lambda}{\alpha_k}$$

where $\alpha_0 = 1$ and $\alpha_k = k$ at $k \neq 0$, rewrite equation (12) in the following form:

$$y_{n,2}^2(\pi,\tau,t) - y_{n,2}^2(0,\tau,t) = 2(-1)^n\sigma_n(\tau,t)h_n(\xi).$$

Putting this expression into identity (11), we obtain (5). Similarly, we can prove (4).

If we replace the Dirichlet boundary conditions with periodic ($y(0,t) = y(\pi,t)$) or antiperiodic ($y(0,t) = -y(\pi,t)$) boundary conditions, then instead of equation (11), we have

$$\frac{\partial\lambda_n(\tau,t)}{\partial t} = 0, \text{ i.e. } \lambda_n(\tau,t) = \lambda_n(\tau,0),\ n\in\mathbb{Z}.$$

Now, in equation $L(\tau,t)\nu_n = \lambda_n(\tau,t)\nu_n,\ n\in\mathbb{Z}\backslash 0$, we obtain $t = 0$. Since the eigenvalues $\lambda_n(\tau) = \lambda_n(\tau,0),\ n\in\mathbb{Z}\backslash 0$ of the periodic or antiperiodic problem for equation $L(\tau,0)\nu_n = \lambda_n(\tau)\nu_n,\ n\in\mathbb{Z}$ do not depend on the parameter $\tau\in\mathbb{R}$, we have $\lambda_n(\tau,t) = \lambda_n(\tau) = \lambda_n,\ n\in\mathbb{Z}$.

Theorem 1 is proved.

Now, using the following trace formulas

$$q(\tau,t) = \sum_{k=-\infty}^{\infty}(-1)^{k-1}\sigma_k(\tau,t)h_k(\xi(\tau,t)), \qquad (13)$$

$$q^2(\tau,t)+q_\tau(\tau,t)=\sum_{k=-\infty}^{\infty}\left(\frac{\lambda_{2k-1}^2+\lambda_{2k}^2}{2}-\xi_k^2(\tau,t)\right),$$

system (5) can be rewritten in the closed form:

$$\frac{\partial\xi_n(\tau,t)}{\partial t}=2(-1)^n\sigma_n(\tau,t)h_n\left(\xi(\tau,t)\right)\left[4\xi_n^3(\tau,t)+2\xi_n(\tau,t)\left(\sum_{k=-\infty}^{\infty}\left(\frac{\lambda_{2k-1}^2+\lambda_{2k}^2}{2}-\xi_k^2(\tau,t)\right)-3\rho^2\right)\right].$$

Next, using the change in variables

$$\xi_n(\tau,t)=\lambda_{2n-1}+(\lambda_{2n}-\lambda_{2n-1})sin^2x_n(\tau,t),\ n\in\mathbb{Z} \tag{14}$$

system of Dubrovin equations (5) can be rewritten as a single equation in the Banach space $\mathbb{K}$:

$$\frac{dx(\tau,t)}{dt}=H(x(\tau,t)),\ x(\tau,t)|_{t=0}=x^0(\tau)\in\mathbb{K} \tag{15}$$

where

$$\mathbb{K}=\left\{x=(...,x_{-1}(\tau,t),x_0(\tau,t),x_1(\tau,t),...):||x||=\sum_{n=-\infty}^{\infty}(1+|n|)(\lambda_{2n}-\lambda_{2n-1})|x_n|<\infty\right\},$$

$$H(x)=(...,H_{-1}(x),H_0(x),H_1(x),...),$$

$$H_n(x)=(-1)^n\sigma_n(0)f_n(...,\lambda_1+(\lambda_2-\lambda_1)sin^2x_1(\tau,t),...)\cdot g_n(...,\lambda_1+(\lambda_2-\lambda_1)sin^2x_1(\tau,t),...)=$$
$$=(-1)^n\sigma_n(0)f_n(x(\tau,t))g_n(x(\tau,t)).$$

From the conditions $q_0(x+\pi)=q_0(x)\in C^5(\mathbb{R})$ and the asymptotics of the eigenvalues of the periodic and antiperiodic problems for the system of Dirac equations, we obtain the following equality:

$$\lambda_{2k-1},\lambda_{2k}=k+\sum_{j=1}^{6}c_jk^{-j}\pm2^{-5}|k|^{-5}|q_{2k}^5|+|k|^{-6}\varepsilon_k^{\pm},\ \gamma_k\equiv\lambda_{2k}-\lambda_{2k-1}=\frac{|q_{2k}^5|}{2^4|k|^5}+\frac{\delta_k}{|k|^6}, \tag{16}$$

$$\sum_{k=-\infty}^{\infty}|q_{2k}^5|^2<\infty,\ \sum_{k=-\infty}^{\infty}(\varepsilon_k^{\pm})^2<\infty,\ \delta_k=\varepsilon_k^+-\varepsilon_k^-.$$

Hence, taking into account $\xi_n(\tau,t)\in[\lambda_{2n-1},\lambda_{2n}]$, we have

$$\inf_{k\neq n}|\xi_n(\tau,t)-\xi_k(\tau,t)|\geq a>0.$$

Now, using inequality in the above and formula (14), the following lemma can be proved:

Lemma 1. The following estimates are valid:

$$c_1\leq|f_n(x)|\leq c_2,\ |\frac{\partial f_n(x)}{\partial x_m}|\leq c_3\gamma_m, \tag{17}$$

$$|g_n(x)|\leq c_4|n|^3,\ |\frac{\partial g_n(x)}{\partial x_m}|\leq c_5\gamma_m|n|(|m|+1) \tag{18}$$

where $c_j>0$, $j=1,2,3,4,5$ does not depend on the parametres m and n.

Lemma 2. If the initial functions $q_0(x)$ satisfy the following condition:

$$q_0(x+\pi)=q_0(x)\in C^5(\mathbb{R}),$$

then the vector function $H(x)$ satisfies the Lipschitz condition in the Banach space $\mathbb{K}$, that is, there exists a constant $L=const>0$ such that the following inequality holds for arbitrary elements$x,y\in\mathbb{K}$:

$$||H(x)-H(y)||\leq L||x-y||,$$

where

$$L = A \sum_{n=-\infty}^{\infty} \gamma_n |n|^3 (1 + |n|) < \infty. \tag{19}$$

Proof. First, using Lemma 1, we estimate the derivative with respect to $x_m(\tau, t)$ of the function $F_n(x) = f_n(x)g_n(x)$:

$$\left|\frac{\partial F_n(x)}{\partial x_m}\right| \le \left|\frac{\partial f_n(x)}{\partial x_m}\right| |g_n(x)| + |f_n(x)| \left|\frac{\partial g_n(x)}{\partial x_m}\right| \le c_3 c_4 \gamma_m |n|^3 + c_2 c_5 \gamma_m |n|(|m| + 1) \le$$

$$\le c_3 c_4 \gamma_m |n|^3 (|m| + 1) + c_2 c_5 \gamma_m |n|(|m| + 1) \le \gamma_m |n|^3 (|m| + 1) \left\{ c_3 c_4 + \frac{c_2 c_5}{|n|^2} \right\} \le$$

$$\le \gamma_m |n|^3 (|m| + 1)\{c_3 c_4 + 1\} = A \gamma_m |n|^3 (|m| + 1).$$

where $A = c_3 c_4 + 1$ does not depend on the parametres m and n.

Next, using the following expression

$$H_n(x(\tau, t)) = (-1)^n \sigma_n^0(\tau) F_n(x(\tau, t)),\ n \in \mathbb{Z}.$$

We obtain the following equality:

$$|H_n(x(\tau, t)) - H_n(y(\tau, t))| = |F_n(x(\tau, t)) - F_n(y(\tau, t))|.$$

Now, let us apply the Lagrange mean value theorem to the following function:

$$\varphi(t) = F_n(x + t(y - x)),$$

on segment $t \in [0; 1]$. Then, we obtain the following equalities:

$$\varphi(1) - \varphi(0) = \varphi'(t^*),\ F_n(x) - F_n(y) = \sum_{k=-\infty}^{\infty} \frac{\partial F_n(\theta)}{\partial x_k} (x_k - y_k),$$

where $\theta = x + t^*(y - x)$. Hence, it follows that

$$|H_n(x(\tau, t)) - H_n(y(\tau, t))| = |F_n(x(\tau, t)) - F_n(y(\tau, t))| \le \sum_{k=-\infty}^{\infty} \frac{\partial F_n(\theta)}{\partial x_k} |x_k(\tau, t) - y_k(\tau, t)| \le$$

$$\le A|n|^3 \sum_{k=-\infty}^{\infty} \gamma_k (|k| + 1)|x_k(\tau, t) - y_k(\tau, t)| \le A|n|^3 ||x - y||.$$

Now, let us estimate the following norm: $||H(x(\tau, t)) - H(y(\tau, t))||$:

$$||H(x) - H(y)|| = \sum_{n=-\infty}^{\infty} \gamma_n (|n| + 1)|H_n(x) - H_n(y)| \le \sum_{n=-\infty}^{\infty} \gamma_n (|n| + 1) A|n|^3 ||x - y|| =$$

$$= ||x - y|| \sum_{n=-\infty}^{\infty} \gamma_n |n|^3 (|n| + 1) \le L||x - y||.$$

where

$$L = A \sum_{n=-\infty}^{\infty} \gamma_n |n|^3 (|n| + 1) < \infty.$$

Thus, the Lipschitz condition is satisfied. Therefore, the solution of the Cauchy problem (6), (7) for all $t > 0$ and $\tau \in \mathbb{R}$ exists and is unique.

Lemma 2 is proved.

Remark 1. Theorem 1 and Lemma 2 give a method for constructing a solution to problems (1)-(2). To do this, first find the spectral data $\lambda_n, \xi_n^0(\tau), \sigma_n^0(\tau) = \pm 1, n \in \mathbb{Z}$, of the Dirac operator $L(\tau, 0)$. Denote the spectral data of operator $L(\tau, t)$ by $\lambda_n, \xi_n(\tau, t), \sigma_n(\tau, t) = \pm 1, n \in \mathbb{Z}$. Then, solving Cauchy problems (5) and (6) for an arbitrary value of τ, we find $\xi_n(\tau, t), \sigma_n(\tau, t), n \in \mathbb{Z}$.

After that, from the trace formula (13), we find the function $q(\tau,t)$, i.e., solving the problem (1)-(2).

Thus, we have proved the following theorem:

Theorem 2. If the initial functions $q_0(x)$ satisfy the following condition:

$$q_0(x+\pi)=q_0(x)\in C^5(\mathbb{R}),$$

then there exists a uniquely determined solution $q(\tau,t)$ of problem (1)-(2) belonging to the class $C_x^3(t>0)\cap C_t^1(t>0)\cap C(t\geq 0)$, which is defined as the sum of series (13).

Remark 2. The function $q(\tau,t)$ constructed using the system of Dubrovin equations (5)-(6) and the trace formula (13) satisfies equation (1).

Proof. First, we use the following second trace formula:

$$q^2(\tau,t)+q_\tau(\tau,t)=\sum_{k=-\infty}^{\infty}\left(\frac{\lambda_{2k-1}^2+\lambda_{2k}^2}{2}-\xi_k^2(\tau,t)\right). \tag{20}$$

Differentiating formula (20) with respect to t, we have

$$2q(\tau,t)q_t(\tau,t)+q_{\tau t}(\tau,t)=-\sum_{k=-\infty}^{\infty}2\xi_k(\tau,t)\frac{\partial\xi_k(\tau,t)}{\partial t}. \tag{21}$$

Furthermore, taking into account the system of Dubrovin equations (5), from (21), we obtain

$$2q(\tau,t)q_t(\tau,t)+q_{\tau t}(\tau,t)=$$

$$=2\sum_{k=-\infty}^{\infty}\{2(-1)^k\sigma_k(\tau,t)h_k(\xi(\tau,t))\xi_k(\tau,t)\}[4\xi_k^3(\tau,t)+2\xi_k(\tau,t)(q^2(\tau,t)+q_\tau(\tau,t)-3\rho^2)]=$$

$$=2\sum_{k=-\infty}^{\infty}\frac{\partial\xi_k(\tau,t)}{\partial\tau}[4\xi_k^3(\tau,t)+2\xi_k(\tau,t)(q^2(\tau,t)+q_\tau(\tau,t)-3\rho^2)]=$$

$$=2\sum_{k=-\infty}^{\infty}\left\{\frac{\partial\xi_k^4(\tau,t)}{\partial\tau}+\frac{\partial\xi_k^2(\tau,t)}{\partial\tau}-3\rho^2\right\}. \tag{22}$$

Next, we calculate the sums of the following functional series:

$$\sum_{k=-\infty}^{\infty}\frac{\partial\xi_k^4(\tau,t)}{\partial\tau},\quad\sum_{k=-\infty}^{\infty}\frac{\partial\xi_k^2(\tau,t)}{\partial\tau}.$$

To do this, we first define the following trace formula:

$$-\frac{q_{\tau\tau\tau}(\tau,t)}{2}-q(\tau,t)q_{\tau\tau}(\tau,t)+2q^2(\tau,t)q_\tau(\tau,t)+q^4(\tau,t)=\sum_{k=-\infty}^{\infty}\left(\frac{\lambda_{2k-1}^4+\lambda_{2k}^4}{2}-\xi_k^4(\tau,t)\right). \tag{23}$$

Differentiate the trace formulas (20) and (23) with respect to τ:

$$-2qq_\tau-q_{\tau\tau}=\sum_{k=-\infty}^{\infty}\frac{\partial\xi_k^2(\tau,t)}{\partial\tau},$$

$$\frac{q_{\tau\tau\tau\tau}}{2}+q_\tau q_{\tau\tau}+qq_{\tau\tau\tau}-4qq_\tau^2-2q^2q_{\tau\tau}-4q^3q_\tau=\sum_{k=-\infty}^{\infty}\frac{\partial\xi_k^4(\tau,t)}{\partial\tau},$$

Using these formulas from (22), we find

$$2qq_t+q_{\tau t}=q_{\tau\tau\tau\tau}+2q_\tau q_{\tau\tau}+2qq_{\tau\tau\tau}-8qq_\tau^2-4q^2q_{\tau\tau}-8q^3q_\tau-\{4qq_\tau+2q_{\tau\tau}\}(q^2+q_\tau-3\rho^2),$$

$$2qq_t+q_{\tau t}=q_{\tau\tau\tau\tau}+2qq_{\tau\tau\tau}-12qq_\tau^2-6q^2q_{\tau\tau}-12q^3q_\tau+12\rho^2qq_\tau+6\rho^2q_{\tau\tau}, \tag{24}$$

Equation (24) can be reduced to the following linear equation:

$$z_\tau(\tau,t)+\frac{1}{2q}z(\tau,t)=\varphi(\tau,t), \tag{25}$$

where $z(\tau,t)=q_t(\tau,t)$, $\varphi(\tau,t)=q_{\tau\tau\tau\tau}+2qq_{\tau\tau\tau}-12qq_\tau^2-6q^2q_{\tau\tau}-12q^3q_\tau+12\rho^2qq_\tau+6\rho^2q_{\tau\tau}$.

It is easy to check that the following function

$$z(\tau,t)=q_{\tau\tau\tau}(\tau,t)-6q^2(\tau,t)q_\tau(\tau,t)+6\rho^2q_\tau(\tau,t),$$

is a solution for equation (25).

From here and from the notation $z(\tau,t)=q_t(\tau,t)$, we obtain equation (1).

Remark 3. The uniform convergence of the series in formulas (13) and (19) follows from estimate (16).

Now we prove the stability of the solution of the Cauchy problem (1). First, we prove the following lemma:

Lemma 3. The solution of the Cauchy problem (15) is stable.

Proof. First we consider (15) and the following Cauchy problems:

$$\frac{d\tilde{x}(\tau,t)}{dt}=H(\tilde{x}(\tau,t)),\ \tilde{x}(\tau,t)|_{t=0}=\tilde{x}^0(\tau)\in\mathbb{K} \tag{26}$$

It is known that if the solution of the Cauchy problem (15) satisfies the following condition, then this solution is called stable:

$$\forall\varepsilon>0,\exists\delta=\delta(\varepsilon)>0,||x^0(\tau)-\tilde{x}^0(\tau)||<\delta\Longrightarrow\forall\tau\in\mathbb{R},\ ||x(\tau,t)-\tilde{x}(\tau,t)||<\frac{\varepsilon}{A}.$$

Equations (15) and (26) are equivalent to the following integral equations, respectively:

$$x(\tau,t)=x^0(\tau)+\int_0^t H(x(\tau,s))ds,\quad \tilde{x}(\tau,t)=\tilde{x}^0(\tau)+\int_0^t H(\tilde{x}(\tau,s))ds.$$

Thus,

$$||x(\tau,t)-\tilde{x}(\tau,t)||=\sum_{n=-\infty}^{\infty}\gamma_n|x_n(\tau,t)-\tilde{x}_n(\tau,t)|\le$$

$$\le\sum_{n=-\infty}^{\infty}\gamma_n\left(|x_n^0(\tau)-\tilde{x}_n^0(\tau)|+\int_0^t|H_n(x(\tau,s))ds-H_n(\tilde{x}(\tau,s))|\,ds\right)\le$$

$$\le\sum_{n=-\infty}^{\infty}\gamma_n|x_n^0(\tau)-\tilde{x}_n^0(\tau)|+\sum_{n=-\infty}^{\infty}L\gamma_n\int_0^t|x_n(\tau,s)-\tilde{x}_n(\tau,s)|ds=||x^0(\tau)-\tilde{x}^0(\tau)||+$$

$$+L\int_0^t\sum_{n=-\infty}^{\infty}\gamma_n|x_n(\tau,s)-\tilde{x}_n(\tau,s)|ds=||x^0(\tau)-\tilde{x}^0(\tau)||+L\int_0^t||x(\tau,s)-\tilde{x}(\tau,s)||ds,$$

According to the Grönwall's inequality, we have

$$||x(\tau,t)-\tilde{x}(\tau,t)||\le||x^0(\tau)-\tilde{x}^0(\tau)||e^{Lt}.$$

If we choose $\delta=\frac{\varepsilon}{A}e^{-Lt}$, then we have the following inequality:

$$||x(\tau,t)-\tilde{x}(\tau,t)||\le||x^0(\tau)-\tilde{x}^0(\tau)||e^{Lt}<\delta e^{Lt}=\frac{\varepsilon}{A}.$$

Lemma 3 is proved.

Theorem 3. The solution of the Cauchy problem (1) is stable.

Proof. First consider (1) and the following Cauchy problems:

$$\begin{cases}\tilde{q}_t=\tilde{q}_{xxx}-6(\tilde{q}^2-\rho^2)\tilde{q}_x,\ 0<\rho<\infty\\ \tilde{q}(x,t)|_{t=0}=\tilde{q}_0(x),\ \tilde{q}_0(x+\pi)=\tilde{q}_0(x)\in C^5(\mathbb{R})\end{cases} \tag{27}$$

It is known that if the solution of the Cauchy problem (1) satisfies the following condition, then this solution is called stable:

$$\forall \varepsilon > 0, \exists \delta_1 = \delta_1(\varepsilon) > 0, |q_0(\tau) - \tilde{q}_0(\tau)| < \delta_1 \Longrightarrow \forall \tau \in \mathbb{R},\ |q(\tau,t) - \tilde{q}(\tau,t)| < \varepsilon.$$

The following relation follows from the correctness of the direct problem for the Dirac operator and substitution (14):

$$\text{if } \exists \delta_1 > 0, |q_0(\tau) - \tilde{q}_0(\tau)| < \delta_1 \text{then } \forall \delta > 0, ||x^0(\tau) - \tilde{x}_0(\tau)|| < \delta.$$

If we put the change of variables (14) in the trace formula (13), then we get the trace formula with respect to x for the solution of the Cauchy problems (1) and (27), respectively:

$$q(\tau,t) = \sum_{n=-\infty}^{\infty} (-1)^{n-1} \sigma_n^0(\tau) \gamma_n sin\left(2x_n(\tau,t)\right) f_n(x(\tau,t))$$

$$\tilde{q}(\tau,t) = \sum_{n=-\infty}^{\infty} (-1)^{n-1} \tilde{\sigma}_n^0(\tau) \tilde{\gamma}_n sin\left(2\tilde{x}_n(\tau,t)\right) f_n(\tilde{x}(\tau,t))$$

$$|q(\tau,t) - \tilde{q}(\tau,t)| = \left| \sum_{n=-\infty}^{\infty} (-1)^{n-1} \left\{ \sigma_n^0(\tau) \gamma_n sin\left(2x_n(\tau,t)\right) f_n(x(\tau,t)) - \tilde{\sigma}_n^0(\tau) \tilde{\gamma}_n sin\left(2\tilde{x}_n(\tau,t)\right) f_n(\tilde{x}(\tau,t)) \right\} \right| \leq$$

$$\leq C \sum_{n=-\infty}^{\infty} \gamma_n \left| sin\left(2x_n(\tau,t)\right) f_n(x(\tau,t)) - sin\left(2\tilde{x}_n(\tau,t)\right) f_n(\tilde{x}(\tau,t)) \right|.$$

Now we define following functions:

$$G_n(x(\tau,t)) = sin\left(2x_n(\tau,t)\right) f_n(x(\tau,t)),\ G_n(\tilde{x}(\tau,t)) = sin\left(2\tilde{x}_n(\tau,t)\right) f_n(\tilde{x}(\tau,t)),$$

Apply the Lagrange mean value theorem to the following function:

$$\psi(z) = G_n(x + z(\tilde{x} - x)),\ \psi(1) - \psi(0) = \psi^{'}(z^*),\ \theta = x + t^*(\tilde{x} - x)$$

on segment $z \in [0;1]$. Then, we have

$$|q(\tau,t) - \tilde{q}(\tau,t)| \leq C \sum_{n=-\infty}^{\infty} \gamma_n \left| G_n(x(\tau,t)) - G_n(\tilde{x}(\tau,t)) \right| =$$

$$= C \sum_{n=-\infty}^{\infty} \gamma_n \left\{ \sum_{m=-\infty}^{\infty} \left| \frac{\partial G_n(\theta(\tau,t))}{\partial x_m(\tau,t)} \right| |x_m(\tau,t) - \tilde{x}_m(\tau,t)| \right\} =$$

$$= C \sum_{n=-\infty}^{\infty} 2\gamma_n |cos 2x_n(\tau,t) f_n(x(\tau,t))| |x_n(\tau,t) - \tilde{x}_n(\tau,t)| +$$

$$+ C \sum_{n=-\infty}^{\infty} \gamma_n \left\{ \sum_{\substack{m=-\infty, \\ m \neq n}}^{\infty} \left| sin(2\theta_n(\tau,t)) \frac{\partial f_n(\theta(\tau,t))}{\partial x_m(\tau,t)} \right| |x_m(\tau,t) - \tilde{x}_m(\tau,t)| \right\} \leq$$

$$\leq A_1 ||x(\tau,t) - \tilde{x}(\tau,t)|| + A_2 \sum_{n=-\infty}^{\infty} \gamma_n ||x(\tau,t) - \tilde{x}(\tau,t)|| \leq A ||x(\tau,t) - \tilde{x}(\tau,t)|| < A \cdot \frac{\varepsilon}{A} = \varepsilon.$$

It is known that following inequality follows from the above inequality:

$$||q(\tau,t) - \tilde{q}(\tau,t)|| \leq \varepsilon$$

Theorem 3 is proved.

Remark 4. Lemma 1, Lemma 3 and (16) estimates were used in the process of deriving the above inequality.

4. Conclusion

The inverse spectral problem method is used to integrate the modified Korteweg-de Vries equation with a finite density in the class of periodic infinite-gap functions. It is proven that if the initial conditions are fivefold differentiable periodic infinite-gap functions, then the Cauchy problem for the mKdV equation with finite density has a smooth solution at all times.

References

[1] Gardner C., Green I., Kruskal M., Miura R. A method for solving the Korteweg-de Vries equation, *Phys. Rev. Letter*, Vol. 19, 1697, pp. 1095-1098. https://doi.org/10.1103/PhysRevLett.19.1095

[2] Faddeev L. D. Properties of the S-matrix of the one-dimensional Schrodinger equation, *Tr.MIAN SSSR*, Vol. 73, 1964, pp. 314–336. https://www.mathnet.ru/links/f792a8563be600a674d64326eca0f1d1/tm1633.pdf

[3] Marchenko V.A. *Sturm-Liouville operators and their applications*, Naukova dumka, Kiev, 1977.

[4] Levitan B. M. *Inverse Sturm-Liouville problems*, Nauka, Moscow, 1984.

[5] Lax P.D. Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure and Appl. Math.*, Vol. 21, 1968, pp.467-490. https://doi.org/10.1002/cpa.3160210503

[6] Zaxarov V. E., Shabat A. B. Tochnaya teoriya dvumernoy samofokusirovki v odnomernoy avtomodulyatsii voln v nelineynix sredax. *JETF*, Vol. 61, 1971, pp. 118-134. https://doi.org/10.31857/S0044466922060072

[7] M. Wadati. The exact solution of the modified Korteweg-de Vries equation. *J. Phys. Soc. Japn.*, Vol. 32, 1972, pp. 44-47.

[8] Its A. R., Matveev V. B. Operatori Shredingera s konechnozonnim spektrom i N - solitonnie resheniya uravneniya Kortevega-de Friza. *Theoretical and Mathematical Physics*, Vol. 23, 1975, pp. 51–68.

[9] Dubrovin B. A., Novikov S. P. Periodicheski i uslovno periodicheski analogi mnogosolitonnix resheniy uravneniya Kortevega-de Friza. *JETF*, Vol. 67, 1974, pp. 2131–2143.

[10] Its A. R., Kotlyarov V. P. Explicit formulas for solutions of a nonlinear Schrodinger equation. *Dokl. Akad. Nauk Ukrain. SSR Ser. A.*, Vol. 11, 1976, pp. 965–968. https://doi.org/10.1070/SM1995v082n02ABEH003575

[11] Matveev V. B., Smirnov A. O. Solutions of the Ablowitz–Kaup–Newell–Segur hierarchy equations of the "rogue wave" type: A unified approach.*Theoretical and Mathematical Physics*, Vol. 186, 2016, pp. 156–182. https://doi.org/10.1134/S0040577916020033

[12] Khasanov A. B., Khasanov T. G. The Cauchy problem for the Korteweg-de Vries equation in the class of periodic infinite-gap functions.*Zapiski Nauchnykh Seminarov POMI*, Vol. 506, 2021, pp. 258–279. https://www.mathnet.ru/rus/znsl7154

[13] Khasanov A. B., Allanazarova T. J. On modified Korteweg – de Vries equation with a loaded term. *Ukr. Math. Journal*, Vol. 32, 2021, pp. 1541-1563. https://doi.org/10.37863/umzh.v73i11.6073

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

A CRITERION FOR CORRECT SOLVABILITY OF A FIRST ORDER QUASILINEAR THE VERHULST LOGISTIC GROWTH DIFFERENCE EQUATION

SEYIT KERIMKHULLE[1,a)], ULZHAN MAKHAZHANOVA[1,b)], AINUR SALIYEVA[1,c)], AISLU KASSEKEYEVA[1,d)], ALIBEK ADALBEK[1,e)], ROMAN TABERKHAN[1,f)]

[1]Department of Information Systems, L.N. Gumilyov Eurasian National University, 2, Satpayev Street, Astana, 010008, Kazakhstan

[a)]Corresponding author: kerimkhulle@gmail.com; [b)]makhazhan.ut@gmail.com; [c)]ainur.s.saliyeva@gmail.com; [d)]aibike2903@gmail.com; [e)]alibek.adalbek@gmail.com; [f)]roman.tn82@gmail.com.

Abstract. In this work we consider the first order quasilinear the Verhulst logistic growth difference equation:

$$-\frac{1-s_{n+1}}{s_{n+1}}+(1+r_n)\frac{1-s_n}{s_n}=f_n\,,\ n=0,\pm1,\pm2,\dots,$$

where $r_n \geq 0$; $(1-s)/s=\{(1-s_n)/s_n\}_{n=0,\pm1,\pm2,\dots}$, $0<s<1$ – values of stock of logistic growth, endogenous factor, desired solution of the equation; $f=\{f_n\}_{n=0,\pm1,\pm2,\dots}\in\ell_p$, $1\leq p\leq\infty$ – values of logistic growth flow, exogenous factor, right side of the equation and $\forall t=1,2,\cdots<\infty\mid\prod_{j=n}^{n+t-1}(1+r_j)<\infty$. The equation is called correctly solvable in the given space ℓ_p, $1\leq p\leq\infty$ if for any $f\in\ell_p$ there is a unique solution $(1-s)/s\in\ell_p$ and the following inequality $\|(1-s)/s\|_{\ell_p}\leq c(p)\|f\|_{\ell_p}$ for all $f\in\ell_p$ holds with absolute constant $0<c(p)<\infty$. We find a criterion for correct solvability of the above difference equation in space ℓ_p.

Keywords: The Verhulst logistic growth; the first order quasilinear; the difference equation; a criterion; correctly solvable; a unique solution.

1. Introduction

We consider the first order quasilinear the Verhulst logistic growth difference equation [1]:

$$-\frac{1-s_{n+1}}{s_{n+1}}+(1+r_n)\frac{1-s_n}{s_n}=f_n\,,\ n=0,\pm1,\pm2,\dots,\tag{1}$$

where $n=0,\pm1,\pm2,\dots$ – grid nodes of integers to calculate $(1-s)/s=\{(1-s_n)/s_n\}_{n=0,\pm1,\pm2,\dots}$, $0<s_n<1$ – stock values of the system of logistic growth – an endogenous factor, the desired solution of the equation (1) and $f=\{f_n\}_{n=0,\pm1,\pm2,\dots}$ – values of flows of the system of logistic growth - an exogenous factor, the right side of the equation (1), $\{r_n\geq 0\}_{n=0,\pm1,\pm2,\dots}$ – logistic growth rate,

$$\forall d=1,2,\cdots<\infty\mid\prod_{j=n}^{n+d-1}(1+r_j)<\infty\,,\tag{2}$$

and

$$f\in\ell_p,\ 1\leq p<\infty\iff\|f\|_{\ell_p}^p=\sum_{n=-\infty}^{+\infty}|f_n|^p<\infty\,,$$

– the space of summable p-th power of numerical sequences f,

$$f \in \ell_p,\, p=\infty \iff \|f\|_{\ell_\infty} = \sup_{n=0,\pm1,\pm2,\dots} |f_n| < \infty\,,$$

– the space of bounded numerical sequences f.

As a solution of equation (1) we assume any absolutely summable sequences $(1-s)/s$ that satisfies (1) everywhere on $n=0,\pm1,\pm2,\dots$. We need the following definition:

Definition 1.1. *We call the equation* (1) *correctly solvable in the given space* ℓ_p *if the following two assertions hold:*

I) for any $f \in \ell_p$, *the equation* (1) *has a unique solution* $(1-s)/s \in \ell_p$,

II) the solution $(1-s)/s \in \ell_p$ *of the equation* (1) *satisfies the inequality*

$$\|(1-s)/s\|_p \le c(p)\|f\|_p\,, \text{for all } f \in \ell_p\,, \tag{3}$$

where $c(p)$ *– is an absolute positive constant.*

We note that the term "correct solvability" can be used as a condition that ensures the sustainability of the system of logistic growth (1). Indeed, from inequality (3) we obtain that a small change in the right side $f \in \ell_p$ implies a small change in the desired solution $(1-s)/s \in \ell_p$ of equation (1).

Throughout the paper in accordance with I)-II), we denote by the symbol y only a solution of equation (1) that belongs to ℓ_p.

Our main goal is to establish a criterion for correct solvability in space ℓ_p of first order quasilinear the Verhulst logistic growth difference equation (1). Then the statement of the main problem has following form:

Theorem 1.1. *Let* $p \in [1,\infty]$. *The first order difference quasilinear the Verhulst logistic growth difference equation* (1) *is correctly solvable in space* ℓ_p *if and only if there is a finite* $d=1,2,\dots$ *such that*

$$r_*(d) > 1,\; r_*(d) := \inf_{n=0,\pm1,\pm2,\dots} \left\{ \prod_{j=n}^{n+d-1} (1+r_j) \right\},\; r_n \ge 0. \tag{4}$$

We note that in this paper will be implemented a variation method for localizing the qualitative characteristics of solutions of equation (1) at infinity. Here we must especially note the fundamental work of Otelbaev M. [2], who, with his averaging function, achieved scientifically significant results considering the differential equations as well as difference equations. In [3], Lukachev M. and Shuster L. obtained a necessary and sufficient condition for the correct solvability of linear differential equation in $L_p(\mathbb{R}), 1 \le p \le \infty$ and the asymptotic evaluation of solutions was also established. A further development for the matrix problem under study was made by Oinarov R. and Taspaganbetova Zh. [4].

2. Proof of Main Result for $1 \le p < \infty$.

Proof of Theorem 1.1. *Necessity.* Below in the necessity part of Theorem, we assume that equation (1) is correctly solvable in space ℓ_p. We need the following lemmas.

Lemma 2.1. *There is an identity*

$$\frac{\Delta s_n}{s_n(1-s_n)} = \frac{\Delta\left(\frac{1-s_n}{s_n}\right)}{\frac{1-s_n}{s_n}}. \tag{5}$$

Proof. Indeed, using factorization

$$\frac{1}{s_n(1-s_n)} = \frac{1}{s_n} + \frac{1}{1-s_n},$$

where $\Delta(\cdot)_n = (\cdot)_{n+1} - (\cdot)_n$ is a difference scheme for calculating the value of the sequence $(\cdot)_n$ at the nodes of the grid of integers n, and the chain of identities, we have

$$\begin{aligned}\frac{\Delta s_n}{s_n(1-s_n)} &= \frac{s_{n+1}-s_n}{s_n(1-s_n)} = \frac{s_{n+1}-s_n}{s_n} - \frac{(1-s_{n+1})-(1-s_n)}{1-s_n} \\ &= \frac{s_{n+1}}{s_n} - \frac{1-s_{n+1}}{1-s_n} = -\frac{(1-s_{n+1})s_n - (1-s_n)s_{n+1}}{s_n(1-s_n)} \\ &= \frac{\frac{1-s_{n+1}}{s_{n+1}} - \frac{1-s_n}{s_n}}{\frac{1-s_n}{s_n}} = \frac{\Delta\left(\frac{1-s_n}{s_n}\right)}{\frac{1-s_n}{s_n}}.\end{aligned}$$

□

Lemma 2.2. *Let $(1-s_{n+\dots})/s_{n+\dots}$ be the solution of homogeneous equation (1) and $d = 1, 2, \dots$. Then*

$$\frac{1-s_{n+d}}{s_{n+d}} = \prod_{j=n}^{n+d-1}(1+r_j)\cdot\frac{1-s_n}{s_n}, \tag{6}$$

and

$$\frac{1-s_n}{s_n} = \frac{1}{\prod_{j=n}^{n+d-1}(1+r_j)}\cdot\frac{1-s_{n+d}}{s_{n+d}}. \tag{7}$$

Proof. Indeed, from the homogeneous equation (1) we have

$$\frac{1-s_{n+d+1}}{s_{n+d+1}}\cdot\frac{s_{n+d}}{1-s_{n+d}} = 1 + r_{n+d},$$

in particular, for $d = 0$ we have

$$\frac{1-s_{n+1}}{s_{n+1}}\cdot\frac{s_n}{1-s_n} = 1 + r_n.$$

That, for d=1 we have

$$\frac{1-s_{n+2}}{s_{n+2}}\cdot\frac{s_{n+1}}{1-s_{n+1}} = 1 + r_{n+1}.$$

Continue for arbitrary $d = 2, 3, \dots$. Further, for $d-1$ we have

$$\frac{1-s_{n+d}}{s_{n+d}}\cdot\frac{s_{n+d-1}}{1-s_{n+d-1}} = 1 + r_{n+d-1}.$$

Whence multiplying the left and right parts of the system of equations

$$\frac{s_n}{1-s_n}\cdot\frac{1-s_{n+1}}{s_{n+1}}\cdot\dots\cdot\frac{1-s_{n+d}}{s_{n+d}} = \prod_{j=n}^{n+d-1}(1+r_j),$$

we obtain (6) and (7). □

Lemma 2.3. *Let $(1-s_{n+\dots})/s_{n+\dots}$ be the solution of equation (1) and $d = 1, 2, \dots$. Then*

$$\frac{1-s_{n+d}}{s_{n+d}} = \prod_{j=n}^{n+d-1}(1+r_j)\cdot\frac{1-s_n}{s_n} - \sum_{k=n}^{n+d-1}\prod_{j=k}^{n+d-1}(1+r_j)f_k, \tag{8}$$

and

$$\frac{1-s_n}{s_n} = \frac{1}{\prod_{j=n}^{n+d-1}(1+r_j)}\cdot\frac{1-s_{n+d}}{s_{n+d}} + \sum_{k=n}^{n+d-1}\frac{f_k}{\prod_{j=n}^{k}(1+r_j)}. \tag{9}$$

Proof. From equation (1) it follows for $d = 1$

$$\frac{1 - s_{n+1}}{s_{n+1}} = (1 + r_n) \cdot \frac{1 - s_n}{s_n} - f_n \,,$$

whence

$$\frac{1 - s_n}{s_n} = \frac{1}{1 + r_n} \cdot \frac{1 - s_{n+1}}{s_{n+1}} + \frac{f_n}{1 + r_n} \,.$$

Further, for $d = 2$, from (1) we have chains of identities

$$\begin{aligned}
\frac{1 - s_{n+2}}{s_{n+2}} &= (1 + r_{n+1}) \cdot \frac{1 - s_{n+1}}{s_{n+1}} - f_{n+1} = (1 + r_n)(1 + r_{n+1}) \frac{1 - s_n}{s_n} - (1 + r_{n+1}) f_n - f_{n+1} \\
&= \prod_{j=n}^{n+1} (1 + r_j) \frac{1 - s_n}{s_n} - \prod_{j=n+1}^{n+1} (1 + r_j) f_n - f_{n+1} \\
&= \prod_{j=n}^{n+1} (1 + r_j) \frac{1 - s_n}{s_n} - \prod_{j=n}^{n+1} (1 + r_j) \sum_{k=n}^{n+1} \frac{f_k}{\prod_{j=n}^{k} (1 + r_j)} \,,
\end{aligned}$$

whence

$$\frac{1 - s_n}{s_n} = \frac{1}{\prod_{j=n}^{n+1} (1 + r_j)} \cdot \frac{1 - s_{n+2}}{s_{n+2}} + \sum_{k=n}^{n+1} \frac{1}{\prod_{j=n}^{k} (1 + r_j)}$$

and so on. Further, for any $d = 1, 2, \ldots$, also from (1) we have chains of identities

$$\begin{aligned}
\frac{1 - s_{n+d}}{s_{n+d}} &= (1 + r_{n+d-1}) \cdot \frac{1 - s_{n+d-1}}{s_{n+d-1}} - f_{n+d-1} \\
&= \prod_{j=n}^{n+d-1} (1 + r_j) \frac{1 - s_n}{s_n} - \prod_{j=n}^{n+d-1} (1 + r_j) \sum_{k=n}^{n+d-1} \frac{f_k}{\prod_{j=n}^{k} (1 + r_j)} \\
&= \prod_{j=n}^{n+d-1} (1 + r_j) \frac{1 - s_n}{s_n} - \sum_{k=n}^{n+d-1} \prod_{j=k}^{n+d-1} (1 + r_j) f_k \,.
\end{aligned}$$

Whence we obtain (8) and (9). □

Lemma 2.4. *Let condition* (2) *hold for some* $p \in [1, \infty)$. *Then*

$$\prod_{n=-\infty}^{\infty} (1 + r_n) = \infty \,. \tag{10}$$

Proof. Assume the contrary, i.e. $\prod_{j=n}^{\infty} (1 + r_j) < \infty$. Let $n = \ldots, -2, -1 < d = 0$ in (9). We obtain

$$\frac{1 - s_n}{s_n} = \frac{1}{\prod_{j=0}^{n-1} (1 + r_j)} \cdot \frac{1 - s_0}{s_0} + \sum_{k=0}^{n-1} \frac{f_k}{\prod_{j=0}^{k} (1 + r_j)} \,. \tag{11}$$

Let $\hat{f} \in \ell_p$ and $\hat{f} > 0$. Then there exists $(1 - s_{-\infty})/s_{-\infty} := \lim_{n \to -\infty} (1 - s_n)/s_n$. Indeed, the convergence of the first term in (11) is obvious. To check the convergence of the second term write it in the form:

$$\sum_{k=0}^{n-1} \frac{\hat{f}_k}{\prod_{j=0}^{k} (1 + r_j)} = \frac{1}{\prod_{j=0}^{n-1} (1 + r_j)} \sum_{k=0}^{n-1} \prod_{j=k+1}^{n-1} (1 + r_j) \hat{f}_k \,. \tag{12}$$

It is clear that the first factor on the right side of (12) converges by our assumption, and the second one (by the same reason) converges as an improper integral:

$$0 \le \sum_{k=0}^{\infty} \prod_{j=k+1}^{\infty} (1 + r_j) \hat{f}_k \le \prod_{j=0}^{\infty} (1 + r_j) \|\hat{f}\|_{\ell_1} < \infty \,. \tag{13}$$

Hence there exists $\lim_{n\to-\infty}(1-s_n)/s_n = (1-s_{-\infty})/s_{-\infty}$. Let us show that $(1-s_{-\infty})/s_{-\infty} = 0$. Indeed, if $(1-s_{-\infty})/s_{-\infty} \neq 0$, then it is clear that $|(1-s_n)/s_n - (1-s_{-\infty})/s_{-\infty}| \leq |(1-s_{-\infty})/s_{-\infty}|/2$ for all $n << -1$,

$$\left|\frac{1-s_n}{s_n}\right| = \left|\frac{1-s_n}{s_n} - \frac{1-s_{-\infty}}{s_{-\infty}} + \frac{1-s_{-\infty}}{s_{-\infty}}\right|$$
$$\geq \left|\left|\frac{1-s_{-\infty}}{s_{-\infty}}\right| - \left|\frac{1-s_n}{s_n} - \frac{1-s_{-\infty}}{s_{-\infty}}\right|\right| \geq \frac{\frac{1-s_{-\infty}}{s_{-\infty}}}{2}\,,\ n << -1\,.$$

But from the last inequality it follows that $\|(1-s)/s\|_{\ell_p} = \infty$. Contradiction. With n tending to infinity in (11) we obtain

$$0 = \frac{1}{\prod_{j=0}^{\infty}(1+r_j)} \cdot \frac{1-s_0}{s_0} + \sum_{k=0}^{\infty} \frac{\hat{f}_k}{\prod_{j=0}^{k}(1+r_j)}\,. \tag{14}$$

It is obvious that the second term in (14) is positive, since

$$\sum_{k=0}^{\infty} \frac{\hat{f}_k}{\prod_{j=0}^{k}(1+r_j)} \geq \frac{\|\hat{f}\|_{\ell_p}}{\prod_{j=0}^{\infty}(1+r_j)} > 0\,, \tag{15}$$

and therefore, (14) implies $(1-s_0)/s_0 < 0$.

Now consider $n = 0 < d = 1, 2, \dots$ in (9):

$$\frac{1-s_0}{s_0} = \frac{1}{\prod_{j=0}^{d-1}(1+r_j)} \cdot \frac{1-s_d}{s_d} + \sum_{k=0}^{d-1} \frac{\hat{f}_k}{\prod_{j=0}^{k}(1+r_j)} \tag{16}$$

Since we assumed that $\prod_{j=0}^{\infty}(1+r_j) < \infty$, from (16) and by choosing $\hat{f}$, it easily follows that there exists $\lim_{d\to\infty}(1-s_d)/s_d$ and it equals to 0. Therefore, by passing to the limit in (16) as $d \to \infty$, we obtain:

$$\frac{1-s_0}{s_0} = \sum_{k=0}^{\infty} \frac{\hat{f}_k}{\prod_{j=0}^{k}(1+r_j)} \geq \sum_{k=0}^{\infty} \frac{1}{\prod_{j=0}^{k}(1+r_j)} \cdot \sum_{k=0}^{\infty} \hat{f}_k > 0\,. \tag{17}$$

Thus, $(1-s_0)/s_0 < 0 < (1-s_0)/s_0$, that is, contradiction. And finally the equality (10) holds. □

For any given $n = 0, \pm1, \pm2, \dots$, we define a function

$$d_n = \inf_{d=1,2,\dots} \left\{ d \mid \prod_{j=n}^{n+d-1} (1+r_j) \geq 2 \right\},\ r_n \geq 0,\ n = 0, \pm1, \pm2, \dots \tag{18}$$

It is clear that the function d_n is defined correctly. The function d_n was first used in [4].

Lemma 2.5. *Denote* $d_* := \sup_{n=0,\pm1,\pm2,\dots} d_n$. *Then*

$$d_* < \infty\,. \tag{19}$$

Proof. Let $\operatorname{supp}\psi(d/d_n) = [0;1]$, i.e.,

$$\psi\left(\frac{d}{d_n}\right) = \begin{cases} 1, & d = 1, \\ 1, & d = 2, \\ \cdots, & \cdots \\ 1, & d = d_n, \\ 0, & d = d_n + 1, \\ \cdots, & \cdots \end{cases}$$

for $n = 0, \pm 1, \pm 2, \dots$ is a cut-off function. Consider the function

$$\frac{1 - z_{n+d}}{z_{n+d}} = \prod_{j=n}^{n+d-1} (1 + r_j) \psi\left(\frac{d}{d_n}\right), \; d = 1, 2, \dots, \tag{20}$$

for fixed $n = 0, \pm 1, \pm 2, \dots$. It is easy to see that z_{n+d} is the solution of the equation

$$-\frac{1 - z_{n+d+1}}{z_{n+d+1}} + (1 + r_{n+d}) \frac{1 - z_{n+d}}{z_{n+d}} = f_{n+d}, \; d = 1, 2, \dots, \tag{21}$$

with the right side

$$f_{n+d} = -\prod_{j=n}^{n+d-1} (1 + r_j) \Delta\left[\psi\left(\frac{d}{d_n}\right)\right], \; d = 1, 2, \dots, \tag{22}$$

To apply assertions (5) to (20), (21) and (22), we need estimates of $\|f_{n+(\cdot)}\|_{\ell_p}$ and $\|z_{n+(\cdot)}\|_{\ell_p}$. To this extent we use (18) and properties of $\psi(d/d_n)$

$$\begin{aligned}
\|f_{n+(\cdot)}\|_{\ell_p}^p &= \sum_{k=-\infty}^{\infty} |f_{n+k}|^p = \sum_{k=n}^{n+d_n-1} \left| \prod_{j=n}^{k} (1 + r_j) \Delta\left[\psi\left(\frac{k-n}{d_n}\right)\right] \right|^p \\
&\le \sum_{k=n}^{n+d_n-1} \left| \prod_{j=n}^{k} (1 + r_j) \right|^p \sum_{k=n}^{n+d_n-1} \left| \Delta\left[\psi\left(\frac{k-n}{d_n}\right)\right] \right|^p \\
&\le \left| \prod_{j=n}^{n+d_n-1} (1 + r_j) \right|^p \frac{d_n}{d_n^p} \le \frac{2^{p+1}}{d_n^{p-1}} = \frac{c}{d_n^{p-1}}, \implies \|f_{n+(\cdot)}\|_{\ell_p} \le c d_n^{-1/p'}, \; p' = \frac{p}{p-1}, \; 1 \le p < \infty.
\end{aligned} \tag{23}$$

Similarly, we obtain an upper estimate for $\|z_{n+(\cdot)}\|_{\ell_p}$:

$$\begin{aligned}
\|z_{n+(\cdot)}\|_{\ell_p}^p &= \sum_{k=-\infty}^{\infty} |z_{n+k}|^p = \sum_{k=n}^{n+d_n-1} \left| \prod_{j=n}^{k} (1 + r_j) \left| \psi\left(\frac{k-n}{d_n}\right) \right| \right|^p \\
&\le \sum_{k=n}^{n+d_n-1} \left| \prod_{j=n}^{k} (1 + r_j) \right|^p \sum_{k=n}^{n+d_n-1} \left| \psi\left(\frac{k-n}{d_n}\right) \right|^p \\
&\le \left| \prod_{j=n}^{n+d_n-1} (1 + r_j) \right|^p \cdot 2 d_n \le 2^{2p+1} d_n = c d_n, \implies \|z_{n+(\cdot)}\|_{\ell_p} \le c d_n^{1/p}, \; 1 \le p < \infty.
\end{aligned} \tag{24}$$

In addition, we can obtain a lower estimate for $\|z_{n+(\cdot)}\|_{\ell_p}$

$$\begin{aligned}
\|z_{n+(\cdot)}\|_{\ell_p}^p &= \sum_{k=-\infty}^{\infty} |z_{n+k}|^p \ge \sum_{k=n}^{n+d_n-1} |z_{n+k}|^p = \sum_{k=n}^{n+d_n-1} \left| \prod_{j=n}^{k} (1 + r_j) \left| \psi\left(\frac{k-n}{d_n}\right) \right| \right|^p \\
&\ge \frac{d_n}{\left| \prod_{j=n}^{n+d_n-1} (1 + r_j) \right|^p} = \frac{d_n}{2^p} = c^{-1} d_n, \implies \|z_{n+(\cdot)}\|_{\ell_p} \ge c^{-1} d_n^{1/p}, \; 1 \le p < \infty.
\end{aligned} \tag{25}$$

Since $d_n = 1, 2, \dots$ for every $n = 0, \pm 1, \pm 2, \dots$ (see (2)), by (20)-(25), and (3), we obtain:

$$c^{-1} d_n^{1/p} \le \|z_{n+(\cdot)}\|_{\ell_p} \le c \|f_{n+(\cdot)}\|_{\ell_p} \le \frac{c}{d_n^{-1/p'}}$$

$$\Longrightarrow d_n \le c\,,\ n = 0, \pm 1, \pm 2, \cdots \Longrightarrow d_* = \sup_{n=0,\pm 1,\pm 2,\dots} d_n \le c < \infty\,.$$

From (19) we obtain (4) with $d = d_*$:

$$r_*(d_*) = \inf_{n=0,\pm 1,\pm 2,\dots} \left\{ \prod_{j=n}^{n+d_*-1} (1 + r_j) \right\} \ge \inf_{n=0,\pm 1,\pm 2,\dots} \left\{ \prod_{j=n}^{n+d_n-1} (1 + r_j) \right\} = 2\,.$$

□

Proof of Theorem 1.1. *Sufficiency.* Let (4) hold. Then

$$\frac{1}{\prod_{j=0}^{\infty}(1 + r_j)} = \infty\,,\ \prod_{j=0}^{\infty}(1 + r_j) = \infty\,. \tag{26}$$

The equalities (26) can be checked in a similar way. Let us check the second one:

$$\prod_{j=0}^{\infty}(1 + r_j) = \prod_{k=0}^{\infty} \left(\prod_{j=kd_*}^{(k+1)d_*-1} (1 + r_j) \right) \ge \prod_{k=0}^{\infty} 2^{k+1} = \infty\,.$$

From (4) by Theorem 1.1, we obtain that in the space ℓ_p for $p \in [1, \infty)$ the problem (1)-(3) is correctly solvable. Let $(1 - s_n)/s_n$, $n = 0, \pm 1, \pm 2, \dots$ be its solution. Then (1) has no other solutions in ℓ_p. Indeed, if $(1 - u_n)/u_n$, $n = 0, \pm 1, \pm 2, \dots$ is another solution of equation (1), $(1 - u)/u \ne (1 - s)/s$ and $(1 - u)/u \in \ell_p$, then

$$(1 - h_n)/h_n := (1 - s_n)/s_n - (1 - u_n)/u_n \in \ell_p\,,\ n = 0, \pm 1, \pm 2, \dots,$$

In addition,

$$-\frac{1 - s_{n+1}}{s_{n+1}} + (1 + r_n)\frac{1 - s_n}{s_n} = f_n\,,\ n = 0, \pm 1, \pm 2, \dots,$$

$$-\frac{1 - u_{n+1}}{u_{n+1}} + (1 + r_n)\frac{1 - u_n}{u_n} = f_n\,,\ n = 0, \pm 1, \pm 2, \dots,$$

yield

$$\frac{1 - h_n}{h_n} = c \prod_{j=0}^{n-1}(1 + r_j) < \infty\,,\ n = 0, \pm 1, \pm 2, \dots. \tag{27}$$

But from (27) and (26) it follows that $(1 - h)/h \in \ell_p$ only for $c = 0$ and therefore $(1 - s_n)/s_n \equiv (1 - u_n)/u_n$, $n = 0, \pm 1, \pm 2, \dots$, contradiction. Hence we obtain the statement of Theorem 1.1. In addition, since criterion (4) does not depend on $p \in [1, \infty)$, only one of the statements I) or II) of Theorem 1 is true (see (3)).

3. Proof of Main Result for $p = \infty$.

Proof of Theorem 1.1. *Necessity.* Let equation (1) be correctly solvable in ℓ_∞. Then equality (10) holds. Indeed, if on the contrary

$$\frac{1}{\prod_{j=0}^{\infty}(1 + r_j)} < \infty\,,\ \prod_{j=0}^{\infty}(1 + r_j) < \infty\,,$$

then let $(1-s_n)/s_n \in \ell_\infty$, $n=0,\pm1,\pm2,\dots$ be a solution of equation (1) with $f_n \in \ell_\infty$, $n=0,\pm1,\pm2,\dots$. Then the function

$$\frac{1-z_n}{z_n} = \frac{1-s_n}{s_n} + \prod_{j=0}^{n-1}(1+r_j),\ n=0,\pm1,\pm2,\dots$$

is also a solution of equation (1) and $(1-z_n)/z_n \in \ell_\infty$, $n=0,\pm1,\pm2,\dots$, which contradicts I) (see (3)), and therefore (10) holds. Let us check that $d_* < \infty$ (see (19)). Below we will use (20), (21) and (22) from Lemma 5. We need the following estimates of $\|f_{n+(\cdot)}\|_{\ell_\infty}$ and $\|z_{n+(\cdot)}\|_{\ell_\infty}$:

$$\begin{aligned}\|f_{n+(\cdot)}\|_{\ell_\infty} &= \sup_{k=0,\pm1,\pm2,\dots}|f_{n+k}| = \sup_{k=1,2,\dots,d_n}\prod_{j=n}^{n+k-1}(1+r_j)\left|\Delta\psi\left(\frac{k-n}{d_n}\right)\right| \\ &\le \prod_{j=n}^{n+d_n}(1+r_j)\frac{1}{d_n} = \frac{c}{d_n}\,.\end{aligned} \tag{28}$$

$$\begin{aligned}\|z_{n+(\cdot)}\|_{\ell_\infty} &= \sup_{k=0,\pm1,\pm2,\dots}|z_{n+k}| = \sup_{k=1,2,\dots,d_n}\prod_{j=n}^{n+k-1}(1+r_j)\psi\left(\frac{k-n}{d_n}\right) \\ &\le \prod_{j=n}^{n+d_n-1}(1+r_j)\frac{1}{d_n} = c\,.\end{aligned} \tag{29}$$

In addition,

$$\begin{aligned}\|z_{n+(\cdot)}\|_{\ell_\infty} &= \sup_{k=0,\pm1,\pm2,\dots}|z_{n+k}| = \sup_{k=1,2,\dots,d_n}\prod_{j=n}^{n+k-1}(1+r_j)\psi\left(\frac{k-n}{d_n}\right) \\ &\ge \frac{\sup_{k=1,2,\dots,d_n}\psi\left(\frac{k-n}{d_n}\right)}{\prod_{j=n}^{n+d_n-1}(1+r_j)} = c^{-1}\,.\end{aligned} \tag{30}$$

Since $d_n = 1,2,\dots$ for $n=0,\pm1,\pm2,\dots$ (see (2)), by (21)-(22), (28)-(29), I)-II) (see (3)) and (30), we obtain:

$$\begin{aligned}c^{-1} \le \|z_{n+(\cdot)}\|_{\ell_\infty} \le c\|f_{n+(\cdot)}\|_{\ell_\infty} \le \frac{c}{d_n} \\ \Longrightarrow\ d_n \le c,\ n=0,\pm1,\pm2,\cdots \Longrightarrow\ d_* = \sup_{n=0,\pm1,\pm2,\dots} d_n \le c < \infty\,.\end{aligned}$$

But then (4) holds because $r_*(d_*) \ge 2$.

Proof of Theorem 1.1. *Sufficiency.* Let $r(d) > 1$ for some $d = 1,2,\dots$. Then

$$\hat{J} = \sup_{n=0,\pm1,\pm2,\dots} J_n < \infty,\ J_n := \sum_{k=n}^{\infty}\frac{1}{\prod_{j=n}^{k}(1+r_j)},\ n=0,\pm1,\pm2,\dots. \tag{31}$$

Indeed, (31) holds because of the following relations:

$$\begin{aligned}J_n = \sum_{k=n}^{\infty}\frac{1}{\prod_{j=n}^{k}(1+r_j)} = \sum_{k=0}^{\infty}\sum_{l=n+kd_n}^{n+(k+1)d_n-1}\frac{1}{\prod_{j=n}^{l}(1+r_j)} \\ \le \sum_{k=0}^{\infty}\sum_{l=n+kd_*}^{n+(k+1)d_*-1}\frac{1}{\prod_{j=n}^{l}(1+r_j)} \le \sum_{k=0}^{\infty}\frac{d_*}{2^{kd_*}} = d_* + \frac{d_*}{2^{d_*}}\sum_{k=1}^{\infty}\frac{1}{2^k} = d_* + \frac{d_*}{2^{d_*}-1} < \infty\,.\end{aligned}$$

Let $f_n \in \ell_\infty$, $n = 0, \pm1, \pm2, \ldots$. Then it is obvious that the function

$$\frac{1-s_n}{s_n} = \sum_{k=n}^{\infty} \frac{f_k}{\prod_{j=n}^{k}(1+r_j)},\ n = 0, \pm1, \pm2, \ldots, \tag{32}$$

is the solution of equation (1), and

$$\left\| \frac{1-s}{s} \right\|_{\ell_\infty} = \|f\|_{\ell_\infty} \cdot \sup_{n=0,\pm1,\pm2,\ldots} J_n = \hat{J}\|f\|_{\ell_\infty}. \tag{33}$$

Under these conditions, the general solution $(1-S_n)/S_n$ of the equation (1) is of the form (see (32) and (33))

$$\frac{1-S_n}{S_n} = \frac{1-s_n}{s_n} + \frac{1-h_n}{h_n},\ n = 0, \pm1, \pm2, \ldots,$$

and therefore, according to (33) and (26), we obtain $(1-S_n)/S_n \in \ell_\infty$ only when $(1-h_n)/h_n \equiv 0$, $n = 0, \pm1, \pm2, \ldots$. Finally, statements I)-II) from (3) hold.

This publication has been supported by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP09259435).

References

[1] A. Tsoularis and J. Wallace, Analysis of logistic growth models *Mathematical biosciences*, 179:1, 2002, pp.21–55.

[2] M. Otelbaev, On summability with a weight of a solution of the Sturm-Liouville equation, *Mathematical notes of the Academy of Sciences of the USSR*, 16:6, 1974, pp. 1172–1179.

[3] M. Lukachev and L. Shuster, Conditions for correct solvability of a first order linear differential equation in space $L_p(\mathbb{R})$ and asymptotic properties of its solutions, *arXiv preprint math/0609752*, 2006.

[4] R. Oinarov and Z. Taspaganbetova, Criteria of boundedness and compactness of a class of matrix operators, *Journal of Inequalities and Applications*, 53, 2012, pp. 1–18.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

SOLUTION OF SYSTEMS OF HIGHER ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

SYRGAK KYDYRALIEV[1], ANARKUL URDALETOVA[2], ELENA BUROVA[1]

[1]American University of Central Asia, Bishkek, Kyrgyz Republic
[2]Kyrgyz-Turkish Manas University, Bishkek, Kyrgyz Republic
e-mails: kydyraliev_s@auca.kg, anarkul.urdaletova@manas.edu.kg, burova_e@auca.kg

Abstract. To integrate systems of linear ordinary differential equations of the first order, the method of Leonard Euler (1707 - 1783), based on the use of characteristic numbers of the matrix of system coefficients, and sometimes the method of integrable combinations of Jean Leron D'Alembert (1717 - 1783) are usually used. Combining these methods involves the use of characteristic numbers to obtain integrable combinations, as shown in the works of the authors, which creates a synergy effect - that can greatly simplify the solution process. It was previously shown by the authors that this effect is especially pronounced in the solution of inhomogeneous systems.

Keywords: systems of linear ordinary differential equations; solution of systems of equations with constant coefficients, Euler's method, d'Alembert's method, higher-order equations, constant coefficients, synergy.

AMS Subject Classification: 34A05, 34A25, 34A30, 34A34.

Ordinary differential equations are one of the most beautiful and studied mathematical disciplines. As one of the most famous scientists of the 20th century, Alan Turing, said: "Science is a differential equation." [1]. His great predecessor Laplace believed that the entire universe from a mathematical point of view is just a huge set of differential equations. .... Therefore, the ability to solve and integrate them, as mathematicians say, was an urgent need of the time. [2]

Of course, mathematical science is developing rapidly. It has long been proven that not all ordinary differential equations, even second-order linear equations with variable coefficients, are quadrature integrable. [3, 4]. This circumstance contributed to the rapid development of methods for the approximate solution of differential equations. At the same time, despite the great progress in this direction, the problem of determining the types of differential equations and their systems that can be integrated with quadratures remains relevant.

In this paper, this problem is solved for a certain type of system of high-order linear ordinary differential equations. The results were obtained based on the method developed by the authors [5-9].

Consider the system of linear ordinary differential equations of the form

$$\begin{cases} y'' + sy' = ay + bz + f(x), \\ z'' + sz' = cy + dz + g(x), \end{cases} \tag{1}$$

where the coefficients a, b, c, d, s are constant numbers, and f and g, are given functions.

By the classical theory [3, 4], the characteristic equation of system (1) is the equation

$$\Delta(\lambda) = 0, \tag{2}$$

where the function $\Delta(\lambda)$ has the form:

$$\Delta(\lambda) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - c \cdot b.$$

The characteristic numbers of the system (1) are the roots of equation (2).

Let p and q be the characteristic numbers of the system. Then, to solve the system, at the first step we subtract the function py from the left and right parts of the 1st equation of system (1) and pz from both parts of the 2nd equation:

$$\begin{cases} y'' + sy' - py = (a-p)y + bz + f(x), \\ z'' + sz' - pz = cy + (d-p)z + g(x). \end{cases} \tag{3}$$

Since the number p is characteristic, the determinant

$$\begin{vmatrix} a-p & b \\ c & d-p \end{vmatrix}$$

equals zero, which means that the rows of the determinant are proportional. Let the coefficient of proportionality be equal to k.

Multiply the 2nd equation by k :

$$\begin{cases} y'' + sy' - py = (a-p)y + bz + f(x), \\ k[z'' + sz' - pz] = kcy + k(d-p)z + kg(x), \end{cases}$$

and subtract the second equation from the first.

Then

$$[y - kz]'' + s[y - kz]' - p[y - kz] = f(x) - kg(x). \tag{4}$$

Equation (4) is a second-order linear ordinary differential equation with respect to the unknown function $y - kz$. After integrating it, we obtain an expression for $y - kz$ in the form

$$y - kz = F(x;\ C_1;\ C_2), \tag{5}$$

where F is some definite function.

We repeat the procedure, taking q instead of p, and we get

$$y - mz = G(x;\ C_3;\ C_4), \tag{6}$$

where G is some definite function.

Let us consider equalities (5) and (6) as a system with respect to the unknowns y and z, and, solving it, we obtain the solution of system (1).

Example 1

Let's solve the system of equations

$$\begin{cases} y'' - 5y' = 3y - 7z + 20e^{7x}, \\ z'' - 5z' = 9y - 13z + 6e^{x}. \end{cases} \tag{7}$$

The characteristic numbers of the coefficient matrix of system (7) are the roots of the equation

$$\begin{vmatrix} 3-\lambda & -7 \\ 9 & -13-\lambda \end{vmatrix} = (3-\lambda)(-13-\lambda) - 9 \cdot (-7) = 0 \ => \ \lambda_1 = -4; \ \ \lambda_2 = -6.$$

1) Using the root $\lambda = -4$, we rewrite system (7) in the form

$$\begin{cases} y'' - 5y' + 4y = 7y - 7z + 20e^{7x}, \\ z'' - 5z' + 4y = 9y - 9z + 6e^{x}. \end{cases}$$

Multiply the first equation by 9, the second equation by 7, subtract and get

$$(9y'' - 7z'') - 5(9y' - 7z') + 4(9y - 7z) = 180e^{7x} - 42e^{x} \tag{8}$$

For clarity, we introduce the notation

$$u = 9y - 7z$$

and obtain a linear ordinary differential equation of the second order

$$u'' - 5u' + 4u = 180e^{7x} - 42e^{x}.$$

Let's solve it using the Chain method [7, 8]. To do this, we find the roots of the corresponding characteristic equation

$$\mu^2 - 5\mu' + 4\mu = 0 \Rightarrow \mu_1 = 1;\ \mu_2 = 4,$$

and use them to rewrite the second-order equation as a chain of first-order equations:

$$u' - 1 \cdot u = w;\ \ w' - 4w = 180e^{7x} - 42e^{x}.$$

Since the equation

$$w' - 4w = 180e^{7x} - 42e^{x}$$

can be written in the form

$$(we^{-4x})'e^{4x} = 180e^{7x} - 42e^{x},$$

its solution

$$w = 60e^{7x} + 14e^{x}.$$

Thus,

$$u' - u = 60e^{7x} + 14e^{x}.$$

This equation can also be solved by folding the left side:

$$(ue^{-x})'e^{x} = 60e^{7x} + 14e^{x}.$$

Therefore, the solution to equation (8) is the function

$$9y - 7z = Ae^{x} + Be^{4x} + 10e^{7x} + 14xe^{x}.$$

2) Repeat the procedure, taking the second characteristic number $\lambda = -6$:

$$\begin{cases} y'' - 5y' + 6y = 9y - 7z + 20e^{7x}, \\ z'' - 5z' + 6y = 9y - 7z + 6e^{x}. \end{cases}$$

The difference in the equations of the system gives a linear equation of the second order:

$$(y'' - z'') - 5(y' - z') + 6(y - z) = 20e^{7x} - 6e^{x}.$$

Integrating it in the same way as equation (8), we obtain

$$y - z = Ce^{2x} + De^{3x} + e^{7x} - 3e^{x}.$$

It remains to find y and z. To do this, we need to solve the system

$$\begin{cases} 9y - 7z = Ae^{x} + Be^{4x} + 10e^{7x} + 14xe^{x}, \\ y - z = Ce^{2x} + De^{3x} + e^{7x} - 3e^{x}. \end{cases}$$

Multiply the second equation by 7 and subtract from the first:

$$2y = Ae^{x} + Be^{4x} - 7Ce^{2x} - 7De^{3x} + 3e^{7x} + (14x + 21)\, e^{x}.$$

Now, multiply the second equation by 9 and subtract from the first:

$$2z = Ae^{x} + Be^{4x} - 9Ce^{2x} - 9De^{3x} + e^{7x} + (14x + 27)\, e^{x}.$$

As a result, it turned out that the solution to system (7) is a pair of functions

$$\begin{cases} y = 0.5\left[Ae^{x} + Be^{4x} - 7Ce^{2x} - 7De^{3x} + 3e^{7x} + (14x + 21)e^{x}\right], \\ z = 0.5\left[Ae^{x} + Be^{4x} - 9Ce^{2x} - 9De^{3x} + e^{7x} + (14x + 27)\, e^{x}\right]. \end{cases}$$

To obtain a system solution (7), we used two linear combinations of solutions $9y - 7z$ and $y - z$, each of which was determined by the corresponding characteristic number of the system.

But what to do in the case when the characteristic equation has only one multiple root? Consider an example in which a "super-multiple system" will appear - a system in which both

the characteristic equation of the coefficient matrix of the system and the characteristic equation of a differential equation determined by a linear combination of solutions have multiple roots.

Example 2

Let's solve the system of equations

$$\begin{cases} y'' + 2y' = 5y - 9z + 106260x^{19}e^{-x}, \\ z'' + 2z' = 4y - 7z. \end{cases} \tag{9}$$

The characteristic numbers of the coefficient matrix of system (9) are the roots of the equation

$$\begin{vmatrix} 5-\lambda & -9 \\ 4 & -7-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = -1.$$

We use the root λ = -1 to rewrite system (9) as

$$\begin{cases} y'' + 2y' + C = 6y - 9z + 106260x^{19}e^{-x}, \\ z'' + 2z' + z = 4y - 6z. \end{cases} \tag{10}$$

Multiplying the second equation by 1.5, and subtracting the result from the first equation, we get

$$(y'' - 1.5z'') + 2(y' - 1.5z') + (y - 1.5z) = 106260x^{19}e^{-x}$$

Let us introduce the notation

$$u = y - 1.5z$$

and obtain a linear ordinary differential equation of the second order

$$u'' + 2u' + u = 106260x^{19}e^{-x}. \tag{11}$$

Let's solve it using the chain method. So, let's find the roots of the corresponding characteristic equation

$$\mu^2 + 2\mu' + \mu = 0 \Rightarrow \mu_1 = \mu_2 = -1,$$

and use them to rewrite the second-order equation as a chain of first-order equations:

$$u' + 1 \cdot u = w; \;\; w' + w = 106260x^{19}e^{-x}.$$

Sequentially integrating these equations, we obtain a solution to the equation (11) is the function

$$y - 1.5z = e^{-x}(253x^{21} + C_1x + C_2).$$

Note that the process of solving equations will be greatly simplified if we note that, for example,

$$u' + 1 \cdot u = (ue^x)'e^{-x}.$$

Now, you can pay attention to the fact that the expression $y - 1.5z$ is available in the right parts of the equations of the system (10). Therefore, the second equation of system (10) can be rewritten as

$$z'' + 2z' + z = 4[e^{-x}(253x^{21} + C_1x + C_2)].$$

This equation can also be rewritten as a chain of first-order equations:

$$z' + 1 \cdot z = v; \;\; v' + v = 4(253x^{21} + C_1x + C_2)e^{-x}.$$

We solve these equations one by one, and we get that

$$z = e^{-x}(2x^{23} + \frac{2C_1x^3}{3} + 2C_2x^2 + C_3x + C_4).$$

It remains to use the equality

$$y - 1.5z = e^{-x}(253x^{21} + C_1x + C_2),$$

and calculate the value of y:

$$y = 1.5z + e^{-x}(253x^{21} + C_1x + C_2) =$$

$$= 1.5e^{-x}(2x^{23} + \frac{2C_1x^3}{3} + 2C_2x^2 + C_3x + C_4) + e^{-x}(253x^{21} + C_1x + C_2) =$$
$$= e^{-x}[3x^{23} + 253x^{21} + C_1x^3 + 3C_2E^2 + (1.5C_3 + C_1)x + 1.5C_4 + C_2].$$

Note that the approach used to solve systems of high-order linear equations can also be used to solve some systems of linear equations with variable coefficients.

Example 3

Let's solve the system of equations

$$\begin{cases} tgx \cdot y' = 4y - z + 4sin^5x, \\ tgx \cdot z' = y + 2z. \end{cases} \tag{12}$$

The characteristic equation of the system

$$\begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = 0$$

has only one multiple root equal to 3. Using this root, we can rewrite system (12) in the form

$$\begin{cases} tgx \cdot y' - 3y = y - z + 4sin^5x, \\ tgx \cdot z' - 3z = y - z. \end{cases} \tag{13}$$

Let us subtract the second from the first equation of the system (13):

$$tgx(y' - z') - 3(y - z) = 4sin^5x,$$

and we obtain a first-order linear ordinary differential equation with respect to the unknown function $y - z$.

Integrate and get

$$y - z = (sin^2x - cos^2x + C)sin^3x.$$

In order to complete the process of solving the system, we substitute the value of the function $y - z$ into the right side of any of the equations of the system (13) (let it be the first one) and integrate the resulting equation:

$$tgx \cdot y' - 3y = (sin^2x - cos^2x + C)sin^3x + 4sin^5x,$$
$$tgx\left(\frac{y}{sin^3x}\right)' = 5sin^2x - cos^2x + C,$$
$$\left(\frac{y}{sin^3x}\right)' = 5sinx \cdot cosx - \frac{cos^3x}{sinx} + C\frac{cosx}{sinx},$$
$$\frac{y}{sin^3x} = 2.5sin^2x - [ln\,|sinx| - 0.5sin^2x] + Cln\,|sinx| + B,$$
$$y = \{3sin^2x + (C-1)ln\,|sinx| + B\}sin^3x.$$

To determine the value of z, it is enough to perform an elementary algebraic operation:

$$z = y - (y - z) = \{3sin^2x + (C-1)ln\,|sinx| + B\}\,sin^3x - (sin^2x - cos^2x + C)sin^3x$$

$$= (sin^2x + 1 - C + (C-1)ln\,|sinx| + B)sin^3x.$$

Thus, it turned out that the solution to system (12) is the functions

$$\begin{cases} y = \{3sin^2x + (C-1)ln\,|sinx| + B\}\,sin^3x, \\ z = (sin^2x + 1 - C + (C-1)ln\,|sinx| + B)sin^3x. \end{cases}$$

We get a solution to system (12) due to the fact that the linear combination of solutions $y - z$ is on the right side of the transformed system (12) in the system (13). Perhaps this is a lucky break, but it turns out not. Repeating in detail the proof of the corresponding statement for systems with constant coefficients [11], we can show that in the case of multiple roots of the characteristic equation, the linear combination of solutions of the system, determined by the characteristic number, will always be on the right-hand side of the transformed system.

Let's consider a system of the form

$$\begin{cases} (\beta x+\gamma)^2 y'' + s(\beta x+\gamma)y' = ay + bz + f(x), \\ (\beta x+\gamma)^2 z'' + s(\beta x+\gamma)z' = cy + dz + g(x), \end{cases} \tag{14}$$

where the coefficients $a, b, c, d, \beta, \gamma, s$ are arbitrary constants, f and g are given functions. As before, the characteristic numbers of the system (14) are the roots of the equation

$$\Delta(\lambda) = 0, \tag{15}$$

where the function $\Delta(\lambda)$ has the form:

$$\Delta(\lambda) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - c \cdot b.$$

Let p and q be the characteristic numbers of the system. Then, in order to solve the system, we subtract the function py from the left and right sides of the 1st equation of the system (14) and pz from both sides of the 2nd equation:

$$\begin{cases} (\beta x+\gamma)^2 y'' + s(\beta x+\gamma)y' - py = (a-p)y + bz + f(x), \\ (\beta x+\gamma)^2 z'' + s(\beta x+\gamma)z' - pz = cy + (d-p)z + g(x). \end{cases} \tag{16}$$

Since the number p is characteristic, the lines of the determinant are proportional. Let the proportionality factor be k.

Multiply the 2nd equation by k :

$$\begin{cases} (\beta x+\gamma)^2 y'' + s(\beta x+\gamma)y' - py = (a-p)y + bz + f(x), \\ k[(\beta x+\gamma)^2 z'' + s(\beta x+\gamma)z' - pz] = kcy + k(d-p)z + kg(x), \end{cases}$$

and subtract the second equation from the first.

Then

$$(\beta x+\gamma)^2 [y-kz]'' + s(\beta x+\gamma)[y-kz]' - p[y-kz] = f(x) - kg(x). \tag{17}$$

Equation (17) is the Euler-Lagrange differential equation for the unknown function $y - kz$. Integrating it, we obtain an expression for $y - kz$ in the form

$$y - kz = F(x; C_1; C_2), \tag{18}$$

where F is some definite function.

We repeat the procedure, taking q instead of p, and obtain

$$y - mz = G(x; C_3; C_4), \tag{19}$$

where G is some definite function.

Consider equalities (18) and (19) as a system with respect to unknowns y and z,, and, having solved it, we obtain a solution to system (14).

Example 4

Let's solve the system of equations

$$\begin{cases} x^2 y'' - xy' = -3y + 4z + 6x^2, \\ x^2 z'' - xz' = -3y + 5z. \end{cases} \tag{20}$$

The characteristic numbers of the matrix of coefficients of the system (20) are the roots of the equation

$$\begin{vmatrix} -3-\lambda & 4 \\ -3 & 5-\lambda \end{vmatrix} = (-3-\lambda)(5-\lambda) - 4 \cdot (-3) = 0 \Rightarrow \lambda_1 = -1; \lambda_2 = 3.$$

1) Using the root $\lambda = -1$, we rewrite system (20) in the form

$$\begin{cases} x^2 y'' - xy' + y = -2y + 4z + 6x^2, \\ x^2 z'' - xz' + z = -3y + 6z. \end{cases}$$

Multiply the first equation by 1.5 and subtract the second from the product:

$$x^2(1.5y'' - z'') - x(1.5y' - z') + (1.5y - z) = 9x^2. \tag{21}$$

So, we introduce the notation $u = 1.5y - z$ and obtain the Euler-Lagrange equation

$$x^2u'' - xu' + u = 9x^2.$$

Let's solve it using the Chain method [10]. For this, we solve the corresponding algebraic system

$$\begin{cases} k + m = 1 - (-1), \\ k \cdot m = 1, \end{cases}$$

and use the roots $k = m = 1$ in order to rewrite equation (21) as a chain of first-order equations:

$$xu' - 1 \cdot u = v; \quad xv' - 1 \cdot v = 9x^2.$$

Equation

$$xv' - 1 \cdot v = 9x^2$$

has a solution

$$v = 9x^2 + Cx.$$

Thus,

$$xu' - u = 9x^2 + Cx.$$

Therefore, the solution to equation (21) is the function

$$1.5y - z = 9x^2 + C_1x\ln x + C_2x.$$

2) We repeat the procedure, taking the second characteristic number $\lambda = 3$.

$$\begin{cases} x^2y'' - xy' - 3y = -6y + 4z + 6x^2, \\ x^2z'' - xz' - 3z = -3y + 2z. \end{cases}$$

Multiply the first equation by 0.5 and subtract the second from the product:

$$x^2(0.5y'' - z'') - x(0.5y' - z') + (0.5y - z) = 6x^2.$$

Integrating it in the same way as equation (21), obtain

$$0.5y - z = -x^2 + C_3x^3 + \frac{C_4}{x}.$$

It remains to find y and z. For this we solve the system

$$\begin{cases} 1.5y - z = 9x^2 + C_1x\ln x + C_2x, \\ 0.5y - z = -x^2 + C_3x^3 + \frac{C_4}{x}, \end{cases}$$

and we find that the solution to system (20) is a pair of functions

$$\begin{cases} y = 10x^2 + C_1x\ln x + C_2x - C_3x^3 - \frac{C_4}{x}, \\ z = 6x^2 + 0.5C_1x\ln x + 0.5C_2x - 1.5C_3x^3 - 1.5\frac{C_4}{x}. \end{cases}$$

Conclusion

The results of the work demonstrate the powerful effect of synergy arising from the combination of the approaches of Euler and d'Alembert to the solution of systems of linear ordinary differential equations. In this case, it is demonstrated how the unified approach allows, without any special theoretical difficulties, to move from systems of ordinary differential equations of the first order to systems of equations of the second order. We hope that it is clear from the content of the article that similar results also take place for systems of equations of higher orders.

Also, no additional theoretical difficulties are caused by the consideration of systems for which the roots of the characteristic equation of the matrix of coefficients have complex roots [5, 6].

References

[1] Quotes and aphorisms. https://quote-citation.com/aphorism/49778 (accessed 23 July 2023).

[2] Tyapkin A.A., Shibanov A.S., *Poincare*, Molodaya Guardia, 1982, 415 p.

[3] Guterman, M.M., Nitecki, Z.H., *Differential Equations. A Fist Course*, Saunders College Publishing, 1988, 689 p.

[4] Boyce W.E., DiPrima R.C., *Elementary Differential Equations and Boundary Value Problems*, Wiley, 2012, 832 p.

[5] Kydyraliev S.K., Urdaletova A.B., Direct Integration of Systems of Linear Differential and Difference Equations, *Filomat*, Vol.33, No.5, 2019, pp.1453—1461.

[6] Kydyraliev S.K., Urdaletova A.B., New method for solving systems of linear differential and difference equations,*Bulletin of KazNU*, Vol.63, No.4, 2009, pp.21–29.

[7] Urdaletova A.B., Kydyraliev S.K., A new Algorithm to Solve Linear Differential Equations,*Analitical and Approximate Methods – Proceeds International Conference/ Shacer Verlag*, 2003.

[8] Urdaletova A.B., Kydyraliev S.K., A new approach to solving linear ordinary differential equations,*Bulletin of KazNU*, No.3, 2008, pp.134–141.

[9] Kydyraliev S.K., Urdaletova A.B., Burova E.S., Solution of some linear systems of ordinary differential equations with variable coefficients, *Advances in Differential Equations and Control Processes* , Vol.29, 2022, pp.1–11.

[10] Kydyraliev S.K., Urdaletova A.B., Burova E.S. Advantages of the Chaining Method for Solving Linear Differential and Difference Equations, *Vestnik KRSU*, Vol.20, No.8, pp. 16–20.

[11] Kydyraliev S.K., Urdaletova A.B., Direct Integration of Systems of Linear Differential and Difference Equations, *Filomat*, Vol.33, No.5, 2019, pp.1453—1461.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

NONLINEAR n – VELOCITY SINGULARLY PERTURBED INVERSE PROBLEM OF KAC-TYPE

TAALAIBEK OMUROV, ZHYLDYZ SARKELOVA

Kyrgyz National University named after Jusup Balasagyn, Bishkek, Kyrgyz Republic
e-mails: omurovtd@mail.ru, sjjyldyzaa@gmail.com

Abstract. There are numerous works in the field of direct and inverse transfer problems (ITP) of the Kac and Boltzmann type in bounded and unbounded domains, for example [1,4,5,7] and others. But what has been said cannot be attributed to singularly perturbed inverse problems (SPIP) in the transfer theory of the Kac type and Boltzmann type, which is the relevance of the study of this work.

Keywords: singularly perturbed Kac-type equation, n-velocity transport problem, inverse problem, unbounded domain.

AMS Subject Classification: 35Q49, 35Q35, 35R30.

1. Introduction

It is known that in the field of direct problems of mathematical physics, which are connected by singularly perturbed equations, there are fundamental works, for example [2,3,6], etc. But the problematic issues that are associated with single-velocity transfer SPIP become more complicated in the field of multi-velocity transfer SPIP since the implementation of most methods for solving multi-velocity transfer equations (TE) requires the construction of coordinate systems that are built based on specific problems and can be rectangular Cartesian, where they are specified, or curvilinear, where they are converted.

At the same time, some additional restrictions are required regarding new functions, with which new variables are associated with certain relationships depending on the given areas. A change of variables of this nature is motivated either by

a) practical interest that new variables represent about the problems under study (for example, the transfer problem has some kind of symmetry, etc.), or:

b) in the case when it is difficult to study the transfer problem in given variables.

Note that the verification of additional restrictions, in particular, which are obtained under the conditions of a change of two or three variables, as it can be indicated (for example: in polar, cylindrical, and spherical coordinate systems). And for larger numbers of speeds, it is difficult to indicate these restrictions, or in the general case, it is not even possible to indicate these restrictions since they are associated not only with the physical values of those functions that are contained in the TE, but with those areas that the transfer problem sets, and even more so in an unlimited area.

Therefore, in this paper, we study the n-velocity Kac-type transfer SPIP in an unbounded domain, while the proposed method for solving the original inverse problem (IP) allows us to remain in Cartesian coordinates, regardless of the number of phase plane variables, i.e., we systematically generalize the idea of the developed algorithm of an asymptotic nature (AN) [8] for n-velocity SPIP.

The results of the studied SPIP are obtained based on the representation of AN, and the estimate of the proximity of the solutions of the SPIP and the degenerate IP is proved in the sense of $W_h^p(\Omega_0 = (0, T_0) \times R^n)$ when a priori information is given from $L^p(R^n)$.

Since $W_h^p(\Omega_0)$ is not a Banach space, we can introduce $\tilde{W}_h^p(\Omega_0)$, $(p \geq 1)$ is the Lebesgue space, by definition, is the completion of the space $W_h^p(\bar{\Omega}_0)$, where $W_h^p(\bar{\Omega}_0)$ is dense in $\tilde{W}_h^p(\Omega_0)$, and the integrals are understood in the sense of Lebesgue.

Consider an n-velocity coefficient-inverse transfer problem of the form:

$$\varepsilon^{\beta}\left[\frac{\partial}{\partial t}\left(E^{1,1,...,1}_{(a_1,...,a_n)}U_{\varepsilon}\right) + U_{\varepsilon}^2(t, x_1, ..., x_n)\right] + \lambda E^{1,1,..,1}_{(a_1,...,a_n)}U_{\varepsilon} + h_0(x_1, ..., x_n)U_{\varepsilon} = \\ = Z_{\varepsilon}(x_1, ..., x_n)f(t), \quad (1)$$

$$\begin{cases} (U^{(i)}_{\varepsilon t}(t, x_1, ..., x_n)|_{t=0} = V^{(i)}_t(0, x_1, ..., x_n) + (\sum_{j=1}^n 2a_j x_j \varepsilon^{-1})^i \exp\left(-\frac{1}{\varepsilon}\sum_{j=1}^n x_j^2\right), \\ \\ V^{(i)}_t(0, x_1, ..., x_n) = \varphi_i(x_1, ..., x_n), \ (i = 0, 1), \ \ \forall (x_1, ..., x_n) \in R^n, \end{cases} \quad (2)$$

$$\begin{cases} (U^{1,1,...,1}_{(a_1,...,a_n)})|_{t=T} \equiv (U_{\varepsilon t} + \sum_{j=1}^n a_j U_{\varepsilon x_j})|_{t=T} = g_0(x_1, ..., x_n) + g_{\varepsilon}(x_1, ..., x_n), \\ \\ (V_t + \sum_{j=1}^n a_j V_{x_j})_{t=T} = g_0(x_1, ..., x_n), \ \ \forall (x_1, ..., x_n) \in R^n, \end{cases} \quad (3)$$

at the same time, information is entered regarding the initial data in the form:

$$\begin{cases} ||g_{\varepsilon}||_{L^p(R^n)} = \left(\iint_{R^n} |g_{\varepsilon}(x_1, ..., x_n)|^p dQ\right)^{\frac{1}{p}} \leq \Delta_1(\varepsilon), (dQ = dx_1...dx_n), \\ \\ ||g_{\varepsilon}(x_1 - a_1(t-s), ..., x_n - a_n(t-s))||_{L^p(0,T)} = \\ \\ = \left(\sup_{\bar{\Omega}_0} \int_0^t |g_{\varepsilon}(x_1 - a_1(t-s), ..., x_n - a_n(t-s))|^p ds\right)^{\frac{1}{p}} \leq \Delta_2(\varepsilon), \ (\Delta_1, \Delta_2 \leq \Delta_0(\varepsilon) \underset{\varepsilon \to 0}{\to} 0), \\ ||U_{\varepsilon}(0, x_1, ..., x_n) - V(0, x_1, ..., x_n)||_{L^p(R^n)} \leq \left(\frac{\pi}{p}\right)^{\frac{1}{p}} \varepsilon^{\frac{1}{p}} = \gamma_0 \varepsilon^{\frac{1}{p}}, \\ \\ \mathrm{E}^{1,1,...,1}_{(a_1,...,a_n)} = \frac{\partial}{\partial t} + \sum_{j=1}^n a_j \frac{\partial}{\partial x_j}, \end{cases} \quad (4)$$

where $(U_{\varepsilon}, Z_{\varepsilon})$ are unknown functions. Under these conditions, it is required to show the closeness of the solutions of the SPIP and the degenerate IP in the class of functions $W_h^p(\Omega_0)$, here:

$$0 < h_0(x_1, ..., x_n), \ f(t), \ \varphi_i(x_1, ..., x_n),$$

$$g_0(x_1, ..., x_n), \ g_{\varepsilon}(x_1, ..., x_n), \ 0 < a_j, \ \lambda = const, \ (j = \overline{1, n}), \ 0 < \beta < \frac{1}{2}$$

are known, and

$$\begin{cases} h_0 \equiv \sum_{j=1}^{n} h_j(x_j) + h(x_1,...,x_n); 0 \le h \le \tilde{h} = const,\ 0 \le h_0 \le \tilde{h}_0 = const,\ \forall (x_1,...,x_n) \in R^n, \\ \left(\int_{R^n} h(x_1,...,x_n)dQ\right)^{\frac{1}{p}} \le \gamma_1 = const, \\ \sup\limits_{[0,T]} |f^{(i)}(t)| \le f_0 = const,\ (i=0,1);\ f(0)=0,\ f(T) \ne 0, \\ f(T) - \int_0^T \exp\left(-\frac{\lambda}{\varepsilon^{\beta}}(T-s)\right) f'(s)ds = M_0(T,\lambda,\varepsilon^{\beta}) \ne 0, \\ \forall \varepsilon \in (0,1),\ (\varepsilon = 0);\ 0 < \frac{1}{\lambda} = const << 1. \end{cases} \tag{5}$$

2. Integralization of the degenerate inverse problem

It is known that $\varepsilon = 0$ for the singularly perturbed inverse problem (1) - (3) follows from the degenerate inverse problem:

$$E^{1,1,...,1}_{(a_1,...,a_n)} V(t,x_1,...,x_n) + \frac{1}{\lambda} h_0(x_1,...,x_n)V = \frac{1}{\lambda}\tilde{Z}(x_1,...,x_n)f(t), \tag{6}$$

$$V_t^{(i)}(0,x_1,...,x_n) = \varphi_i(x_1,...,x_n),\ (i=0,1),\ \forall (x_1,...,x_n) \in R^n, \tag{7}$$

$$\left(E^{1,1,...,1}_{(a_1,...,a_n)} V\right)|_{t=T} = g_0(x_1,...,x_n),\ \forall (x_1,...,x_n) \in R^n,$$

where the vector function $\Phi = \left(V, \tilde{Z}\right)$ is unknown.
Discussing Similarly, under conditions (5), (7), (8) from equation (6) the following system follows:

$$\begin{cases} E^{1,1,...,1}_{(a_1,...,a_n)} V + \frac{1}{\lambda} h_0(x_1,...,x_n)V = (f(T))^{-1} f(t)\left[g_0(x_1,...,x_n) + \frac{1}{\lambda}h_0 V\right] \equiv \\ \equiv (B_0 V)(t,x_1,...,x_n), \\ \tilde{Z}(x_1,...,x_n) = (f(T))^{-1}\left[\lambda g_0(x_1,...,x_n) + h_0 V(t,x_1,...,x_n)\right]. \end{cases} \tag{8}$$

Therefore, based on

$$V = Q(t,x_1,...,x_n)\exp\left[-\left(\sum_{j=1}^{n} \frac{1}{\lambda a_j}\int_{-\infty}^{x_j} h_j(\tau_j)d\tau_j\right)\right], \tag{9}$$

with the condition

$$Q|_{t=0} = \varphi_0(x_1,...,x_n)\exp\left[\sum_{j=1}^{n} \frac{1}{\lambda a_j}\int_{-\infty}^{x_j} h_j(\tau_j)d\tau_j\right] \equiv \psi_0(x_1,...,x_n),\ \forall (x_1,...,x_n) \in R^n, \tag{10}$$

from (6) we have an equation of the form:

$$E^{1,1,...,1}_{(a_1,...,a_n)} Q = \left\{-\frac{1}{\lambda}h(x_1,...,x_n)V + (B_0V)(t,x_1,...,x_n)\right\}\exp\left(\sum_{j=1}^{n} \frac{1}{\lambda a_j}\int_{-\infty}^{x_j} h_j(\tau_j)d\tau_j\right), \tag{11}$$

that is (11), (12) are the Cauchy problem with respect to the function $Q(t, x_1, ..., x_n)$. Then from (2) it follows:

$$Q = \psi_0(x_1 - a_1 t, ..., x_n - a_n t) + \int_0^t \left(\exp\left(\sum_{j=1}^{n} \frac{1}{\lambda a_j} \int_{-\infty}^{x_j - a_j t} h_j(\tau_j) d\tau_j \right) \right) \times$$

$$\times \left\{ -\frac{1}{\lambda} h(x_1 - a_1(t-s), ..., x_n - a_n(t-s)) V(s, x_1 - a_1(t-s), ..., x_n - a_n(t-s)) + \right.$$

$$\left. + (B_0 V)\,(s, x_1 - a_1(t-s), ..., x_n - a_n(t-s)) \right\} ds. \quad (12)$$

Therefore, by substituting (13) with (10), we have

$$V = \varphi_0(x_1 - a_1 t, ..., x_n - a_n t) \exp\left[- \left(\sum_{j=1}^{n} \frac{1}{\lambda a_j} \int_{x_j - a_j t}^{x_j} h_j(\tau_j) d\tau_j \right) \right] +$$

$$+ \int_0^t \exp\left[- \left(\sum_{j=1}^{n} \frac{1}{\lambda a_j} \int_{x_j - a_j (t-s)}^{x_j} h_j(\tau_j) d\tau_j \right) \right] \{ - \frac{1}{\lambda} h(x_1 - a_1(t-s), ..., x_n - a_n(t-s)) \times$$

$$\times V(s, x_1 - a_1(t-s), ..., x_n - a_n(t-s)) + (B_0 V)\,(s, x_1 - a_1(t-s), ..., x_n - a_n(t-s)) \} ds \equiv$$

$$\equiv (BV)(t, x_1, ..., x_n), \quad (13)$$

where (14) is a loaded integral equation of the second kind with respect to the function $V(t, x_1, ..., x_n)$. Since

$$0 < \frac{1}{\lambda} = const << 1,$$

then the operator B admits the conditions of the Banach principle [4], i.e., equation (14) is uniquely solvable in $C(\bar{\Omega}_0)$. And this means that the function $V \in C^{1,1,...,1}(\bar{\Omega}_0)$ is known. Then, taking into account (9), the function $\tilde{Z}(x_1, ..., x_n)$ is considered to be known. In this case, we assume that the functions:

$$(V_t)_t,\ (V_{x_j})_t \in L^p(0, T), (j = \overline{1, n})$$

for all fixed that is:

$$\begin{cases} ||V_{t^2}||_{L^p} = \left(\int_0^T |V_{t^2}(t, x_1, ..., x_n)|^p dt \right)^{\frac{1}{p}} \le C_{00}, \\ ||V_{tx_j}||_{L^p} = \left(\int_0^T |V_{tx}(t, x_1, ..., x_n)|^p dt \right)^{\frac{1}{p}} \le C_{0j}, (j = \overline{1, n}), \forall (x_1, ..., x_n) \in R^n, \\ C_0 = \max(C_{00}, C_{0j}), (j = \overline{1, n}). \end{cases} \quad (14)$$

Lemma 1. *When conditions (5), (7), (8) are satisfied, the degenerate transport equation (6), taking into account (9), unique solvability in $C^{1,1,...,1}(\bar{\Omega}_0)$, and condition (15) is allowed for the functions $V_{t^2},\ V_{tx_j}, (j = \overline{1, n})$.*

3. Unique solvability of the singularly perturbed inverse problem (SPIP) and proximity of solutions to the (SPIP) and the degenerate inverse problem

First, in order to find out the unambiguous solvability of the transfer (SPIP) under study, we apply an asymptotic representation, that is:

$$\begin{cases} U_\varepsilon(t,x_1,...,x_n) = V + \xi_\varepsilon + \exp\left(-\sum_{j=1}^{n} \frac{(x_j - a_j t)^2}{\varepsilon}\right), \\ Z_\varepsilon(x_1,...,x_n) = \tilde{Z} + \eta_\varepsilon(x_1,...,x_n). \end{cases} \tag{15}$$

Then from (1), taking into account (6), (16) it follows:

$$\varepsilon^\beta \left\{ \frac{\partial}{\partial t}\left(E^{1,1,...,1}_{(a_1,...,a_n)}\xi_\varepsilon\right) + \left[V + \xi_\varepsilon + \exp\left(-\sum_{j=1}^{n}\frac{(x_j - a_j t)^2}{\varepsilon}\right)\right]^2 \right\} + \lambda E^{1,1,1}_{(a_1,a_2,a_3)}\xi_\varepsilon +$$
$$+h_0(x,y,z)\left(\xi_\varepsilon + \exp\left(-\sum_{j=1}^{n}\frac{(x_j - a_j t)^2}{\varepsilon}\right)\right) = \eta_\varepsilon(x_1,...,x_n)f(t) - \varepsilon^\beta\left(V_{t^2} + \sum_{j=1}^{n} a_j V_{tx_j}\right), \tag{16}$$

where $\left(V, \tilde{Z}\right)$ is the solution of the degenerate inverse problem (6) - (8), $(\xi_\varepsilon, \eta_\varepsilon)$ are the residual functions that are contained in equation (17) with the conditions:

$$\xi_t^{(i)}(t,x_1,...,x_n)|_{t=0} = 0,\ (i = 0,1), \tag{17}$$

$$\left(E^{1,1,...,1}_{(a_1,...,a_n)}\xi_\varepsilon\right)|_{t=T} = g_\varepsilon(x_1,...,x_n),\ \ \forall (x_1,...,x_n) \in R^n. \tag{18}$$

The residual functions $(\xi_\varepsilon, \eta_\varepsilon)$ are determined from the inverse problem (17) - (19), where (19) is additional information for this problem. Therefore, to find out the unique solvability of this problem, first, we transform (17) to the form:

$$E^{1,1,...,1}_{(a_1,...,a_n)}\xi_\varepsilon = \frac{1}{\varepsilon^\beta}\int_0^t \exp\left(-\frac{\lambda}{\varepsilon^\beta}(t-s)\right)\{-h_0(x_1,...,x_n)\xi_\varepsilon(s,x_1,...,x_n) + \eta_\varepsilon(x_1,...,x_n)f(s) -$$
$$-\varepsilon^\beta\left[2\xi_\varepsilon(s,x_1,...,x_n)\left(V(s,x_1,...,x_n) + \exp\left(-\sum_{j=1}^{n}\frac{(x_j - a_j s)^2}{\varepsilon}\right)\right) + \xi_\varepsilon^2(s,x_1,...,x_n)\right]\}ds +$$
$$+Y_1(t,x_1,...,x_n,\varepsilon) \equiv \left(H_0\xi_\varepsilon\right)(t,x_1,...,x_n) + \frac{1}{\varepsilon^\beta}\int_0^t \exp\left(-\frac{\lambda}{\varepsilon^\beta}(t-s)\right)f(s)ds \times$$
$$\times\eta_\varepsilon(x_1,...,x_n) + Y_1(t,x_1,...,x_n,\varepsilon), \tag{19}$$

where the function $Y_1(t,x_1,...,x_n,\varepsilon)$ is defined by the formula:

$$\begin{cases} Y_1 \equiv \int_0^t \exp\left(-\frac{\lambda}{\varepsilon^\beta}(t-s)\right)\{-(\ V_{s^2}(s,x_1,...,x_n)+\sum_{j=1}^n a_j V_{sx_j}(s,x_1,...,x_n))- \\ -\left(V(s,x_1,...,x_n)+\exp\left(-\sum_{j=1}^n \frac{(x_j-a_j s)^2}{\varepsilon}\right)\right)^2\Big\}ds, \\ |Y_1| \le \varepsilon^{\frac{\beta}{q}}\left(C_0+C_0(\sum_{j=1}^n a_j)\right)\left(\frac{1}{\lambda q}\right)^{\frac{1}{q}}+(T_0+1)^2\frac{1}{\lambda}\varepsilon^\beta=\delta_1(\varepsilon)\underset{\varepsilon\to 0}{\to}0;\ \frac{1}{q}+\frac{1}{p}=1, \\ \sup\limits_{\bar\Omega_0}|V|\le T_0. \end{cases} \tag{20}$$

From equation (20), based on (19), the following equation follows:

$$\eta_\varepsilon(x_1,...,x_n)=M_0^{-1}(T,\lambda,\varepsilon^\beta)\left\{g_\varepsilon(x_1,...,x_n)-Y_1(T,x_1,...,x_n,\varepsilon)+(H_0\xi_\varepsilon)(T,x_1,...,x_n)\right\}. \tag{21}$$

Therefore, by substituting (22) into (20) we get:

$$E^{1,1,...,1}_{(a_1,...,a_n)}\xi_\varepsilon = Y_2(t,x_1,...,x_n,\varepsilon)-(H_0\xi_\varepsilon)(t,x_1,...,x_n)+\frac{1}{\varepsilon^\beta}\int_0^t\exp\left(-\frac{\lambda}{\varepsilon^\beta}(t-s)\right)f(s)ds\times \\ \times M_0^{-1}(H_0\xi_\varepsilon)(T,x_1,...,x_n), \tag{22}$$

where

$$Y_2\equiv Y_1+\frac{1}{\varepsilon^\beta}\int_0^t\exp\left(-\frac{\lambda}{\varepsilon^\beta}(t-s)\right)f(s)ds\times M_0^{-1}\left\{g_\varepsilon(x_1,...,x_n)-Y_1(T,x_1,...,x_n,\varepsilon)\right\}. \tag{23}$$

Therefore, taking into account (18) from (23) we have:

$$\xi_\varepsilon(t,x_1,...,x_n)=\int_0^t\{-(H_0\xi_\varepsilon)(s,x_1-a_1(t-s),...,x_n-a_n(t-s))+ \\ +\frac{1}{\varepsilon^\beta}\int_0^s\exp\left(-\frac{\lambda}{\varepsilon^\beta}(s-s')f(s')ds'\times M_0^{-1}(H_0\xi_\varepsilon)(T,x_1-a_1(t-s),...,x_n-a_n(t-s)\ )\right\}ds+ \\ +Y(t,x_1,...,x_n,\varepsilon)\equiv(P\xi_\varepsilon)(t,x_1,...,x_n), \tag{24}$$

here

$$\begin{cases} Y\equiv\int_0^t\{Y_1(s,x_1-a_1(t-s),...,x_n-a_n(t-s),\varepsilon)+\frac{1}{\varepsilon^\beta}\int_0^s\exp\left(-\frac{\lambda}{\varepsilon^\beta}(s-s')\right)f(s')ds'\times \\ \times M_0^{-1}[g_\varepsilon(x_1-a_1(t-s),...,x_n-a_n(t-s))-Y_1(T,x_1-a_1(t-s),...,x_n-a_n(t-s),\varepsilon)]\}ds, \\ |Y|\le\delta_1(\varepsilon)\left(T+|M_0^{-1}|f_0\frac{1}{\lambda}\right)+|M_0^{-1}|f_0\frac{1}{\lambda}\int_0^t|g_\varepsilon(x_1-a_1(t-s),...,x_n-a_n(t-s))|ds\le \\ \le\gamma_2\delta_1(\varepsilon)+\gamma_3\left(\int_0^t|g_\varepsilon|^p ds\right)^{\frac{1}{p}}\le\gamma_2\delta_1(\varepsilon)+\gamma_3\Delta_0(\varepsilon)=\delta_2(\varepsilon), \\ \gamma_2=T+|M_0^{-1}|f_0\frac{1}{\lambda};\ \ \gamma_3=|M_0^{-1}|f_0\frac{1}{\lambda}(T)^{\frac{1}{q}}. \end{cases} \tag{25}$$

Lemma 2. *Under the conditions of Lemma 1 and the system*

$$\begin{cases} L_P<1, \\ P:S_{r_0}\to S_{r_0},\ \ \left(S_{r_0}=\left\{\xi_\varepsilon:|\xi_\varepsilon|\le r_0=const,\ \forall(t,x_1,...,x_n)\in\bar\Omega_0\right\}\right), \end{cases} \tag{26}$$

equation (25) is uniquely solvable in $C\left(\bar{\Omega}_0\right)$*, and*

$$||\xi_\varepsilon||_C \leq (1-L_p)^{-1}\,\delta_2(\varepsilon). \tag{27}$$

Therefore, based on (22) we have:

$$\begin{cases} ||\eta_\varepsilon(x_1,...,x_n)||_{L_h^p(R^n)} \leq |M_0^{-1}|\left(\tilde{h}\right)^{\frac{1}{p}}\Delta_0(\varepsilon) + |M_0^{-1}|\delta_0(\varepsilon)\gamma_1 + \gamma_1|M_0^{-1}|\gamma_4(1-L_p)^{-1}\times \\ \times\delta_2(\varepsilon) = \delta_3(\varepsilon), \\ \frac{1}{\lambda}\left(\tilde{h}_0 + 2(T_0+1) + (1-L_p)^{-1}\delta_2(\varepsilon)\right) \leq \gamma_4, \\ ||Z_\varepsilon - \tilde{Z}||_{L_h^p(R^n)} = ||\eta_\varepsilon||_{L_h^p(R^n)}. \end{cases} \tag{28}$$

Indeed, the results of Lemma 2 are obvious, since when (27) holds for the operator P, the conditions of the Banach principle are realized, which means that Eq. (25) has a unique and continuous solution in $C\left(\bar{\Omega}_0\right)$. Then, taking into account (22), we have an estimate of the form (29). Q.E.D.

Further, when representations of an asymptotic nature (16) are used to study the original multi-velocity SPIP, then the conclusions of Lemmas 1;2 hold for all terms of functions (16). Then, based on (16), an estimate of the form follows:

$$|U_\varepsilon - V| \leq ||\xi_\varepsilon||_C + \exp\left(-\sum_{j=1}^{n}\frac{(x_j - a_j t)^2}{\varepsilon}\right). \tag{29}$$

Therefore, taking into account the conditions of Lemmas 1;2 and estimating (30) in the sense of the norm $L_h^p(\Omega_0)$, we obtain:

$$||U_\varepsilon - V||_{L_h^p} \leq (1-L_p)^{-1}\delta_2(\varepsilon)\gamma_1(T)^{\frac{1}{p}} + (\tilde{h}T)^{\frac{1}{p}}\gamma_0\varepsilon^{\frac{1}{p}} = \delta_4(\varepsilon) \underset{\varepsilon\to 0}{\to} 0. \tag{30}$$

As a result, considering the totality of results (29) and (31), and at the same time taking into account

$$\psi = \left(U_\varepsilon - V;\ Z_\varepsilon - \tilde{Z}\right),$$

we have an estimate:

$$\begin{cases} W_h^p(\Omega_0) = \left\{(t,x_1,...,x_n) \in \Omega_0 : \psi_1(t,x_1,...,x_n) \in L_h^p(\Omega_0),\ \psi_2(x_1,...,x_n) \in L_h^p(R^n)\right\}, \\ ||\psi||_{W_h^p(\Omega_0)} = ||U_\varepsilon - V||_{L_h^p(\Omega_0)} + ||Z - \tilde{Z}||_{L_h^p(R^n)} \leq \delta_\varepsilon(\varepsilon) + \delta_4(\varepsilon) = \Delta(\varepsilon) \underset{\varepsilon\to 0}{\to} 0. \end{cases} \tag{31}$$

Theorem 1. *Under the conditions of Lemmas 1, 2, and (32), SPIP (1) - (4) has a unique solution according to rule (16), and the admissible error between the solutions of SPIP and the degenerate inverse problem in* $W_h^p(\Omega_0)$ *will be of the order of* $\Delta(\varepsilon)$

4. Conclusion

In this work, the Kac-type SPIP in an unbounded domain is studied and the results are obtained in a weighted space $W_h^p(\Omega_0)$ when a priori information about the input data is given in $L^p(R^n)$. In this case, the solution is constructed based on a modification of the developed method of asymptotic nature [8], where the estimate of the proximity of the solutions of the SPIP and the degenerate inverse problem is proved in the same space $W_h^p(\Omega_0)$. The obtained conclusions may allow the future to solve multi-velocity transport SPIP in an unlimited area when electrostatic accelerations are non-negative functions.

References

[1] Akhiezer A.I., *Kinetic equations of Boltzmann*, ITF AN SSR, 1973, 29 p.

[2] Vasil'eva A.B., Butuzov V.F., *Asymptotic expansions of solutions of singularly perturbed equations*, Nauka, 1973, 272 p.

[3] Imanaliev M.I., *Asymptotic methods in the theory of singularly perturbed integro-differential systems*, Ilim, 1972, 356 p.

[4] Omurov T.D., Tuganbaev M.M., *Direct and inverse problems of one-speed transport theory*, Ilim, 2010, 116 p.

[5] Sultangazin U.M., *Discrete nonlinear models of the Boltzmann equation*, Science, 1985, 192 p.

[6] Tikhonov A.N., On the dependence of solutions of differential equations on a small parameter, *Mathematical collection*, Vol.22(64), No.2, 1948, pp.193–204.

[7] Frosali, van der Mee, Paveri-Fontana, Conditions for runaway phenomena in the kinetic theory of particle swams, *Journal Math. Phys.*, Vol.30, No.5, 1989, pp. 1177 – 1186.

[8] Omurov T.D., Sarkelova Zh. Zh., Inverse problems for singularly perturbed transfer equations of Kac-Boltzman type, *Advances in Differential Equations and Control Processes*, Vol.21, No.2, 2019, pp.159-170.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ANALYTICAL SOLUTIONS OF INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

VENERA MURATALIEVA[1]

[1]Institute of Mathematics of NAS of KR, Bishkek, Kyrgyzstan
e-mail: vmuratalieva@mail.ru

Abstract. Given a scalar or multidimensional (matrix-vector) equation of Volterra type with power coefficients by integral summands, algorithms (in generalized sense) are built to detect existence of solutions in analytic functions and occurrence of arbitrary constants in it. Also, given an equation with a coefficient, specific values (spectrum) of the coefficient are found. Various types of equations are investigated in the paper.

Keywords: algorithm, spectrum, integral equation, integro-differential equation, analytical function

AMS Subject Classification: 45J05

1. Introduction

As a consequence of the analyticity effect (see result [1], results obtained since 2000s [2] and [3]), we supposed [4], [5] that significant new results on equations with analytical coefficients may be found. Our hopes were justified. Briefly, we constructed and implemented the following algorithms on a computer. Given an equation of Volterra type with power coefficients by integral summands, the algorithm presents data to detect existence of solutions and occurrence of arbitrary constants in it. Given an equation with a coefficient, specific values (spectrum) of the coefficient are found.

Section 2 contains denotations. A new notion "power-add" as a general characteristic of a linear operator in the space of analytical functions is introduced.

Section 3 presents description of methods and algorithms proposed by us.

Sections 4 and 5 present examples on spectra of scalar and multidimensional (vector-matrix) linear integral and integro-differential equations with a parameter.

Section 6 presents examples of existence of analytical solutions of integral equations with non-analytical right hand parts.

2. Denotations

Denote

$N := \{1, 2, 3, ...\}$; $N_0 := \{0, 1, 2, 3, ...\}$; $Z := \{-3, -2, -1, 0, 1, 2, 3...\}$;

$R := (-\infty, \infty)$; $R_+ := [0, \infty)$;

A be the space of analytical functions with real coefficients $R \to R$; A_+ be the space of functions $R_+ \to R$;

E is the unit matrix.

An unknown function

$$u(t) = \sum_{k=0}^{\infty} u_k t^k = u_0 + u_1 t + u_2 t^2 + ... \tag{1}$$

A given function

$$f(t) = \sum_{k=0}^{\infty} f_k t^k = f_0 + f_1 t + f_2 t^2 + ... \tag{2}$$

Remark 2.1. *In theorems proven and announced below radius of convergence of the function $u(t)$ coincides with one of the function $f(t)$.*

Consider a linear operator $L : A \to A$. If for large $k \in N$

$$L(t^k) = const_k \cdot t^{k+w}, w \in Z \tag{3}$$

then the number w is said to be the "power-add" Add of the operator L. Obviously,

$$Add(L_1 L_2) = Add(L_1) + Add(L_2). \tag{4}$$

We will use the following three operators:

$$Add(t^p) = p, p \in N; \tag{5}$$

the differential operator

$$D := \frac{d}{dt};\ Add(D) = -1; \tag{6}$$

the integral operator

$$D^{-1}(u(s) : s) := \int_0^t u(s)ds; Add(D^{-1}) = 1. \tag{7}$$

3. Methods and Algorithms

Conditions of some theorems in their primary form are too difficult for a human to make definite conclusions. Meanwhile, sometimes same conditions being transformed in any way became evident to formulate theorems. We mention graphs, tables, diagrams as such means.

Applying the method of power series to integro-differential equations yields vast infinite systems of algebraic equations for coefficients of these series. Their first, second, ... equations differ from others and it is practically impossible to construct such equations and make a definite conclusion. We propose to use computer for this purpose.

We consider equations for unknown analytical functions (1) with terms being compositions of operators (5), (6), (7), of types $H(b, p, m)u(t) := bt^p D^m u(t), b \in R, b \neq 0, p \in N_0, m \in Z$ in the left hand part, and the known analytical function (2).

Applying the operator (6) to terms t^k with small k may yield 0. Hence, the following function of integer arguments is introduced:

$$A(m, n) := \begin{cases} 0 & (((m \geq 0) \bigwedge (m > n)) \bigvee (n < 0)), \\ \frac{n!}{(n-m)!} & (((m \geq 0) \bigwedge (m \leq n)) \bigvee (m < 0)). \end{cases} \tag{8}$$

Substituting (1) and (2) into the equation we obtain that the term $H(b, p, m)u(t)$ makes the following contribution to the equation with $f[n], n \in N_0$:

$$G(b, p, m, n)u_k := bA(m, n - p + m)u_{n-p+m}.$$

Hence, the following different types of algebraic equations arise:

(A1) equations where some $A(m, n - p + m) = 0$;

(A2) equations where sum of $bA(m, n - p + m)$ with the same $(p - m)$ equals zero;

(A3) equations where all terms are non-zero.

Boundaries for n in the case (A1) can be *a priori* estimated algorithmically.

If all b are integer then boundaries for n to check possible cases (A2) can be estimated algorithmically too.

The full list of the cases (A1) and (A2) and an image of the "general case" (A3) prepared by the computer would assist to the human to formulate a theorem on existence and uniqueness of solution of the equation given.

For example, consider the following equation

$$t^4u''(t) + 4u(t) - 3\int_0^t (t-s)u(s)ds = f(t), t \in R_+. \tag{9}$$

The corresponding system of algebraic equations:

$$A(2, n-2)u_{n-2} + 4A(0, n)u_n - 3A(-2, n-3)u_{n-3} = f_n, n \in N_0. \tag{10}$$

If $n - 2 < 2$ then the first term vanishes.

If $n - 3 < 0$ then the third term vanishes.

Hence, "the general case" begins with $n = 4$; u_n can be expressed by means of u_{n-2} and u_{n-3}.

4. Spectral Properties of Linear Integro-Differential Equations of the Third Kind of the First and Second Order

Considered the following equations for entire analytical functions

Example 4.1.

$$t^2u'(t) + \int_0^t \lambda u(s)ds = f(t), t \in R_+, \tag{11}$$

Example 4.2.

$$t^2u'(t) + t^2u(t) + \int_0^t \lambda u(s)ds = f(t), t \in R_+, \tag{12}$$

Example 4.3.

$$t^3u''(t) + \int_0^t \lambda u(s)ds = f(t), t \in R_+, \tag{13}$$

and equations with a polynomial $K(t)$:

Example 4.4.

$$t^2u'(t) + \lambda\int_0^t K(t)u(s)ds = f(t), t \in R_+, \tag{14}$$

Example 4.5.

$$t^3u''(t) + \lambda\int_0^t K(t)u(s)ds = f(t), t \in R_+; f(0) = 0, K(0) \neq 0. \tag{15}$$

Theorem 4.1. *If λ is not in $\{0, -2, -6, -12, ..., -n(n+1), ...\}$ then the equation (11) has a unique solution otherwise it has either none solution or many solutions.*

Theorem 4.2. *If λ is not in $\{0, -4, -9, ..., -n^2, ...\}$ then the equation (12) has a unique solution ...*

Theorem 4.3. *If λ is not in $\{0, -4, -9, ..., n(n-1)(n+1), ...\}$ then the equation (13) has a unique solution ...*

Theorem 4.4. *(Non-formally). If λ is not in any sequence $\{\lambda_1 = 0, \lambda_2, \lambda_3, ...\}$ then the equation (14) (or the equation (15)) has a unique solution ....*

Hypothesis 1. *If power-adds of at least two operators of the left hand part of an equation of types similar to (11)-(14) are equal 1 and power-adds of other operators are greater then such equation has spectral properties (with corresponding conditions on right hand parts).*

Examples: For equations (11) and (14) we have power-adds: 2-1; 1.

For equations (13) and (15) we have power-adds: 3-2; 1.

For equation (12) we have power-adds: 2-1; 2-0; 1.

5. Spectral Properties Of Linear Multidimensional Integral Equations

In this section the series (1) and (2) are considered for coefficients $u_k, f_k \in R^\omega, \omega > 1$.
We consider equations of type

$$P(t)u(t) + \lambda \int_0^t Q(t,s)u(s)ds = f(t), t \in R_+ \tag{16}$$

where $P(t)$ and $Q(t,s)$ are given polynomials with $R^{\omega\times\omega}$ matrix-coefficients.

Example 5.1.

$$tu(t) + \lambda \int_0^t Qu(s)ds = f(t), t \in R_+. \tag{17}$$

Substituting (1) and (2) we obtain

$$u_0 t + u_1 t^2 + u_2 t^3 + u_3 t^4 +$$

$$\lambda \int_0^t Q(u_0 + u_1 s + u_2 s^2 + u_3 s^3 + ...)ds =$$

$$f_0 + f_1 t + f_2 t^2 + f_3 t^3 + ...,$$

$$u_0 t + u_1 t^2 + u_2 t^3 + u_3 t^4 + ... + \lambda Q(u_0 t + u_1 t^2/2 + u_2 t^3/3 + u_3 t^4/4 + ...) =$$

$$f_0 + f_1 t + f_2 t^2 + f_3 t^3 + ...$$

We obtain the following system of linear vector algebraic equations:

$$0 = f_0; u_0 + \lambda Q u_0 = f_1; (E + \lambda Q)u_0 = f_1;$$

$$u_1 + \lambda Q u_1/2 = f_2; (E + \lambda Q/2)u_1 = f_2;$$

$$u_2 + \lambda Q u_2/3 = f_3; (E + \lambda Q/3)u_2 = f_3;$$

$$...$$

$$u_n + \lambda Q u_n/(n+1) = f_{n+1}; (E + \lambda Q/(n+1))u_n = f_{n+1}, ...$$

Let $q_1, ..., q_k \in R$ be characteristic numbers of the matrix $Q : det(E - q_j Q) = 0$.

Theorem 5.1. *If $f(0) = 0$ and $\lambda \neq nq_j, n \in N, j = 1..k$, then the equation (4) has a unique analytical solution.*

Proof. If $n > 2|\lambda| \cdot \|Q\|$ then $\|\lambda Q/(n+1)\| < 1/2$;

$$\|E + \lambda Q/(n+1)\| \geq \|E\| - |\lambda Q/(n+1)|| > 1 - 1/2 = 1/2;$$

$$\|u_n\| \leq 2\|f_{n+1}\|.$$

For all $n > 2|\lambda| \cdot \|Q\|$ we have $det(E + \lambda Q/(n+1)) \neq 0$.

Hence, all u_n are defined uniquely. The series (1) converges as well as the series (2) does. Theorem is proven.

6. Analytical Solutions of Volterra Integral Equations with Non-Analytical Right Hand Part

Consider

Example 6.1.

$$tu(t) + \int_0^t u(s)ds = \sqrt{f}(t),\ t \in R. \tag{18}$$

Substitute power series (1) and (2):

$$(t(u_0 + u_1t + u_2t^2 + ...) + \int_0^t (u_0 + u_1s + u_2s^2 + ...)ds)^2 = f_0 + f_1t + f_2t^2 + ...$$

The necessary conditions for existing a solution are

$$(*)\ f_0 = 0;\ f_1 = 0.$$

We have:

$$(u_0 + u_1t + u_2t^2 + ... + u_0 + u_1t/2 + u_2t^2/3 + ...)^2 = f_2 + f_3t + f_4t^2 + ...;$$

$$(2u_0 + 3/2 \cdot u_1t + 4/3 \cdot u_2t^2 + ... + (n+2)/(n+1) \cdot u_nt^n + ...)^2 = f_2 + f_3t + f_4t^2 + ...$$

Denote $v_n = (n+2)/(n+1) \cdot u_n, n \in N_0$*:*

$$(v_0 + v_1t + v_2t^2 + ...)^2 = f_2 + f_3t + f_4t^2 + ...$$

If the right hand part is positive then this equation has two analytical solutions $\pm g(t)$*. We have* $|u_n| \leq |v_n|, n \in N_0$*.*

Theorem 6.1. *If (*) and* $f(t) = t^2 f_p(t)$ *where* $f_p(t)$ *is a positive analytical function then the equation (18) has two analytical solutions.*

Consider

Example 6.2.

$$tu(t) + \int_0^t su(s)ds = \sqrt{f}(t),\ t \in R. \tag{19}$$

Substitute (1) and (2):

$$(t(u_0 + u_1t + u_2t^2 + ...) + u_0t^2/2 + u_1t^3/3 + u_2t^4/4 + ...)^2 = f_0 + f_1t + f_2t^2 + ...$$

Suppose (*):

$$(u_0 + u_1t + u_2t^2 + ... + u_0t/2 + u_1t^2/3 + u_2t^3/4 + ...)^2 = f_2 + f_3t + ...$$

Denote $w_0 = u_0; w_n = u_n + u_{n-1}/(n+1), n \in N$*.*

As well as above, if the analytical function with coefficients $\{w_n : n \in N_0\}$ *exists then* $u_0 = w_0; u_n = w_n - u_{n-1}/(n+1), n =\in N$*.*

Theorem 6.2. *If (*) and* $f(t) = t^2 f_p(t)$ *where* $f_p(t)$ *is a positive analytical function then the equation (19) has two analytical solutions.*

7. Conclusion

We demonstrated that various types of integral and integro-differential equations with analytical functions can be successfully investigated by a uniform method. The next purpose is to find not only sufficient but also necessary conditions for existing of solutions with any properties, to substitute Hypothesis.

References

[1] Pankov P.S., Imanaliev T.M. Convergence of Finite Difference Method for First-Order Partial Differential Equations with Analytical Initial Conditions. *Analytical and Approximate Methods: International Conference at the Kyrgyz-Russian Slavic University*. - Shaker Verlag, Aachen, Germany, 2003, pp. 185-193.

[2] Alybaev K.S., Pankov P.S., Tampagarov K.B., Narbaev M.P. B*oundary layer lines in the theory of singularly perturbed ordinary differential equations with analytical functions*, Jalal-Abad, 2023.

[3] Kenenbaeva G. *Effect of analyticity for differential and integral equations (in Russian)*. Lap Lambert Academic Publishing, Saarbrücken, Germany, 2015.

[4] Muratalieva V.T. Algorithm for investigation of spectral properties of linear tasks with analytical functions (in Russian), *Bulletin of Jalal-Abad State University*, 2016, 1:32, pp. 55-59.

[5] Muratalieva V. Spectral properties of Volterra linear integro-differential equations of the third kind of the first and second order, *Abstracts of the V International Scientific Conference "Asymptotical, Topological and Computer Methods in Mathematics"*, Bishkek-Bozteri, 2016, p. 34.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON THE DISCRETE SPECTRUM OF THE SCHRÖDINGER OPERATOR OF THE SYSTEM OF THREE-PARTICLES WITH MASSES $m_1 = m_2 = \infty$ AND $m_3 < \infty$ ON THE THREE DIMENSIONAL LATTICE

ZAHRIDDIN MUMINOV

Tashkent State University of Economics, Islam Karimov street 49, 100066, Tashkent, Uzbekistan
e-mail: zimuminov@gmail.com

Abstract. In this paper, we study the spectrum of the discrete Schrödinger operator H corresponding to a system of three particles, with masses $m_1 = m_2 = \infty$ and $m_3 < \infty$, and interacting via short-range pair potentials in the three dimensional lattice. The discrete spectrum of the Schrödinger operators corresponding to the system of particles, where two of them are of an infinite mass, is shown to posses some distinct properties not observed for finite mass particles. We show that the essential spectrum of such operators consists of a finite disjoint union of real segments and infinitely many eigenvalues may exist at the gaps of these segments depending on the interaction energies among the particles (a type of Efimov's effect).

Keywords: Schrödinger operator, dispersion functions, zero-range pair potentials, discrete spectrum, essential spectrum, cluster operators, bound states.

AMS Subject Classification: Primary: 81Q10, Secondary:35P20, 47N50

Introduction

The study of systems of three particles of various masses and interaction energies has always been of particular interest both in physics and mathematics. Three-particle systems have some distinguished physical features not observed for systems of two particles. One of them is Efimov's effect [1], when three identical bosons can have bound states despite the interactions between any two of the particles are not enough to form a pair. Consequently, under particular conditions, there may exist infinitely many eigenvalues for the three-body Schrödinger operators accumulating at the boundaries of the essential spectrum. The first thorough mathematical proof of this phenomenon was presented by D. Yafaev [2] using Faddeev's integral equations, and further proofs were provided in the works [3, 4, 5, 6]. Efimov's theoretical predictions remained experimentally unproven for several decades, until a group of researchers demonstrated the existence of a bound state in ultracold gas of caesium atoms [7].

The presence of Efimov's effect for the three-particle discrete Schrödinger operators, lattice counterparts of the continuous Schrödinger operators, can also be naturally expected due to similarity of the spectra in the lattice and continuous cases [2]. This was shown using the physical methods in [8, 9]. Later, S. N. Lakaev [6] proved the existence of the effect for a system of three bosons on a three-dimensional lattice, which is represented by a discrete Schrödinger operator parametrized by the so-called quasi-momentum. It was shown that Efimov's effect occurs for specific values of the quasi-momentum only.

Spectral properties of the discrete Schrödinger operators on lattices have been intensively studied in recent years (see e.g., [16, 26, 37, 36, 23, 27, 35]). Unlike for the continuous case,

This work was partially supported by the Foundation for Basic Research of the Republic of Uzbekistan (Grant No. FZ–20200929224).

the discrete three-particle Hamiltonian operator $H = H_0 - V$, where H_0 is a non-perturbative operator and V is a potential function, is not rotationally invariant. However, using the technique of separation of variables, it can be represented as a direct integral of a family of bounded discrete Schrödinger operators $H(K)$, $K \in \mathbb{T}^3$, called the fiber operators [6, 10, 11]. As a result, the study of the spectrum of the three-particle Schrödinger operators is reduced to the study of the fiber operator $H(K)$ for each quasi-momentum K. According to the boundedness of the operator $H(K)$, its essential spectrum, $\sigma_{ess}(H(K))$, consists of a union of at most a countably many bounded segments (see [12, 30]). In this case, Efimov's effect may appear not only at the threshold of the essential spectrum, but also at the edges of the gaps between those segments (see [13]).

In this paper, we study the spectrum of the discrete Schrödinger operator H corresponding to a system of three particles of masses $m_1 = m_2 = \infty$ and $m_3 < \infty$, which interact via short-range pair potentials on the three-dimensional lattice, $\mathbb{Z}^3$. Using the integral decomposition method mentioned above, we obtain the fiber operators $H(K)$, $K \in \mathbb{T}^3$ and investigate their spectral properties. It is proven that the discrete spectrum of the Schrödinger operator H is infinite (Theorem 4.9). Moreover, it is shown that for certain values of the interaction energy, infinitely many eigenvalues may appear in the gaps of the essential spectrum. This reveals some properties of the three-particle system with two of the particles being of an infinite mass, not specific for the case where all particles have finite masses (see e.g., [31]). In the recent works of [24] and [25], the infiniteness of the discrete spectrum were demonstrated in the one-dimensional space, and in [28] in the two-dimensional case. In the three-dimensional space, the problem becomes rather challenging due to the possibility for the convergence of the considered three-dimensional integrals over $\mathbb{T}^3$, and the existence of the threshold resonances of the operator H, in addition to the eigenvalues.

The plan of the present manuscript is as follows. In section 1, a momentum representation of the Schrödinger operator associate with Hamiltonian of a system of three-particles with masses $m_1 = m_2 = \infty$ and $m_3 < \infty$ is described as bounded self-adjoint operators in the corresponding Hilbert spaces. The channel operators, two-particle discrete Schrödinger operators $h_\alpha(k), k \in \mathbb{T}^3$, $\alpha = 1, 2, 3$, corresponding to the three two-particle subsystems of the three-particle system and their properties are presented in section 2. In section 3, we describe the essential spectrum of $H(K)$ in terms of the spectra of the channel operators. The main results of the paper (including Theorem 4.9) are stated and proven in Section 4. The last section is conclusion.

Throughout the paper we adopt the following notations: $\mathbb{Z}^3$ is the three-dimensional lattice, $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3 = (-\pi, \pi]^3$ is the three-dimensional torus (the first Brillouin zone, i.e., the dual group of $\mathbb{Z}^3$) equipped with the Haar measure, the subscripts $\alpha, \beta, \gamma \in \{1, 2, 3\}$ are pairwise different numbers.

1. Three-particle discrete Schrödinger operator on the lattice $\mathbb{Z}^3$

1.1. **Momentum representation.** The operator $H(K)$, $K \in \mathbb{T}^3 = (-\pi, \pi]^3$ associated with the current system is of the form

$$H(K) = H_0(K) - V_{12} - V_{23} - V_{31}, \tag{1}$$

where the operators $H_0(K)$ and $V_{\beta\gamma}$ are defined on the Hilbert space $L_2((\mathbb{T}^3)^2)$ by

$$(H_0(K)f)(p,q) = E(K;p,q)f(p,q), \quad f \in L_2((\mathbb{T}^3)^2), \tag{2}$$

$$\begin{gathered}(V_1 f)(p,q) = \frac{\mu_1}{(2\pi)^3}\int\limits_{\mathbb{T}^3} f(p,t)dt, \quad (V_2 f)(p,q) = \frac{\mu_2}{(2\pi)^3}\int\limits_{\mathbb{T}^3} f(t,q)dt,\\ (V_3 f)(p,q) = \frac{\mu_3}{(2\pi)^3}\int\limits_{\mathbb{T}^3} f(t,p+q-t)dt, \quad f \in L_2((\mathbb{T}^3)^2),\end{gathered} \tag{3}$$

where

$$E(K;p,q)=\varepsilon_1(p)+\varepsilon_2(q)+\varepsilon_3(K-p-q),$$

and the real-valued continuous function

$$\varepsilon_\alpha(p)=\frac{1}{m_\alpha}\epsilon(p),\quad \epsilon(p)=\sum_{j=1}^{3}(1-\cos p^{(j)}),\quad p=(p^{(1)},p^{(2)},p^{(3)})\in\mathbb{T}^3, \tag{4}$$

is called the *dispersion relation of the α-th normal mode* associated with the free particle α ($\alpha=1,2,3$).

Since $m_1=m_2=\infty$ and $0<m_3<\infty$, we obtain

$$\varepsilon_1(p)=\varepsilon_2(q)=0,\quad \text{and}\quad E(K;p,q)=\frac{1}{m_3}\epsilon(p). \tag{5}$$

In the physical literature, the parameter $K\in\mathbb{T}^3$ is called the *three-particle quasi-momentum.*

2. Channel Operator

Consider the operators $H_\alpha(K), K\in\mathbb{T}^3$, $\alpha=1,2,3$, acting on the Hilbert space $L^2((\mathbb{T}^3)^2)$ as

$$H_\alpha(K)=H_0(K)-V_\alpha,\quad \alpha=1,2,3,$$

where $H_0(K)$ and V_α are defined in (2) and (3), respectively. The operators $H_\alpha(K), K\in\mathbb{T}^3$ are called *the cluster operators* corresponding to the cluster decomposition $\{\{\alpha\},\{\beta,\gamma\}\}$, $\alpha,\beta,\gamma\in\{1,2,3\}$ [30].

The decomposition of the space $L^2((\mathbb{T}^3)^2)$ into the direct integral

$$L^2((\mathbb{T}^3)^2)=\int\limits_{p\in\mathbb{T}^3}\oplus L^2(\mathbb{T}^3)dp$$

allows decomposing $H_\alpha(K)$ into the direct integral

$$H_\alpha(K)=\int\limits_{p\in\mathbb{T}^3}\oplus\left(h_\alpha(K-p)+\varepsilon_\alpha(p)\mathrm{I}\right)dp, \tag{6}$$

where $h_\alpha(k)$, $k\in\mathbb{T}^3$, $\alpha=1,2,3$ is the two-particle Schrödinger operator corresponding to the subsystem $\{\beta,\gamma\}$ of the three-particle system defined by

$$h_\alpha(k)=h_\alpha^0(k)-v_\alpha,\quad k\in\mathbb{T}^3. \tag{7}$$

The operators $h_\alpha^0(k)$ and v_α are defined on the Hilbert space $L_2(\mathbb{T}^3)$ as

$$(h_\alpha^0(k)f)(p)=E_k^{(\alpha)}(p)f(p),\quad f\in L^2(\mathbb{T}^3),$$

where

$$E_k^{(\alpha)}(p)=\varepsilon_\beta(p)+\varepsilon_\gamma(k-p),\quad p\in\mathbb{T}^3, \tag{8}$$

and

$$(v_\alpha f)(p)=\frac{\mu_\alpha}{(2\pi)^3}\int\limits_{\mathbb{T}^3}f(q)dq,\quad f\in L^2(\mathbb{T}^3),\ p\in\mathbb{T}^3.$$

2.1. **Spectral properties of the two-particle discrete Schrödinger operator.** In this section, we study spectral properties of the operators $h_\alpha(k)$, $k \in \mathbb{T}^3$.

Since $m_1 = m_2 = \infty$ and $0 < m_3 < \infty$ and the equality (4), the functions (8) take the forms

$$E_k^{(1)}(p) = \epsilon(k-p)/m_3, \quad E_k^{(2)}(p) = \epsilon(k-p)/m_3, \quad E_k^{(3)}(p) = 0, \quad p \in \mathbb{T}^3. \tag{9}$$

Consequently, since the potentials v_α, $\alpha = 1, 2, 3$, has a convolution type property, all three two-particle Schrödinger operators depend on the quasi-momentum $k \in \mathbb{T}^3$, formally, and moreover

$$h_1 := h_1(k), \quad h_2 := h_2(k), \quad h_3 := h_3(k) \quad \text{and} \quad h_1 = h_2 \quad \text{if} \quad \mu_1 = \mu_2.$$

Then, the operators $h_1(k)$, $h_2(k)$ and $h_3(k)$ act on $f \in L^2(\mathbb{T}^3)$ as

$$h_1(k)f(p) = \varepsilon_3(p)f(p) - v_1 f(p), \quad h_2(k)f(p) = \varepsilon_3(p)f(p) - v_2 f(p), \quad h_3(k)f(p) = -v_3 f(p).$$

As v_α is a finite rank operator, according to the Weyl theorem, the essential spectrum $\sigma_{\text{ess}}(h_\alpha(k))$ of the operator $h_\alpha(k)$ in (7) coincides with the spectrum $\sigma(h_\alpha^0(k))$ of the non-perturbed operator $h_\alpha^0(k)$. More specifically,

$$\sigma_{\text{ess}}(h_\alpha(k)) = [E_{\min}^{(\alpha)}(k),\, E_{\max}^{(\alpha)}(k)],$$

where

$$E_{\min}^{(\alpha)}(k) \equiv \min_{p \in \mathbb{T}^3} E_k(p), \quad E_{\max}^{(\alpha)}(k) \equiv \max_{p \in \mathbb{T}^3} E_k^{(\alpha)}(p). \tag{10}$$

Moreover,

$$\sigma_{\text{ess}}(h_1(k)) = \sigma_{\text{ess}}(h_1(k)) = [0, 6/m_3] \quad \text{and} \quad \sigma_{\text{ess}}(h_3(k)) = \{0\} \tag{11}$$

Since the non-perturbed part of $h_3(k)$ is zero, we define a Fredholm determinant associated with the operator $h_1(k)$ and $h_2(k)$ as

$$\Delta_\alpha(z) = 1 - \frac{\mu_\alpha}{(2\pi)^3} \int\limits_{\mathbb{T}^3} \frac{ds}{\varepsilon_3(s) - z}, \quad z \in \mathbb{C} \setminus [0, 6/m_3], \quad \alpha = 1, 2.$$

Set

$$\mu_1^0 = \mu_2^0 = \left(\frac{1}{(2\pi)^3} \int\limits_{\mathbb{T}^3} \frac{ds}{\varepsilon_3(s)} \right)^{-1}.$$

Lemma 2.1. *Let $\alpha = 1, 2$. The number $z \in \mathbb{C} \setminus [0, 6/m_3]$ is an eigenvalue of $h_\alpha(k)$ if and only if*

$$\Delta_\alpha(z) = 0.$$

Proof. The equation

$$h_\alpha(k)f = zf \quad \text{i.e.,} \quad f = \mu_\alpha(h_\alpha^0(k) - z)^{-1} v f$$

has a non-trivial solution if and only if

$$\Delta_\alpha(z)C = 0, \quad C \in \mathbb{C},$$

has a non-trivial solution.

Therein, the solutions $C \in \mathbb{C}$ and $f \in L^2(\mathbb{T}^3)$ are connected by the relations

$$C = v_\alpha f \quad \text{and} \quad f = \mu_\alpha(h_\alpha^0(k) - z)^{-1} C.$$

□

Lemma 2.2. *Let $\alpha = 1, 2$.*

(a) If $\mu_\alpha > \mu_\alpha^0$, then there exists a unique simple eigenvalue $z = z_\alpha^0$ of $h_\alpha(k)$ in the interval $(-\infty, E_{\min}(K))$, i.e. $\Delta_\alpha(z_\alpha^0) = 0$. Moreover, z_α^0 does not depend on $k \in \mathbb{T}^3$.

(b) If $0 < \mu_\alpha \le \mu_\alpha^0$, then $h_\alpha(k)$ has no eigenvalues in the interval $(-\infty, 0)$.

Proof. We note that the function $\Delta_\alpha(z)$ is monotonic decreasing with respect to z in $(-\infty, 0)$. Also we have the relations

$$\lim_{z\to-\infty}\Delta_\alpha(z)=1,\quad \lim_{z\to 0-}\Delta_\alpha(z)=1-\mu_\alpha/\mu_\alpha^0,$$

(a) If $\mu_\alpha > \mu_\alpha^0$, i.e. $1-\mu_\alpha/\mu_\alpha^0 < 0$, then the intermediate theorem implies existence of a unique zero $z=z_\alpha^0\in(-\infty,0)$ of the function $\Delta_\alpha(\cdot)$.

(b) If $0<\mu_\alpha\le\mu_\alpha^0$, then the intermediate theorem confirms that $\Delta_\alpha(\cdot)$ has no any zero in $(-\infty,0)$. □

Now we can summarize the result of this section

Lemma 2.3.

$$\sigma_{\rm disc}(h_\alpha(k))=\begin{cases}\varnothing, & \text{if}\quad 0<\mu_\alpha\le\mu_\alpha^0,\\ \{z_\alpha^0\}, & \text{if}\quad \mu_\alpha>\mu_\alpha^0,\end{cases},$$

$$\sigma(h_\alpha(k))=\begin{cases}[0,6/m_3], & \text{if}\quad 0<\mu_\alpha\le\mu_\alpha^0,\\ \{z_\alpha^0\}\cup[0,6/m_3], & \text{if}\quad \mu_\alpha>\mu_\alpha^0\end{cases},\quad \alpha=1,2,$$

$$\sigma_{\rm disc}(h_3(k))=\{-\mu_3\}\quad \textit{and}\quad \sigma(h_3(k))=\{-\mu_3\}\cup\{0\}.$$

2.2. **The spectrum of $H_\alpha(K)$.** The direct integral representation (6) of the operator $H_\alpha(K)$ gives (see Refs. [31]) the relation

$$\sigma(H_\alpha(K))=\sigma_{\rm three}(H_\alpha(K))\cup\sigma_{\rm two}(H_\alpha(K)),\tag{12}$$

where

$$\sigma_{\rm three}(H_\alpha(K))=\bigcup_{p\in\mathbb{T}^3}\{\sigma_{\rm ess}(h_\alpha(K-p))+\varepsilon_\alpha(p)\}\quad\text{and}\quad \sigma_{\rm two}(H_\alpha(K))=\bigcup_{p\in\mathbb{T}^3}\{\sigma_{\rm disc}(h_\alpha(K-p)))+\varepsilon_\alpha(p)\}.$$

As $\sigma_{ess}(h_\alpha(K-p))=[0,6/m_3]$, $\sigma_{\rm disc}(h_\alpha(K-p))=\{z_\alpha^0\}$, $\alpha=1,2$, and $\sigma_{ess}(h_3(K-p))=\{0\}$, $\sigma_{\rm disc}(h_3(K-p))=\{-\mu_3\}$ (see Lemma 2.3), and (5) we obtain

$$\sigma_{\rm three}(H_\alpha(K))=[0,6/m_3],\alpha=1,2,3$$

and

$$\sigma_{\rm two}(H_\alpha(K))=\begin{cases}\varnothing, & \text{if}\quad 0<\mu_\alpha\le\mu_\alpha^0\\ \{z_\alpha^0\}, & \text{if}\quad \mu_\alpha>\mu_\alpha^0\end{cases},\ \alpha=1,2,\quad\text{and}\quad \sigma_{\rm two}(H_3(K))=\left[-\mu_3,\frac{6}{m_3}-\mu_3\right].$$

Thus we get

Lemma 2.4. *Let $\mu_\alpha>0$, $\alpha=1,2$, and $\mu_3\ge 0$. Then, for every $K\in\mathbb{T}^3$, the following relations hold:*

$$\sigma(H_\alpha(K))=\begin{cases}\left[0,\frac{6}{m_3}\right], & \textit{if}\quad 0<\mu_\alpha\le\mu_\alpha^0\\ \{z_\alpha^0\}\cup\left[0,\frac{6}{m_3}\right], & \textit{if}\quad \mu_\alpha>\mu_\alpha^0\end{cases},\ \alpha=1,2,$$

$$\sigma(H_3(K))=\left[-\mu_3,\frac{6}{m_3}-\mu_3\right]\cup\left[0,\frac{6}{m_3}\right],$$

where $\{z_\alpha^0\}$, $\alpha=1,2$, is an eigenvalue of $h_\alpha(k)$, $\alpha=1,2$.

3. The essential spectrum of $H(K)$

One of the remarkable results in the spectral theory of multi-particle continuous Schrödinger operators is the description of the essential spectrum of the Schrödinger operators in terms of cluster operators (the HVZ-theorem. For the continuous case, see Refs. [34, 32, 33] and for the discrete case Refs. [31] and [30]).

Lemma 3.1. *For every $K \in \mathbb{T}^3$, the essential spectrum $\sigma_{ess}(H(K))$ of $H(K)$ is the union of the spectra of the channel operators $H_\alpha(K) = H_0(K) - \mu_\alpha V_\alpha$, $\alpha = 1, 2, 3$, i.e.,*

$$\sigma_{ess}(H(K)) = \sigma(H_1(K)) \cup \sigma(H_2(K)) \cup \sigma(H_3(K)).$$

Proof. (a) The proof of Lemma 2.3 can be found in [31] and [30]. □

The structure of the essential spectrum of the operator $H(K)$ ca be described by the following lemma.

Lemma 3.2. *Let $\mu_\alpha > 0$, $\alpha = 1, 2$, and $\mu_3 \geq 0$. For the essential spectrum of the main operator $H(K)$, we have*

$$\sigma_{ess}(H(K)) = \left[0, \frac{6}{m_3}\right] \cup \left(\Lambda_1 \cup \Lambda_2 \cup \left[-\mu_3, \frac{6}{m_3} - \mu_3\right]\right),$$

where

$$\Lambda_\alpha = \begin{cases} \varnothing, & \text{if} \quad 0 \leq \mu_\alpha \leq \mu_\alpha^0, \\ \{z_1^0\}, & \text{if} \quad \mu_\alpha > \mu_\alpha^0, \end{cases} \qquad \alpha = 1, 2.$$

Moreover, we have

$$\sigma_{ess}(H(K)) = (\Lambda_1 \cup \Lambda_2) \cup \left[-\mu_3, \frac{6}{m_3}\right], \quad \text{if} \quad 0 \leq \mu_3 \leq \frac{6}{m_3},$$

$$\sigma_{ess}(H(K)) = \left(\Lambda_1 \cup \Lambda_2 \cup \left[-\mu_3, \frac{6}{m_3} - \mu_3\right]\right) \cup \left[0, \frac{6}{m_3}\right], \quad \text{if} \quad \mu_3 > \frac{6}{m_3}.$$

Proof. Lemmas 2.4 and 3.1 imply the proof. □

4. The discrete spectrum of $H(K)$

Let $U_K : L^2(\mathbb{T}^3) \to L^2(\mathbb{T}^3)$ be a unitary operator defined as

$$(U_K f)(p) = f(-K/2 + p, -K/2 + q). \tag{13}$$

It establishes the equivalence between $H(K)$ and $H(0)$.

The subspaces

$$\mathcal{H}_0 = \{f \in L^2((\mathbb{T}^3)^2) | f(p, q) = g(p + q), g \in L^2(\mathbb{T}^3)\}$$

and $\mathcal{H}_0^\perp = L^2((\mathbb{T}^3)^2) \ominus \mathcal{H}_0$ are invariant under the operator $H(0)$. Moreover, we have

$$\sigma_{disc}(H(K)) = \sigma_{disc}(H^0) \cup \sigma_{disc}(H^1(K)), \tag{14}$$

where $H^0(K) = PH(K)P$ and $H^1(K) = P^\perp H(K)P^\perp$, with P being an orthogonal projection on $\mathcal{H}_0$, and $P^\perp = I - P$.

Since $\mathcal{H}_0$ and $L^2(\mathbb{T}^3)$ are isomorphic, the operator $H^0(K)$ is unitarily equivalent to the operator

$$E_0(0) - \mu_3 I - (\mu_1 + \mu_2)v,$$

where $E_0(0)$ denotes the multiplication by $\varepsilon_3(K - \cdot)$, I is the identity operator, and v is defined by

$$(vf)(p) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{T}^3} f(q)dq, \quad f \in L^2(\mathbb{T}^3),\ p \in \mathbb{T}^3. \tag{15}$$

The operator $H^1(0)$ acts on $\mathcal{H}_0^\perp$ as

$$H^1(K) = H_0(K) - \mu_1 V_1 - \mu_2 V_2.$$

The following assertion plays important role in our proofs and can be proven by elementary methods, as in Ref. [17, Lemma 2.1].

Lemma 4.1. *Let $d \geq 3$. For any fixed $n \in \mathbb{Z}^3$, the function*

$$d_n(z) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{T}^3} \frac{e^{i(n,t)}}{\varepsilon_3(t) - z} dt, \quad z \in \mathbb{C} \setminus [E_{\min}(K), E_{\max}(K)] \tag{16}$$

is positive and monotonically increasing in the interval $(-\infty, E_{\min}(K))$ as a function of z.

Additionally, we have the following limit:

$$d_n(0) = \lim_{z \to E_{\min}(K)-} d_n(z) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{T}^3} \frac{e^{i(n,t)}}{\varepsilon_3(t) - E_{\min}(K)} dt. \tag{17}$$

Since $(\varepsilon_3(t))^{-1}$ is a L^1-function, we get

Lemma 4.2. *For any $z \in (-\infty, 0]$, the relation $\lim\limits_{n \to \infty} d_n(z) = 0$ holds.*

Recall that

$$(V_1 f)(p,q) = \frac{\mu_1}{(2\pi)^3} \int\limits_{\mathbb{T}^3} f(p,t) dt, \quad (V_2 f)(p,q) = \frac{\mu_2}{(2\pi)^3} \int\limits_{\mathbb{T}^3} f(t,q) dt, \quad f \in L_2((\mathbb{T}^3)^2.$$

We represent the operators V_1 and V_2 in the forms $V_1 = \mu_1 \varphi_1^* \varphi_1$ and $V_2 = \mu_2 \varphi_2^* \varphi_2$, where the operators $\varphi_1, \varphi_2 : L^2((\mathbb{T}^3)^2) \to L^2(\mathbb{T}^3)$ are defined as

$$(\varphi_1 f)(p) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int\limits_{\mathbb{T}^3} f(p,t) dt \quad \text{and} \quad (\varphi_2 f)(q) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int\limits_{\mathbb{T}^3} f(t,q) dt, \tag{18}$$

respectively.

Lemma 4.3. *The number $z \in \mathbb{C} \setminus \sigma_{ess}(H^1(K))$ is an eigenvalue of the operator $H^1(K)$ if and only if $D(z) = 0$, where*

$$D(z) = \prod_{n \in \mathbb{Z}^3} D_n(z), \tag{19}$$

with

$$D_n(z) = 1 - \frac{\mu_1}{\Delta_2(z)} \frac{\mu_2}{\Delta_1(z)} d_n^2(z). \tag{20}$$

If $z_n \in \mathbb{C} \setminus \sigma_{ess}(H^1(K))$ is an eigenvalue of $H^1(K)$ and $D_n(z_n) = 0$, $n \in \mathbb{Z}^3$, then the corresponding eigenfunction is of the form

$$f_n(p,q) = \frac{\mu_1}{\varepsilon_3(K-p-q) - z} \left[e^{i(n,(-K/2+p))} + \frac{\mu_2 d_n(z_n)}{\Delta_2(z_n)} e^{i(n,(K/2-q))} \right]. \tag{21}$$

Proof. Since $H(K)$ and $H^1(K)$ are unitarily equivalent to $H(0)$ and $H^1(0)$, respectively, we prove the assertion for the latter operator.

Let $z \in \mathbb{C} \setminus \sigma_{ess}(H^1(0))$ be an eigenvalue of $H^1(0)$, and let f be the corresponding eigenfunction, then

$$f = R_0(z)[\mu_1 V_1 + \mu_2 V_2] f. \tag{22}$$

This equation has a non-trivial solution if and only if the system of two linear equations

$$\tilde{\varphi}_\alpha = \varphi_\alpha \left(R_0(z)[\mu_1 \varphi_1^* \tilde{\varphi}_1 + \mu_2 \varphi_2^* \tilde{\varphi}_2] \right), \ \alpha = 1, 2, \ \tilde{\varphi}_1, \tilde{\varphi}_2 \in L^2(\mathbb{T}^3) \tag{23}$$

on the space $L^2(\mathbb{T}^3) \oplus L^2(\mathbb{T}^3)$ has a non-zero solution.

Solutions of the equations (22) and (23) are related by

$$f(p,q) = R_0(z)[\mu_1 \varphi_1^* \tilde{\varphi}_1 + \mu_2 \varphi_2^* \tilde{\varphi}_2], \tag{24}$$

and

$$\tilde{\varphi}_\alpha = \varphi_\alpha f, \quad \alpha = 1, 2. \tag{25}$$

We note that the function

$$\Delta_\alpha := I - \mu_\alpha \varphi_\alpha R_0(z)\varphi_\alpha^*, \quad z \in \mathbb{C} \setminus \sigma_{ess}(H_0), \quad \alpha = 1, 2,$$

is nonzero for any $z \in \mathbb{C} \setminus \sigma_{ess}(H^1(0))$, therefore there exists its inverse

$$\Delta_\alpha^{-1} = (I - \mu_\alpha \varphi_\alpha R_0(z)\varphi_\alpha^*)^{-1}.$$

Then, the solutions $\tilde{\varphi}_\alpha, \alpha = 1, 2$, of the equation (23) satisfy the following system of integral equations

$$\begin{cases} \tilde{\varphi}_1 = \mu_2 \Delta_1^{-1} Q \tilde{\varphi}_2, \\ \tilde{\varphi}_2 = \mu_1 \Delta_2^{-1} Q^* \tilde{\varphi}_1, \end{cases} \tag{26}$$

where Q^* is the adjoint of Q and

$$Q = \varphi_1 R_0(z) \varphi_2^* \tag{27}$$

is the integral operator on $L^2(\mathbb{T}^3)$ defined as

$$(Q\varphi)(p) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{T}^3} \frac{\varphi(t)dt}{\varepsilon_3(-p-t) - z}. \tag{28}$$

Using the substitution method, the system (26) can be reduced into the form of an one unknown function

$$\tilde{\varphi}_1 = Q(z)\tilde{\varphi}_1, \quad \text{i.e.,} \quad Q(z) = \mu_1\mu_2 \Delta_1^{-1}\Delta_2^{-1} Q Q^*. \tag{29}$$

Moreover, if $\Phi = (\tilde{\varphi}_1, \tilde{\varphi}_2)$ is a solution to (26), then $\tilde{\varphi}_1$ is an eigenfunction of $Q(z)$ corresponding to the eigenvalue 1. Conversely, let $\tilde{\varphi}_1$ be an eigenfunction of $Q(z)$ corresponding to the eigenvalue 1. Then $\Phi = (\tilde{\varphi}_1, \tilde{\varphi}_2)$, with $\tilde{\varphi}_2 = \frac{\mu_1}{\Delta_2(z)} Q\tilde{\varphi}_1$ is a solution to (26) (i.e. (23)). Notice that the multiplicities of the linearly independent eigenvectors $\tilde{\varphi}_1$ and Φ coincide.

Additionally, the function f defined in (24) is an eigenfunction of $H^1(0)$ corresponding to an eigenvalue $z \in \mathbb{C} \setminus \sigma_{ess}(H^1(0))$. Moreover, the multiplicities of eigenvalues z of $H^1(0)$ and $\lambda = 1$ of $Q(z)$ coincide.

The operator $Q(z)$ is a convolution-type trace-class integral operator. The standard Fourier transform $\mathcal{F}_1 : L^2(\mathbb{T}^3) \to \ell^2(\mathbb{Z}^3)$,

$$(\mathcal{F}_1 g)(n) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{T}^3} e^{i(n,p)} g(p)dp, \quad g \in L^2(\mathbb{T}^3),\ n \in \mathbb{Z}^3,$$

establishes that its Fourier transform $\widehat{Q}(z) := \mathcal{F}_1 Q(z) \mathcal{F}_1^*$ acts as a multiplication operator on the space $\ell^2(\mathbb{Z}^3)$ by the function

$$\lambda_n(z) = \frac{\mu_1\mu_2}{\Delta_1(z)\Delta_2(z)} d_n(z) d_{-n}(z), \quad n \in \mathbb{Z}^3.$$

Thus, the spectrum of $\widehat{Q}(z)$ consists of the following union

$$\sigma(\widehat{Q}(z)) = \{0\} \cup \bigcup_{n \in \mathbb{Z}^3} \{\lambda_n(z)\}$$

with the space of eigenfunctions

$$\hat{\varphi}_n(m) = \delta_{n,m}, \quad n, m \in \mathbb{Z}^3,$$

where $\delta_{\cdot,\cdot}$ is the Kroneker delta function on $\mathbb{Z}^3$.

Note that the compact operator $Q(z)$ has eigenvalues $\lambda_n(z)$, $n \in \mathbb{Z}^3$ with the corresponding eigenfunctions

$$\psi_n(q) = e^{i(n,p)}, \quad n \in \mathbb{Z}^3,\ p \in \mathbb{T}^3. \tag{30}$$

Therefore, the determinant of the operator $I - Q(z)$ can be written as the following product

$$\det(I - Q(z)) = \prod_{n \in \mathbb{Z}^3} (1 - \lambda_n(z)). \tag{31}$$

Let $\lambda_n(z_n) = 1$ be an eigenvalue of $Q(z_n)$, then ψ_n is the first component of the solution $\Phi_n = (\psi_n, \tilde{\varphi}_2)$ of equation (23), and the second component is defined by (26)

$$\tilde{\psi}_n(q) := \tilde{\varphi}_2(q) = \mu_1 \Delta_2^{-1} Q^* \psi_n(q) = \frac{\mu_1 d_n(z_n)}{\Delta_2(z_n)} e^{-i(n,q)}. \tag{32}$$

Subsequently, the number z_n is an eigenvalue of $H^1(0)$ and the corresponding eigenfunction can be found by (24) as

$$f_n^0 = R_0(z_n)[\mu_1 \varphi_1^* \psi_n + \mu_2 \varphi_2^* \tilde{\psi}_n], \tag{33}$$

i.e.,

$$f_n^0(p,q) = \frac{\mu_1}{\varepsilon_3(-p-q) - z} \left[e^{i(n,p)} + \mu_2 \frac{d_n(z_n)}{\Delta_2(z_n)} e^{i(n,-q)} \right].$$

According to the relation $H(0) = U_K^* H(K) U_K$, the number z_n is also an eigenvalue of $H(K)$ with the eigenfunction $f_n = U_K f_n^0$, i.e.,

$$f_n(p,q) = \frac{\mu_1}{\varepsilon_3(K-p-q) - z} \left[e^{i(n,(-K/2+p))} + \frac{\mu_2 d_n(z_n)}{\Delta_2(z_n)} e^{i(n,(K/2-q))} \right],$$

where the unitary operator U_K is defined in (13). □

Note that if $z_n = E_{\min}(K)$ and $\mu_1 = \mu_2 = (d_0 \pm d_n)^{-1}$, then z_n is a threshold resonance.

Let us recall some properties of the function $\Delta_\alpha(z)$, $\alpha = 1, 2$.

Lemma 4.4. *(a) If $z \in (-\infty, 0)$, then $\Delta'_\alpha(z) < 0$.*
(b) If $0 < \mu_\alpha \le \mu_0$, then $\Delta_\alpha(z) > 0$
(c) If $\mu_\alpha > \mu_0$, then there exists a unique zero of $\Delta_\alpha(z)$ with a multiplicity of 1.

Proof. The part (a) easily follows from the definition of $\Delta_\alpha(z)$, as it is a decreasing function in $(-\infty, 0)$. The relation $\Delta'_\alpha(z) < 0$ and the Intermediate Value Theorem yield other assertions. □

For any $\mu_1, \mu_2 > 0$, instead of the function $D_n(z)$, $n \in \mathbb{Z}^3$, we consider

$$\widetilde{D}_n(z) = \Delta_1(z)\Delta_2(z) - \mu_1\mu_2 d_n^2(z), \quad n \in \mathbb{Z}^3. \tag{34}$$

Let z_1^0 and z_2^0 be the zeros of the functions $\Delta_1(z)$ and $\Delta_2(z)$, respectively, in the interval $(-\infty, 0)$. Without loss of generality, we assume $z_1^0 \le z_2^0$ whenever z_1^0 and z_2^0 exist. The following lemma is straightforward from the definition of $\widetilde{D}_n(z)$.

Lemma 4.5. *(a) $\widetilde{D}_n(z)$ is strictly decreasing in $(-\infty, z_1^0)$ or in $(-\infty, 0)$, if z_1^0 does not exist.*
(b) $\widetilde{D}_n(z)$ is negative in $(z_1^0, 0)$ or in (z_1^0, z_2^0), if z_2^0 exists.
(a) $\widetilde{D}_n(z_n)$ is strictly increasing in $(z_2^0, 0)$, whenever z_2^0 exists.

Proof. (a) The derivative $\widetilde{D}'_n(z) = \Delta'_1(z)\Delta_2(z) + \Delta_1(z)\Delta'_2(z) - 2\mu_1\mu_2 d_n(z) d'_n(z)$ of $\widetilde{D}_n(z)$ is negative, since $\Delta_1(z) > 0$, $\Delta_2(z) > 0$, $\Delta'_1(z) < 0$, $\Delta'_2(z) < 0$ (see Lemma 4.4) and $d_n(z) > 0$, $d'_n(z) > 0$ for $z, z \in (-\infty, z_1^0)$.

(b) Since $\Delta_1(z) < 0$, $\Delta_2(z) > 0$ and $d_n(z) > 0$ for any $z \in (z_1^0, 0)$ ($z \in (z_1^0, z_2^0)$), the function $\widetilde{D}_n(z)$ is negative in this interval.

(c) Let z_2^0 exist.

$$\widetilde{D}_n(z) = \mu_1\mu_2 \left(g_1(z) g_2(z) - (\mu_1^{-1} - \mu_2^{-1})^2/4 \right),$$

where

$$g_1(z) = d_0(z) - (\mu_1^{-1} + \mu_2^{-1})/2 - d_n(z), \quad g_2(z) = d_0(z) - (\mu_1^{-1} + \mu_2^{-1})/2 + d_n(z)$$

are increasing functions.

Thus the derivative $\widetilde{D}'_n(z)$ is positive in $(z_2^0, 0)$. □

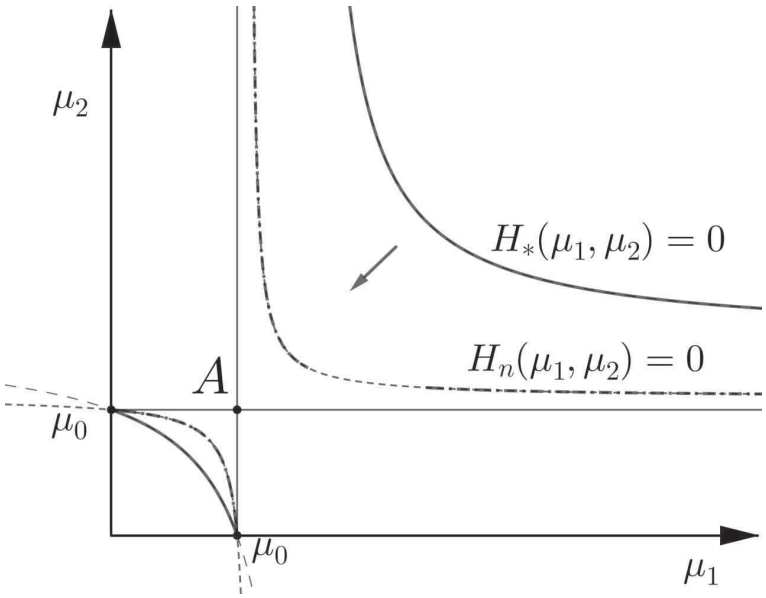


FIGURE 1. Hyperbolas

For any $n \in \mathbb{Z}^3 \setminus \{0\}$, let us introduce the hyperbolas on the $\mu_1\mu_2$-plane.

$$H_n := H_n(\mu_1, \mu_2) = 1 - (\mu_1 + \mu_2)/\mu_0 + \mu_1\mu_2(1/\mu_0^2 - d_n(0)^2) = 0, \tag{35}$$

$$H_* = H_*(\mu_1, \mu_2) = 1 - (\mu_1 + \mu_2)/\mu_0 + \mu_1\mu_2(1/\mu_0^2 - d_*^2) = 0, \tag{36}$$

where

$$d_n(0) = \lim_{z\to 0-} d_n(z) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{T}^3} \frac{e^{i(n,t)}}{\varepsilon_3(t)} dt, \quad d_* = \sup_{n\in\mathbb{Z}^3\setminus\{0\}} d_n(0), \quad \mu_0 = 1/d_0(0),$$

and

$$H_n(\mu_1, \mu_2) = \lim_{z\to 0-} \widetilde{D}_n(z). \tag{37}$$

Since $d_n(0) \to 0+$ as $n \to \infty$ the supremum $d_* = \sup_{n\in\mathbb{Z}^3\setminus\{0\}} d_n(0)$ reaches at some $n \in \mathbb{Z}^3$.

Remark 4.1. *While H_n is defined for $n = 0 \in \mathbb{Z}^3$ it is not hyperbola, but it is the straight line of the form*

$$H_0 = 1 - (\mu_1 + \mu_2)/\mu_0 = 0.$$

One may see that the equation of the hyperbola $H_n, n \in \mathbb{Z}^3 \setminus \{0\}$ equivalent to the equation of the hyperbola

$$(\mu_1 - \mu_1^\infty)(\mu_2 - \mu_2^\infty) - \nu_0^2 = 0$$

by the asymptotes

$$\mu_1^\infty = \frac{\mu_0}{(1 - (\mu_0 d_n(0))^2)} \quad \text{and} \quad \mu_2^\infty = \frac{\mu_0}{(1 - (\mu_0 d_n(0))^2)},$$

and the vertices $(\mu_1^\infty - \nu_0, \mu_1^\infty - \nu_0)$ and $(\mu_1^\infty + \nu_0, \mu_1^\infty + \nu_0)$, where

$$\nu_0 = \frac{\mu_0 d_n(0)}{\sqrt{1 - (\mu_0 d_n(0))^2}}.$$

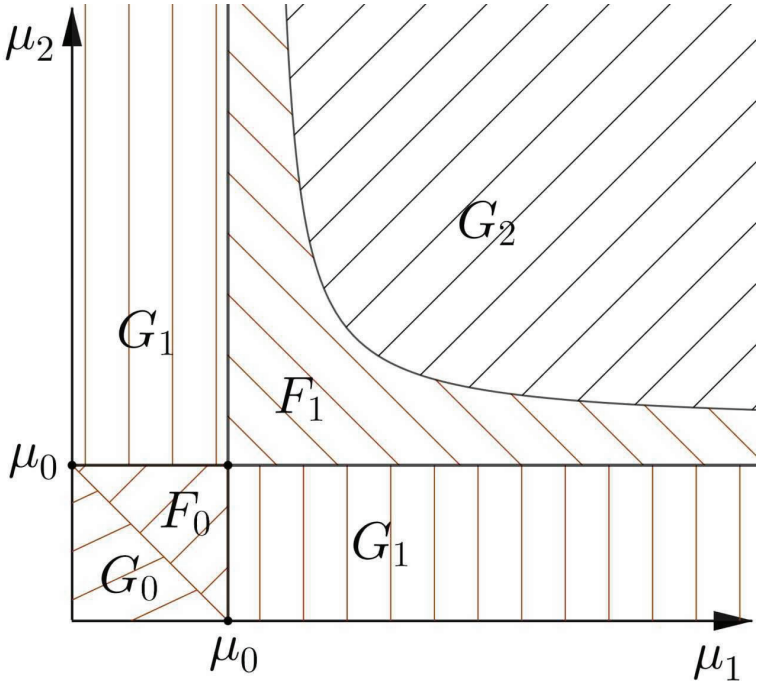


FIGURE 2. The connected components

The curves

$$\Gamma_0 = \left\{(\mu_1, \mu_2) \in \mathbb{R}^2_{>0} | \mu_1 + \mu_2 = \mu_0\right\}, \tag{38}$$

$$\Gamma_n^l = \left\{(\mu_1, \mu_2) \in \mathbb{R}^2_{>0} | H_n(\mu_1, \mu_2) = 0,\ \mu_1 < \mu_1^\infty\right\}, \tag{39}$$

$$\Gamma_n^r = \left\{(\mu_1, \mu_2) \in \mathbb{R}^2_{>0} | H_n(\mu_1, \mu_2) = 0,\ \mu_1 > \mu_1^\infty\right\}, \tag{40}$$

$$\Gamma_*^r = \left\{(\mu_1, \mu_2) \in \mathbb{R}^2_{>0} | H_*(\mu_1, \mu_2) = 0,\ \mu_1 > \mu_1^\infty\right\}, \tag{41}$$

lie on the first quadrant of the $\mu_1\mu_2$-plane, and are the boundary of G_0, the left branch of the hyperbola H_n, the right branch the hyperbola H_n and the right branch of the hyperbola H_*, respectively.

The expressions of the asymptotes and vertices imply that for any $n, m \in \mathbb{Z}^3 \setminus \{0\}$, $n \neq m$, the hyperbolas Γ_n^l and Γ_m^l have common points $(\mu_0, 0)$ and $(0, \mu_0)$ on the $\mu_1\mu_2$-plane, and their right branches Γ_n^r and Γ_m^r do not intersect (see Figure 1). Moreover, if $d_m(0) < d_n(0)$, then the right branch Γ_m^r becomes closer than the right branch Γ_n^r, since the vertices approach the point $A(\mu_0, \mu_0)$ as $d_n(0) \to 0$.

We can claim the following assertion.

Lemma 4.6. *(a) If $d_m(0) < d_n(0)$, the Γ_m^r lies to the left Γ_n^r;*
(b) For any $n \in \mathbb{Z}^3 \setminus \{0\}$, the curve Γ_n^r lies in F_1.

The right branch of the hyperbola $H_*(\mu_1, \mu_2) = 0$, the line segment $H_0(\mu_1, \mu_2) = 0$, and half lines $\mu_1 = \mu_0$ and $\mu_2 = \mu_0$ divide the $\mu_1\mu_2$-plane into the regions

$$G_0 = \{(\mu_1, \mu_2) | \mu_1 >, \mu_2 > 0, \quad \mu_1 + \mu_2 \leq \mu_0\}, \tag{42}$$

$$F_0 = \{(\mu_1, \mu_2) | 0 < \mu_1 < \mu_0, 0 < \mu_2 < \mu_0, \quad \mu_1 + \mu_2 \geq \mu_0\}, \tag{43}$$

$$G_1 = \{(\mu_1, \mu_2) | (0 < \mu_2 \leq \mu_0 \& \mu_0 \geq \mu_1) \quad \text{or} \quad (0 < \mu_1 \leq \mu_0 \& \mu_0 \geq \mu_2)\}, \tag{44}$$

$$F_1 = \{(\mu_1, \mu_2) | \mu_2 > \mu_0,\ \mu_1 > \mu_0,\ H_*(\mu_1, \mu_2) \leq 0\}, \tag{45}$$

$$G_2 = \{(\mu_1, \mu_2) | \mu_2 > \mu_0,\ \mu_1 > \mu_0,\ H_n(\mu_1, \mu_2) > 0\}. \tag{46}$$

$$\tag{47}$$

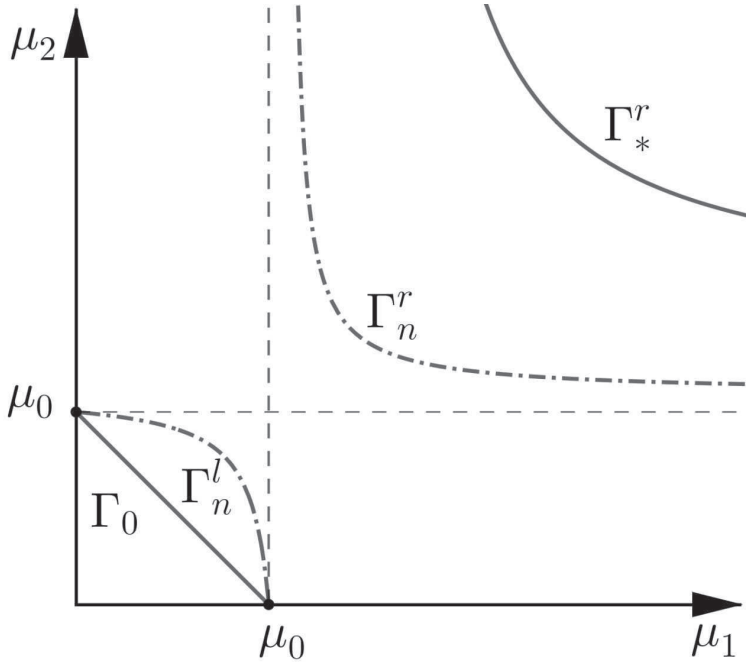


FIGURE 3. The connected components

The following set

$$P_* = \{m \in \mathbb{Z}^3 \setminus \{0\} | d_m(0) = d_*\}, \quad \text{where} \quad d_* = \sup_{n \in \mathbb{Z}^3 \setminus \{0\}} d_n(0), \tag{48}$$

has a finite elements, since $d_n(0) \to 0$ as $n \to \infty$ (see Lemma 4.2).

Lemma 4.7. *(a) Let $(\mu_1, \mu_2) \in G_0$, then for any $n \in \mathbb{Z}^3$ the function $\widetilde{D}_n(z)$ has no zeros in $(-\infty, 0)$.*

(b) Let $(\mu_1, \mu_2) \in F_0$, then for finite number of $n \in \mathbb{Z}^3 \setminus \{0\}$ (at least for $0 \in \mathbb{Z}^3$) the function $\widetilde{D}_n(z)$ has a unique zero in $(-\infty, 0)$.

(c) Let $(\mu_1, \mu_2) \in G_1$, then for any $n \in \mathbb{Z}^3$ (at least for $0 \in \mathbb{Z}^3$) the function $\widetilde{D}_n(z)$ has a unique zero in $(-\infty, z_1^0)$ and it has no eigenvalues in $(z_1^0, 0)$.

(d) Let $(\mu_1, \mu_2) \in F_1$, then exists $C > 0$ such that $n \in \mathbb{Z}^3$, $|n| > C$ the $\widetilde{D}_n(z)$ has two zeros $z_n \in (-\infty, z_1^0)$ and $z_n \in (z_2^0, 0)$. Moreover, for finite number of $n \in \mathbb{Z}^3$, $|n| \le C$ (at least for $n \in P_* \cup \{0\}$), the function $\widetilde{D}_n(z)$ has a unique zero in $(-\infty, z_1^0)$, but it has no any zeros in $(z_2^0, 0)$.

(f) Let $(\mu_1, \mu_2) \in G_2$, then for any $n \in \mathbb{Z}^3 \setminus \{0\}$, the function $\widetilde{D}_n(z)$ has two zeros $z_n \in (-\infty, z_1^0)$ and $z_n \in (z_2^0, 0)$.

(g) The $\widetilde{D}_n(z)$ has no zeros in $[z_1^0, z_2^0]$.

Proof. As Lemma 4.5, the Intermediate Value Theorem and the limit

$$\lim_{z \to -\infty} \widetilde{D}_n(z) = 1$$

play the key role in our substantiation, we do not mention them in the parts of the proof.

(a) Since $\widetilde{D}_n(z)$ is strictly decreasing and $\widetilde{D}_n(0) = H_0(\mu_1, \mu_2) \ge 0$, the proof is obvious.

(b) There existence of z_0 for $\widetilde{D}_0(z) = 0$ yields from $\widetilde{D}_n(0) < 0$.

Let $n \in \mathbb{Z}^3 \setminus \{0\}$. The part of the left branch of H_n on the $\mu_1\mu_2$-plane lies in F_0. If $H_n(\mu_1, \mu_2) < 0$, then there exists a unique $z_n \in (-\infty, 0)$ as $\widetilde{D}_n(z)$ is strictly decreasing.

As long as the part of the left branch of H_n gets close to the lines $\mu_1 = \mu_0$ and $\mu_2 = \mu_0$ as $n \to \infty$, the inequality $\widetilde{D}_n(0) < 0$ holds for finite number of n.

In particular, if $H_*(\mu_1,\mu_2)\geq 0$, then the statement is true for only $\widetilde{D}_0(z)$.

(c) The root z_1^0 of either equation $\Delta_1(z)=0$ or $\Delta_2(z)=0$ exists, and $\widetilde{D}_n(z_1^0)<0$ and $\widetilde{D}_n(z)$ is negative in $(z_1^0,0)$. These facts imply the proof.

(d) In this case both functions $\Delta_1(z)$ and $\Delta_2(z)$ have zeros below 0, and assume $z_1^0\leq z_2^0$.

If $m\in P_*$, then the point (μ_1,μ_2) lies to left of the right branch of H_m and $H_m(\mu_1,\mu_2)<0$.

It is clear that the set $B_*:=B(\mu_1,\mu_2)=\{n\in\mathbb{Z}^3|H_n(\mu_1,\mu_2)\leq 0\}$ contains the set P_* or equals to it.

As $n\to\infty$ the right branch of H_n gets closed to the lines $\mu_1=\mu_0$ and $\mu_2=\mu_0$, so $H_n(\mu_1,\mu_2)>0$ for infinitely many $n\in\mathbb{Z}^3$ and the set B_* is finite set.

As a result, for $n\in\mathbb{Z}^3\setminus(B_*\cup\{0\})$ the function $\widetilde{D}_n$ satisfies $\widetilde{D}_n(z_1^0)<0$, $\widetilde{D}_n(z_2^0)<0$ and $\widetilde{D}_n(0)>0$, which give that $\widetilde{D}_n$ have simple zeros $z_n\in(-\infty,z_1^0)$ and $\mathbb{Z}eta_n\in(z_2^0,0)$.

Note that $\widetilde{D}_n(z_1^0)<0$ holds for any $n\in\mathbb{Z}^3$, which implies z_n exists for any $n\in\mathbb{Z}^3$.

(f) If $(\mu_1,\mu_2)\in G_2$, then $\widetilde{D}_n(z_1^0)<0$ for $n\in\mathbb{Z}^3$, and $\widetilde{D}_n(z_2^0)<0$ and $\widetilde{D}_n(z_2^0)<0$ for $n\in\mathbb{Z}^3\setminus\{0\}$. Therefore $z_n\in(-\infty,z_1^0)$, $n\in\mathbb{Z}^3$ and $\zeta_n\in(z_2^0,0)$, $n\in\mathbb{Z}^3\setminus\{0\}$ exist.

(g) $\widetilde{D}_n(z)$ is negative in $[z_1^0,z_2^0]$, thus the statement is true. □

Lemma 4.8. *If infinitely many z_n or ζ_n exist, then*

$$\lim_{n\to\infty}z_n=z_1^0\quad or\quad \lim_{n\to\infty}\zeta_n=z_2^0.$$

Proof. Without loss of generality we assume z_2^0 exists.

Since $\widetilde{D}_n(z_n)=0$, i.e. $d_n(z)=\sqrt{\Delta_1(z_n)\Delta_2(z_n)/(\mu_1\mu_2)}$, and $d_n(z)\to 0$ as $n\to\infty$ due to Lemma 4.2, the convergence $\Delta_1(z_n)\to 0$ or $\Delta_2(z_n)\to 0$ as $n\to\infty$ holds. The continuity of $\Delta_1(z)$ and $\Delta_2(z)$ imply $\lim\limits_{n\to\infty}z_n=z_1^0$ or $\lim\limits_{n\to\infty}z_n=z_2^0$.

Since $z_n<z_1^0\leq z_2^0$, we get the proof of the first limit.

In a similar way, we may prove the second limit. □

The following theorem follows from Lemma 4.3 and 4.7.

Theorem 4.9. *(a) Let $(\mu_1,\mu_2)\in G_0$, then $\sigma_{\rm disc}(H)=\varnothing$.*

(b) Let $(\mu_1,\mu_2)\in F_0$, then $\sigma_{\rm disc}(H)=\bigcup_{n\in A_*}\{z_n\}$, where

$$A_*:=A(\mu_1,\mu_2)=\{n\in\mathbb{Z}^3|\widetilde{D}_n(0)<0\}$$

is a finite set.

(c) Let $(\mu_1,\mu_2)\in G_1$, then $\sigma_{\rm disc}(H)=\bigcup_{n\in\mathbb{Z}^3}\{z_n\}$.

(d) Let $(\mu_1,\mu_2)\in F_1$, then $\sigma_{\rm disc}(H)=\bigcup_{n\in\mathbb{Z}^3}\{z_n\}\cup\bigcup_{n\in B_*}\{\zeta_n\}$, where

$$B_*:=B(\mu_1,\mu_2)=\{n\in\mathbb{Z}^3|\widetilde{D}_n(0)>0\}$$

is an infinite set.

(f) Let $(\mu_1,\mu_2)\in G_2$, then $\sigma_{\rm disc}(H)=\bigcup_{n\in\mathbb{Z}^3}\{z_n\}\cup\bigcup_{n\in\mathbb{Z}^3\setminus\{0\}}\{\zeta_n\}$

Conclusions

The discrete Hamiltonian of a system of three arbitrary particles was considered on the three-dimensional lattice, $\mathbb{Z}^3$. The interactions between the pairs of particles were described via zero-range (contact) attractive potentials. Using the direct-integral decomposition, the problem was reduced to investigating the properties of the simpler two-particle Schrödinger operators, called the channel operators. We applied certain properties of the set of parameter-dependent hyperbolas to show infiniteness of the discrete spectrum of the corresponding Schrödinger operator provided masses of any two particles are infinite.

Acknowledgements. The first two authors acknowledge support from the Foundation for Basic Research of the Republic of Uzbekistan (Grant No. FZ–20200929224).

References

[1] V. N. Efimov, "Weakly-bound states of three resonantly-interacting particles", Sov. J. Nuclear Phys., **12** (1971) 589.

[2] D. Yafaev, "On the theory of the discrete spectrum of the three-particle Schrödinger operator", Math. USSR-Sb., **23** (1974) 535–559.

[3] Yu. N. Ovchinnikov, I. M. Sigal, "Number of bound states of three-body systems and Effemov's effect", Ann. Phys., **123** (1989) 274–295.

[4] A. Sobolev, "The Efimov's effect. Discrete spectrum asymptotics", Commun. Math. Phys., **156** (1993) 127–168.

[5] H. Tamura, "The Efimov's effect of three-body Schrödinger operator", J. Funct. Anal., **95** (1991) 433–459.

[6] S. Lakaev, "The Efimov's effect of a system of three identical quantum lattice particles", Funct. Anal. Its Appl., **27** (1993) 15–28.

[7] T. Kraemer, M. Mark, P. Waldburger, J. G. Danzl, C. Chin, B. Engeser, A. D. Lange, K. Pilch, A. Jaakkola; H.-C. Nägerl, R. Grimm, "Evidence for Efimov quantum states in an ultracold gas of caesium atoms", Nature, **440** (2006) 315–318.

[8] D. Mattis, "The few-body problem on a lattice", Rev. Mod. Phys., **58** (2), (1986) 361–379.

[9] A. Mogilner, "Hamiltonians in solid-state physics as multiparticle discrete Schrödinger operators: problems and results", Adv. in Sov. Math., **5** (1991) 139–194.

[10] G. Graf, D. Schenker, "2-magnon scattering in the Heisenberg model", Ann. Inst. Henri Poincaré, Phys. Théor., **67** (1997) 91–107.

[11] D. Yafaev, *Scattering Theory: Some Old and New Problems.* Lecture Notes in Mathematics **1735**. Springer-Verlag, Berlin, 2000.

[12] S. Albeverio, S. Lakaev, Z. Muminov, "On the structure of the essential spectrum for the three-particle Schrödinger operators on lattices", Math. Nachr. **280**, (2007) 699–716.

[13] M. Muminov,"The infiniteness of the number of eigenvalues in the gap in the essential spectrum for the three-particle Schrödinger operator on a lattice", Theor. Math. Phys, **159** (2009) 299–317.

[14] S. N. Lakaev, Z. I. Muminov, "The asymptotics of the number of eigenvalues of a three-particle lattice Schrödinger operator", Funct. Anal. its Appl., **37** (2003) 228–231.

[15] S. Albeverio, S. Lakaev, Z. Muminov "Schrödinger operators on lattices. The Efimov effect and discrete spectrum asymptotics", Ann. Inst. H. Poincaré Phys. Theor. **5** (2004) 743–772.

[16] V. S. Rabinovich, S. Roch, "The essential spectrum of Schrödinger operators on lattice", J. Phys. A.:Math. Gen., **39** (2006) 8377.

[17] Z. Muminov, Sh. Alladustov, Sh. Lakaev, "Spectral and threshold analysis of a small rank perturbation of the discrete Laplacian", J. Math. Anal. Appl., **496** (2021)124827.

[18] F. Hiroshima, J. Lőrinci, "The spectrum of non-local discrete Schrödinger operators with a δ-potential", Pac. J. Math. Ind. **6** (2014) 7.

[19] S. N. Lakaev, S. K. Kurbanov, S. U. Alladustov, "Convergent Expansions of Eigenvalues of the Generalized Friedrichs Model with a Rank-One Perturbation", Complex Anal. Oper. Theory **15** (2021) 121.

[20] Sh. Nakamura, Y. Tadano, "On a continuum limit of discrete Schrödinger operators on square lattice", J. Spectr. Theory, **11** (1) (2021) 355–367.

[21] I. Hiroshi, M. Hisashi, "Inverse scattering at a fixed energy for Discrete Schrödinger Operators on the square lattice", Ann. Inst. Fourier, **65** (3) (2015) 1153–1200.

[22] Sh. U. Alladustov, O. I. Kurbonov, H. Z. Axralov, "On the negative eignvalues of the discrete Schrödinger operator with non-local potential in three-dimentional case. "Lob. J. Math., **43**:11 (2022) 3039–3047.

[23] Z. I. Muminov, Sh. Alladustov, Sh. Lakaev, "Threshold Analysis of the Three Dimensional Lattice Schrodinger Operator with Non-local Potential", Lob. J. Math., **41**(6) (2020) 1094–1102.

[24] M. I. Muminov, N. M. Aliev, "Spectrum of the three-particle Schrödinger operator on a one-dimensional lattice", Theor. Math. Phys, **171** (3) (2012) 754–768.

[25] N. M. Aliev, M. E. Muminov, "On the spectrum of the three-particle Hamiltonian on a unidimensional lattice", Siberian Adv. Math., **17** (3) (2015) 3–22.

[26] G. Dell'Antonio, Z.I. Muminov, Y.M. Shermatova. "On the number of eigenvalues of a model operator related to a system of three-particles on lattices", J. Phys. A: Math. Theor., **44** (2011) 315302.

[27] F. Hiroshima, Z. I. Muminov, U. Kuljanov. "Threshold of discrete Schrödinger operators with delta potentials on N-dimensional lattice", Linear Multilinear Algebra, **70** (5) (2022) 919–954 .

[28] Z. I. Muminov, N. M. Aliev, T. Radjabov, "On the discrete spectrum of the three-particle Schrödinger operator on a two-dimensional lattice. "Lob. J. Math., **43**:11 (2022) 3239–3251.

[29] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics. Vol. IV: Analysis of Operators.* Academic Press, New York, 1978.

[30] Sh. Yu. Kholmatov, Z. Muminov, "The essential spectrum and bound states of N-body problem in an optical lattice."J. Phys. A: Math. Theor. **51** (2018) 265202.

[31] S. Albeverio, S. N. Lakaev, Z. I. Muminov, “Schrödinger operators on lattices. The Efimov effect and discrete spectrum asymptotics, ”Ann. Inst. H. Poincaré Phys. Theor., **5** (2004) 743–772 .
[32] C. Van Winter, “Theory of finite systems of particles”, I. Mat.-Fys. Skr. Danske Vid. Selsk, **1**(8), (1964) 1–60.
[33] G. Zhislin, “Discussion of the spectrum of the Schrödinger operator for systems of many particles”, Tr. Mosk. Mat. Obs., **9** (1960) 81–128 .
[34] W. Hunziker “On the spectra of Schrödinger multiparticle Hamiltonians”, Helv. Phys. Acta, **39**, (1966) 451–462 .
[35] Sh. Yu. Kholmatov, S. N. Lakaev, F. M. Almuratov, “On the spectrum of Schrödinger-type operators on two dimensional lattices”, J. Math. Anal. Appl., **514** (2022)126363.
[36] K. Ando, H. Isozaki, H. Morioka, “Spectral Properties of Schrödinger Operators on Perturbed Lattices”, Ann. Henri Poincaré, **17** (2016) 2103–2171.
[37] S. Albeverio, S. N. Lakaev, Z. I. Muminov, “The threshold effects for a family of Friedrichs models under rank one perturbations”, J. Math. Anal. Appl., **330**(2), (2007) 1152–1168.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

CONSTRUCTION OF HYPERGEOMETRIC TYPE INHOMOGENEOUS SECOND-ORDER SYSTEM SOLUTIONS

ZHANAR UBAYEVA, MEIRAMGUL TALIPOVA, RYSKUL ZHAKHINA, ALTYNZER BAIGANOVA

Aktobe Regional University named after K. Zhubanov, Aktobe, Kazakhstan

e-mail: Zhanar_Ubaeva@mail.ru

Abstract. The possibilities of constructing solutions to an inhomogeneous special system consisting of two partial differential equations of the third order have been studied. A number of specific particular cases of a generalized system hypergeometric type are derived whose solutions are generalized hypergeometric functions and orthogonal polynomials of two variables. The general method for constructing solutions near regular and irregular singularities is established, the number of linearly independent particular solutions is determined, the regular and irregular singularities are classified, the conditions of compatibility and integrability are clarified. Application of the uncertain coefficients method to the construction of heterogeneous systems particular solution is shown by using concrete examples. The number of linearly independent particular solutions for Euler and Kampe de Ferriet type systems and particular solution of the inhomogeneous differential equation theorems were proved.

Keywords: system, generalized hypergeometric function, shown, regular, irregular, heterogeneous system, features.

1. Introduction

In the theory of ordinary differential equations all special functions and classic orthogonal polynomials are particular cases of the Gauss hypergeometric function and each of them are solutions corresponding to hypergeometric type second-order homogeneous equation. Such approach facilitated the study of their various properties. Application in various technical problems stimulated the study of the corresponding inhomogeneous equations. This study aims to investigate the possibility of building special solutions inhomogeneous system consisting of two differential equations of third order and highlight a number of specific individual cases of generalized hypergeometric type system solutions which are generalized hypergeometric functions and orthogonal polynomials of two variables. This study aims to establish a common method for constructing solutions near regular and irregular features to classify these features and provide examples for the plausible application of uncertain coefficients method for constructing common solution of inhomogeneous systems the one inhomogeneous differential equation in partial derivatives of the second and third orders.

2. Construction of hypergeometric type inhomogeneous second-order system solutions

Statement of the problem. A regular system consisting of two differential equations in second-order partial derivatives of the form $(0,0)$ is investigated

$$\begin{aligned} x^2 g_0 p_{2,0} + xy g_1 p_{1,1} + x g_3 p_{1,0} + y g_4 p_{0,1} + g_5 p_{0,0} &= g(x,y), \\ y^2 q_0 p_{0,2} + xy q_1 p_{1,1} + x q_3 p_{1,0} + y q_4 p_{0,1} + q_5 p_{0,0} &= q(x,y), \end{aligned} \tag{1}$$

where the coefficients of the system are polynomials in form

$$g_i(x,y) = r_{i,0} - \alpha_{i,0}x^h, q_i(x,y) = t_{0,i} - \beta_{0,i}y^h \tag{2}$$

$g(x,y)$ and $q(x,y)$ analytic functions of two variables; $p_{0,0} = Z(x,y)$ total unknown, $r_{j,k}, \alpha_{j,k}, t_{j,k}$, $\beta_{j,k}(j,k = \overline{0,2})$ unknown constants.

It is required to study the possibilities of constructing inhomogeneous system solutions (1) in form of orthogonal polynomials of two variables, when the corresponding homogeneous system is hypergeometric or reducible to them.

3. On the solution of homogeneous systems with solutions in the form of orthogonal polynomials of two variables

Theorem 1. The general solution of the inhomogeneous system (1) is represented as the sum of the general solution

$$\overline{Z} = \sum_{j=1}^{\infty} C_j Z_j(x,y), (j = 1,2,3,4)$$

corresponding homogeneous system

$$\begin{gathered} x^2(r_{2,0} - \alpha_{2,0}x^h)p_{2,0} + xy(r_{1,1} - \alpha_{1,1}x^h)p_{1,1} + x(r_{1,0} - \alpha_{1,0}x^h)p_{1,0} + y(r_{0,1} - \alpha_{0,1}x^h)p_{0,1} + \\ +(r_{0,0} - \alpha_{0,0}x^h)p_{0,0} = 0 \\ y^2(t_{0,2} - \beta_{0,2}y^h)p_{0,2} + xy(t_{1,1} - \beta_{1,1}y^h)p_{1,1} + x(t_{1,0} - \beta_{1,0}y^h)p_{1,0} + y(t_{0,1} - \beta_{0,1}y^h)p_{0,1} + \\ +(t_{0,0} - \beta_{0,0}y^h)p_{0,0} = 0 \end{gathered} \tag{3}$$

and partial solutions $Z_0(x,y)$ heterogeneous system (1), (2):

$$Z(x,y) = \overline{X}(x,y) + Z_0(x,y) = \sum_{j=1}^{4} C_j Z_j(x,y) + Z_0(x,y). \tag{4}$$

According to the general theory of homogeneous systems in form (3), when the compatibility condition and the integrability condition are satisfied it has up to four linearly independent partial solutions. Now, we'll show using a specific example interesting us systems with solutions in form of orthogonal polynomials of two variables are derived from the system (3) when $h \geq 2$. Moreover, the transformation plays a large role

$$x^h = u, y^h = \nu$$

which bring homogeneous system (3) to a second-order hypergeometric type system with respect to new variables u and ν type (1.6).

The system

$$\begin{gathered} u(1-u)p_{2,0} - u\nu p_{1,1} + [\gamma - (\alpha + \beta + 1)u]p_{1,0} - \beta\nu p_{0,1} - \alpha\beta p_{0,0} = 0, \\ \nu(1-\nu)p_{0,2} - u\nu p_{1,1} + [\gamma^{'} - (\alpha + \beta^{'} + 1)\nu]p_{0,1} - \beta^{'}up_{1,0} - \alpha\beta^{'}p_{0,0} = 0, \end{gathered} \tag{5}$$

when its solution is known as the Appell function

$$F_2(\alpha, \beta, \beta^{'}; \gamma, \gamma^{'}; u, \nu) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta^{'})_n}{(\gamma)_m(\gamma^{'})_n m!n!} u^m \nu^n$$

can be deduced from the hypergeometric system using the Kampe de Ferriet method [1, pp. 155–169].

Example As an example, the system

$$(1-x^2)p_{2,0} - xyp_{1,1} + (n-2)xp_{1,0} - (m+1)yp_{0,1} + (m+n)(m+1)p_{0,0} = 0,$$

$$(1-y^2)p_{0,2} - xyp_{1,1} - (n+1)xp_{1,0} + (m-2)yp_{0,1} + (m+n)(n+1)p_{0,0} = 0, \qquad (6)$$

where $p_{0,0} = U(x,y)$ general unknown, considered by M.Ch. Hermite in the study of orthogonal polynomials of two variables. As you can see, in the system (6) $h = 2$. Conversion $x^2 = u, y^2 = \nu$ leads (6), to the Horn hypergeometric system (5):

$$u(1-u)p_{2,0} - u\nu p_{1,1} + [\frac{1}{2} - (\frac{1}{2} - \frac{n}{2} + 1)u]p_{1,0} - \frac{m+1}{2}\nu p_{0,1} - \frac{m+n}{2}\frac{m+1}{2}p_{0,0} = 0,$$

$$\nu(1-\nu)p_{0,2} - u\nu p_{1,1} - \frac{n+1}{2}up_{1,0} + [\frac{1}{2} - (\frac{1}{2} - \frac{m}{2} + 1)\nu]p_{0,1} - \frac{m+n}{2}\frac{n+1}{2}p_{0,0} = 0, \qquad (7)$$

The compatibility of the Horn hypergeometric type system (5) is ensured by applying the Kampe de Ferriet method and the integrability condition is also satisfied. Therefore, it has up to four linearly independent particular solutions. Indeed, this can be verified using the Frobenius – Latysheva method [3].

And the system of defining equations of system (7) with respect to the singularity $0, 0$ has pairs of roots:$(\rho_1 = 0, \sigma_1 = 0), (\rho_1 = 0, \sigma_2 = \frac{1}{2}), (\rho_2 = \frac{1}{2}, \sigma_1 = 0), (\rho_2 = \frac{1}{2}, \sigma_2 = \frac{1}{2})$.

These pairs are indicators of four linearly independent particular solutions in the form of generalized power series of two variables [3]. Considering conversation $x^2 = u, y^2 = \nu$ formulate the following statement.

Theorem 2. The general solution of the original system (6) is representable in the form

$$\begin{aligned} Z(x,y) = C_{1,1}F_2(-\frac{m+n}{2}, \frac{m+1}{2}, \frac{n+1}{2}; \frac{1}{2}, \frac{1}{2}; x^2, y^2) + \\ C_{1,2}yF_2(-\frac{m+n-1}{2}, \frac{m+1}{2}, \frac{n+2}{2}; \frac{1}{2}, \frac{1}{2}; x^2, y^2) + \\ + C_{1,3}xF_2(-\frac{m+n-1}{2}, \frac{m+2}{2}, \frac{n+1}{2}; \frac{3}{2}, \frac{1}{2}; x^2, y^2) + \\ C_{1,4}xyF_2(-\frac{m+n-2}{2}, \frac{m+2}{2}, \frac{n+2}{2}; \frac{3}{2}, \frac{3}{2}; x^2, y^2). \end{aligned} \qquad (8)$$

However, the Appell function $F_2(\alpha, \beta, \beta'; \gamma, \gamma'; x, y)$ at $\alpha = -p$ turns into a polynomial degree p. Therefore, in this case, the first particular solution will be represented as a polynomial

$$F_2(-\frac{m+n}{2}, \frac{m+1}{2}, \frac{n+1}{2}; \frac{1}{2}, \frac{1}{2}; x^2, y^2) = \sum_{m,n=0}^{\infty} \frac{(-\frac{m+n}{2})_{m+n}(\frac{m+1}{2})_m(\frac{n+1}{2})_n}{(\frac{1}{2})_m(\frac{1}{2})_n m!n!} x^{2m}y^{2n}$$

.

This technique is also applicable for the construction of biorthogonal polynomials of Appell: $J_{mn}(\alpha, \gamma, \gamma'; x, y)$, $F_{mn}(\gamma, \gamma'; x, y), E_{mn}(\gamma, \gamma'; x, y)$. Applying the conversion first $x^2 = u, y^2 = \nu$ we obtain a system of hypergeometric type (5), Further, using the Frobenius – Latysheva method, we construct solutions as considered in example of M.Ch. Hermite. The advantage of this technique has been demonstrated in the work [3].

Another property of the polynomial $U_{mn}(x,y)$ distinguish in form of the following theorem:

Theorem 3. Polynomial $U_{mn}(x,y)$ is a partial system solution (6) and one homogeneous second-order partial differential equation

$$(1-x^2)U_{xx} - 2xyU_{xy} + (1-y^2)U_{yy} - 3(xU_x + yU_y) + (m+n)(m+n+2)U = 0, \qquad (9)$$

obtained by adding the two equations of the system (6).

The simplest properties of two variable polynomials orthogonal in a domain with arbitrary weight were considered [4]. They established that in some cases the orthogonal two variable polynomials are eigenfunctions of linear partial differential equations of the second order (9). Such equations are called admissible.

4. Construction of hypergeometric type inhomogeneous second-order system solutions

Many properties of inhomogeneous systems (1) with coefficients in form (2) remain poorly studied. Particularly heterogeneous systems associated with orthogonal polynomials of many variables. The relationship of the corresponding hypergeometric type homogeneous system with generalized two variables hypergeometric functions was studied. Let us return to the example of M.Ch. Hermite and move on to construct solutions to his heterogeneous system.

Theorem 4. An inhomogeneous system

$$(1-x^2)p_{2,0} - xyp_{1,1} + (n-2)xp_{1,0} - (m+1)yp_{0,1} + (m+n)(m+1)p_{0,0} = g(x,y),$$
$$(1-y^2)p_{0,2} - xyp_{1,1} - (n+1)xp_{1,0} + (m-2)yp_{0,1} + (m+n)(n+1)p_{0,0} = q(x,y), \qquad (10)$$

where the right side of the system

$$g(x,y) = (m+n)(m+1) + \frac{(n-2)+(m+n)(m+1)}{n+1}x + (m+n-1)y + \frac{n-m-4+(m+n)(m+1)}{m+1}xy,$$
$$q(x,y) = (m+n)(n+1) + (m+n-1)x + \frac{(m-2)+(m+n)(n+1)}{m+1}y + + \frac{m-n-4+(m+n)(n+1)}{(n+1)(m+1)}xy,$$

has a general solution (4), as the sum of the general decision corresponding to homogeneous system (6) and the particular solution $Z_0(x,y)$ of the inhomogeneous system (10).

Evidence. General solution for a homogeneous system (6) already built as (8). To this end, we are using transformation $x^2 = u, y^2 = \nu$, homogeneous system (6) led to the form (7), which solutions are expressed in terms of the hypergeometric functions of Appell F_2. Therefore, instead of the inhomogeneous system (10), it would be possible to construct a particular solution of the inhomogeneous system

$$u(1-u)p_{2,0} - u\nu p_{1,1} + [\frac{1}{2} - (\frac{1}{2} - \frac{n}{2} + 1)u]p_{1,0} - \frac{m+1}{2}\nu p_{0,1} - \frac{m+n}{2}\frac{m+1}{2}p_{0,0} = g(u,\nu),$$
$$\nu(1-\nu)p_{0,2} - u\nu p_{1,1} - \frac{n+1}{2}up_{1,0} + [\frac{1}{2} - (\frac{1}{2} - \frac{m}{2} + 1)\nu]p_{0,1} - \frac{m+n}{2}\frac{n+1}{2}p_{0,0} = q(u,\nu), \quad (11)$$

where $g(u,\nu)$ and $q(u,\nu)$ right side of system (10). However, we restrict ourselves directly to the construction of a particular solution (10).

We use the method of indefinite coefficient in order to construct a particular solution of system (10), based on the representation of the right side $g(x,y)$ and $q(x,y)$ we are looking for a particular solution in the form of a polynomial

$$Z_0(x,y) = A_{0,0} + A_{1,0}x + A_{0,1}y + A_{1,1}xy. \qquad (12)$$

Substituting (12) into a heterogeneous system (10), define unknown coefficients $A_{j,k}(j,k = \overline{0,1})$. A partial solution is a polynomial

$$Z_0(x,y) = 1 + \frac{1}{n+1}x + \frac{1}{m+1}y + \frac{xy}{(n+1)(m+1)}. \qquad (13)$$

Therefore, taking into account the general solution (8), the corresponding homogeneous system (6), we represent the general solution of the inhomogeneous system (10) in form

$$Z(x,y) = \overline{Z}(x,y) + Z_0(x,y) = C_{1,1}F_2(-\frac{m+n}{2}, \frac{m+1}{2}, \frac{n+1}{2}; \frac{1}{2}, \frac{1}{2}; x^2, y^2) +$$
$$+C_{1,2}yF_2(-\frac{m+n-1}{2}, \frac{m+1}{2}, \frac{n+2}{2}; \frac{1}{2}, \frac{1}{2}; x^2, y^2) +$$
$$+C_{1,3}xF_2(-\frac{m+n-1}{2}, \frac{m+2}{2}, \frac{n+1}{2}; \frac{3}{2}, \frac{1}{2}; x^2, y^2) + C_{1,4}xyF_2(-\frac{m+n-2}{2}, \frac{m+2}{2}, \frac{n+2}{2}; \frac{3}{2}, \frac{3}{2}; x^2, y^2) +$$

$$+1+\frac{1}{n+1}x+\frac{1}{m+1}y+\frac{1}{(m+1)(n+1)}xy. \tag{14}$$

Solution (14) is a general solution of system (11). The theorem is proved.

Along with constructing a solution to an inhomogeneous system, their relationship with one second-order partial differential equation remains unstudied; therefore, it is some interest to construct a particular solution to the second-order partial differential equation obtained by adding the two equations of the inhomogeneous system.

Theorem 5. Second-order inhomogeneous partial differential equation

$$(1-x^2)p_{2,0}-2xyp_{1,1}+(1-y^2)p_{0,2}-3(xp_{1,0}+yp_{0,1})+(m+n)(m+n+2)p_{0,0}=g(x,y)+q(x,y) \tag{15}$$

obtained by adding the two equations of the inhomogeneous system (10) with the right-hand side $g(x,y)$ and $q(x,y)$ has a partial solution (13).

Indeed, by indefinite coefficients method it is easy to verify that the polynomial (13) is a particular solution of the inhomogeneous equation (15).

Taking into account, the inhomogeneous second-order partial differential equation (15) can be represented as

$$(1-x^2)U_{xx}-2xyU_{xx}+(1-y^2)U_{yy}-3(xU_x+yU_y)+(m+n)(m+n+2)U=g(x,y)+q(x,y).$$

Therefore, (13) is a particular solution of this inhomogeneous equation.

These studies should also be extended to more general systems consisting of two third-order equations based on known properties of second-order systems, then establish distinctive features of third-order hypergeometric systems.

5. Conclusion

The previously unstudied inhomogeneous special system, consisting of two third-order partial differential equations with a homogeneous part in the form of a generalized hypergeometric type system, was introduced for the consideration. The study consists of two parts.

In the first part, we studied the possibilities of constructing solutions to the special case of an inhomogeneous special system, consisting of two second-order partial differential equations system with coefficients of the form). The previously published solutions for the hypergeometric functions of two variables, in particular the orthogonal polynomials of two variables were explored, with focus placed on the distinctive features of specific hypergeometric type homogeneous system of the third order. A number of specific interesting particular cases of this system were distinguished for various values: $h=0, h=1$ and $h\geq 2$. Special attention is paid to the compatibility conditions and integrability, as well as the substantiation of the Frobenius – Latysheva method.

References

[1] Appell P. and Kampe de Feriet J., Fonctons hypergeometriges of hyperspheriges Polynomes d'Hermite, *Paris, Gauthier -Villars*, 1926, 434 p.

[2] Slater L.J., Lit D., Ph.D. Generalized Hypergeometric functions, *Cambridge At the university Press*, 1966, 292 p.

[3] Tasmambetov Zh.N., Construction of normal and normallyregular solutions of special systems of partial equations of second order, *Aktobe: IP Zhanadilov S.T.*, 2015, 463 p.

[4] Babister A.W., Trancendental function satisfying nonhomogeneous linear differential equations, *New York-London*, 1967, 414 p.

[5] Ubayeva Zh.K., Tasmambetov Zh.N., Rajabov N., Features of Constructing a Solution Heterogeneous Equation and Clausen-Type Systems, *Mathematical Modelling of Engineering Problems*, No.4, 2022, pp.906-918.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

MULTIDIMENSIONAL NORMAL-REGULAR SOLUTIONS OF DEGENERATE SYSTEMS ASSOCIATED WITH BESSEL FUNCTIONS OF MANY VARIABLES

ZHAXYLYK TASMAMBETOV

West Kazakhstan University named after M. Utemisov , Uralsk, Kazakhstan

e-mail: tasmam@rambler.ru

Abstract. Degenerate hypergeometric systems related to the Horn system are established, the solutions of which are multidimensional Bessel functions. The possibilities of constructing normal-regular solutions for related systems such as Horn, Whittaker, and Bessel types are investigated. The properties of solutions for the Bessel-type system obtained through a double transformation from the Whittaker system are revealed. The possibilities of constructing solutions for Horn-type systems obtained from the Lauricella system through limit transitions in the form of multidimensional Bessel functions are studied, as well as the normal-regular solutions of multivariable systems and their connection.

Keywords: degenerate, hypergeometric systems, related, constructions, multidimensional Bessel functions, normal-regular, properties.

The study of hypergeometric functions of many variables has been greatly developed in the works of P. Appel, M. Lauricella, A. Erdei, M. Humbert, Kampe de Ferriet, J. Horn, etc. In addition to 14 complete hypergeometric functions $F_k(k=1,2,3,4)$, $G_l(l=1,2,3)$, $H_t(t=\overline{1,7})$, by the efforts of J. Horn, M. Humbert and L. Borngesser, 20 degenerate hypergeometric functions of two variables were established [1-3]. J. Horn established systems of partial differential equations of the second order, which satisfy the known 34 hypergeometric functions of two variables [4]. M. Lauricella generalized four P. Appel's functions F_1, F_2, F_3, F_4 into four functions F_A, F_B, F_C, F_D respectively, and established four systems whose solutions are these functions [5-6]. On the basis of these classical results, degenerate hypergeometric systems were developed and studied. Thus, in [7], 205 hypergeometric functions of three variables are given and investigated.

Studying degenerate hypergeometric systems derived from and Lauricella systems, F_B and F_D V.I. Khudozhnikov established two new functions $\Phi_{B,n}^{k,l}\left(\begin{matrix}(\alpha_k)\,,\left(\alpha_l'\right)\,,(\beta_k)\\ \gamma\end{matrix}|(z_n)\right)$ and Φ_D [8]. Representatives of the Uzbek mathematical school Salakhitdinova M.S.: T.K. Yuldashev, A.H. Hasanov, M.T. Irgashev are engaged in applications of properties of hypergeometric functions of many variables in studies of degenerate equations [9-12].

However, research on many issues of hypergeometric functions of many variables is not at the proper level. Especially, the connection between related systems such as Horn, Whittaker, Bessel. So far, no analogue of the Bessel equation has been established in the form of degenerate hypergeometric systems whose solutions are multidimensional Bessel functions. It is also required to establish numerous applications of these functions as in the one-dimensional case [13-15].

In some of our works [16-17], we have used a Horn-type system for this purpose, where Humbert's main theorem is valid [2].

Theorem 1.1. Horn type system

$$x_j\frac{\partial^2 W}{\partial x_j^2}+(\gamma_j-x_j)\frac{\partial W}{\partial x_j}-\sum_{(k\neq j)}\frac{\partial W}{\partial x_k}-\lambda W=0, j=\overline{1,n} \tag{1.1}$$

has linearly independent partial solutions 2^n

$$W_1(x_1,...,x_n)=\Psi_2^{(n)}(\lambda;\gamma_1,...,\gamma_n;x_1,...,x_n),$$

$$W_2(x_1,...,x_n)=x^{1-\gamma_1}\Psi_2^{(n)}(\lambda+1-\gamma_1,2-\gamma_1,\gamma_2,...,\gamma_n;x_1,...,x_n),$$

$$-\ -\ -\ -

1. Two-dimensional Bessel functions as solutions of degenerate hypergeometric systems of the Horn type

Problem statement. The properties of multidimensional Bessel functions obtained by limiting the transition from regular solutions of systems of the Horn type (1.1), as well as their relationship with normally regular solutions of the form (1.5), are investigated. A number of special cases of the system (1.1) are considered. So, if $n = 2$ the statement [6] is true.

Theorem 2.1. A degenerate system consisting of two partial differential equations of the second order

$$\begin{aligned} x_1 Z_{x_1x_1} + (\gamma_1 - x_1)Z_{x_1} - x_2 Z_{x_2} - \lambda Z = 0, \\ x_2 Z_{x_2x_2} + (\gamma_2 - x_2)Z_{x_2} - x_1 Z_{x_1} - \lambda Z = 0, \end{aligned} \tag{2.1}$$

where $(\gamma_j)(j = 1, 2)-$some permanent, $Z(x, y)-$ common unknown, has 2^2 linearly independent partial solutions

$$\begin{aligned} Z_1(x_1, x_2) = \Psi_2(\lambda; \gamma_1, \gamma_2; x_1, x_2), \\ Z_2(x_1, x_2) = x_1^{1-\gamma_1}\Psi_2(\lambda + 1 - \gamma_1, \gamma_2; x_1, x_2), \\ Z_3(x_1, x_2) = x_2^{1-\gamma_2}\Psi_2(\lambda + 1 - \gamma_2, \gamma_1; x_1, x_2), \\ Z_4(x_1, x_2) = x_1^{1-\gamma_1}x_2^{1-\gamma_2}\Psi_2(\lambda + 2 - \gamma_1 - \gamma_2; 2 - \gamma_1, 2 - \gamma_2; x_1, x_2). \end{aligned} \tag{2.2}$$

The particular solution (2.2) is a degenerate hypergeometric Humbert function and is defined using a series

$$\Psi_2(\lambda; \gamma_1, \gamma_2; x_1, x_2) = \sum_{m_1,m_2=0}^{\infty} \frac{(\lambda)_{m_1+m_2}}{(\gamma_1)_{m_1}(\gamma_2)_{m_2}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!}. \tag{2.2_1}$$

obtained by limiting the transition from the Appel function F_2 [3, p.124]:

$$\begin{aligned} \lim_{\epsilon\to 0} F_2(\lambda, \frac{1}{\epsilon}, \frac{1}{\epsilon}; \gamma_1, \gamma_2; \epsilon x_1, \epsilon x_2) = \sum_{m_1,m_2=0}^{\infty} \frac{(\lambda)_{m_1+m_2}}{(\gamma_1)_{m_1}(\gamma_2)_{m_2}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} = \\ = \Psi_2(\lambda; \gamma_1, \gamma_2; x_1, x_2). \end{aligned} \tag{2.2_2}$$

The series (2.2_1) converges absolutely and uniformly at $|x_1| < \epsilon, |x_2| < \epsilon$.

It should be noted that the generalization F_2- of the Appel function is the Lauricell function F_A. Therefore, a limiting transition of the form (2.2_2) is also valid between the Lauricell F_A and Humbert functions Ψ_2. In addition, using the transformation (1.3), it is possible to construct a normally regular solution of the Horn system (2.1).

2. The Bessel function of two variables as solutions of a degenerate hypergeometric Horn system

In this paper, we have established two degenerate systems whose solutions are degenerate hypergeometric functions resulting in Bessel functions. The first system is directly connected to the Horn system. Later we will see that all multidimensional Bessel functions are defined using the Humbert function of two or more variables. We prove a number of theorems.

Theorem 2.2. A degenerate hypergeometric system of the Horn type

$$\begin{aligned} x_1 Z_{x_1x_1} + (2\alpha - x_1)Z_{x_1} - x_2 Z_{x_2} - \lambda Z = 0, \\ x_2 Z_{x_2x_2} + (2\beta - x_2)Z_{x_2} - x_1 Z_{x_1} - \lambda Z = 0, \end{aligned} \tag{2.3}$$

obtained from the Horn system (2.1) by substitution $\gamma_1 = 2\alpha$, $\gamma_2 = 2\beta$ has four linearly independent partial solutions

$$\begin{aligned} Z_1(x_1, x_2) = \Psi_2(\lambda; 2\alpha, 2\beta; x_1, x_2), \\ Z_2(x_1, x_2) = x_1^{-\alpha}\Psi_2(\lambda - 2\alpha + 1, 2 - 2\alpha, 2\beta; x_1, x_2), \\ Z_3(x_1, x_2) = x_2^{-\beta}\Psi_2(\lambda - 2\beta + 1, 2\alpha, 2\beta - 2; x_1, x_2), \end{aligned}$$

$$Z_4(x_1,x_2)=x_1^{-\alpha}x_2^{-\beta}\Psi_2(\lambda-2\alpha-2\beta+2;2\alpha-2,2\beta-2;x_1,x_2), \tag{2.4}$$

where the limiting transition in the function (2.4) defines a degenerate function resulting in a Bessel function of two variables

$$\lim_{\lambda\to\infty}\Psi_2(\lambda,2\alpha,2\beta;\frac{x_1}{\lambda},\frac{x_2}{\lambda})=J(2\alpha,2\beta;x_1,x_2)=$$

$$=(\frac{x_1}{2})^{2\alpha}(\frac{x_2}{2})^{2\beta}\sum_{m_1,m_2=0}^{\infty}\frac{(-1)_{m_1+m_2}(\frac{x_1}{2})^{2m_1}(\frac{x_2}{2})^{2m_2}}{m_1!m_2!\Gamma(2\alpha+m_1+1)\Gamma(2\beta+m_2+1)}=$$

$$=\Psi_2(\lambda;\gamma_1,\gamma_2;x_1,x_2). \tag{2.5}$$

The remaining linearly independent partial solutions are $Z_l(x_1,x_2)(l=2,3,4)$ also expressed in terms of the Humbert function (2.4). Indeed, from the Humbert function (2.21), taking into account the substitution $\gamma_1=2\alpha$, $\gamma_{12}=2\beta$ we obtain that the particular solution (2.4) is represented as

$$\Psi_2(\lambda;2\alpha,2\beta;x_1,x_2)=\sum_{m_1,m_2=0}^{\infty}\frac{(\lambda)_{m_1+m_2}}{(2\alpha)_{m_1}(2\beta)_{m_2}}\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}. \tag{2.6}$$

Next, we will use the properties of the function reducing to the Bessel function

$${}_0F_1(\gamma,x)=J(\gamma,x)=\sum_{m=0}^{\infty}\frac{1}{(\gamma)_m}\frac{x^m}{m!}, \tag{2.7}$$

$$J_m(x)=\frac{(\frac{x}{2})^{\mu}}{\Gamma(\mu+1)}J(\mu+1,-\frac{x^2}{2^2}), \tag{2.8}$$

$$\lim_{\epsilon\to0}F_2(\frac{1}{\epsilon},\frac{1}{\epsilon},\frac{1}{\epsilon};\gamma_1,\gamma_2;\epsilon^2x,\epsilon^2y)=\sum_{m_1,m_2=0}^{\infty}\frac{1}{(\gamma_1)_{m_1}(\gamma_2)_{m_2}}\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}=J(\gamma_1,x_1)J(\gamma_2,x_2). \tag{2.9}$$

The limiting transition (2.9) allows us, using (2.7) and (2.8), to verify [20] the validity of the representation of the Bessel function of two variables in the form

$$J_{\mu,\nu}(x_1,x_2)=J(\gamma_1,x_1)J(\gamma_2,x_2)=J_\mu(x_1)Z_\nu(x_2)=\frac{(\frac{x_1}{2})^{\mu}}{\Gamma(\mu+1)}J(\mu+1;-\frac{x_1^2}{2^2})$$

$$\frac{(\frac{x_2}{2})^{\nu}}{\Gamma(\nu+1)}J(\nu+1;-\frac{x_2^2}{2^2})=$$

$$=(\frac{x_1}{2})^{\mu}(\frac{x_2}{2})^{\nu}\sum_{m_1,m_2=0}^{\infty}\frac{(-1)_{m_1+m_2}(\frac{x_1}{2})^{2m_1}(\frac{x_2}{2})^{2m_2}}{m_1!m_2!\Gamma(2\alpha+m_1+1)\Gamma(2\beta+m_2+1)} \tag{2.5_1}$$

From here we get (2.5):$\gamma_1=2\alpha$, $\gamma_{12}=2\beta$

$$J_{\mu,\nu}(x_1,x_2)=(\frac{x_1}{2})^{2\alpha}(\frac{x_2}{2})^{2\beta}\sum_{m_1,m_2=0}^{\infty}\frac{(-1)_{m_1+m_2}(\frac{x_1}{2})^{2m_1}(\frac{x_2}{2})^{2m_2}}{m_1!m_2!\Gamma(2\alpha+m_1+1)\Gamma(2\beta+m_2+1)}$$

In addition to the four linearly independent partial solutions found, it has two more normally regular solutions. This makes it possible to set some additional properties of solutions (2.6).

2.1. **Normal-regular solutions of the Horn system.** To construct normally regular solutions of a Bessel-type system, we will again use the connection between the Horn and Bessel systems.

The following statement is true.

Theorem 2.3. Auxiliary system

$$x_1U_{x_1x_1}+[2\alpha_{10}x_1+\gamma_1-x_1]U_{x_1}-x_2U_{x_2}+[\alpha_{10}^2x_1+(\gamma_1-x_1)\alpha_{10}-\alpha_{10}x_2-\lambda]U=0,$$

$$x_2U_{x_2x_2}+[2\alpha_{01}x_2+\gamma_2-x_2]U_{x_2}-x_1U_{x_1}+[\alpha_{01}^2x_2+(\gamma_2-x_2)\alpha_{01}-\alpha_{10}x_1-\lambda]U=0, \quad (2.10)$$

obtained from the Horn system (2.1) using the transformation

$$Z(x_1,x_2)=exp(\alpha_{10}x_1+\alpha_{01}x_2)U(x_1,x_2) \quad (2.11)$$

when two necessary conditions are met

$$\alpha_{10}^2-\alpha_{10}=0,\alpha_{01}^2-\alpha_{01}=0, \quad (2.12)$$

and

$$f_{00}^{(1)}(\rho_1,\rho_2)\doteq\rho_1(\rho_1-1+\gamma_1)=0,$$
$$f_{00}^{(2)}(\rho_1,\rho_2)\doteq\rho_2(\rho_2-1+\gamma_2)=0, \quad (2.13)$$

has two normally-regular solutions

$$Z_{5,j}(x_1,x_2)=exp(x_j)U_{5,j}(x_1,x_2) \quad (2.14)$$

where $U_{5,j}(x_1,x_2)(j=1,2)$ two degenerate hypergeometric series of two variables

$$U_{5,1}(x_1,x_2)=1-\frac{\gamma_1-\lambda}{\gamma_1}x_1+\frac{\lambda}{\gamma_2}x_2+\frac{\lambda(\lambda+1-\gamma_1)}{\gamma_1\gamma_2}x_1x_2+$$
$$+\frac{(\gamma_1-\lambda)(\gamma_1+1-\lambda)}{\gamma_1(\gamma_1+1)}\frac{x_1^2}{2!}\frac{\lambda(\lambda+1)}{\gamma_2(\gamma_2+1)}\frac{x_2^2}{2!}+... \quad (2.15)$$

$$U_{5,2}(x_1,x_2)=1+\frac{\lambda}{\gamma_1}x_1+\frac{(\gamma_1-\lambda)}{\gamma_2}x_2+\frac{\lambda(\lambda+1-\gamma_2)}{\gamma_1\gamma_2}x_1x_2+$$
$$+\frac{(\lambda(\lambda+1)}{\gamma_1(\gamma_1+1)}\frac{x_1^2}{2!}\frac{(\gamma_2-\lambda)(\gamma_2+1-\lambda)}{\gamma_2(\gamma_2+1)}\frac{x_2^2}{2!}+... \quad (2.16)$$

Proof. To prove this, we will use the Frobenius-Latysheva method [16]. The fulfillment of the first necessary condition (2.12) defines four pairs of roots: $I.(\alpha_{10}=0,\alpha_{01}=0)$, $II.(\alpha_{10}=1,\alpha_{01}=0)$, $III.(\alpha_{10}=0,\alpha_{01}=1)$, $IV.(\alpha_{10}=1,\alpha_{01}=1)$.

At $I.(\alpha_{10}=0,\alpha_{01}=0)$ we get the original system (2.1) with a solution in the form of (2.2). Further at $II.(\alpha_{10}=1,\alpha_{01}=0)$ and $III.(\alpha_{10}=0,\alpha_{01}=1)$ two connected systems are defined

$$x_1U_{x_1x_1}+(\gamma_1+x_1)Ux_1-x_2U_{x_2}+(\gamma_1-\lambda)U=0,$$
$$x_2U_{x_2x_2}+(\gamma_2-x_2)Ux_2-x_1U_{x_1}+(x_1+\lambda)U=0, \quad (2.17)$$

and

$$x_1U_{x_1x_1}+(\gamma_1-x_1)Ux_1-x_2U_{x_2}+(x_2+\lambda)U=0,$$
$$x_2U_{x_2x_2}+(\gamma_2+x_2)Ux_2-x_1U_{x_1}+(\gamma_2-\lambda)U=0. \quad (2.18)$$

To construct their solution, the Frobenius-Latysheva method is used. Their solution is sought in the form of a generalized power series of two variables

$$U(x_1,x_2)=x_1^{\rho_1}x_2^{\rho_2}\sum_{m_1,m_2=0}^{\infty}A_{m_1,m_2}x_1^{m_1}x_2^{m_2},A_{0,0}\neq0 \quad (2.19)$$

where $\rho_j(j=1,2)$, $A_{m_1,m_2}(m_1,m_2=0,1,2,...)-$unknown constants.

Unknown constants $\rho_j(j=1,2)$ the series (2.19) are determined from the system of defining equations (2.13). The unknown coefficients $A_{m_1,m_2}(m_1,m_2=0,1,2,...)-$ of the series are determined from the system of recurrent sequences

$$\sum_{m_1,m_2=0}^{\infty} A_{m_1-\mu,m_2-\nu} f_{\mu,\nu}^{(j)}(\rho_1+m_1-\mu,\rho_2+m_2-\nu)=0, \tag{2.20}$$

$(m,n=0,1,...); j=1,2; \mu,\nu=0,1,2,...$

As a result, from the attached systems (2.17) and (2.18), we define two degenerate hypergeometric series of two variables (2.15) and (2.16), respectively. Therefore, substituting them in (2.14) we get two normally regular solutions of the system (2.1).

We will reveal some properties of the constructed solutions of the Horn system:

Theorem 2.4. There are relations

$$\Psi_{2,j}(\lambda,\gamma_1,\gamma_2;x_1,x_2)=exp(x_j)U_{5,j}(x_1,x_2), \tag{2.21}$$

$$exp(-x_j)\Psi_{2,j}(\lambda,\gamma_1,\gamma_2;x_1,x_2)=U_{5,j}(x_1,x_2), \tag{2.22}$$

where $\Psi_{2,1}=\Psi_{2,2}=\Psi_2$ and $U_{5,j}(x_1,x_2)(j=1,2)$ rows (2.15) and (2.16). Let's prove one of these relations. Let $j=1$ and it is required to prove the relation

$$\Psi_{2,1}(\lambda,\gamma_1,\gamma_2;x_1,x_2)=exp(x_1)U_{5,1}(x_1,x_2). \tag{2.23}$$

Let's open the right part (2.23) using decompositions of the given functions

$$exp(x_1)U_{5,1}(x_1,x_2)=(1+\frac{x}{1!}+\frac{x^2}{2!}+...+\frac{x^n}{n!}+...)$$

$$1-\frac{\gamma_1-\alpha}{\gamma_1}x_1+\frac{\lambda}{\gamma_2}x_2+\frac{\lambda(\lambda+1-\gamma_1)}{\gamma_1\gamma_2}x_1x_2+\frac{(\gamma_1-\lambda)(\gamma_1+1-\lambda)}{\gamma_1(\gamma_1+1)}\frac{x_1^2}{2!}+\frac{\lambda(\lambda+1)}{\gamma_2(\gamma_2+1)}\frac{x_2^2}{2!}+...=$$

$$=1+[\frac{1}{1!}]x_1+\frac{\lambda}{\gamma_2}x_2+[\frac{\lambda(\lambda+1-\gamma_1)}{\gamma_1\gamma_2}+\frac{\lambda}{\gamma_2}]x_1x_2+[\frac{(\gamma_1-\lambda)(\gamma_1+1-\lambda)}{\gamma_1(\gamma_1+1)}+\frac{1}{2!}-\frac{\gamma_1-\lambda}{\gamma_1}\frac{1}{1!}]x_2+$$

$$+\frac{\lambda(\lambda+1)}{\gamma_2(\gamma_2+1)}\frac{x_2^2}{2!}+...=1+\frac{\lambda}{\gamma_1}x_1+\frac{\lambda}{\gamma_2}x_2+\frac{\lambda(\lambda+1)}{\gamma_1\gamma_2}x_1x_2+\frac{\lambda(\lambda+1-\gamma_1)}{\gamma_1(\gamma_1+1)}\frac{x_1^2}{2!}+\frac{\lambda(\lambda+1)}{\gamma_2(\gamma_2+1)}\frac{x_2^2}{2!}+...$$

$$=\Psi_{2,1}(\lambda,\gamma_1,\gamma_2;x_1,x_2),$$

that is, they received the left part, which was required to be proved. Similarly, other relations are proved.

It should be noted that at IV. $(\alpha_{10}=1,\alpha_{01}=1)$ we get an incompatible attached system. Therefore, it does not have a normally-regular solution.

2.2. **Relationship between normal-regular solutions of Horn systems and Bessel type.** In the previous paragraph, we proved two theorems 2.3 and 2.4 on the construction of normally regular solutions of a Horn-type system.

Similar theorems hold in the case of a Bessel type system.

Theorem 2.5. Auxiliary system

$$x_1U_{x_1x_1}+[2\alpha_{10}x_1+2\alpha-x_1]U_{x_1}-x_2U_{x_2}+[\alpha_{10}^2x_1+(2\alpha-x_1)\alpha_{10}-\alpha_{10}x_2-\lambda]U=0,$$

$$x_2U_{x_2x_2}+[2\alpha_{01}x_2+2\beta-x_2]U_{x_2}-x_1U_{x_1}+[\alpha_{01}^2x_2+(2\beta-x_2)\alpha_{01}-\alpha_{10}x_1-\lambda]U=0, \tag{2.24}$$

derived from the Bessel system (2.3) with the help of transformation (2.1) when two necessary conditions are met (2.12) and (2.13) have two normally regular solutions

$$Z_{6,j}(x_1,x_2)=exp(x_j)U_{6,j}(x_1,x_2) \tag{2.25}$$

where $U_{6,j}(x_1,x_2)(j=1,2)$ two degenerate hypergeometric functions of two variables resulting in Bessel functions

$$U_{6,1}(x_1,x_2)=1-\frac{2\alpha_1-\lambda}{2\alpha}x_1+\frac{\lambda}{2\beta}x_2+\frac{\lambda(\lambda+1-2\alpha)}{2\alpha2\beta}x_1x_2+$$

$$+\frac{(2\alpha-\lambda)(2\alpha+1-\lambda)}{2\alpha(2\alpha+1)}\frac{x_1^2}{2!}+\frac{\lambda(\lambda+1)}{2\beta(2\beta+1)}\frac{x_2^2}{2!}+...$$

$$U_{6,2}(x_1,x_2)=1+\frac{\lambda}{2\alpha}x_1+\frac{\lambda}{2\beta}x_2-\frac{(2\alpha-\lambda)}{2\beta}x_2+\frac{\lambda(\lambda+1-2\beta)}{2\alpha 2\beta}x_1x_2+$$

$$+\frac{(\lambda(\lambda+1)}{2\alpha(2\alpha+1)}\frac{x_1^2}{2!}\frac{(2\beta-\lambda)(2\beta+1-\lambda)}{2\beta(2\beta+1)}\frac{x_2^2}{2!}+...$$

The proof of the above theorem 2.5 is similar to the proof of Theorem 2.3. After the transformation (2.11) from auxiliary systems

$$x_1U_{x_1x_1}+(2\alpha+x_1)Ux_1-x_2U_{x_2}+(2\alpha-\lambda)U=0,$$

$$x_2U_{x_2x_2}+(2\beta-x_2)Ux_2-x_1U_{x_1}+(x_1+\lambda)U=0,$$

and

$$x_1U_{x_1x_1}+(2\alpha-x_1)Ux_1-x_2U_{x_2}+(x_2+\lambda)U=0,$$

$$x_2U_{x_2x_2}+(2\beta+x_2)Ux_2-x_1U_{x_1}+(2\beta-\lambda)U=0.$$

with solutions in the form of (2.26) and (2.27), respectively. As a result, two normally regular solutions of the form (2.25) will be determined. The statement is also true for them.

Theorem 2.6. There are relations

$$\Psi_{2,j}(\lambda,2\alpha,2\beta;x_1,x_2)=exp(x_j)U_{6,j}(x_1,x_2), \tag{2.28}$$

$$exp(-x_j)\Psi_{2,j}(\lambda,2\alpha,2\beta;x_1,x_2)=U_{6,j}(x_1,x_2), \tag{2.29}$$

where $\Psi_{2,1}=\Psi_{2,2}=\Psi_2$ and $U_{5,j}(x_1,x_2)(j=1,2)$ rows (2.28) and (2.29).

We will also prove one of these relations. Let, $j=2$ and it is required to prove the relation

$$exp(-x_2)\Psi_{2,2}(\lambda,2\alpha,2\beta;x_1,x_2)=U_{6,2}(x_1,x_2). \tag{2.30}$$

Proof. For the proof, we will open the left part (2.30) using decompositions of the given functions. Then:

$$exp(x_2)\Psi_{2,2}(\lambda,2\alpha,2\beta;x_1,x_2)=(1+\frac{(-x_2)}{1!}+\frac{(-x_2^2)}{2!}+...+\frac{(-x_2^n)}{n!}+...)$$

$$(1+\frac{\lambda}{2\alpha}\frac{x_1}{1!}+\frac{\lambda}{2\beta}\frac{x_2}{1!}+\frac{\lambda(\lambda+1)}{2\alpha,2\beta}\frac{x_1x_2}{1!}+\frac{\lambda(\lambda+1)}{2\alpha(2\alpha+1)}\frac{x_1^2}{2!}+\frac{\lambda(\lambda+1)}{2\beta(2\beta+1)}\frac{x_2^2}{2!}+...)=$$

$$=1+\frac{\lambda}{2\alpha}\frac{x_1}{1!}+[-\frac{1}{1!}+\frac{\lambda}{2\beta}]x_2+[\frac{\lambda(\lambda+1)}{2\alpha 2\beta}-\frac{\lambda}{2\alpha}]x_1x_2+\frac{\lambda(\lambda+1)}{2\alpha(2\alpha+1)}\frac{x_1^2}{2!}+\frac{\lambda(\lambda+1)}{2\beta(2\beta+1)}+\frac{1}{2!}-$$

$$-\frac{\lambda}{2\beta}x_2^2+...=1+\frac{\lambda}{2\alpha}\frac{x_1}{1!}-\frac{2\beta-\lambda}{2\beta}\frac{x_2}{1!}+\frac{\lambda(\lambda+1-2\beta)}{2\alpha 2\beta}\frac{x_1x_2}{1!}+\frac{\lambda(\lambda+1)}{2\alpha(2\alpha+1)}\frac{x_1^2}{2!}+$$

$$+\frac{(2\beta-\lambda)(2\beta+1-\lambda)}{2\beta(2\beta+1)}\frac{x_2^2}{2!}+...=U_{6,1}(x_1,x_2).$$

The other relations given in (2.28) and (2.29) are proved in the same way.

3. Multidimensional Bessel functions and their relation to normally regular solutions

Theorem 3.1. A degenerate Bessel type system

$$x_j\frac{\partial^2 W}{\partial x_j^2}+(2\alpha_j-x_j)\frac{\partial W}{\partial x_j}-\sum_{(k\neq j)}\frac{\partial W}{\partial x_k}-\lambda W=0, j=\overline{1,n} \tag{3.1}$$

obtained from the Horn system (1.1) by substitution $\gamma_j=2\alpha_j j=\overline{1,n}$ has 2^n linearly independent partial solutions, one of which is expressed in terms of a degenerate Humbert function resulting in a Bessel function of n variables.

$$\lim_{\lambda\to\infty}\Psi_2^n(\lambda,2\alpha_1,...,2\alpha_n;x_1,x_2)=J(2\alpha_1,...,2\alpha_n;x_1,x_2)=$$

$$=(\frac{x_1}{2})^2...(\frac{x_n}{2})^2\sum_{m_1,...,m_n=0}^{\infty}\frac{(-1)^{m_1+...+m_n}(\frac{x_1}{2})^{2m_1}...(\frac{x_n}{2})^{2m_n}}{m_1!...m_n!\Gamma(2\alpha_1+m_1+1)...\Gamma(2\alpha_n+m_n+1)}. \tag{3.2}$$

The case $n=2$ was considered in paragraph 2.1. (3.2) is a generalization of (2.51), that is, the Bessel function of n variables is obtained.

In paragraph 2.2, it was also proved that in the case $n=2$ there are only two normally regular solutions of the form (2.14). Similarly, it can be verified that when $n=3$ there are only three normally regular solutions. The statement is true.

Theorem 3.2. Auxiliary system derived from a Bessel type system (3.1) when $n=3$, using the transformation

$$Z(x_1,x_2,x_3)=exp(\alpha_{1,0,0}x_1+\alpha_{0,1,0}x_2+\alpha_{0,0,1}x_3)U(x_1,x_2,x_3), \tag{3.3}$$

when two necessary conditions are met

$$\alpha_{1,0,0}^2-\alpha_{1,0,0}=0,\alpha_{0,1,0}^2-\alpha_{0,1,0}=0,\alpha_{0,0,1}^2-\alpha_{0,0,1}=0, \tag{3.4}$$

and

$$\begin{aligned}f_{00}^{(1)}(\rho_1,\rho_2)&=\rho_1(\rho_1-1+\gamma_1)=0,\\ f_{00}^{(2)}(\rho_1,\rho_2)&=\rho_2(\rho_2-1+\gamma_2)=0,\\ f_{00}^{(3)}(\rho_1,\rho_2)&=\rho_3(\rho_1-1+\gamma_3)=0,\end{aligned} \tag{3.5}$$

has three normally-regular solutions

$$Z_{9,t}(x_1,x_2,x_3)=exp(x_t)U_{9,t}(x_1,x_2,x_3) \tag{3.6}$$

where $U_{9,t}(x_1,x_2,x_3)(t=1,2,3)$ two degenerate hypergeometric series of two variables

$$U_{9,1}(x_1,x_2,x_3)=1-\frac{2\alpha_1-\lambda}{2\alpha_1}x_1+\frac{\lambda}{2\alpha_2}x_2+\frac{\lambda}{2\alpha_3}x_3+\frac{\lambda(\lambda+1-2\alpha_1)}{2\alpha_1 2\alpha_2}x_1x_2+\frac{\lambda(\lambda+1-2\alpha_1)}{2\alpha_1 2\alpha_3}x_1x_3+$$
$$+\frac{\lambda(\lambda+1)}{2\alpha_2 2\alpha_3}x_2x_3+\frac{\lambda(\lambda+1)}{2\alpha_2(2\alpha_2+1)}\frac{x_2^2}{2!}+\frac{(2\alpha_1-\lambda)(2\alpha_1+1-\lambda)}{2\alpha_1(2\alpha_1+1)}\frac{x_1^2}{2!}+\frac{\lambda(\lambda+1)}{2\alpha_3(2\alpha_3+1)}\frac{x_3^2}{2!}+... \tag{3.7}$$

$$U_{9,2}(x_1,x_2,x_3)=1+\frac{\lambda}{2\alpha_1}x_1-\frac{2\alpha_2-\lambda}{2\alpha_2}x_2+\frac{\lambda}{2\alpha_3}x_3+\frac{\lambda(\lambda+1-2\alpha_2)}{2\alpha_1 2\alpha_2}x_1x_2+\frac{\lambda(\lambda+1)}{2\alpha_1 2\alpha_3}x_1x_3+$$
$$+\frac{\lambda(\lambda+1-2\alpha_2)}{2\alpha_2 2\alpha_3}x_2x_3+\frac{\lambda(\lambda+1)}{2\alpha_1(2\alpha_1+1)}\frac{x_1^2}{2!}+\frac{(2\alpha_2-\lambda)(2\alpha_2+1-\lambda)}{2\alpha_2(2\alpha_2+1)}\frac{x_2^2}{2!}+\frac{\lambda(\lambda+1)}{2\alpha_3(2\alpha_3+1)}\frac{x_3^2}{2!}+... \tag{3.8}$$

$$U_{9,3}(x_1,x_2,x_3)=1+\frac{\lambda}{2\alpha_1}x_1+\frac{\lambda}{2\alpha_2}x_2-\frac{2\alpha_3-\lambda}{2\alpha_3}x_3+\frac{\lambda(\lambda+1)}{2\alpha_1 2\alpha_2}x_1x_2+\frac{\lambda(\lambda+1-2\alpha_3)}{2\alpha_1 2\alpha_3}x_1x_3+$$

$$+\frac{\lambda(\lambda+1-2\alpha_3)}{2\alpha_2 2\alpha_3}x_2x_3+\frac{\lambda(\lambda+1)}{2\alpha_1(2\alpha_1+1)}\frac{x_1^2}{2!}+\frac{\lambda(\lambda+1)}{2\alpha_2(2\alpha_2+1)}\frac{x_2^2}{2!}+\frac{(2\alpha_3-\lambda)(2\alpha_3+1-\lambda)}{2\alpha_3(2\alpha_3+1)}\frac{x_3^2}{2!}+... \tag{3.9}$$

Here it is important to fulfill two necessary conditions (3.4) and (3.5). They have 2^3 up to three roots. However, of these, only four triples: define the so-called joint connected systems. In case $I.(\alpha_{1,0,0}=0,\alpha_{0,1,0}=0),\alpha_{0,0,1}=0$, $II.(\alpha_{1,0,0}=1,\alpha_{0,1,0,}=0),\alpha_{0,0,1}=0$, $III.(\alpha_{1,0,0}=0,\alpha_{0,1,0}=1),\alpha_{0,0,1}=0)$, $IV.(\alpha_{1,0,0}=0,\alpha_{0,1,0}=0,\alpha_{0,0,1}=1)$ define the so-called joint connected systems.

In case $I.(\alpha_{1,0,0}=0,\alpha_{0,1,0}=0),\alpha_{0,0,1}=0$ you will get the original system (3.1) with a solution in the form of

$$\Psi_{2,t}^{(3)}(\lambda,2\alpha_1,2\alpha_2,2\alpha_3;x_1,x_2,x_3)=\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(\lambda)_{m_1+m_2+m_3}}{(2\alpha_1)!(2\alpha_2)!(2\alpha_3)!}\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}\frac{x_3^{m_3}}{m_3!}. \tag{3.10}$$

In other cases $II-IV$, the resulting connected systems have three normally regular solutions of the form (3.6), where the series (3.7)-(3.9) are their solutions, respectively, when the necessary condition (3.5) is met. The relations between the solution (3.10) and normal-regular solutions are valid.

Theorem 3.3. There are relations:

$$\Psi_{2,t}^{(3)}(\lambda,2\alpha_1,2\alpha_2,2\alpha_3;x_1,x_2,x_3)=exp(x_t)U_{9,t}(x_1,x_2,x_3),t=1,2,3$$

where $U_{9,t}(x_1,x_2,x_3),t=1,2,3$ degenerate hypergeometric series of three variables (3.7)-(3.9).

Theorem 3.4. There are relations

$$exp(-x_t)\Psi_{2,t}^{(3)}(\lambda,2\alpha_1,2\alpha_2,2\alpha_3;x_1,x_2,x_3)=U_{9,t}(x_1,x_2,x_3).$$

The proof of the above relations is similar to the proof of (2.28) and (2.29).

Here is the most general theorem.

Theorem 3.5. An auxiliary system obtained from a Bessel type system using a transformation of the form (1.3) when two necessary conditions are met

$$\alpha_{1,0,0}^2-\alpha_{1,0,0}=0,...,\alpha_{0,0,1}^2-\alpha_{0,0,1}=0,$$

and

$$f_{00}^{(1)}(\rho_1,\rho_2,...,\rho_n)=0,$$
$$---------$$
$$f_{00}^{(n)}(\rho_1,\rho_2,...,\rho_n)=0,$$

has n normal-regular solutions of the form

$$Z_{n+1,t}(x_1,x_2,...,x_n)=exp(x_t)U_{n+1,t}(x_1,x_2,...,x_n) \tag{3.11}$$

where $U_{n+1,t}(x_1,x_2,...,x_n)(t=1,...,n)$ n degenerate hypergeometric series from n variables

$$U_{n+1,1}(x_1,...,x_n)=1-\frac{2\alpha_1-\lambda}{2\alpha_1}x_1+\frac{\lambda}{2\alpha_2}x_2+...+\frac{\lambda}{2\alpha_n}x_n+\frac{\lambda(\lambda+1-2\alpha_1)}{2\alpha_1 2\alpha_2}x_1x_2+...$$
$$\frac{\lambda(\lambda+1-2\alpha_1)}{2\alpha_1 2\alpha_2}x_1x_n+\frac{\lambda(\lambda+1)}{2\alpha_1 2\alpha_3}x_2x_3+...+\frac{\lambda(\lambda+1)}{2\alpha_{n-1}2\alpha_n}x_{n-1}x_n+\frac{(2\alpha_1-\lambda)(2\alpha_1+1-\lambda+1)}{2\alpha_1(2\alpha_1+1)}\frac{x_1^2}{2!}+...$$
$$\frac{\lambda(\lambda+1)}{2\alpha_2(2\alpha_2+1)}\frac{x_2^2}{2!}+...+\frac{\lambda(\lambda+1)}{2\alpha_n(2\alpha_n+1)}\frac{x_n^2}{2!}+...+\frac{(2\alpha_1-\lambda)(2\alpha_1+1-\lambda)(2\alpha_1+m_1-\lambda)}{2\alpha_1(2\alpha_1+1)...(2\alpha_1+m_1-1)}\frac{x_1^{m_1}}{m_1!}+$$
$$+\frac{\lambda(\lambda+1)...(\lambda+m_n-1)}{2\alpha_n(2\alpha_n+1)...(2\alpha_n+m_n-1)}\frac{x_n^{m_n}}{m_n!}+... \tag{3.12$_1$}$$
$$---------$$

$$U_{n+1,1}(x_1,...,x_n)=1+\frac{\lambda}{2\alpha_1}x_1+...+\frac{\lambda}{2\alpha_{n-1}}x_{n-1}-\frac{2\alpha_n-\lambda}{2\alpha_n}x_n+\frac{\lambda(\lambda+1)}{2\alpha_1 2\alpha_2}x_1x_2+$$
$$+...+\frac{\lambda(\lambda+1-2\alpha_n)}{2\alpha_1 2\alpha_n}x_1x_n+...+\frac{(2\alpha_n-\lambda)(2\alpha_n+1-\lambda)}{2\alpha_1(2\alpha_1+1)}\frac{x_n^2}{2!}+...+$$
$$+\frac{(2\alpha_n-\lambda)(2\alpha_n+1-\lambda)...(2\alpha_n+m_n-1)}{2\alpha_n(2\alpha_n+1)...(2\alpha_n+m_n-1)}\frac{x_n^{m_n}}{m_n!}+... \qquad (3.12_n)$$

There is also a generalization of theorems 3.3 and 3.4.

Theorem 3.6. There are relations:

$$\Psi_{2,t}^{(n)}(\lambda;2\alpha_1,...,2\alpha_n;x_1,...,x_n)=exp(x_t)U_{n+1,t}(x_1,...,x_n),t=1,2,3 \qquad (3.13)$$

where $U_{n+1,t}(x_1,...,x_n),t=1,2,3$ degenerate hypergeometric series of variables (3.12_1) – (3.12_n).

Theorem 3.7. There are relations

$$exp(-x_t)\Psi_{2,t}^{(n)}(\lambda;2\alpha_1,...,2\alpha_n;x_1,...,x_n)=U_{n+1,t}(x_1,...,x_n). \qquad (3.14)$$

Thus we have established degenerate hypergeometric systems associated with the Horn system (1.1) whose solutions are multidimensional Bessel functions. The possibilities of constructing normally regular solutions of such systems are investigated, the connections between regular solutions of the form (1.4) and normally regular solutions of the form (1.5) are proved. A number of properties related to Humbert functions of two or more variables are also proved.

The possibilities of constructing solutions of Horn-type systems obtained from the Lauricella system F_A using limit transitions in the form of multidimensional Bessel functions are studied. Solutions of degenerate hypergeometric systems obtained from the Lauricella system F_B, using limit transitions are also constructed. It is proved that their solutions are degenerate hypergeometric functions associated with multidimensional Bessel functions and normally regular solutions of many variables.

3.1. **Properties of the Bessel functions of two variables obtained from the Horn system using a double transformation.** In [17] it was shown that the degenerate system

$$x_1^2U_{x_1x_1}-x_1x_2U_{x_2}+[\frac{1}{4}x_1^2-\frac{1}{2}x_1x_2+kx_1+\alpha(1-\alpha)]U=0,$$
$$x_2^2U_{x_2x_2}-x_1x_2U_{x_1}+[\frac{1}{4}x_2^2-\frac{1}{2}x_1x_2+kx_2+\beta(1-\beta)]U=0, \qquad (3.15)$$

obtained from the Horn system (2.1) using the transformation

$$Z_t(x_1,x_2)=exp(\frac{1}{2}x_1+\frac{1}{2}x_2)x_1^{-\frac{y_1}{2}}x_2^{-\frac{y_2}{2}}U(x_1,x_2) \qquad (3.16)$$

at $\gamma_1=2\alpha$, $\gamma_2=2\beta$ also refers to a Bessel-type system. System (3.15) is interesting because it also refers to a Whittaker-type system [6] and has four linearly independent partial solutions

$$U_1(x_1,x_2)=exp(-\frac{1}{2}x_1-\frac{1}{2}x_2)\Phi_1(x_1,x_2),$$
$$U_2(x_1,x_2)=exp(-\frac{1}{2}x_1-\frac{1}{2}x_2)\Phi_2(x_1,x_2),$$
$$U_3(x_1,x_2)=exp(-\frac{1}{2}x_1-\frac{1}{2}x_2)\Phi_3(x_1,x_2),$$
$$U_4(x_1,x_2)=exp(-\frac{1}{2}x_1-\frac{1}{2}x_2)\Phi_4(x_1,x_2), \qquad (3.17)$$

where

$$\Phi_1(x_1,x_2)=x^\alpha y^\beta\Psi_2(\lambda,2\alpha,2\beta;x_1,x_2), \qquad (3.18_1)$$

$$\Phi_2(x_1,x_2)=x^{\alpha}y^{-\beta+\frac{1}{2}}\Psi_2(\lambda-2\beta+1,2\alpha,2\beta-2;x_1,x_2), \quad (3.18_2)$$

$$\Phi_3(x_1,x_2)=x^{-\alpha+\frac{1}{2}}y^{\beta}\Psi_2(\lambda-2\alpha+1,2\alpha-2,2\beta;x_1,x_2), \quad (3.18_3)$$

$$\Phi_4(x_1,x_2)=x^{1-\alpha}y^{1-\beta}\Psi_2(\lambda-2\alpha-2\beta+2,2\alpha-2,2\beta-2;x_1,x_2). \quad (3.18_4)$$

This shows that the systems of the type of Horn (2.1), Bessel (2.1) and (3.15), Whittaker given in [6] are related. Therefore, they have common properties. Can make sure that all of them are expressed through the Humbert function $\Psi_2(\lambda,\gamma_1,\gamma_2;x_1,x_2)$ 2.2_1. Let's consider some properties of the constructed functions (3.17) and (3.18).

1. The solutions $U_t(x_1,x_2)(t=1,2,3,4)$ in (3.17) are normally regular of the form (2.14).

2. The second property is formulated in the form of the following statement.

Theorem 3.8. Partial solutions $(3.18_1)-(3.18_4)$ are solutions of the attached system

$$x_1^2\Phi_{x_1x_1}+x_1^2\Phi_{x_1}-x_1x_2\Phi_{x_2}+[kx_1+\alpha(1-\alpha)]\Phi=0,$$
$$x_2^2\Phi_{x_2x_2}+x_2^2\Phi_{x_2}-x_1x_2\Phi_{x_1}+[kx_2+\beta(1-\beta)]\Phi=0. \quad (3.19)$$

All solutions are expressed in terms of the Humbert hypergeometric function $\Psi_2(\lambda;\gamma_1,\gamma_2;x_1,x_2)$, where $\gamma_1=2\alpha,\gamma_2=2\beta$.

Proof. To construct them, the Frobenius-Latysheva method is used [14]. We are looking for the solution (3.19) in the form of a generalized power series of two variables where $\rho_j(j=1,2)$, $A_{m_1,m_2}(m_1,m_2=0,1,2,...)-$unknown constants. Unknown constants $\rho_j(j=1,2)$ of rows (2.19) are determined from the system of defining equations (2.13). Unknown coefficients $A_{m_1,m_2}(m_1,m_2=0,1,2,...)$ of the series are determined from the system of recurrent sequences (2.20).

3. On the basis of (2.5) and (2.6), the ratio is valid

$$\lim_{\lambda\to\infty}\Phi_1(x_1,x_2)=\lim_{\lambda\to\infty}[x_1^{\alpha}x_2^{\beta}\sum_{m_1,m_2=0}^{\infty}\frac{(\lambda)_{m_1+m_2}}{(2\alpha)_{m_1}(2\beta)_{m_2}}\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}]=$$

$$=x_1^{\alpha}x_2^{\beta}\lim_{\lambda\to\infty}\Psi_2(\lambda,2\alpha,2\beta;x_1,x_2)=x_1^{\alpha}x_2^{\beta}J(2\alpha,2\beta;x_1,x_2), \quad (3.20)$$

that is, the limit for a $\lambda\to\infty$ degenerate hypergeometric function of two variables is expressed in terms of a function resulting in Bessel functions (2.51), which is the solution of the system (2.3).

4.**Theorem 3.9.** Normal-regular solutions (3.17) are partial solutions of the auxiliary system

$$x_1^2\Phi_{x_1x_1}+2\alpha_{10}^2x_1^2\Phi_{x_1}-x_1x_2\Phi_{x_2}[(\alpha_{10}^2-\frac{1}{4})x_1^2-(\alpha_{01}^2-\frac{1}{2})x_1x_2+kx_1+\alpha(1-\alpha)]\Phi=0,$$

$$x_2^2\Phi_{x_2x_2}+x_1x_2\Phi_{x_1}+2\alpha_{01}^2x_2^2\Phi_{x_2}+[(\alpha_{01}^2-\frac{1}{4})x_2^2-(\alpha_{10}^2-\frac{1}{2})x_1x_2+kx_2+\beta(1-\beta)]\Phi=0, \quad (3.21)$$

obtained by converting

$$U(x_1,x_2)=exp(\alpha_{10}x_1+\alpha_{01}x_2)\Phi(x_1,x_2) \quad (3.22)$$

where α_{10} and $\alpha_{01}-$ indefinite coefficients that are determined from a system of characteristic equations

$$b_{10}^{(1)}(\alpha_{10},\alpha_{01})=\alpha_{10}^2-\frac{1}{4}=0,$$
$$b_{01}^{(1)}(\alpha_{10},\alpha_{01})=\alpha_{01}^2-\frac{1}{4}=0. \quad (3.23)$$

The proof of the above theorem 3.9 is similar to the proof of Theorem 2.3.

5. Condition (3.23) is the first necessary condition for the existence of normally regular solutions (3.17). From here $\alpha_{10}=-\frac{1}{2},\alpha_{01}=-\frac{1}{2}$ the pair defines the attached system (3.19).

6. As a result of the double transformation (3.16) and (3.22), when $\gamma_1=2\alpha$, $\gamma_2=2\beta$ we obtain a degenerate function resulting in Bessel functions.

7.From the representation of this function in the form (2.6), it is not difficult to make sure that in genera $\lambda = -n$ it turns into a polynomial

$$\Psi_2(\lambda; 2\alpha, 2\beta; x_1, x_2) = \sum_{m_1,m_2=0}^{\infty} \frac{(\lambda)_{m_1+m_2}}{(2\alpha)_{m_1}(2\beta)_{m_2}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!}. \tag{3.24}$$

In a number of papers, we have studied the relationship of Humbert functions with Laguerre polynomials in two variables [21-22]:

$$\Psi_2(\lambda; \alpha_1 + 1, \alpha_2 + 1; x_1, x_2) = \sum_{m_1,m_2=0}^{\infty} \frac{(-n)_{m_1+m_2}}{(\alpha_1 + 1)_{m_1}(\alpha_2 + 1)_{m_2}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!}$$

obtained from the Humbert function (2.21) for $\lambda = -n(n > 0 - integer)$, $\gamma_1 = \alpha_1 + 1, \gamma_2 = \alpha_2 + 1$, $(\alpha_1 > -1, \alpha_2 > -1)$.

It follows from this that the Humbert, Bessel, Whittaker and Laguerre functions are related functions.

8. **Theorem 3.10.** The general derivative of the Bessel $\Psi_2(\lambda; 2\alpha, 2\beta; x_1, x_2)$ function is represented as

$$\frac{\partial^{m_1+m_2}}{\partial x_1^{m_1} \partial x_2^{m_2}} \Psi_2(\lambda; 2\alpha, 2\beta; x_1, x_2) = \frac{\lambda(\lambda + 1)...(\lambda + m_1 + m_2 - 1)}{2\alpha(2\alpha + 1)...(2\alpha + m_1 - 1)2\beta(2\beta + 1)...(2\beta + m_2 - 1)} \cdot$$
$$\cdot \Psi_2(\lambda + m_1 + m_2; 2\alpha + m_1, 2\beta + m_2; x_1, x_2). \tag{3.25}$$

From here it is possible to deduce special cases by variable x_1 and x_2:

$$\frac{\partial^{m_1}}{\partial x_1^{m_1}} \Psi_2(\lambda; 2\alpha, 2\beta; x_1, x_2) = \frac{\lambda(\lambda + 1)...(\lambda + m_1 - 1)}{2\alpha(2\alpha + 1)...(2\alpha + m_1 - 1)} \Psi_2(\lambda + m_1; 2\alpha + m_1, 2\beta; x_1, x_2).$$
$$\frac{\partial^{m_2}}{\partial x_2^{m_2}} \Psi_2(\lambda; 2\alpha, 2\beta; x_1, x_2) = \frac{\lambda(\lambda + 1)...(\lambda + m_2 - 1)}{2\beta(2\beta + 1)...(2\beta + m_2 - 1)} \Psi_2(\lambda + m_2; 2\alpha, 2\beta + m_2; x_1, x_2).$$

The first partial derivative of x_1:

$$\frac{\partial}{\partial x_1} \Psi_2(\lambda; 2\alpha, 2\beta; x_1, x_2) = \frac{\lambda}{2\alpha} \Psi_2(\lambda + 1; 2\alpha + 1, 2\beta; x_1, x_2), \tag{3.26}$$

and by variable x_2:

$$\frac{\partial}{\partial x_2} \Psi_2(\lambda; 2\alpha, 2\beta; x_1, x_2) = \frac{\lambda}{2\beta} \Psi_2(\lambda + 1; 2\alpha, 2\beta + 1; x_1, x_2). \tag{3.27}$$

For the proof, we will use the decomposition of functions $\Psi_2(\lambda; 2\alpha, 2\beta; x_1, x_2)$.

The relations for other particular solutions are also valid

$$\frac{\partial}{\partial x_1} \Psi_2(\lambda + 1 - 2\alpha, 2 - 2\alpha, 2\beta; x_1, x_2) = \frac{\lambda + 1 - 2\alpha}{2 - 2\alpha} \Psi_2(\lambda + 2 - 2\alpha, 3 - 2\alpha, 2\beta; x_1, x_2),$$

$$\frac{\partial}{\partial x_2} \Psi_2(\lambda + 1 - 2\beta; -2\beta + 2, 2\alpha; x_1, x_2) = \frac{\lambda + 1 - 2\beta}{2 - 2\beta} \Psi_2(\lambda + 2 - 2\beta; 2\alpha, 3 - 2\beta + 1; x_1, x_2).$$

$$\frac{\partial}{\partial x_1} [x_1^{1-2\alpha} \Psi_2(\lambda + 1 - 2\alpha, 2 - 2\alpha, 2\beta; x_1, x_2)] = (1 - 2\alpha) x^{-2\alpha} \Psi_2(\lambda - 2\alpha + 2, 2 - 2\alpha, 2\beta; x_1, x_2),$$

$$\frac{\partial}{\partial x_2} [x_1^{1-2\alpha} \Psi_2(\lambda + 1 - 2\alpha, 2 - 2\alpha, 2\beta; x_1, x_2)] = x_1^{1-2\alpha} \frac{\partial}{\partial x_2} \Psi_2(\lambda + 1 - 2\beta; 2 - 2\alpha, 2\beta; x_1, x_2) =$$
$$= x_1^{1-2\alpha} \frac{\lambda + 1 - 2\beta}{2\beta} \Psi_2(\lambda + 2 - 2\beta; 2 - 2\alpha, 2\beta + 1; x_1, x_2).$$

9.Limit transition

$$\lim_{\lambda\to\infty}\Psi_2(\lambda;\gamma_1,\gamma_2;x_1,x_2)=J(\gamma_1,x_1)J(\gamma_2,x_2)=\sum_{m_1,m_2=0}^{\infty}\frac{1}{(\gamma_1)_{m_1}(\gamma_2)_{m_2}}\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}=J(\gamma_1,\gamma_2;x_1,x_2). \tag{3.28}$$

We will use to reveal the differential properties of a function that reduces to a Bessel function of two variables. Indeed, when $\gamma_1=2\alpha,\gamma_2=2\beta$ equality is true

$$J(\gamma_1,\gamma_2;x_1,x_2)=J(2\alpha,2\beta;x_1,x_2)=\sum_{m_1,m_2=0}^{\infty}\frac{1}{(2\alpha)_{m_1}(2\beta)_{m_2}}\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}=$$
$$=J(2\alpha,x_1)J(2\beta,x_2) \tag{3.29}$$

Then, based on (2.5), the Bessel function of two variables of the first kind, we obtain in the form

$$J_{2\alpha,2\beta}(x_1,x_2)=\sum_{m_1,m_2=0}^{\infty}\frac{(-1)^{m_1+m_2}(\frac{x_1}{2})^{2m_1+2\alpha}(\frac{x_2}{2})^{2m_2+2\beta}}{\Gamma(2\alpha+m_1+1)\Gamma(2\beta+m_2+1)m_1!m_2!}.$$

10. The first derivatives of functions reduced to the Bessel function of two variables are represented as

$$\frac{\partial}{\partial x_1}[J(2\alpha;x_1)J(2\beta;x_2)]=\frac{1}{2\alpha}[J(2\alpha+1;x_1)J(2\beta;x_2)],$$
$$\frac{\partial}{\partial x_2}[J(2\alpha;x_1)J(2\beta;x_2)]=\frac{1}{2\beta}[J(2\alpha;x_1)J(2\beta+1;x_2)].$$

Second derivatives:

$$\frac{\partial}{\partial x_1\partial x_2}[J(2\alpha;x_1)J(2\beta;x_2)]=\frac{1}{2\alpha 2\beta}[J(2\alpha+1;x_1)J(2\beta+1;x_2)],$$
$$\frac{\partial^2}{\partial x_1^2}[J(2\alpha;x_1)J(2\beta;x_2)]=\frac{1}{(2\alpha)_2}[J(2\alpha+2;x_1)J(2\beta;x_2)],$$
$$\frac{\partial^2}{\partial x_2^2}[J(2\alpha;x_1)J(2\beta;x_2)]=\frac{1}{(2\beta)_2}[J(2\alpha;x_1)J(2\beta+1;x_2)].$$

Higher derivatives:

$$\frac{\partial^{m_1}}{\partial x_1^{m_1}}[J(2\alpha;x_1)J(2\beta;x_2)]=\frac{1}{(2\alpha)_{m_1}}[J(2\alpha+m_1;x_1)J(2\beta;x_2)],$$
$$\frac{\partial^{m_2}}{\partial x_2^{m_2}}[J(2\alpha;x_1)J(2\beta;x_2)]=\frac{1}{(2\beta)_{m_2}}[J(2\alpha;x_1)J(2\beta+m_2;x_2)].$$
$$\frac{\partial^{m_1+m_2}}{\partial x_1^{m_1}\partial x_2^{m_2}}[J(2\alpha;x_1)J(2\beta;x_2)]=\frac{1}{(2\alpha)_{m_1}(2\beta)_{m_2}}[J(2\alpha+m_1;x_1)J(2\beta+m_2;x_2)].$$

11. There is a limit transition [23]:

$$\lim_{\epsilon\to 0}F_2(\frac{1}{\epsilon},\frac{1}{\epsilon},\frac{1}{\epsilon};\gamma;\epsilon^2x_1,\epsilon^2x_2)=\sum_{m_1,m_2=0}^{\infty}\frac{1}{(\gamma)_{m_1+m_2}}\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}=J(\gamma,x_1+x_2).$$

12. The degenerate hypergeometric function, $J(\gamma,x_1+x_2)$ of two variables, is a particular solution of the system [23]:

$$x_1Z_{x_1x_1}+x_2Z_{x_1x_2}+\gamma_1Z_{x_1}-Z=0,$$

$$x_2 Z_{x_2 x_2} + x_1 Z_{x_1 x_2} + \gamma_2 Z_{x_2} - Z = 0.$$

13. The above and other properties can similarly be derived for solving a general Bessel-type system [21].

$$x_j^2 U_{x_1 x_1} - x_j \sum_{(\gamma \neq j)} x_\gamma U_\gamma + [\frac{x_j^2}{4} - \frac{x_j}{2} \sum_{(\gamma \neq j)} x_\gamma + k x_j + \alpha_j (1 - \alpha_j)] U = 0, (j = 1, 2, ..., n)$$

where $k = \alpha_+ \beta - \lambda; \alpha_j (j = 1, 2, ..., n), \lambda -$ some parameter, general unknown.

14.In [21] it is proved that a particular solution of this system is the limit of the Humbert function of n variables

$$\lim_{\lambda \to \infty} \Psi_2^n (\lambda, 2\alpha_1, ..., 2\alpha_n; \lambda x_1, ..., \lambda x_n) = \sum_{m_1, ..., m_n = 0}^{\infty} \frac{(1)_{m_1 + ... + m_n} {x_1}^{m_1} ... {x_n}^{m_n}}{(2\alpha_1)_{m_1} ... (2\alpha_n)_{m_n} m_1! ... m_n!} = J(2\alpha_1, ..., 2\alpha_n; x_1, ..., x_n).$$

References

[1] Humbert P., The confluent hypergeometric functions of two variables, *Proc. Roy. Soc.Edin* , 1922, pp.73–96.

[2] Horn J., Hypergeometrische Funktionen zweier veranderlichen, *Math.Ann. 115*, 1931, pp.381–407.

[3] Borngasser L., Uber hypergeometrische funkionen zweier Veranderlichen, Dissertation, *Darmstadt* , 1933.

[4] Beitmen G., Erdeii A., Higher A., Transcendental Functions. Part I. Hypergeometric functions, *The functions of the Lezhanders. M: Science* , 1965.

[5] Lauricella G., Sulle funzioni ipergeometriche a piu variabili, *Rend. Circ. Matem.*, 1897, t.7, pp.111–158.

[6] Appell P. and Kampe de Feriet J., Fonctons hypergeometriges of hyperspheriges Polynomes d'Hermite, *Paris, Gauthier -Villars* , 1926, 434 p.

[7] Srivastava H.M., Karlsson P.W., Multiple Gaussian hypergeometric series, *Ellis Harwood Limited, Chichester*, 1985, 426 p.

[8] Khudozhnikov V.Y., Two new degenerated hypergeometric functions of many variables and integral equations with them, *Diff. equations*, 2003, No6(39), pp.835-843. bibitem1 Khudozhnikov V.Y., Two new degenerated hypergeometric functions of many variables and integral equations with them, *Diff. equations* , 2003, No39(6), pp.835-843.

[9] Hasanov A., Choi Junesang, *Note on Euler-Bernoulli Equation. Sohag J. Math.*, 2020, No7(2), pp.33-36.

[10] Salakhitdinov M.S., Hypergeometric functions and their applications to the study of boundary and spectral problems for partial differential equations with singular singularities, *Second International Russian-Uzbek Symposium "Mixed Type Equations and Related Problems of Analysis and Informatics", Proceedings. KBR, Elbrus* , 2012, 239 p.

[11] Hasanov A., Yuldashev T. K., Analytic Continuation Formulas for the Hypergeometric Functions in Three Variables of Second Order, *Lobachevskii J. of Mathematics*, 2022, No 43(5), pp.1134 -1141.

[12] Ergashev T.G., Tulakova, The Neumann Problem for a Multidimensional Elliptic Equation with Several Singular Coefficients in an Infinite Domain, *Lobachevskii J. of Mathematics*, 2022, No 43(1), pp.199 -206.

[13] Watson G.N., Theory of Bessel functions, *New York: Cambridge U.P.*, 1944.

[14] Korenev B.G., Introduction to the theory of Bessel functions, *Moscow: "Nauka"*, 1971.

[15] Jeffreys G., Swirls B., Methods of Mathematical Physics, *Moscow: "Mir"*, 1970, 344 p.

[16] Tasmambetov Zh.N., Construction of normal and normally-regular solutions of special systems of partial equations of second order, *Aktobe.*, 2015, 384 p.

[17] Tasmambetov Zh.N., On Irregular Singular Curves of Whittaker Type Systems, *Bulletin of Sam.Gos.Tekh. university. A series of physical and mathematical* , 2013, No4(11), pp. 25-33.

[18] Ubayeva Zh.K., Tasmambetov Zh.N., Rajabov N., Features of Constructing a Solution Heterogeneous Equation and Clausen-Type Systems, *Mathematical Modelling of Engineering Problems,(Q2)* 2022, No4 pp. 906-918.

[19] Ubayeva Zh.K., Tasmambetov Zh.N., Exceptions of formulating the normal-regular solutions of confluent hypergeometric systems obtained from the Lauricella system, *Lobachevskii journal of mathematics, (Q2)* , 2022, No8(43) pp. 2309-2321.

[20] Tasmambetov Zh.N., Studying the features of constructing solutions of a degenerate system in the form of a Bessel function of many variables, *Traditional April International Mathematical Conference in honor of the Day of Science Workers. Abstracts of reports. - Almaty*, 2023, pp. 167-168.

[21] Issenova A., Tasmambetov Zh., Rajabov N., On general properties of degenerate systems of second order partial differential equations of hypergeometric type, *European journal of pure and applied mathematics, (Q3)* , 2021, No3(14,) pp. 1024-1043.

[22] TasmambetovZh.N., Confluent hypergeometric functions and two variables Laguerre polynomials as a solution of Wilczynski type system, *AIP Conference Proceeding 1779, 020137(2016); doi.org /10.1063/1/495975*

[23] Tasmambetov Zh.N., Issenova A.A., Bessel functions of two variables as solutions for systems of the second order differential equations, *Bulletin of the Karaganda university. Mathematics series.*, 2020, No2(98), pp. 141-152.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ОБ ОДНОЙ ЗАДАЧЕ ДЛЯ УРАВНЕНИЯ ТРЕТЬЕГО ПОРЯДКА С КРАТНЫМИ ХАРАКТЕРИСТИКАМИ В ТРЕХМЕРНОМ ПРОСТРАНСТВЕ В ПОЛУОГРАНИЧЕННОЙ ОБЛАСТИ

АЗИЗБЕК ХАМИТОВ[1]

[1]Наманганский инженерно-строительный институт, Наманган, Узбекистан
e-mail:azizbek.khamitov.93@mail.ru

Аннотация. В работе для уравнения третьего порядка с кратными характеристиками рассмотрена краевая задача в трехмерном пространстве в полуограниченной области. Единственность решения поставленной задачи доказана методом интегралов энергии. Существование решения доказано методом разделения переменных. Решение построено явно в виде бесконечного ряда, обоснована возможность почленного дифференцирования ряда по всем переменным.

Ключевые слова: Дифференциальное уравнение с частными производными, уравнение третьего порядка, кратные характеристики, краевая задача, единственность, существование, ряд, полуограниченная область, абсолютная и равномерная сходимость.

Предметная классификация AMS: 35G15

1. Введение

Дифференциальные уравнения в частных производных третьего порядка рассматриваются при решении задач теории нелинейной акустики и в гидродинамической теории космической плазмы, фильтрации жидкости в пористых средах. С точки зрения физических приложений представляют большой интерес и дифференциальные уравнения третьего порядка (см., например, [1-11]).

В работе [12], учитывая свойства вязкости и теплопроводности газа, из системы Навье-Стокса было получено уравнение третьего порядка с кратными характеристиками, содержащее вторую производную по времени

$$u_{xxx} + u_{yy} - \frac{\nu}{y}u_y = u_x u_{xx},\ \nu = const.$$

Это уравнение при $\nu = 1$ описывает осесимметричный поток, а при $\nu = 0$ описывает плоско - параллельный поток [13].

Первые результаты по уравнению третьего порядка с кратными характеристиками были получены в работах H. Block [14], E. Del Vecehio[15]. L. Catabriga в работе [16] для уравнения $D_x^{2n+1}u - D_y^2u = 0$ построил фундаментальное решение в виде двойного несобственного интеграла и изучил свойства потенциала, решил краевые задачи.

В работах [17-18] построены фундаментальные решения уравнения третьего порядка с кратными характеристиками, содержащие вторые производные по времени, выраженные через вырожденные гипергеометрические функции, изучены их свойства, найдены оценки при $|t| \to \infty$.

2. Постановка задачи

В области $D^- = \{(x,y,z) : -\infty < x < 0,\ 0 < y < q,\ 0 < z < r\}$ рассмотрим уравнения

$$L[u] \equiv \frac{\partial^3 u}{\partial x^3} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = 0, \tag{1}$$

где $q > 0,\ r > 0$ - постоянные вещественные числа, и для него исследуем следующую задачу.

Задача B. Найти решение уравнения (1) в области D^- из класса $C^{3,2,2}_{x,y,z}(D^-) \cap C^{2,1,1}_{x,y,z}(D^- \cup \Gamma)$, имеющего ограничение первой производной по x, y и z, второй производной по x при $x \to -\infty$, и $u_y, u_z \in L_2(D^-)$, удовлетворяющего следующими краевыми условиями

$$u_y(x,0,z) = u_y(x,q,z) = 0, \quad u_z(x,y,0) = u_z(x,y,r) = 0,\ -\infty < x < 0, \tag{2}$$

$$\begin{gathered} u(0,y,z) = \psi_1(y,z), u_x(0,y,z) = \psi_2(y,z), \\ \lim_{x\to-\infty} u(x,y,z) = 0, 0 \le y \le q,\ 0 \le z \le r, \end{gathered} \tag{3}$$

где $\Gamma = \partial D^-$ - граница области D^-, $\psi_i(y,z), i = 1,2$ - заданные достаточно гладкие функции, причем

$$\begin{cases} \dfrac{\partial \psi_i(0,z)}{\partial y} = \dfrac{\partial \psi_i(q,z)}{\partial y} = 0, \dfrac{\partial^4 \psi_i(y,0)}{\partial y^3 \partial z} = \dfrac{\partial^4 \psi_i(y,r)}{\partial y^3 \partial z} = 0, \\ \dfrac{\partial \psi_i(y,0)}{\partial z} = \dfrac{\partial \psi_i(y,r)}{\partial z} = 0, \dfrac{\partial^4 \psi_i(0,z)}{\partial z^3 \partial y} = \dfrac{\partial^4 \psi_i(q,z)}{\partial z^3 \partial y} = 0, \end{cases} \quad i = 1,2. \tag{4}$$

Отметим, что в плоскости полуограниченных областях изучены в работах [19-22], а в трехмерном пространстве для уравнения второго порядка в работах [23-24] исследованы некоторые корректные краевые задачи. А также в работах [25-29] в конечные области изучены краевые задачи в трехмерном пространстве.

3. Единственность решения

Теорема 1. Если задача B имеет решение, то оно единственно.

Доказательство. Предположим, обратное пусть задача B имеет два решения $u_1(x,y,z)$ и $u_2(x,y,z)$. Тогда функция $u(x,y,z) = u_1(x,y,z) - u_2(x,y,z)$ удовлетворяет уравнению (1) с однородными краевыми условиями. Докажем, что $u(x,y,z) \equiv 0$ в D^-.

Для этого уравнения (1) умножим на u, тогда получим

$$u\,L[u] \equiv \frac{\partial}{\partial x}\left(u\,u_{xx} - \frac{1}{2}u_x^2\right) - \frac{\partial}{\partial y}(u\,u_y) + u_y^2 - \frac{\partial}{\partial z}(u\,u_z) + u_z^2 = 0. \tag{5}$$

Интегрируя тождество (5) по области $D_c = \{(x,y,z) : c < x < 0,\ 0 < y < q,\ 0 < z < r\}$, где $c < 0$, имеем

$$\begin{aligned} &\int_0^q\int_0^r u(0,y,z)\,u_{xx}(0,y,z)dydz - \int_0^q\int_0^r u(c,y,z)\,u_{xx}(c,y,z)dydz - \frac{1}{2}\int_0^q\int_0^r u_x^2(0,y,z)dydz + \\ &+\frac{1}{2}\int_0^q\int_0^r u_x^2(c,y,z)dydz - \int_c^0\int_0^r u(x,q,z)u_y(x,q,z)\,dxdz + \int_c^0\int_0^r u(x,0,z)u_y(x,0,z)\,dxdz - \\ &-\int_c^0\int_0^q u(x,y,r)u_z(x,y,r)\,dxdy + \int_c^0\int_0^q u(x,y,0)u_z(x,y,0)\,dxdy + dxdydz + \\ &+dxdydz = 0. \end{aligned} \tag{6}$$

Если $c \to -\infty$, то $D_c \to D^-$. При этом, учитывая однородные краевые условия задачи B, т.е. $\psi_i(y,z) = 0, i = 1,2$, свойства функции $u(x,y,z)$ при $x \to -\infty$, и $u_y, u_z \in L_2(D^-)$, из (6) получим

$$\frac{1}{2}\lim_{c\to-\infty}\int_0^q\int_0^r u_x^2(c,y,z)dydz + \iiint\limits_{D^-} u_y^2(x,y,z)\,dxdydz + \iiint\limits_{D^-} u_z^2(x,y,z)\,dxdydz = 0.$$

Отсюда следует, что $u_y(x,y,z)=0$ и $u_z(x,y,z)=0$, тогда $u(x,y,z)=f(x)$ в D^-. Поставляя в уравнение (1), имеем $f'''(x)=0$. Отсюда, $f(x)=C_1x^2+C_2x+C_3$. Из условии (3), получим $f(0)=f'(0)=0$, $\lim\limits_{x\to-\infty}f(x)=0$, тогда $C_1=0, C_2=0, C_3=0$, отсюда имеем что $f(x)=0$. Следовательно, $u(x,y,z)=0$ в $D^-\cup\Gamma$. В силу последнего, получим $u_1(x,y,z)=u_2(x,y,z)$.

Теорема 1 доказана.

4. Существование решения

Теорема 2. Если функции $\dfrac{\partial^6\psi_i(y,z)}{\partial y^3\partial z^3}\in L_2[0<y<q,0<z<r], i=1,2$ и выполняются условия согласования (4), то решение задачи B существует.

Доказательство. Решение задачи B ищем в виде

$$u(x,y,z)=X(x)V(y,z) \tag{7}$$

Поставляя (7) в уравнение (1) и разделяя переменные, относительно функции $X(x)$ получим уравнение:

$$X'''+\lambda X=0, \tag{8}$$

а для функции $V(y,z)$ - следующую краевую задачу:

$$\begin{cases} V_{yy}+V_{zz}+\lambda V=0, \\ V_y(0,z)=V_y(q,z)=0, \\ V_z(y,0)=V_z(y,r)=0, \end{cases} \tag{9}$$

где λ - параметр разделения.

Найдем собственные значения и собственные функции задачи (9).
Положим

$$V(y,z)=Y(y)Z(z). \tag{10}$$

Подставляя (10) в уравнение (9), разделяя переменные, имеем задачи

$$\begin{cases} Y''+\nu Y=0, \\ Y'(0)=Y'(q)=0, \end{cases} \tag{11}$$

$$\begin{cases} Z''+\mu Z=0, \\ Z'(0)=Z'(r)=0, \end{cases} \tag{12}$$

где ν и μ - неотрицательные постоянные, связанные соотношением $\lambda=\nu+\mu$.

Решение задачи (11), (12) имеет следующий вид:

$$\begin{cases} Y(y)=A_1\sin\sqrt{\nu}y+B_1\cos\sqrt{\nu}y, \\ Z(z)=A_2\sin\sqrt{\mu}z+B_2\cos\sqrt{\mu}z. \end{cases} \tag{13}$$

С учетом граничных условий, из (13) находим

$$\begin{cases} Y_n(y)=B_n\cos\dfrac{n\pi y}{q}, \\ Z_m(z)=B_m\cos\dfrac{m\pi z}{r}, \end{cases} \tag{14}$$

где $\nu_n=\left(\dfrac{n\pi}{q}\right)^2$, $\mu_m=\left(\dfrac{m\pi}{r}\right)^2$, $n,m\in N\cup\{0\}$ - собственные значения.

Тогда, в качестве решения спектральной задачи (11), (12) возьмем функции

$$V_{n,m}(y,z)=\begin{cases} \dfrac{1}{\sqrt{qr}}, \ n,m=0, \\ \dfrac{2}{\sqrt{qr}}\cos\frac{n\pi y}{q}\cos\dfrac{m\pi z}{r}, \ n,m\in N, \end{cases} \tag{15}$$

которые соответствуют собственным значениям

$$\lambda_{n,m} = \left(\frac{n^2}{q^2} + \frac{m^2}{r^2}\right)\pi^2,\ n,m \in N \cup \{0\}.$$

Отметим, что система собственных функций (15) задачи (11), (12) является полной и ортонормированной в пространстве $L_2(D^-)$ и образует там базис [26].

Решение уравнения (8) имеет вид:

$$X_{n,m}(x) = \begin{cases} C_{1n,m}x^2 + C_{2n,m}x + C_{3n,m}, \quad \lambda_{n,m} = 0, \\ C_{1n,m}e^{-k_{n,m}x} + e^{\frac{1}{2}k_{n,m}x}\left(C_{2n,m}\cos\left(\frac{\sqrt{3}}{2}k_{n,m}x\right) + C_{3n,m}\sin\left(\frac{\sqrt{3}}{2}k_{n,m}x\right)\right),\ \lambda_{n,m} \neq 0, \end{cases} \tag{16}$$

где

$$k_{n,m} = \sqrt[3]{\lambda_{n,m}} = \sqrt[3]{\nu_n + \mu_m} = \sqrt[3]{\left(\frac{n\pi}{q}\right)^2 + \left(\frac{m\pi}{r}\right)^2}.$$

Далее, по постановке задачи B следует, что

$$\lim_{x\to-\infty} X_{n,m}(x) = 0.$$

Следовательно, в (16) необходимо считать, что $C_{1n,m} = 0$. Тогда функция (16) примет вид

$$X_{n,m}(x) = \begin{cases} 0, \quad \lambda_{n,m} = 0, \\ e^{\frac{1}{2}k_{n,m}x}\left(C_{2n,m}\cos\left(\frac{\sqrt{3}}{2}k_{n,m}x\right) + C_{3n,m}\sin\left(\frac{\sqrt{3}}{2}k_{n,m}x\right)\right), \quad \lambda_{n,m} \neq 0. \end{cases} \tag{17}$$

В силу (7), решение задачи

$$u(x,y,z) = \frac{2}{\sqrt{qr}}\sum_{n,m=1}^{+\infty} e^{\frac{1}{2}k_{n,m}x}\left[C_{2n,m}\cos\left(\frac{\sqrt{3}}{2}k_{n,m}x\right) + C_{3n,m}\sin\left(\frac{\sqrt{3}}{2}k_{n,m}x\right)\right] \times \cos\frac{n\pi y}{q}\cos\frac{m\pi z}{r}. \tag{18}$$

Требуя от ряда (18) выполнения краевых условий (3), получим

$$u(0,y,z) = \psi_1(y,z) = \frac{2}{\sqrt{qr}}\sum_{m,n=1}^{+\infty} C_{2n,m}\cos\frac{n\pi y}{q}\cos\frac{m\pi z}{r},$$
$$u_x(0,y,z) = \psi_2(y,z) = \frac{2}{\sqrt{qr}}\sum_{m,n=1}^{+\infty}\frac{1}{2}k_{n,m}\left[C_{2n,m} + \sqrt{3}C_{3n,m}\right]\cos\frac{n\pi y}{q}\cos\frac{m\pi z}{r},$$

где

$$\left.\begin{aligned} C_{3n,m} &= \frac{2}{\sqrt{3}k_{n,m}}\psi_{2n,m} - \frac{1}{\sqrt{3}}\psi_{1n,m}, \\ \psi_{in,m} &= \frac{2}{\sqrt{qr}}\int_0^q\int_0^r \psi_i(y,z)\cos\frac{n\pi y}{q}\cos\frac{m\pi z}{r}dydz,\ i = 1,2, \end{aligned}\right\} \tag{19}$$

$\psi_{1n,m}$ и $\psi_{2n,m}$ - коэффициенты Фурье функций $\psi_1(y,z)$ и $\psi_2(y,z)$.

Поставляя значения $C_{2n,m}$ и $C_{3n,m}$ в ряд (18), получим

$$u(x,y,z) = \frac{2}{\sqrt{qr}}\sum_{n,m=1}^{+\infty} e^{\frac{1}{2}k_{n,m}x}\left[\psi_{1n,m}\frac{2}{\sqrt{3}}\cos\left(\frac{\sqrt{3}}{2}k_{n,m}x + \frac{\pi}{6}\right) + \psi_{2n,m}\frac{2}{\sqrt{3}k_{n,m}}\sin\left(\frac{\sqrt{3}}{2}k_{n,m}x\right)\right] \times \cos\frac{n\pi y}{q}\cos\frac{m\pi z}{r}. \tag{20}$$

Теперь докажем, что ряд (20) и его производные u_{xxx}, u_{yy} и u_{zz} сходятся абсолютно и равномерно в области $D^- \cup \Gamma$, то функция $u(x,y,z)$, определяемая этим рядом, дает решение задачи B.

Докажем абсолютную и равномерную сходимость ряда (20). Из (20) имеем

$$|u(x,y,z)| = \frac{2}{\sqrt{qr}} \sum_{n,m=1}^{+\infty} \left| e^{\frac{1}{2}k_{n,m}x} \left[\psi_{1n,m} \frac{2}{\sqrt{3}} \cos\left(\frac{\sqrt{3}}{2} k_{n,m} x + \frac{\pi}{6} \right) + \psi_{2n,m} \frac{2}{\sqrt{3} k_{n,m}} \sin\left(\frac{\sqrt{3}}{2} k_{n,m} x \right) \right] \times \right.$$
$$\left. \times \cos\frac{n\pi y}{q} \cos\frac{m\pi z}{r} \right| \le \frac{2}{\sqrt{qr}} \sum_{n,m=1}^{+\infty} \left[|\psi_{1n,m}| + \frac{1}{k_{n,m}} |\psi_{2n,m}| \right]. \tag{21}$$

Интегрируя по частям (19) и принимая во внимание условие (4), получим

$$\psi_{in,m} = \left(\frac{qr}{\pi^2} \right)^3 \frac{\psi^{(6)}_{in,m}}{n^3 m^3}, \ i = 1,2, \tag{22}$$

где

$$\psi^{(6)}_{in,m} = \frac{2}{\sqrt{qr}} \int_0^q \int_0^r \frac{\partial^6 \psi_i(y,z)}{\partial y^3 \partial z^3} \sin\frac{n\pi y}{q} \sin\frac{m\pi z}{r} dy dz, \ i = 1,2.$$

Учитывая условия на заданные функции, из (21) и $k_{n,m} = \sqrt[3]{\lambda_{n,m}} = \sqrt[3]{\nu_n + \mu_m} = \sqrt[3]{\left(\frac{n\pi}{q}\right)^2 + \left(\frac{m\pi}{r}\right)^2} \ge \sqrt[3]{\frac{2\pi^2}{qr}} \sqrt[3]{nm}$, имеем

$$|u(x,y,z)| = M_1 \sum_{n,m=1}^{+\infty} \frac{\left|\psi^{(6)}_{1n,m}\right|}{n^3 m^3} + M_2 \sum_{n,m=1}^{+\infty} \frac{\left|\psi^{(6)}_{2n,m}\right|}{n^{\frac{10}{3}} m^{\frac{10}{3}}} < \infty,$$

где $M_i = \dfrac{2\sqrt{(qr)^5}}{\pi^6} = const > 0, \ i = 1,2.$

Отсюда следует, что ряд (20) сходится абсолютно и равномерно.

Теперь докажем, что производные ряда (20) входящий в уравнение (1), также сходятся абсолютно и равномерно в области $D^- \cup \Gamma$. Для этого вычисляем производные по y и по z, из (20) получим

$$\frac{\partial^2 u}{\partial y^2} = -\frac{2}{\sqrt{qr}} \left(\frac{\pi}{q} \right)^2 \sum_{n,m=1}^{+\infty} n^2 e^{\frac{1}{2}k_{n,m}x} \left[\psi_{1n,m} \frac{2}{\sqrt{3}} \cos\left(\frac{\sqrt{3}}{2} k_{n,m} x + \frac{\pi}{6} \right) + \right.$$
$$\left. + \psi_{2n,m} \frac{2}{\sqrt{3} k_{n,m}} \sin\left(\frac{\sqrt{3}}{2} k_{n,m} x \right) \right] \cos\frac{n\pi y}{q} \cos\frac{m\pi z}{r},$$

$$\frac{\partial^2 u}{\partial z^2} = -\frac{2}{\sqrt{qr}} \left(\frac{\pi}{r} \right)^2 \sum_{n,m=1}^{+\infty} m^2 e^{\frac{1}{2}k_{n,m}x} \left[\psi_{1n,m} \frac{2}{\sqrt{3}} \cos\left(\frac{\sqrt{3}}{2} k_{n,m} x + \frac{\pi}{6} \right) + \right.$$
$$\left. + \psi_{2n,m} \frac{2}{\sqrt{3} k_{n,m}} \sin\left(\frac{\sqrt{3}}{2} k_{n,m} x \right) \right] \cos\frac{n\pi y}{q} \cos\frac{m\pi z}{r}.$$

Оценим полученные равенства и учитывая (22), при $x < 0$, имеем

$$\left| \frac{\partial^2 u}{\partial y^2} \right| \le M_3 \sum_{n,m=1}^{+\infty} \frac{\left|\psi^{(6)}_{1n,m}\right|}{nm^3} + M_4 \sum_{n,m=1}^{+\infty} \frac{\left|\psi^{(6)}_{2n,m}\right|}{n^{\frac{4}{3}} m^{\frac{10}{3}}},$$

$$\left|\frac{\partial^2 u}{\partial z^2}\right| \leq M_5 \sum_{n,m=1}^{+\infty} \frac{\left|\psi_{1n,m}^{(6)}\right|}{n^3 m} + M_6 \sum_{n,m=1}^{+\infty} \frac{\left|\psi_{2n,m}^{(6)}\right|}{n^{\frac{10}{3}} m^{\frac{4}{3}}},$$

где $M_3, M_4, M_5, M_6 = const > 0$.

Используя неравенства Коши- Буняковского и Бесселя, получим

$$\left|\frac{\partial^2 u}{\partial y^2}\right| \leq M_3 \sqrt{\sum_{m,n=1}^{+\infty} \left|\psi_{1n,m}^{(6)}\right|^2} \sqrt{\sum_{n,m=1}^{+\infty} \left(\frac{1}{nm^3}\right)^2} + M_4 \sum_{n,m=1}^{+\infty} \frac{\left|\psi_{2n,m}^{(6)}\right|}{n^{\frac{4}{3}} m^{\frac{10}{3}}} \leq \overline{M_3} \left\|\psi_1^{(6)}\right\| + M_4 \sum_{n,m=1}^{+\infty} \frac{\left|\psi_{2n,m}^{(6)}\right|}{n^{\frac{4}{3}} m^{\frac{10}{3}}} < \infty,$$

$$\left|\frac{\partial^2 u}{\partial z^2}\right| \leq M_5 \sqrt{\sum_{m,n=1}^{+\infty} \left|\psi_{1n,m}^{(6)}\right|^2} \sqrt{\sum_{n,m=1}^{+\infty} \left(\frac{1}{n^3 m}\right)^2} + M_6 \sum_{n,m=1}^{+\infty} \frac{\left|\psi_{2n,m}^{(6)}\right|}{n^{\frac{10}{3}} m^{\frac{4}{3}}} \leq \overline{M_5} \left\|\psi_1^{(6)}\right\| + M_6 \sum_{n,m=1}^{+\infty} \frac{\left|\psi_{2n,m}^{(6)}\right|}{n^{\frac{10}{3}} m^{\frac{4}{3}}} < \infty,$$

так как

$$\sum_{n,m=1}^{+\infty} \left|\psi_{1n,m}^{(6)}\right|^2 \leq \left\|\psi_1^{(6)}\right\|^2_{L_2[0<y<q,0<z<r]}, \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \sum_{m=1}^{+\infty} \frac{1}{m^2} = \frac{\pi^2}{6}, \overline{M_3}, \overline{M_5} = const > 0.$$

Следовательно, ряд, соответствующий функции $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial z^2}$ сходится абсолютно и равномерно. Абсолютная и равномерная сходимость третьей производной по x ряда (20) следует из $\left|\frac{\partial^3 u}{\partial x^3}\right| \leq \left|\frac{\partial^2 u}{\partial y^2}\right| + \left|\frac{\partial^2 u}{\partial z^2}\right|$ и доказанного выше.

Список литературы

[1] O. K. Abdullaev, A. A. Matchanova, On a problem for the third order equation with parabolic-hyperbolic operator including a fractional derivative, *Lobachevskii Journal of Mathematics*, Vol.43, 2022 y., pp. 275 - 283.

[2] Андреев А. А., Яковлева Ю. О., Характеристическая задача для системы гиперболических дифференциальных уравнений третьего порядка общего вида с некратными характеристиками, *Вестн. Самар. тех. ун-та, сер.: Физ.-мат. науки*, No.30, 2013 г., -С. 31-36.

[3] O. S. Zikirov, On the Dirichlet problem for third-order hyperbolic equations, *Russ. Math.*, No.58(7), 2014 y., pp. 53 - 60.

[4] Репин О. А., Кумыкова С. К., Задача со сдвигом для уравнения третьего порядка с разрывными коэффициентами, *Вестн. Самар. тех. ун-та, сер.: Физ.-мат. науки*, No.29(4), 2012 г., -С. 17 - 25.

[5] Сабитов К.Б., Задача Дирихле для уравнения смешанного типа третьего порядка в прямоугольной области, *Дифферен. урав.*, No.47, 2011 г., -С. 706 - 714.

[6] Сопуев А., Аркабаев Н.К., Задачи сопряжения для линейных псевдопараболических уравнений третьего порядка, *Вестн. Томск. ун-та, мат. мех*, No.21(1), 2013 г., -С. 16 - 23.

[7] Шхануков М. Х., О некоторых краевых задачах для уравнения третьего порядка, возникающих при моделировании фильтрации жидкости в пористых средах, *Дифференц. урав.*, No.18, 1982 г., -С. 689 - 699.

[8] Юлдашев Т. К., Обратная задача для одного интегро - дифференциального уравнения Фредгольма в частных производных третьего порядка, *Вестн. Сам. гос. техн. ун-та. Сер. Физ.-мат. науки, - Самара*, No.1(34), 2014 г., -С. 56 - 65.

[9] T. K. Yuldashev, On Fredholm partial integro-differential equation of the third order, *Russ. Math.*, No.59(9), 2015 y., pp. 62 - 66.

[10] T. K. Yuldashev, Yu. P. Apakov, A. Kh. Zhuraev, Boundary value problem for third order partial integro-differential equation with a degenerate kernel, *Lobachevskii Journal of Mathematics*, Vol.42, 2021 y., pp. 1317 - 1327.

[11] T. K. Yuldashev, B. I. Islomov, E. K. Alikulov, Boundary-value problems for loaded third-order parabolic-hyperbolic equations in in?nite three-dimensional domains, *Lobachevskii Journal of Mathematics*, Vol.41, 2020 y., pp. 926 - 944.

[12] Рыжов О.С., Асимптотическая картина обтекания тел вращения со звуковым потоком вязкого и теплопроводящего газа, *Прикл. Матем. и механ., - Москва*, 1965. - Т. 29. Вып. 6. - С. 1004 - 1014.

[13] Диесперов В.Н., О функции Грина линеаризованного вязкого трансзвукового уравнения, *Журнал вычисл. матем. и матем. физики. - Москва*, 1972. - Т. 12. - No.5. - С. 1265-1279.

[14] Block H., Sur les equations lineaires aux derives parielles a carateristiques multiples, *Ark. Mat. Astron. Fus. Note 1*, 1912, 7(13), pp. 1-34; Note 2, 1912, ibid. 7(21), pp. 1-30; Note 3, 1912 - 1913, ibid. 8(23). pp. 1-51.

[15] Del Vicchio E., Sulleequazioni $z_{xxx} - z_y + \varphi_1(x, y) = 0$, $z_{xxx} - z_{yy} + \varphi_2(x, y) = 0$, *Memorie R. Accad. Sci. Ser.2. - Torino*, 1915, 66. pp. 1-41.

[16] Cattabriga L., Potenziali di linea e di dominio per equazioni non paraboliche in due variabilia caratteristiche multiple, *Rendiconti del seminario matimatico della univ. di Padava.*, 1961, 31, pp. 1-45.

[17] Джураев Т.Д, Апаков Ю.П., Об автомодельном решении одного уравнения третьего порядка с кратными характеристиками, *Вестник Сам. гос. техн. ун-та. Сер. Физ.-мат. науки, - Самара*, No.2(15), 2007 г., - С. 18 - 26.

[18] Джураев Т.Д, Апаков Ю.П., К теории уравнения третьего порядка с кратными характеристиками, содержащего вторую производную по времени, *Украинский математический журнал. - Киев*, No.1. том 62., 2010 г., - С. 40 - 51.

[19] Апаков Ю.П., К решению краевых задач для уравнения $U_{xxx} - U_{yy} = 0$ в неограниченных областях, *Доклады Академии Наук Республики Узбекистан. -Ташкент*, No.3, 2006 г., -С. 17 - 20.

[20] Апаков Ю.П., Решение краевых задач для уравнения третьего порядка с кратными характеристиками в неограниченных областях, *Известия Кабардино-Балкарского научного центра РАН. - Нальчик*, No.2(22), 2008 г., -С. 147-151.

[21] Апаков Ю.П., Решение краевых задач для уравнения третьего порядка с кратными характеристиками методом Фурье в областях с некомпактными границами, *Узбекский математический журнал. -Ташкент*, No.1, 2008 г., - С. 14-22.

[22] Апаков Ю.П., К теории уравнений третьего порядка с кратными характеристиками, - *T.: "Fan va texnologiya"*, 2019, -156 с.

[23] Апаков Ю.П., Краевые задачи для параболо-гиперболического уравнений в конечной призматической области трехмерного пространства, *Тез. докл. Всесоюзной школы молодых ученых. "Функциональные методы в прикладной математике и математической физике"2-часть, -Ташкент*, ТошДУ.1988. - С. 5.

[24] Апаков Ю.П., О некоторых нелокальных задачах для параболо-гиперболического уравнения в трехмерном пространстве, *"Прямые и обратные краевые задачи математической физики". -Тошкент, Фан,*, 1987, -С. 80-95.

[25] Апаков Ю.П., Хамитов А.А., О решения одной краевой задачи для уравнения третьего порядка с кратными характеристиками в трехмерном пространстве, *Научный вестник Наманганского государственного университета. -Наманган*, No.4, 2020 г., -С. 21-31.

[26] Сабитов К. Б., Начально-граничные задачи для уравнения колебаний прямоугольной пластины, *Известия вузов. Математика*, No.10, 2021 г., -С. 60-70.

[27] Юлдашев Т. К., Об одной краевой задаче для интегро-дифференциального уравнения в частных производных четвертого порядка с вырожденным ядром, *Итоги науки и техн. Сер. Соврем. мат. и ее прил. Темат. обз.*, 2018, том 145, -С. 95–109.

[28] Sidorov S. N., Inverse Problem of Finding a Factor Depending on the Spatial Variables on the Right-Hand Side in a Parabolic-Hyperbolic Equation, *Differential Equations*, Vol. 57, No. 12, 2021, pp 1585-1597.

[29] Yu. P. Apakov, A. A. Hamitov, Third Boundary Value Problem for an Equation with the Third Order Multiple Characteristics in Three Dimensional Space, *Lobachevskii Journal of Mathematics*, Vol. 44, No 2, 2023, pp. 523-532.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ПРИМЕНЕНИЕ СИСТЕМ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ В ЭПИДЕМИОЛОГИИ

БАЛЖАН ЖУМАРТОВА[1], РОЗА ЫСМАГУЛ[1]

[1]КРУ им. А. Байтурсынова, . Костанай, Казахстан
e-mail: balzhan.zhumartova@mail.ru)

Хорошо известно, что математика универсальная наука, невозможно найти область, в которой она не применяется, она также играет важную роль в эпидемиологии, предоставляя инструменты для моделирования и анализа распространения инфекционных заболеваний. Приведем лишь некоторые основные примеры применений математики в эпидемиологии:

- Моделирование эпидемий: математические модели позволяют прогнозировать, как будет распространяться заболевание среди в населения в зависимости от различных факторов, таких как инфекционность, восприимчивость населения, эффективность мер по борьбе с заболеванием и других факторов.

- Анализ данных: эпидемиологические данные часто подвергаются статистическому анализу, который помогает исследователям выявлять связи между различными факторами и возбудителем.

- Оценка эффективности мер по борьбе с заболеванием: математические модели могут использоваться для оценки эффективности мер по борьбе с заболеванием, таких как вакцинация или карантин.

- На основе данных о предыдущих эпидемиях и различных факторах, таких как климатические условия или наличие мутаций вируса, математические модели могут помочь прогнозировать вероятность будущих эпидемий. Также математические модели могут помочь исследователям разрабатывать стратегии борьбы с эпидемиями, такие как оптимальное использование ресурсов, медицинского оборудования или лекарств. В целом, математические методы играют важную роль в анализе и борьбе с инфекционными заболеваниями.

Особое значение в математическом моделировании имеют дифференциальные уравнения и их системы. В целом, системы дифференциальных уравнений имеют широкое применение во многих областях науки. Например, системы дифференциальных уравнений могут использоваться для моделирования численности популяций животных и растений в зависимости от различных факторов, таких как доступность пищи, конкуренция, хищники и климатические условия; системы дифференциальных уравнений могут использоваться для моделирования роста растительности в зависимости от различных факторов, таких как доступность воды и питательных веществ, температура и свет; системы дифференциальных уравнений могут использоваться для моделирования распространения загрязнения окружающей среды в атмосфере, воде и почве. Эти модели могут помочь определить оптимальные стратегии борьбы с загрязнением, такие как использование фильтров и ограничение выбросов; системы дифференциальных уравнений могут использоваться для моделирования динамики экосистем в целом, включая взаимодействия между популяциями животных и растительности, изменения климата и влияние человеческой деятельности на экосистемы, и наконец, системы дифференциальных уравнений могут использоваться для моделирования распространения инфекций в популяциях животных и людей. Эти модели могут

помочь определить оптимальные стратегии противодействия эпидемии, такие как вакцинация и карантин. В данной статье подробнее остановимся в моделировании распространения инфекционных заболеваний и предсказании их динамики в эпидемиологии. Обычно в таких моделях используются системы дифференциальных уравнений, которые описывают взаимодействие между заболевшими, носителями заболевания и здоровыми людьми в определенной популяции. Например, одной из самых известных моделей является модель SIR, которая описывает динамику распространения инфекции среди людей в популяции. В этой модели существуют три группы людей: S (susceptible) – восприимчивые, I (infected) – зараженные, R (recovered) – выздоровевшие. Уравнения, описывающие динамику этих трех групп, выглядят следующим образом:

$$\begin{aligned} \frac{dS}{dt} &= -\beta \cdot S \cdot I \\ \frac{dI}{dt} &= \beta \cdot S \cdot I - \gamma \cdot I \\ \frac{dR}{dt} &= \gamma \cdot I \end{aligned} \tag{1}$$

где β – коэффициент заражаемости, γ – коэффициент выздоровления.

Данная модель имеет свои плюсы и минусы, которые мы рассмотрим ниже:

Плюсы:

(1) Простота: Эта математическая модель может быть легко использована для изучения распространения заболевания в населении. Это позволяет быстро создавать предварительные прогнозы и сценарии распространения заболевания.
(2) Универсальность: Модель может быть применена для изучения широкого спектра инфекционных заболеваний, таких как грипп, коронавирус, эбола и т.д. Она также может быть использована для изучения эпидемических процессов в различных контекстах, таких как различные страны, регионы и т.д.
(3) Предсказательность: Может использоваться для прогнозирования будущего распространения заболевания. Это позволяет правительствам и другим организациям разрабатывать и принимать меры по борьбе с заболеванием заранее.

Минусы:

(1) Упрощенность: Модель упрощает эпидемический процесс, предполагая, что все люди в населении одинаково подвержены заболеванию. На самом деле существуют различные факторы, такие как возраст, пол, здоровье и т.д., которые могут повлиять на риск заражения.
(2) Ограниченность: Модель предназначена только для изучения трех состояний: восприимчивости, инфицирования и выздоровления. Это означает, что она не учитывает другие факторы, такие как смертность и иммунитет.
(3) Необходимость точных данных: Требуются точные данные для функционирования модели. Это означает, что если данные недоступны или неточны, результаты моделирования могут быть неточными.

В целом, SIR -модель является полезным инструментом для изучения распространения инфекционных заболеваний. Зная время выздоровления возможно найти коэффициент заражаемости,здесь применяется дискретная модель SIR.

Также еще один важный пункт, на котором стоит остановиться, и который играет важную роль при интерпретации данных, полученных в результате решения систем дифференциальных уравнений, описывающихся эпидемиологический процесс – это фазовый портрет.

Фазовый портрет системы дифференциальных уравнений показывает, как траектории изменения состояния системы располагаются в фазовом пространстве. Фазовый портрет может быть найден аналитически или графически.

Аналитический метод нахождения фазового портрета заключается в нахождении решений системы дифференциальных уравнений и построении графика траекторий этих решений в фазовом пространстве.

Графический метод нахождения фазового портрета заключается в использовании фазового портрета как инструмента для визуализации и анализа поведения системы. Для этого можно использовать различные методы, такие как метод фазовых траекторий, метод фазовой плоскости, метод фазовых портретов, метод Ляпунова и другие.

Метод фазовых траекторий заключается в построении графика траекторий изменения состояния системы во времени. Метод фазовой плоскости используется для визуализации фазового портрета в двумерном пространстве, показывая расположение и форму траекторий изменения состояния системы. Метод фазовых портретов используется для анализа более сложных систем, где требуется построение фазовых портретов в трехмерном пространстве. Метод Ляпунова используется для анализа устойчивости и неустойчивости фазовых траекторий и нахождения точек равновесия системы.

В модели SIR фазовый портрет может помочь предсказать поведение инфекции в зависимости от начальных условий и параметров модели. Например, фазовый портрет может показать, как изменится количество людей в каждой из трех групп в зависимости от уровня заражаемости и скорости выздоровления.

Еще одна интересная модель, на которой хочется остановиться – это модель «хищник-жертва». Она используется для описания динамики взаимодействия между двумя видами в экосистеме: хищниками и их жертвами. В этой модели предполагается, что численность хищников зависит от количества доступных жертв, а численность жертв зависит от численности хищников. Обычно модель хищник-жертва описывается с помощью системы дифференциальных уравнений, где одно уравнение описывает изменение численности жертв со временем, а другое уравнение описывает изменение численности хищников со временем. В общем виде эта модель выглядит следующим образом:

$$\begin{aligned} \frac{dN}{dt} &= rN - cNP \\ \frac{dP}{dt} &= acNP - bP \end{aligned} \tag{2}$$

где N - численность жертв, P - численность хищников, r - скорость рождаемости жертв, c - скорость убыли жертв от хищников, a - эффективность превращения жертвы в хищника, b - скорость убыли хищников. Модель хищник-жертва позволяет исследовать динамику взаимодействия между хищниками и жертвами в экосистеме, а также предсказывать, как изменения в одном из параметров могут повлиять на численность обеих популяций. Она может быть полезна для понимания эволюции живых организмов и поддержания экологического баланса в природе.

Интерес фазового портрета модели «хищник-жертва» заключается в том, что он позволяет легко визуализировать и анализировать динамику взаимодействия между хищниками и жертвами. В частности, фазовый портрет может показать, как изменение параметров модели может влиять на стабильность или нестабильность экосистемы. Например, если фазовый портрет показывает устойчивую точку равновесия, то это означает, что экосистема может существовать в относительной стабильности, когда численность хищников и жертв остается примерно постоянной. Однако, если фазовый портрет показывает неустойчивую точку равновесия, то малейшие изменения в численности хищников или жертв могут привести к необратимым изменениям в экосистеме. Система (2) имеет единственную точку покоя:

$$(N_0, P_0) = (\frac{b}{ac}, \frac{r}{c}) \tag{3}$$

На рисунке 1 представлен фазовый портрет данной модели, нарисованный в Python-е, рассматриваем только первую четверть координатной плоскости, так значения не могут быть отрицательными.

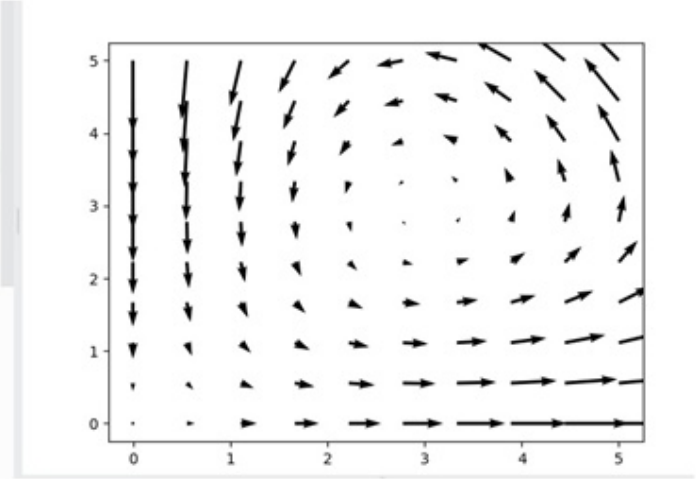


Рис. 1. Фазового портрета модели «хищник-жертва»

В данном случаем, мы в качестве примера взяли следующие параметры: b=6, ac=2, r=6, c=2. Видно, что центра портрета, точка с координатами (3;3) соответствует формуле (3), если подставить значения. Это значит, что данная точка является устойчивым циклом системы. Это означает, что система будет двигаться вокруг центра вдоль замкнутой кривой (цикла) без изменения ее формы. Таким образом, фазовый портрет данной модели является важным инструментом для анализа и понимания динамики взаимодействия между хищниками и жертвами в экосистеме. Он может помочь ученым предсказывать, какие изменения в окружающей среде и в поведении живых организмов могут привести к изменениям в экосистеме и как эти изменения могут повлиять на жизнь других видов.

Но кроме перечисленного, данную модель можно применить для развития инфекции, точно для описания взаимосвязи зараженного и патогена, поэтому данную модель можно модифицировать для применения в эпидемиологии, ведь идёт «борьба» между организмом человека, животного и возбудителем, вирусом.

В данном случае модель «хищник-жертва» примет следующий вид:

$$\begin{aligned} \frac{dS}{dt} &= -\beta \cdot S \cdot I \\ \frac{dI}{dt} &= \beta \cdot S \cdot I - \gamma \cdot I \end{aligned} \tag{4}$$

здесь S - количество подтвержденных риску заразиться людей, I - количество инфицированных,β – коэффициент инфицирования, γ – коэффициент заболеваемости.

Первое уравнение описывает изменение числа подверженных риску заразиться людей, уменьшающееся с течением времени за счет заболевания или выздоровления. Второе уравнение описывает изменение числа инфицированных людей, которые могут выздороветь или умереть.

Надо отметить, что кроме вышеупомянутых моделей, существует множество других моделей, использующих дифференциальные уравнения, для описания распространения инфекционных заболеваний в различных условиях. Эти модели позволяют анализировать различные аспекты эпидемиологии, такие как влияние мер по борьбе с инфекцией на ее динамику, оптимизацию стратегий вакцинации и т.д. Одну из таких моделей, состоящую из двух нелинейных дифференциальных уравнений мы рассмотрим далее.

Итак, наша задача сводится к нахождению первого интеграла следующей системы нелинейных дифференциальных уравнений (5), которая задает общую модель эпидемии:

$$\begin{aligned} \frac{dx}{dt} &= -\ 2 \cdot x \cdot y \\ \frac{dy}{dt} &= \ 2 \cdot x \cdot y - y \end{aligned} \tag{5}$$

где x - число восприимчивых к болезни, y - число заболевших в соответствующих масштабах. Решаем данную систему следующим образом:

$$\frac{dy}{dt} \ \div \frac{dx}{dt} = \frac{2 \cdot x \cdot y - y}{-\ 2 \cdot x \cdot y} \tag{6}$$

$$\frac{dy}{dx} = -1 + \frac{1}{2 \cdot x} \tag{7}$$

$$dy = (-1 + \frac{1}{2 \cdot x})dx \tag{8}$$

$$y = -x + \frac{1}{2} lnx + c_1 \tag{9}$$

Таких образом пока мы нашли только первый интеграл (9), который для системы дифференциальных уравнений имеет важное значение в динамической теории систем. Первый интеграл представляет собой функцию, которая сохраняется в течение времени эволюции системы. Таким образом, нахождение первого интеграла может помочь в анализе поведения системы и понимании ее динамики.

Кроме того, нахождение первого интеграла может помочь в решении системы дифференциальных уравнений аналитически, что может быть полезно при проведении исследований в научных областях, таких как физика, математика, механика и других. Также нахождение первого интеграла может помочь в упрощении системы дифференциальных уравнений, что позволяет существенно ускорить вычислительные процессы при численном решении системы. В целом, нахождение первого интеграла для системы дифференциальных уравнений позволяет получить глубокое понимание динамики системы, аналитически решить систему и упростить ее численное решение.

Итак, из формулы (9) видно, что число восприимчивых убывает, а число заболевших достигает некого максимума и убывает до нуля. Что отражает реальное протекание любой эпидемии. Вычислим, при каком же число зараженных достигает максимума.

Используя начальные условия, а именно, что в начальный момент времени возникновения вирусного заболевания, $x = 1$ так как всё население является восприимчивым, и $y = 0$ заболевшие отсутствуют. Значит, мы имеем следующее c_1:

$$y = -x + \frac{1}{2} lnx + c_1 = 0 \Rightarrow c_1 = 1 \tag{10}$$

Формула (9), примет следующий вид:

$$y = -x + \frac{1}{2} lnx + 1 \tag{11}$$

А дальше легко найти максимум:

$$y' = -1 + \frac{1}{2x} \tag{12}$$

Далее приравняем производную к нулю:

$$-1 + \frac{1}{2x} = 0 \tag{13}$$

$$x = \frac{1}{2} \tag{14}$$

Руководствуясь формулой (9) имеем:

$$y_max = -\frac{1}{2} + \frac{1}{2}ln\frac{1}{2} + c_1 = c_1 - \frac{1}{2}(1 + ln2) \tag{15}$$

Имеем, что число заболевших достигает максимума при (15), при$x = 1/2$, и где c_1 - общее число больных и восприимчивых при x=1. То есть пик эпидемия приходится ровно на тот момент, когда половина популяции уже поражены.

Далее на рисунках 2,3 представлен фазовый портрет системы (5) при $x, y \geq 0$ с разным масштабом и шагом.

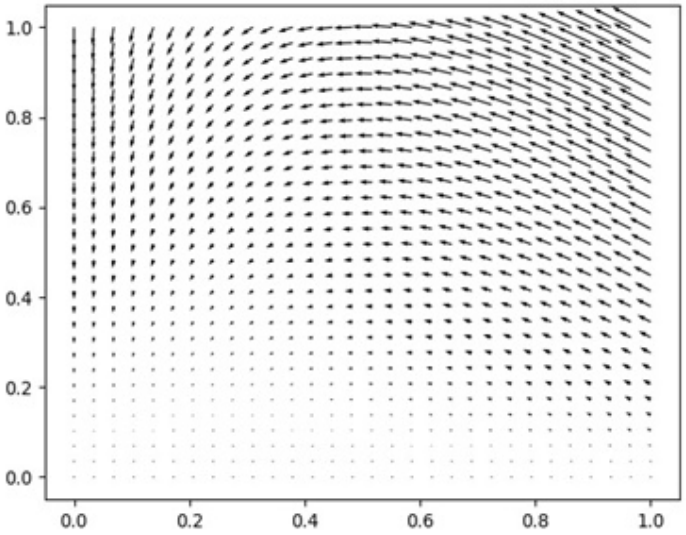


Рис. 2. Фазовый портрет системы (5) с шагом 0,2

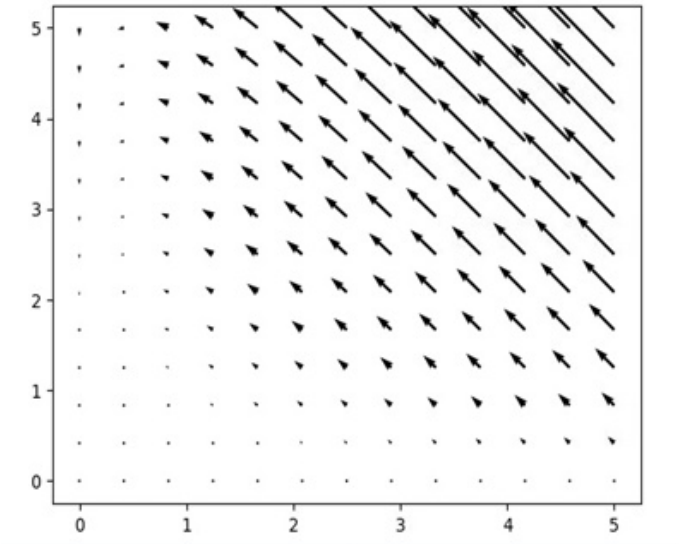


Рис. 3. Фазовый портрет системы (5) с шагом 1

В последние годы возрастает интерес к применению дифференциальных уравнений в эпидемиологии в связи с появлением новых инструментов и технологий для сбора и анализа данных. Одним из основных преимуществ использования дифференциальных уравнений в эпидемиологии является возможность моделирования сложных систем и взаимодействий между различными факторами. Это позволяет ученым исследовать не только сам процесс распространения заболевания, но и его влияние на популяцию. Однако, использование дифференциальных уравнений в эпидемиологии также имеет свои ограничения. Например, многие модели основаны на упрощенных предположениях о поведении популяции и могут не учитывать все факторы, влияющие на распространение заболевания. Кроме того, точность прогнозов может быть ограничена недостаточной информацией о популяции и заболевании.

Тем не менее, развитие новых методов и технологий в области сбора и анализа данных может улучшить прогнозирование и более точно описать процессы в популяции. В целом, использование дифференциальных уравнений в эпидемиологии имеет большой потенциал для более эффективного контроля и управления заболеваниями.

Список литературы

[1] Д.Эрроусмит, К. Плейс, Обыкновенные дифференциальные уравнения. Качественая теория с приложением: Пер. с англ. – М.:Мир, 1986. – 243 с.

[2] Жумартова Б.О., Ысмагул Р.С. Применение SIR модели в моделировании эпидемий // Международный журнал гуманитарных и естественных наук. – 2021. – № 12-2 (63). Новосибирск. – С.6-8.

[3] Сергеев И.Н, Дифференциальные уравнения. – М.:Академия, 2013. – 288 с.

[4] Братусь А.С., Новожилов А.С., Платонов А.П. Динамические системы и модели биологии / А.С. Братусь, А.С.Новожилов, А.П.Платонов. – М.: Физматлит, 2010. – 436 с.

[5] Математическая модель безыммунной эпидемии Жумартова Б.О.// 52-54 с. «Ломоносов – 2023»: XVIII Международная научная конференция студентов, магистрантов и молодых ученых: материалы конференции в 2-х частях. I часть. – Астана: Казахстанский филиал МГУ имени М.В.Ломоносова, 2023. – 214 с.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

РЕАЛИЗАЦИЯ МЕТОДА МАЛОГО ПАРАМЕТРА ДЛЯ ИССЛЕДОВАНИЯ МНОГОПЕРИОДИЧЕСКИХ РЕШЕНИЙ СИСТЕМ С ОПЕРАТОРОМ ДИФФЕРЕНЦИРОВАНИЯ ПО ДИАГОНАЛИ

ЖАЙШЫЛЫК САРТАБАНОВ

Актюбинский региональный университет имени К.Жубанова, Актобе, Казахстан
e-mail: sartabanov-42@mail.ru

В данном докладе представлен метод малого параметра для исследования многопериодических решений систем с оператором дифференцирования по диагонали. Он основан на идеи метода Пуанкаре по изучению периодических решений систем обыкновенных дифференциальных уравнений. До сего времени представляемый метод был применим только к вопросам многопериодических решений систем с указанным оператором дифференцирования, когда соответсвующие линейные системы обладали свойством дихотомичности [1-6].

Такое существенное обобщение стало возможным в связи с введением периодических характеристик оператора дифференцирования по диагонали путем рассмотрения характеристических систем на цилиндрической поверхности пространства временных переменных. Таким образом, в качестве поля дифференцирования по диагонали 1) взяв многообразие вида евклидовой плоскости имели линейные δ-характеристики, паралельные диагонали плоскости временных переменных и 2) взяв многообразие вида цилиндрической поверхности имели характеристики вида винтовой линии. Эти два подхода эквивалентные, но их следует использовать исходя из их удобности при изучении задач.

Например, при изучении вопросов устойчивость решений систем, в основном, следует пользоваться δ-характеристическим методом. В случае исследования задач, связанных с многопериодичностью решений более целесообразным является β- характеристический метод. В этом убеждаемся при распространении классического метода малого параметра Пуанкаре на случай многопериодических решений систем с операторами дифференцирования по направлениям векторных полей, в частности, вдоль главной диагонали пространство временных переменных.

В связи с этим рассмотрим систему с малым параметром μ вида

$$Dx = f(\tau, t, x, \mu) \tag{1}$$

с оператором дифференцирования по диагонали

$$D = \frac{\partial}{\partial \tau} + \left\langle e, \frac{\partial}{\partial t} \right\rangle,$$

где $e = (1, ..., 1)$-m-вектор, $\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t_1}, ..., \frac{\partial}{\partial t_m}\right)$- вектор, $\left\langle e, \frac{\partial}{\partial t} \right\rangle$ - скалярное произведение векторов e и $\frac{\partial}{\partial t}$; $\tau \in (-\infty, +\infty) = R$, $t = (t_1, ..., t_m) \in R \times ... \times R = R^m$, $x = (x_1, ..., x_n) \in R \times ... \times = R^n$; малый параметр $\mu \in (0, \mu^0) = M_0$, $\mu^0 - const > 0$; $f(\tau, t, x, \mu) = (f_1(\tau, t, x, \mu), ..., f_n(\tau, t, x, \mu))$ - вектор-функция, обладающая свойствами непрерывности по

всем аргументам τ, t, x, μ, аналитичности по t, x, μ и (θ, ω)-периодичности по (t, τ) с рационально несоизмеримыми периодами $\omega_0 = \theta, \omega_1, ..., \omega_m$, при этом $\omega = (\omega_1, ..., \omega_m)$. Предполагается выполненным условие продолжимости решений системы (1) на всю её область определения $R \times R^m$.

В качестве поля дифференцирования оператора D берем цилиндрическую поверхность $(\tau, t) \in R \times T^m = M$, где $T^m = S_\theta \times ... \times S_\theta$, S_θ-окружность длины $2\pi r = \theta$ с центром $(0, r)$.

Рассматривая характеристическую систему $\frac{dt}{d\tau} = e$ на этой поверхности определим её характеристику $\beta(\tau, \tau^0, t^0) = \beta(\tau + \theta, \tau^0, t^0) = \beta(\tau, \tau^0 + \theta, t^0)$ с начальной точкой $(\tau^0, t^0) \in M$. Очевидно, что $\beta(\sigma, \tau, t)$ с парметром $\sigma \in R$ является первым интегралом характеристической системы.

Задача заключается в отыскании (θ, ω)-периодических по (τ, t) решений системы (1):

$$x = \varphi(\tau, t, \mu) \equiv \varphi(\tau + \theta, t + \omega, \mu) \tag{2}$$

в предположении, что система

$$Dx^0 = f(\tau, t, x^0, 0) \equiv f^0(\tau, t, x^0) \tag{3}$$

допускаем единственное (θ, ω) - периодическое решение

$$x^0 = \varphi^0(\tau, t) \equiv \varphi(\tau + \theta, t + \omega) \tag{4}$$

И показать, что решение (2) обладает свойством

$$lim_{\mu \to 0} \varphi(\tau, t, \mu) = \varphi^0(\tau, t) \tag{5}$$

С этой целью рассмотрим решение x системы (1) вида

$$x = x(\tau, t, u(\beta(0, \tau, t), \mu), \mu) \tag{6}$$

определенное начальным условием

$$x(0, t, u(\beta(t, \mu), \mu)) = \varphi^0(0, t) + u(t, \mu) \tag{6^0}$$

и выберем возмущение начальной функции $u(t, \mu)$ так, чтобы решение (6) оказалось (θ, ω)-периодическим.

Чтобы показать решение (6)-(6^0) обладает свойствами (2)-(5) предварительно доказываются следующие теоремы.

Теорема 1 (об условии многопериодичности решений системы (1)).

Решение $x(\tau, t, v(\beta(\tau^0, \tau, t), \mu), \mu)$ с начальным условием $x(\tau^0, t, v(t, \mu), \mu) = v(t, \mu)$ было (θ, ω)- периодическим необходимо и достаточно, чтобы начальная функция $v(t, \mu)$ была ω-периодическим дифференциальным решением функционального уравнения

$$v(t, \mu) = x(\tau^0 + \theta, t, v(t, \mu), \mu) \tag{$*$}$$

Теорема 2 (О периодических неявных функциях).

Предположим что вектор-функция $\Phi(t, u, \mu)$ при $(t, u, \mu) \in R^m \times R^n \times M_0$ удовлетворяет условиям

$$\Phi(t, u^0(t), 0) = 0, det \frac{\partial \Phi(t, u^0(t), 0)}{\partial u} \neq 0, \tag{$**$}$$

где $\Phi(t, u, \mu)$ и $u^0(t)$-аналитические по всем аргументам ω-периодические по t функции. Тогда уравнение

$$\Phi(t, u, \mu) = 0$$

при достаточно малом μ определяет единственную аналитическую ω-периодическую по t функцию $u = u(t, \mu)$, удовлетворяющую условию $u(t, 0) = u^0(t)$.

Далее, на основе решения (6)-(6^0) и условия (*) составляется уравнение $\psi = x(\theta, t, u(t,\mu),\mu) - x(0,t,u(t,\mu),\mu) = 0$ которое обращается в тождество при $u = \varphi^0(0,t)$ и $\mu = 0$ и требуется выполнение условия

$$det\frac{\partial\psi}{\partial u}\Big|_{\substack{u=\varphi^0(0,t)\\ \mu=0}} \neq 0 \qquad (***)$$

Затем пользуясь условием (**), а следовательно, условием (***) доказывается следующая теорема 3.

Теорема 3 (обобщение теоремы Пуанкаре).

Если выполнено условие (***), то система (1) допускает единственное решение (2), обладающее свойством (5) для решения (4) уравнения (3).

Таким образом, здесь реализована идея метода Пуанкаре [7] для периодических решений систем обыкновенных дифференциальных уравнений на систему многопериодических уравнений с D - оператором и опровергнуто утверждение [8, стр.232] о неприменимости этого метода к задачам о почти-периодических решениях, так как оператор D вдоль β - характеристик переходит в обыкновенное дифференцирование.

Также заметим, что реализуя идею теоремы 1 в работе [4] на основе δ-характеристического подхода вместо системы (*) была получена функциональная система уравнений в конечных разностях. Естественно, что исследования вопроса существования дифференцируемых ω-периодических решений таких систем относится к разряду достаточно сложных задач. В данном случае на основе периодических β-характеристик получена система обычных функциональных уравнений. Для исследования решений пространства гладких периодических функций можно пользоваться различными широко известными методами функционального анализа, отличными от приведенного здесь способа.

Работа выполнена при финансовой поддержке Министерства науки и высшего образования Республики Казахстан, грант №AP 19676629.

Ключевые слова: многопериодичность, оператор дифференцирования, метод характеристик, векторное поле.

Предметная классификация AMS: 35E15, 17B66.

Список литературы

[1] Харасахал В.Х., *Почти-периодические решения обыкновенных дифференциальных уравнений*, Алма-Ата: Наука, 1970, 200 р.

[2] Умбетжанов Д.У., *Почти многопериодические решения дифференциальных уравнений в частных производных*, Алма-Ата, 1979, 211 р.

[3] Умбетжанов Д.У., *Почти периодические решения эволюционных уравнений*, Алма-Ата:Наука, 1990, 184 р.

[4] Умбетжанов Д.У., Сартабанов Ж.А., О необходимом и достаточном условии многопериодических решения одной системы уравнений в частных производных с одинаковый главный частью, *Мат. И мех., Алма-Ата*, Vol.2, No.7, 1972, pp.22-27.

[5] Мухамбетова А.А., Сартабанов Ж.А., *Устойчивость решений систем дифференциальных уравнений с многомерным временем*, Актобе: Принт-А, 2007, 168 р.

[6] Кульжумиева А.А., Сартабанов Ж.А., *Периодические решения систем дифференциальных уравнений с многомерным временем*, Уральск: РИЦ ЗКГУ, 2013, 167 р.

[7] Малкин И.Г., *Методы Ляпунова и Пуанкаре в теории нелинейных колебаний*, М.: ГИТТЛ, 1949, 243 р.

[8] Малкин И.Г., *Некоторые задачи теории нелинейных колебаний*, М.:ГИТТЛ, 1956, 491 р.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

МУЛЬТИСТАБИЛЬНОСТЬ В НЕГЛАДКИХ СИСТЕМАХ

ЖАНЫБАЙ ЖУСУБАЛИЕВ[1,2], АЙТИБЕК АБДИРАСУЛОВ[2], УЛАНБЕК СОПУЕВ [2]

[1] Кафедра вычислительной техники, Международная научная лаборатория динамики негладких систем, Юго-Западный государственный университет, Курск, Россия
[2] Факультет математики и информационных технологий, Ошский государственный университет, Ош, Кыргызстан
e-mail: zhanybai@gmail.com, aitibek.osh@gmail.com, ulansopuev@mail.ru

1. Введение

Мультистабильность или одновременная устойчивость нескольких режимов — широко распространенное явление в нелинейных динамических системах [1–3]. Она проявляется в том, что в при одних и тех же значениях параметров существуют несколько устойчивых режимов.

Как известно, глобальная устойчивость является редким свойством нелинейных систем, поскольку из этого свойства вытекает, что существует единственный аттрактор, который притягивает все траектории фазового пространства. Реальные же системы демонстрируют мультистабильное поведение.

Особенностью мультистабильных систем является высокая их чувствительность к помехам: сколь угодно малые случайные помехи или вариации параметров могут приводить к непрогнозируемым изменениям динамики.

В данной работе исследутся особый тип мультистабильности в негладких отображениях, когда сосуществуют несколько вложенных друг в друга притягивающих замкнутых инвариантных кривых, отвечающих устойчивым двухчастотным колебаниям. Изначально этот феномен был обнаружен авторами работ [4–6] в конкретном классе моделей реальных физических систем и долгое время считался экзотическим примером динамического поведения. В [7] на примере кусочно-линейного непрерывного отображения показано, что этот вид мультистабильности типичное свойство широкого класса негладких динамических систем (см. также [8]).

Напомним, что негладкие отображения возникают при моделировании многих задач теории управления, механики, физики, биологии и медицины. Такие отображения «сшиваются» из отдельных гладких функций, области определения которых разделены так называемыми многообразиями переключения. При вариации параметров инвариантное множество, например, неподвижная/периодическая точка дискретной системы сталкивается с одним из многообразий переключения (бифуркации «граничного столкновения», «border collision bifurcations») [9, 10].

Это приводит к нарушению условий существования неподвижной/периодической точки и, как следствие, вызывает необычные нелинейные явления, например, непрерывный переход от одного типа периодического движения к другому, «умножение» периода колебаний, рождение из неподвижной точки квазипериодических или хаотических режимов [11–13].

Бифуркации «граничного столкновения» [9]) не имеют аналогов в гладких системах и не связаны с нарушением условия гиперболичности неподвижной/периодической точки, поэтому они не поддаются анализу методами классической теории бифуркаций.

Основная цель представленной статьи – изучение механизмов рождения и трансформаций вложенных друг в друга замкнутых инвариантных кривых в негладких отображениях с большим числом многообразий переключения.

2. Постановка задачи

Рассмотрим дифференциальное уравнение с разрывной правой частью

$$\dot{x} = \lambda_1(x - S(t,x,y)); \quad \dot{y} = \lambda_2(y - S(t,x,y)); \quad S(t,x,y) = S(t+1,x,y). \tag{1}$$

Здесь $S(t,x,y)$ – кусочно-постоянная функция:

$$S = \frac{S_1 + S_2 + S_3 + S_4}{4},$$
$$S_1 = \operatorname{sign}(\varphi - \eta_1),\ S_2 = \operatorname{sign}(\varphi + \eta_1),\ S_3 = \operatorname{sign}(\varphi - \eta_2),\ S_4 = \operatorname{sign}(\varphi + \eta_2),$$
$$\eta_1 = \frac{q}{\alpha\Gamma}(t - \lfloor t \rfloor),\ \eta_2 = \frac{q}{\alpha\Gamma}(1 - t + \lfloor t \rfloor),\ \varphi = \frac{q}{\Gamma} + x(t) - \vartheta y(t)\Big|_{t=\lfloor t \rfloor},\ \vartheta = \lambda_1/\lambda_2,$$

где $\lfloor\cdot\rfloor$ – функция, выделяющая целую часть аргумента. Это уравнение является математической моделью нагревательной установки для выращивания монокристаллов сапфира. Физический смысл переменных x, y, S и параметров λ_1, λ_2, q, α, Γ пояснен в [14]. Несложно доказать, что скользящих решений в (1) не может быть.

Дифференциальное уравнение (1) сводится к негладкому непрерывному отображению $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ вида

$$F : (x,y) \mapsto F(x,y), \quad F(x,y) = \begin{cases} F_{\mathcal{L}}(x,y), & (x,y) \in S_{\mathcal{L}}; \\ F^{(1)}_{\mathcal{M}}(x,y), & (x,y) \in S^{(1)}_{\mathcal{M}}; \\ F^{(2)}_{\mathcal{M}}(x,y), & (x,y) \in S^{(2)}_{\mathcal{M}}; \\ F^{(3)}_{\mathcal{M}}(x,y), & (x,y) \in S^{(3)}_{\mathcal{M}}; \\ F^{(4)}_{\mathcal{M}}(x,y), & (x,y) \in S^{(4)}_{\mathcal{M}}; \\ F_{\mathcal{R}}(x,y), & (x,y) \in S_{\mathcal{R}}. \end{cases} \tag{2}$$

Здесь $F(x,y)$ кусочно-гладкая вектор-функция. Не ограничивая общности, будем рассматривать конкретное отображение, относящееся к (2), в котором

$$F_{\mathcal{L}}(x,y) = \begin{bmatrix} \mathrm{e}^{\lambda_1}(x-1) + 1 \\ \mathrm{e}^{\lambda_2}(y-1) + 1 \end{bmatrix}, \quad F^{(1)}_{\mathcal{M}}(x,y) = \begin{bmatrix} \mathrm{e}^{\lambda_1}(x-1) + \frac{1}{2}\mathrm{e}^{\lambda_1(2-z)} + \frac{1}{2} \\ \mathrm{e}^{\lambda_2}(y-1) + \frac{1}{2}\mathrm{e}^{\lambda_2(2-z)} + \frac{1}{2} \end{bmatrix},$$

$$F^{(2)}_{\mathcal{M}}(x,y) = \begin{bmatrix} \mathrm{e}^{\lambda_1}(x-1/2) + \frac{1}{2}\mathrm{e}^{\lambda_1(1-z)} \\ \mathrm{e}^{\lambda_2}(y-1/2) + \frac{1}{2}\mathrm{e}^{\lambda_2(1-z)} \end{bmatrix}, \quad F^{(3)}_{\mathcal{M}}(x,y) = \begin{bmatrix} \mathrm{e}^{\lambda_1}(x+1/2) - \frac{1}{2}\mathrm{e}^{\lambda_1(1+z)} \\ \mathrm{e}^{\lambda_2}(y+1/2) - \frac{1}{2}\mathrm{e}^{\lambda_2(1+z)} \end{bmatrix},$$

$$F^{(4)}_{\mathcal{M}}(x,y) = \begin{bmatrix} \mathrm{e}^{\lambda_1}(x+1) - \frac{1}{2}\mathrm{e}^{\lambda_1(2+z)} - \frac{1}{2} \\ \mathrm{e}^{\lambda_2}(y+1) - \frac{1}{2}\mathrm{e}^{\lambda_2(2+z)} - \frac{1}{2} \end{bmatrix}, \quad F_{\mathcal{R}}(x,y) = \begin{bmatrix} \mathrm{e}^{\lambda_1}(x+1) - 1 \\ \mathrm{e}^{\lambda_2}(y+1) - 1 \end{bmatrix},$$

$$z = \frac{2\alpha\Gamma}{q}\varphi, \quad \varphi = \frac{q}{\Gamma} + x - \vartheta y, \quad \vartheta = \lambda_1/\lambda_2$$

и области определения функций $F_{\mathcal{L}}(x,y)$; $F^{(i)}_{\mathcal{M}}(x,y)$, $i=1,2,3,4$; $F_{\mathcal{R}}(x,y)$:

$$S_{\mathcal{L}}=\left\{(x,y):\frac{q}{\Gamma}+x-\vartheta y>\frac{q}{\alpha\Gamma}\right\},\ S^{(1)}_{\mathcal{M}}=\left\{(x,y):\frac{q}{2\alpha\Gamma}\leqslant\frac{q}{\Gamma}+x-\vartheta y\leqslant\frac{q}{\alpha\Gamma}\right\},$$
$$S^{(2)}_{\mathcal{M}}=\left\{(x,y):0\leqslant\frac{q}{\Gamma}+x-\vartheta y<\frac{q}{2\alpha\Gamma}\right\},\ S^{(3)}_{\mathcal{M}}=\left\{(x,y):-\frac{q}{2\alpha\Gamma}\leqslant\frac{q}{\Gamma}+x-\vartheta y<0\right\},$$
$$S^{(4)}_{\mathcal{M}}=\left\{(x,y):-\frac{q}{\alpha\Gamma}\leqslant\frac{q}{\Gamma}+x-\vartheta y<-\frac{q}{2\alpha\Gamma}\right\},\ S_{\mathcal{R}}=\left\{(x,y):\frac{q}{\Gamma}+x-\vartheta y<-\frac{q}{\alpha\Gamma}\right\}.$$

Каждая из функций $F_{\mathcal{L}}(x,y)$, $F^{(i)}_{\mathcal{M}}(x,y)$, $i=1,2,3,4$ и $F_{\mathcal{R}}(x,y)$ – класса C^1 и монотонна во всей своей области определения $S_{\mathcal{L}}$; $S^{(i)}_{\mathcal{M}}$, $i=1,2,3,4$; $S_{\mathcal{R}}$. Границы этих областей называют многобразиями переключения, где производные первого порядка $F(x,y)$ по x и y не существуют.

Отображения вида (2) описывают двухчастотные колебания в системах с мультистабильным поведением (см., например, [8]).

Двухчастотные режимы широко распространены в природе и технике. Такие колебания характеризуются двумя независимыми частотами. В фазовом пространстве динамической системы (2) двухчастотным колебаниям отвечает аттрактор в форме замкнутой инвариантной кривой. Характер движения на инвариантной кривой определяется числом вращения, которое представляет собой отношение частот. Когда оно иррационально, инвариантная кривая плотно заполняется траекториями и динамика квазипериодична.

При рациональном числе вращения на замкнутой кривой лежит пара периодических орбит: устойчивая и седловая, а сама инвариантная кривая образована неустойчивыми многообразиями седловой орбиты. Это относится к случаю захвата частоты, когда происходит синхронизация колебаний с двумя независимыми частотами в одно периодическое колебание с общим периодом. Причем отношение частот постоянно и рационально на интервале значений параметров.

Рассматриваемое отображение является дискретной моделью импульсной системы управления температурным полем нагревательной установки. Исследования проводились при следующих значениях параметров: $\lambda_1=-0.3125$, $\lambda_2=-0.03125$, $q=13.727882855$, $\Gamma=7.0$; $19.0<\alpha<21.0$.

3. Бифуркационный анализ

На рис. 1 изображена бифуркационная диаграмма, рассчитанная численно для $19.0<\alpha<21.0$ и $\Gamma=7.0$. Диаграмма иллюстрирует основные стадии рождения и трансформаций сосуществующих аттракторов.

Бифуркационная диаграмма на рис. 1 содержит несколько ветвей. Ветвь, которая начинается с неподвижной точки, назовем основной. При увеличении α возникает устойчивая замкнутая инвариантная кривая через суперкритическую бифуркацию Неймарка-Сакера в точке α_{NS} из теряющей устойчивость неподвижной точки (см. первую ветвь на рис. 1).

Вторая ветвь возникает жестко через седло-узловую квазипериодическую бифуркацию [15] в точке $\alpha^1_{\mathrm{QFlip}}$, при которой рождается пара замкнутых инвариантных кривых – устойчивая и неустойчивая. До точки бифуркации Неймарка-Сакера α_{NS} устойчивая инвариантная кривая сосуществует с устойчивой неподвижной точкой и другим аттрактором, который в зависимости от параметров может быть периодическим, квазипериодическим или хаотическим.

На рис. 2(а) показан фазовый портрет отображения для значения α, правее от точки бифуркации Неймарка-Сакера α_{NS}. Фазовый протрет иллюстрирует случай сосуществования трех аттракторов, два из которых – устройчивые замкнутые кривые, а третий – устойчивая периодическая орбита с периодом 15. Границей бассейнов притяжения сосуществующих инвариантных кривых является неустойчивая замкнутая кривая с периодической динамикой. Для наглядности фазовые портреты приводятся в исходных переменных

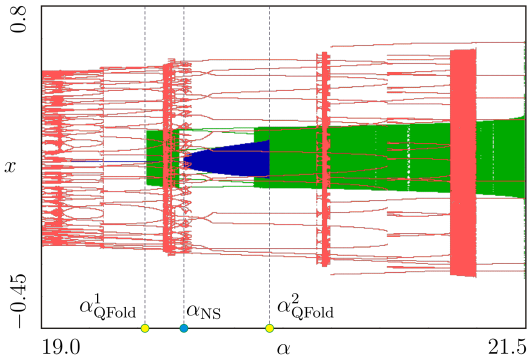
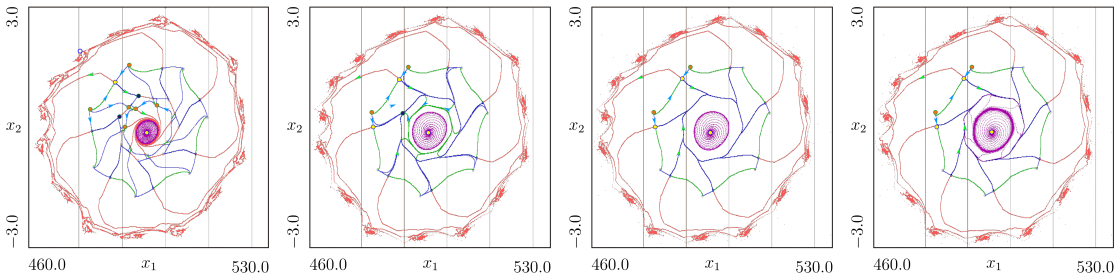


Рис. 1. Бифуркационная диаграмма при $\Gamma = 7.0$.



(а) (б) (в) (г)

Рис. 2. (а) Фазовый портрет для зоны резонанса после бифуркации Неймарка-Сакера, $\alpha = 19.87$. (б) Фазовый портрет при $\alpha = 20.0$. (в) Фазовый портрет после седло-узловой бифуркации «граничного столкновения» точек устойчивого и седлового 7-циклов, лежащих на замкнутой кривой, $\alpha = 20.12$, (г) Фазовый портрет после квазипериодической седло-узловой бифуркации, $\alpha = 20.176165$.

$x_1 = -254.95 \cdot (x - \vartheta y), \quad x_2 = 7.967361 \cdot (x - y)$. Здесь x_1– температура нагревательной установки, x_2 – производная x_1 по времени t. Пять вертикальные линии на фазовых портретах отвечают многобразиям переключения.

Из рис. 2(а) видно, что бассейны притяжения замкнутой резонансной инвариантной кривой и устойчивого 15-цикла (обозначен светло-малиновым цветом на рис. 2(а)-(г)), разделены неустойчивой замкнутой кривой, которая образована устойчивыми многообразиями седлового 7- цикла и точками 7-цикла, имеющего тип неустойчивого фокуса.

Бассейны же притяжения замкнутой инвариантной кривой с квазипериодической динамикой, возникающей через бифуркацию Неймарка-Сакера (см. основную ветвь на рис. 1) и резонансной кривой периода 7 (вторая зеленая ветвь на 1), разделены другой неустойчивой инвариантной кривой. Неустойчивая замкнутая кривая образована устойчивыми многообразиями седлового 7- цикла и точками неустойчивого 7-цикла (репеллера) другого типа. Как видно из рис. 2(а) устойчивая замкутая инвариантная кривая с периодической динамикой (резонансная кривая периода 7) образована неустойчивыми многообразиями седлового 7-цикла.

При дальнейшем увеличении α, неустойчивая замкнутая кривая с периодической динамикой становится неустойчивой квазипериодической (зеленая кривая на рис. 2(б)) через

бифуркацию «граничного столкновения». Затем, с увеличением α, аналогичную бифурукацию претерпевают устойчивая замкнутая кривая с периодической динамикой. В результате такой бифуркации она становится устойчивой кривой с квазипериодической динамикой (рис. 2(г)). Бассейны притяжения сосуществующих квазипериодических аттракторов разделены неустойчивой квазипериодической инвариантной кривой.

И, наконец, устойчивая (темно-малиновая на рис. 2(б), принадлежащей первой ветви бифуркационной диаграммы) и неустойчивая (зеленая на рис. 2(б)) квазипериодические замкнутые кривые исчезают через седло-узловую квазипериодическую бифуркацию [15] при $\alpha = \alpha^2_{\text{QFlip}}$. На рис. 2(д) приведен фазовый портрет отображения после этой бифуркации, когда отображение демонстрирует бистабильную динамику: квазипериодический аттрактор второй ветви, который на бифуркационной диаграмме выделен зеленым цветом, сосуществует с устойчивым периодическим решением периода 15, обозначенным светло-малиновым цветом. Границей бассейнов притяжения сосуществующих аттракторов является неустойчивая периодическая замкнутая кривая.

4. Заключение

Выполнен бифуркационный анализ негладкого отображения с большим числом многообразий переключения с мультистабильной динамикой. Рассмотренное отображение являтся дискретной моделью системы управления температурным полем нагревательной установки для выращивания кристаллов искусственного сапфира. Показано, что отображение демонстрирует особый тип мультистабильного поведения, когда в фазовом пространстве сосуществуют несколько вложенных друг в друга замкнутых инвариантных кривых, отвечающих двухчастотным колебаниям.

Исследованы механизмы рождения и исчезновения сосуществующих замкнутых инвариантных кривых через суперкритическую бифуркацию Неймарка-Сакера, последовательность бифуркаций граничного столкновения и квазипериодической седло-узловой бифуркации.

Ключевые слова: Мультистабильность, бифуркации граничного столкновения, негладкое непрерывное отображение, квазипериодическая седло-узловая бифуркация, замкнутая инвариантная кривая, двухчастотные колебания.

Предметная классификация AMS: 05.45.Gg, 05.45.Pq

Список литературы

[1] Feudel U., Grebogi C., Hunt B. R., Yorke J. A. *Map with More than 100 Coexisting Low-Period Periodic Attractors*, Phys. Rev. E 54, 1996, pp. 71—81.

[2] Feudel U. *Complex Dynamics in Multistable Systems*, Int. J. Bifurcation and Chaos 18, 2008, pp. 1607-–1626.

[3] Feudel U., Pisarchik A., Showalter K. *Multistability and Tipping: From Mathematics and Physics to Climate and Brain — Minireview and preface to the focus issue*, Chaos 28, 2018, pp. 033501-1– 033501-2.

[4] Zhusubaliyev Zh. T., Mosekilde E., Yahochkina O.O. *Torus Bifurcations in Multilevel Converter Systems*, Int. J. Bifurcation and Chaos 21, 2011, pp. 2343–2356.

[5] Zhusubaliyev Zh.T., Mosekilde E., Pavlova E.V. *Multistability and Torus Reconstruction in a DC-DC Converter with Multilevel Control.* IEEE Trans. on Industrial Informatics, 9, pp. 1936-1946. 2013.

[6] Zhusubaliyev Zh. T., Mosekilde E. *Multistability and Hidden Attractors in a Multilevel DC/DC Converter*, Math. Comput. Simul. 109, 2015, pp. 32-–45.

[7] Avrutim V., Zhusubaliyev Zh. T. *Nested Closed Invariant Curves in Piecewise Smooth Maps*, Int. J. Bifurcation and Chaos 29(7), 2019, pp. 193001721-1– 193001721-16.

[8] Zhusubaliyev Zh. T., Avrutin V., Gardini L., Sushko I. *Border Collision Bifurcation of a Resonant Closed Invariant Curve*, Chaos: An Interdisciplinary Journal of Nonlinear Science, 32, 2022, pp. 043101-1 – 043101-10.

[9] Feigin M. I. *Doubling of the Oscillation Period with C-Bifurcations in Piecewise Continuous Systems*, PMM J. Appl. Math. Mech. 34, 1970, pp. 861–869.
[10] Nusse H. E., Yorke J. A. *Border-Collision Bifurcations Including "Period Two to Period Three" Bifurcation for Piecewise Smooth Systems*, Physica D 57, 1992, pp. 39–57.
[11] Banerjee S., Verghese G. C. (eds.) *Nonlinear Phenomena in Power Electronics*, IEEE Press, NY 2001, 645 p.
[12] Zhusubaliyev Zh.T., Mosekilde E. *Bifurcations and Chaos in Piecewise-Smooth Dynamical Systems*, World Scientific, 2003, 363 p.
[13] Di Bernardo M., Budd C.J., Champneys A.R., Kowalczyk P. *Piecewise-Smooth Dynamical Systems: Theory and Applications*, Springer, 2008, 483 p.
[14] Zhusubaliyev Zh. T., Avrutin V., Bastian F. *Transformations of Closed Invariant Curves and Closed-Invariant-Curve-Like Chaotic Attractors in Piecewise Smooth Systems*, Int. J. Bifurcation and Chaos, 31(3), 2021, pp. 2130009-1 – 2130009-15.
[15] Kuznetsov Yu. A., Meijer H.G.E. *Numerical Bifurcation Analysis of Maps: From Theory to Software.* Cambridge University Press, 2019, 753 p.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

СИММЕТРИЯ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С РАЗРЫВНОЙ ПРАВОЙ ЧАСТЬЮ ВРИВОДИТ К БИСТАБИЛЬНОСТИ

ЖАНЫБАЙ ЖУСУБАЛИЕВ[1,2], УЛАНБЕК СОПУЕВ[2], ДИЛМУРАТ ТУРСУНОВ[2] , НУРСУЛТАН БЕКНАЗАРОВ[2]

[1] Юго-Западный государственный универстиет, Курск, Россия,
[2] Ошский государственный университет, Ош, Кыргызстан,
e-mail: ulansopuev@mail.ru, zhanybai@gmail.com, tdaosh@gmail.com, beknazarovnursultan@gmail.com

1. Введение

Рассмотрим систему автоматического управления, состоящую из модулятора M и непрерывной линейной части, структурная схема которой изображена на рис. 1. Здесь x – выход системы; $\sigma(t)$ – внешний периодический сигнал с периодом 2π; $\varphi = \sigma(t) - x$, S – сигналы на входе и на выходе модулятора, соответственно; $W(s) = \dfrac{1}{T_0 s + 1}$ – передаточная функция линейной части (объекта управления), где T_0 – постоянная времени; $s = d/dt$ – оператор дифференцирования. Состояние такой системы описывается дифференциальным уравне-

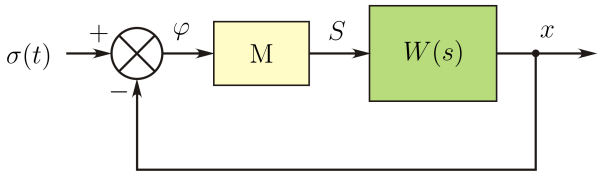


Рис. 1. Структурная схема системы управления с частотно-импульсной модуляцией

нием с разрывной правой частью [1]

$$\dot{x} = \mathrm{g}(t,x),\ \mathrm{g}(t,x) = \lambda(x - S(\varphi)),\ \varphi = \sigma(t) - x,\ \sigma(t+\pi) = -\sigma(t) \tag{1}$$

где t – время; x – неизвестная функция времени t из $\mathbb{R}$ (выход системы); $\dot{x}$ – производная x по t; $\lambda = -\dfrac{1}{T_0}$ – величина обратно пропорциональная по модулю постоянной времени; $\sigma(t) = q\ \cos\ t$, где q – амплитуда задающего сигнала $\sigma(t)$.

Модулятор M осуществляет частотно-импульсную модуляцию

$$S = \begin{cases} +1, & t_k \leqslant t < t'_k, \\ -1, & t'_k \leqslant t < t_{k+1}. \end{cases}$$

Здесь

$$t_{k+1} = t_k + 2\tau_0 + \begin{cases} 0, & q\cdot\cos(t_k+\tau_0) + 1 - (x_k+1)\,\mathrm{e}^{\lambda\tau_0} > 0 \quad \text{и} \\ & q\cdot\cos(t_k+z_0) - 1 - (x_k-1)\,\mathrm{e}^{2\lambda\tau_0} + 2\cdot\mathrm{e}^{2\lambda\tau_0} \leqslant 0, \\ z_k^-, & q\cdot\cos(t_k+\tau_0) + 1 - (x_k+1)\,\mathrm{e}^{\lambda\tau_0} \leqslant 0 \\ z_k^+, & q\cdot\cos(t_k+z_0) + 1 - (x_k+1)\,\mathrm{e}^{\lambda\tau_0} > 0 \quad \text{и}. \\ & q\cdot\cos(t_k+2\tau_0) - 1 - (x_k-1)\,\mathrm{e}^{2\lambda\tau_0} + 2\cdot\mathrm{e}^{2\lambda\tau_0} > 0, \end{cases}$$
$$k = 0, 1, 2, \ldots.$$

В первом случае $t'_k - t_k = t_{k+1} - t'_k = \tau_0$; во втором – $t'_k - t_k = z_k^- + \tau_0$, $t_{k+1} - t'_k = \tau_0$ и в третьем – $t'_k - t_k = \tau_0$, $t_{k+1} - t'_k = z_k^+ + \tau_0$, где $\tau_0 = \text{const}$, $\tau_0 > 0$; $x_k = x(t)|_{t=t_k}$. Здесь $z_k^\pm$ – ширина положительного и отрицательного импульсов, определяемые соответственно как наименьшие неотрицательные корни уравнений

$$1 + (x_k - 1) \cdot \mathrm{e}^{\lambda(2\tau_0 + z_k^+)} - 2 \cdot \mathrm{e}^{\lambda(\tau_0 + z_k^+)} - q \cdot \cos(\theta + 2\tau_0 + z_k^+) = 0$$

и

$$1 - (x_k + 1) \cdot \mathrm{e}^{\lambda(\tau_0 + z_k^-)} - q \cdot \cos(t_k + \tau_0 + z_k^-) = 0.$$

Параметры: $\lambda - 3.2951333973$; $0.06 < \tau_0 < 0.1$; $0.926 < q < 1.19$.

Численные эксперименты показывают, что рассматриваемая система (1) демонстрирует два типа периодический динамики – бистабильную и моностабильную. Бистабильность проявляется в том, что при одних и тех же значениях параметров сосуществует пара устойчивых периодических решений. Выявлено, что бистабильное поведение является следствием нарушения симметрии правой части уравнения движения (1).

Можно показать, что асимптотическая динамика (1) является либо периодической или казипериодической. Других режимов не может быть. Доказательстово выходит за рамки этой работы.

В фазовом пространсте системы (1) таким решениям в сечении Пуанкаре соответствуют замкнутая инвариантная кривая, топологически эквивалентная окружности единичного радиуса [2]. Характер движения изображающей точки на окружности определяется числом вращения Пуанкаре $r = n/m$, где n, m – целые числа. Если оно рацинальное, то решение на окружности является периодическим. Если же r – иррациональное, то соответствующее решение квазипериодическое [2–5].

Известно, что периодические решения с рациональными числами вращения n/m, отвечают синхронизации автоколебательного решения и внешнего возбуждения, когда в периоде возникшего периодического движения укладываются m периодов автоколебаний и n периодов внешней силы. В этом случае говорят, что происходит резонанс порядка $n{:}m$ [5].

На плоскости управляющих параметров, области резонансов с числом вращения $r = n/m$ упорядочены в соответствии с деревом Фейри и образуют так называемую (языки Арнольда) структуру добавления периодов («adding structure») [2, 6–8].

В данной статье мы преследуем цель изучить свойства симметрии правой части уравнения (1), нарушение которых приводит к появлению пары устойчивых сосуществующих резонансных периодических решений, ответственных за бистабильное поведение и численно показать, что резонансные языки на плоскости параметров, соответствующие первому ярусу дерева Фейри с числами вращения $r = 1/m$, где m – четные числа, отвечают областям бистабильности, а с нечетными m – областям с моностабильным периодическим поведением.

2. Численные эксперименты

Основные результаты представлены в виде трех утверждений.

Утверждение 2.1. *Уравнение* (1) *симметричное, если*

$$\begin{gathered} \mathrm{g}^+(t,x) = -\mathrm{g}^-(t+\pi, x+\pi), \\ \mathrm{g}^-(t,x) = -\mathrm{g}^+(t+\pi, x+\pi), \\ \mathrm{g}(t,x) = \begin{cases} \mathrm{g}^+(t,x) = \lambda(x-1),\ S=1, \\ \mathrm{g}^-(t,x) = \lambda(x+1),\ S=-1. \end{cases} \end{gathered} \tag{2}$$

Доказательство (2.1) опускаем, оставляя читателям.

Утверждение 2.2. *Если выполняются условия симметрии* (2), *то* (1) *имеет единственное устойчивое периодическое решение. При нарушении условий* (2) *уравнение* (1) *обладает двумя сосуществующими устойчивыми периодическими решениями, симметричными друг к другу.*

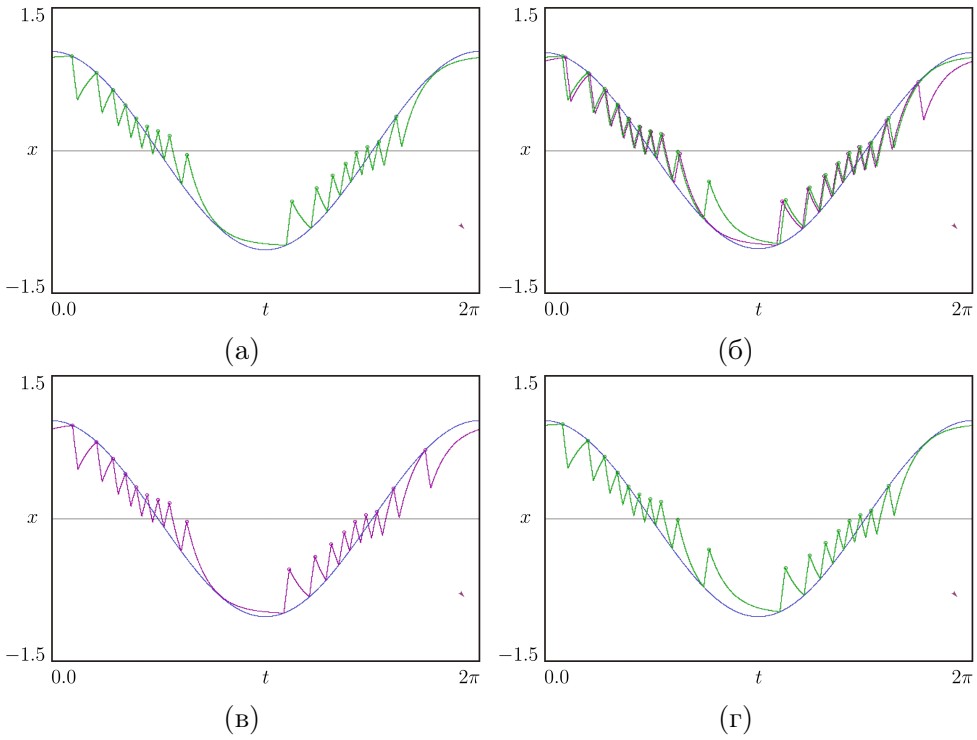


Рис. 2. (а) Периодическое решение, которое является единственным при $q = 1.041666$. (б)– (г) Сосуществующие устойчивые периодические решения при $q = 1.0294650404$, $\tau_0 = 0.08$

Мы здесь ограничились численными экспериментами. Для примера, рис. 2(а) иллюстрирует ситуацию, когда условия (2.1) выполняются. Численные эксперименты показывают, что тогда периодическое решение единственно. Рис. 2(б)-(д) отвечают случаю нарушения (2.1). В этом случае динамическая система (1) демонстрирует бистабильное поведение, когда при одних и тех же значениях параметров сосуществуют имеет два устойчивых периодических решения (рис. 2(б)). На рис. 2(в),(г) показано каждое периодическое решение отдельно. Как можно видеть из этих рисунков, эти решения симметричны друг к другу (см. (2.2)).

Утверждение 2.3. *Языки синхронизации с числами вращения* $1/m$ *отвечают областям бистабильности, если* m *есть нечетное число, в которых уравнение* (1) *имеет два устойчивых периодических решения.*

Если же m *– четное, то существуют единственное периодическое решение.*

На рис. 3 показана двупараметрическая бифуркационная диаграмма, на которой хорошо различимы резонансные языки относительно большой площади. В прямоугольниках указаны соответствующие им числа вращения $1/m$, $m = 13, 14, 15, 16, 17, 18, 19, 20, 21, 23$. Языки, отвечающие числам вращения с четными m отвечают областям бистабильности, в которых сосуществуют два устойчивых периодических решения (на рис.3 $1:14$, $1:16$, $1:18$, $1:20$). Если m – нечетные числа, то языки, соответствуют областям моностабильности (см. утверждение (2.3)).

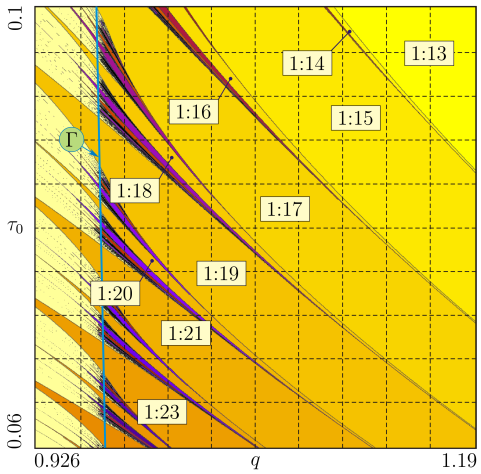


Рис. 3. Двупараметрическая бифуркационная диаграмма на плоскости параметров (q, τ_0)

Для исследования явления синхронизации обычно вводят в рассмотрение отображение первого возвращения Пуанкаре [2, 5]. Детальное рассмотрение методики получения такого отображения для (1) и изучение его свойств, аналогично тому, как это сделано в [9], выходят за рамки представленной статьи. Поэтому мы ограничимся упоминанием лишь некоторых деталей.

• В зависимости от параметров отображение Пуанкаре может быть диффеоморфизимом (классическим отображением окружности) или разрывным. В обоих случаях отображение демонстрирует так называемую бифуркационную структуру «добавления периода» (см. рис. 3).

• Граница, разделяющая плоскость параметров на области существования разрывного отображения и диффеоморфизма, обозначена на рис. 3 через Γ.

Слева от границы Γ отображение Пуанкаре является диффеоморфизмом, свойства которого подробно изучены, например, в [2–4,6,7]. Как известно в этой области языки синхронизации располагаются всюду плотно: между любыми двумя квазипериодическими режимами с различными r существует резонансное периодическое решение [2–4, 6, 7].

Справа от Γ отображение является разрывным [8, 9]. Здесь динамика системы только периодическая [8]. Резонансные языки ограничены не кривыми седло-узловаой бифуркации, как в случае гладких систем [2–4, 6, 7], а линиями бифуркаций граничного столкновения («border collision bfurcations») [9–11].

3. Заключение

В данной работе мы представили результаты численно-аналитического бифуркационного анализа системы управления с частотно-импульсной модуляцией. Поведение такой системы описывается неавтономным дифференциальным уравнением с разрывной правой частью, которое в зависимости от параметров является симметричным или асимметричным.

Показано, что нарушение симметрии уравнения движения приводит к бистабильному поведению, когда возникает пара устойчивых периодических решений с одинаковым периодом, но с разными рациональными числами вращения r.

Известно, что периодические решения с рациональными числами вращения $r = n/m$ отвечают резонансу порядка $n{:}m$. На плоскости управляющих параметров, области резонансов с числом вращения упорядочены в соответствии с деревом Фейри и образуют так называемую структуру добавления периода («adding structure» или структуру языков Арнольда).

Выявлено, что языки синхронизации, отвечающие первому ярусу дерева Фейри с числами вращения $1/m$, соответствуют областям бистабильности, если m четные числа. Если же m – нечетные, то — областям моностабильности. Это утверждение справедливо только для резонансных периодических решений с числами вращения $r = 1/m$.

Вопрос, как различать области существования единственных и сосуществующих резонансных периодических решений с произвольными числами вращения, остается открытым.

Ключевые слова: Дифференциальные уравнения с разрывной правой частью, симметрия, периодическое решение, бистабильность, структура добавления периода, языки Арнольда, число вращения.

AMS SubjectClassification: 05.45.Gg, 05.45.Pq

Список литературы

[1] Filippov A. F. *Differential Equations with Discontinuous Right-Hand Sides*, Kluwer Academic, Dordrecht, The Netherlands, 1988, 304 p.

[2] Arnold V.I. *Small Denominators. I. Mappings of the Circumference into Itself*, Izv. Akad. Nauk SSSR Ser. Mat. 25 (1) (1961) 21–86, (in Russian); Trans. Amer. Math. Soc. Ser. 2 (46) (1965) 213–284.

[3] Glass L. *Cardiac arrhythmias and circle maps: A classical problem*, Chaos 1 (1) (1991) 13-–19.

[4] Glass L., Mackey M.C., *A Simple Model for Phase Locking of Biological Oscillators*, J. Math. Biol. 7 (1979) 339–352.

[5] Iooss G., Joseph D.D. *Elementary Stability and Bifurcation Theory*, Springer-Verlag, New York, 1989, 325.

[6] Glass L., Perez R. *Fine Structure of Phase Locking*, Phys. Rev. Lett. 48 (26) (1982) 1772–1775.

[7] Cvitanovic P., Shraiman B., Söderberg B. *Scaling Laws for Mode Lockings in Circle Maps*, Phys. Scr. 32 (4) (1985) 263—270.

[8] Avrutin V., Gardini L., Sushko I., Tramontana F. *Continuous and Discontinuous Piecewise-Smooth One-Dimensional Maps: Invariant Sets and Bifurcation Structures,* World Scientific, 2019, 637 p.

[9] Zhusubaliyev Zh. T., Avrutin V., Kucherov A. S., Haroun R., El Aroudi A. *Period Adding with Symmetry Breaking/Becovering in a Power Inverter with Hysteresis Control*, Physica D: Nonlinear Phenomana, 444 (2023) 133600 (16 pages).

[10] Zhusubaliyev Zh.T., Mosekilde E. *Bifurcations and Chaos in Piecewise-Smooth Dynamical Systems*, World Scientific, 2003, 363 p.

[11] Zhusubaliyev Zh. T., Mosekilde E. *Direct Transition From a Stable Equilibrium to Quasiperiodicity in Non-Smooth Systems.* Physics Letters A, 372(13), 2237–2246, 2008.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ОЦЕНКА ЧИСЛА ПРЕДЕЛЬНЫХ ЦИКЛОВ УРАВНЕНИЯ $\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$, ГДЕ P И Q ПОЛИНОМЫ.

МАММЕТ Г. ХУДАЙ-ВЕРЕНОВ

Международный университет нефти и газа, Ашхабад, Туркменистан

Для уравнения указанного в заглавии, где P(x,y),Q(x,y) полиномы n-ой степени с действительными коэффициентами доказано, что число предельных циклов не больше чем $2(2n^2+n+1)$. При этом число неалгебрических предельных циклов не превышает $2n^2+n+1$. Библиография 5 названий

Всюду плотность, униформизация, псевдопериодическая функция, гомотопные циклы.

Целью нашего исследования является оценка числа комплексных циклов, лежащих на интегральных кривых уравнения

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}, \quad (1)$$

рассматриваемого в комплексном пространстве C^2 переменных x, y.P,Q - полиномы n-ой степени с комплексными коэффициентами. Интегральную кривую Ф уравнения (1) можно рассматривать как двумерную поверхность в четырёхмерном пространстве со своей группой гомологий. Вместе с (1) будем рассматривать и его параметрические представления

$$\frac{dx}{dt} = \frac{P(x,y)}{N(x,y)}, \ \frac{dy}{dt} = \frac{Q(x,y)}{N(x,y)}, \quad (2)$$

$$\frac{dx}{dz} = \mathrm{P(x,y)}, \ \frac{dy}{dz} = \mathrm{Q(x,y)}, \quad (2')$$

где N многочлен n-ой степени с комплексными коэффициентами. Мы будем говорить, что многочлены P Q,N - общего положения, если коэффициенты этих многочленов таковы, что выполняются следующие условия :

(1) Каждое решение (1) продолжаемо на всю плоскость x за исключением счётного числа особых точек.
(2) Каждое решение (1) всюду плотно в C2.
(3) В окрестности особой точки уравнения (1), как конечной, так и бесконечной, имеется разложение Пуанкаре.
(4) Каждое решение (2) продолжаемо на всю плоскость t за исключением счётного числа алгебраических особых точек.

Не уточняя подробностей можно сказать, что множество многочленов P,Q,N не удовлетворяющих этим условиям составляют множество меньшей размерности, чем само множество всех P,Q,R (см, [1], [2]). Рассмотрим конкретное решение (1)-y=y(x) и соответствующее ему решение x=x(t), y=y(t) системы (2). Пусть на y=y(x) лежит не гомотопный нулю комплексный цикл. Как было показано в [3] компоненты решения x=x(t), y=y(t) будут псевдопериодическими функциями. Если же Z_1, Z_2,..,Z_3- не гомотопные нулю и между собой циклы на y(x), то x(t), y(t) будут псевдопериодическими функциями с периодами l_1,...,l_n, т.е. каждому циклу соответствует свой период. При этом как следует из системы (2),

$$l_i = \int_z \frac{N}{P}\, dx$$

Для цикла Z совпадающего с Z_i или свободно гомотопного Z_i цикла, может случиться, что при параметризации решений с помощью системы (2) или (2'), проекция цикла будет замкнутой кривой на плоскости t или z. Очевидно, это может произойти только для исключительных P и Q. Мы будем предполагать, что не гомотопные нулю циклы на изучаемом решении $y = y(x)$ имеют незамкнутые проекции на t и z. Другими словами, соответствующее $y = y(x)$ решение (2) или (2') имеет не равные нулю периоды.

1. Униформизация решения (1) и соответствующего ему решения (2)

Рассмотрим $y = y(x)$ - решение (1), соответствующее решению $x = x(t)$, $y = y(t)$ системы (2).

Униформизируем $y = y(x)$ функциями:

$$x = \varphi(\tau), \quad y = \Psi(\tau),$$

где φ и Ψ однозначные мероморфные функции в области K. K - либо круг единичного радиуса, либо конечная плоскость, либо расширенная плоскость. Как известно [4,5], φ и Ψ автоморфные функции, инвариантные относительно группы G дробно-линейных преобразований. При этом G изоморфно фундаментальной группе $\pi_1(\Phi)$ - римановой поверхности $y = y(x)$. Если Φ односвязна, то G тривиальная группа, состоящая из тождественного преобразования. Следовательно, каждому не гомотопному нулю циклу на Φ соответствует не тривиальный элемент группы G. Пусть T элемент группы G. Имеем $\forall \tau \in K$:

$$\varphi(T\tau) = \varphi(\tau), \quad \Psi(T\tau) = \Psi(\tau).$$

Подставим в систему (2) $x = \varphi(\tau)$, $y = \Psi(\tau)$.

$$\frac{d\varphi}{d\tau} = \frac{dt}{d\tau} \cdot \frac{P(\varphi, \Psi)}{N(\varphi, \Psi)}, \quad \frac{d\Psi}{d\tau} = \frac{dt}{d\tau} \cdot \frac{Q(\varphi, \Psi)}{N(\varphi, \Psi)} \quad (3)$$

Отсюда находим

$$\frac{dt}{d\tau} = \frac{d\varphi}{d\tau} \cdot \frac{N(\varphi, \Psi)}{P(\varphi, \Psi)}, \quad \frac{dt}{d\tau} = \frac{d\Psi}{d\tau} \cdot \frac{N(\varphi, \Psi)}{Q(\varphi, \Psi)} \quad (4)$$

Следовательно, $\frac{dt}{d\tau}$ есть мероморфная в K функция.

Рассмотрим случай, когда некоторая ветвь $y = y(x)$ регулярна в окрестности M_0 - конечной или бесконечной особой точки уравнения (1). M_0 рассматривается как обыкновенная точка $y = y(x)$ и, следовательно, ей соответствует значение $\tau_0 \in K$, где $\varphi(\tau)$ и $\Psi(\tau)$ являются мероморфными функциями. Кроме того, в силу того, что M_0 особая точка уравнения (1), $t(\tau)$ и $t'(\tau) \to \infty$ при $\tau \to \tau_0$.

Верно и обратное. Если $t'(\tau) \to \infty$ при $\tau \to \tau_0$, то непременно некоторая ветвь $y = y(x)$ регулярна в окрестности особой точки уравнения (1). В самом деле, пусть φ и Ψ регулярны в окрестности τ_0:

$$x - x_0 = \alpha(\tau - \tau_0)^s + \ldots, \quad y - y_0 = \beta(\tau - \tau_0)^m + \ldots,$$

Отсюда

$$y = y_0 + \sum_{i=m}^{\infty} \gamma_i (x - x_0)^{i/s}$$

Кроме того, из (4) следует, что если $P(x_0, y_0) \neq 0$ или $Q(x_0, y_0) \neq 0$, то $\frac{dt}{d\tau}$ регулярна в окрестности τ_0, что противоречит предположению.

В случае, когда φ, Ψ не ограничены в окрестности τ_0, исследуется также. Таким образом, верна Лемма 1.

Лемма 1: Если $\exists \tau_0 \in K$ такая, что $t'(\tau) \to \infty$ при $\tau \to \tau_0$, то решение $y = y(x)$ непременно регулярно в окрестности одной из особых точек, т.е. как говорят, $y = y(x)$ является сепаратрисным решением (1).

Лемма 2: Если ни одна из ветвей решения $y = y(x)$ не является сепаратрисой особой точки уравнения (1), то $x(\tau)$, $y(\tau)$, $t(\tau)$ регулярные в K функции.

В самом деле, из (4) следует, что $t'(\tau)$ мерофорфная в K функция, а из Леммы 1 следует, что $t'(\tau)$ регулярная в K функция. Отсюда следует регулярность $t(\tau)$ в K. Поскольку $\varphi(\tau)$, $\Psi(\tau)$ униформизирующие функции, то они являются мероморфными функциями. Если при $\tau \to \tau_0$ $\varphi(\tau)$ или $\Psi(\tau)$ или они обе стремятся к бесконечности, то они имеют при $\tau = \tau_0$.

Пусть $\varphi(\tau) = \frac{\varphi_1(\tau)}{(\tau - \tau_0)^k}$, $\Psi(\tau) = \frac{\Psi_1(\tau)}{(\tau - \tau_0)^m}$, $\varphi_1(\tau)$, $\Psi_1(\tau)$ регулярные по τ функции, $m_1 \geq 0$, $m_2 \geq 0$, $m_1 + m_2 > 0$. Если $m_1 \neq m_2$, то $\frac{P(\varphi,x)}{N(\varphi,x)}$ и $\frac{Q(\varphi,x)}{N(\varphi,x)}$ можно считать регулярными в окрестности τ_0 функциями. Тогда $\varphi'(\tau)$ и $\Psi'(\tau)$ в силу (3) тоже будут регулярными в этой окрестности функциями. Следовательно, и сами функции $\varphi(\tau)$ и $\Psi(\tau)$ будут регулярными в окрестности τ_0 функциями. Если же $m_1 = m_2 \neq 0$, то $\lim_{\tau \to \tau_0} \frac{\varphi(\tau)}{\Psi(\tau)} = \eta_0 \neq 0$ и как известно $(0, \eta_0)$ будет бесконечно удаленной особой точкой уравнения (1). Поскольку всех таких точек конечное число, то всегда можно считать, что $\lim_{\tau \to \tau_0} N(\varphi, \Psi)(\tau - \tau_0)^{m_1 n} = \text{const} \neq 0$. Тогда в силу (3) $\varphi(\tau)$ и $\Psi(\tau)$ опять будут регулярными в окрестности τ_0 функциями. Следовательно, $\varphi(\tau)$, $\Psi(\tau)$ помощью иметь не могут, т.е. они являются регулярными в K функциями.

Выясним теперь свойство $t(\tau)$, связанное с группой G. Напомним, что $t(\tau)$ определяется из (4). Найдем $\forall A \in G$ $t(A\tau)$.

$$[t(A\tau)]' = t'(\tau)(A\tau)'$$

В силу (4)

$$[t(A\tau)]' = \frac{\varphi'(A\tau)N(\varphi(A\tau), \Psi(A\tau))}{P(\varphi(A\tau), \Psi(A\tau))}(A\tau)' = \frac{\varphi'(A\tau)N(\varphi(\tau), \Psi(\tau))}{P(\varphi(\tau), \Psi(\tau))}(A\tau)'$$

Поскольку

$$\varphi(A\tau) = \varphi(\tau), \quad \Psi(A\tau) = \Psi(\tau)$$

имеем

$$[\varphi(A\tau)]' = \varphi'(\tau)(A\tau)' = \varphi'(\tau)$$

$$[\Psi(A\tau)]' = \Psi'(\tau)(A\tau)' = \Psi'(\tau)$$

Следовательно

$$[t(A\tau)]' = \frac{\varphi'(\tau)N(\varphi, \Psi)}{P(\varphi, \Psi)} = t'(\tau)$$

или

$$[t(A\tau)]' - t'(\tau) = 0$$

$$[t(A\tau) - t(\tau)]' = 0$$

$$t(A\tau) - t(\tau) = l_A$$

Нами доказана

ЛЕММА 3. Пусть $y(x)$ решение (1), $x(t)$, $y(t)$ соответствующее решение (2). Тогда параметр t определяется как функция τ из системы (4) и удовлетворяет соотношению:

$$t(A\tau) = t(\tau) + l_A, \quad \forall A \in G.$$

Уточним постоянное l_A. Пусть Z цикл на $y(x)$, а соответствующий элемент G. Это означает следующее: когда переменная τ пробегает отрезок $[\tau_0, A\tau_0]$ начиная с $\tau_0 \in K$, x, y пробегут замкнутую кривую Z^* свободно гомотопную циклу Z на $y(x)$. Следовательно,

$$\int_{[\tau_0, A\tau_0]} t'(\tau)d\tau = \int_{Z^*} \frac{N(x,y)}{P(x,y)}dx$$

$$t(A\tau_0) - t(\tau_0) = \int_{Z^*} \frac{N(x,y)}{P(x,y)}dx = l_A.$$

Таким образом, число l_A не зависит от способа униформизации решения $y(x)$. Обозначим через S множество сепаратрисных решений (1). Из леммы 1 и 2 следует ЛЕММА 4.

ЛЕММА 4: Если $y(x) \in S$, то три функции $x = \varphi(\tau)$, $y = \Psi(\tau)$, $t = t(\tau)$ униформизируют одновременно $y = y(x)$, $x = x(t)$, $y = y(t)$.

Заметим, что в случае $y(x) \in S$ функция $t(\tau)$ имеет, вообще говоря, логарифмическую особенность. Для сепаратрисных решений считается, что особая точка в окрестности которой регулярна ветвь этого решения, лежит на $y(x)$ и циклы разделённые этой точкой считаются не гомотопными.

2. Свойства функции $\omega(\tau) = \frac{t'(\tau)}{N(\varphi,\Psi)}$

Теорема 1. Пусть $y(x) \in S$. Тогда $\omega(\tau) \neq 0$, $\omega(\tau) \neq \infty$ для всех $\tau \in K$.

Доказательство. По лемме 4, $x(\tau)$, $y(\tau)$, $t(\tau)$ являются голоморфными функциями в K. Отсюда следует, что ω - мероморфная функция в K. Пусть при $\tau = \tau_0$ $\omega(\tau)$ имеет полюс ($\omega(\tau_0) = \infty$).

Это возможно только если $N(\varphi(\tau_0), \Psi(\tau_0)) = 0$. Тогда необходимо, чтобы $t'(\tau_0) = 0$. В самом деле, если $t'(\tau_0) \neq 0$, то как следует из (3), для голоморфности $\varphi(\tau)$, $\Psi(\tau)$ в точке τ_0 необходимо, чтобы

$$P(\varphi(\tau_0), \Psi(\tau_0)) = 0,\ Q(\varphi(\tau_0), \Psi(\tau_0)) = 0.$$

Но это исключено условием теоремы. По этой же причине, если $N(\varphi, \Psi)$ имеет в τ_0 нуль k-того порядка, то и $t'(\tau)$ имеет нуль не менее k-того порядка, и $\omega(\tau) = \frac{t'(\tau)}{N(\varphi,\Psi)}$ остается конечной в τ_0. Небольшой деформацией $N(x, y)$ можно добиться, чтобы $N(\varphi, \Psi) \equiv 0$, т. е. $N(x, y) = 0$ не является интегралом уравнения (1), что не нарушает общности предположения. В силу этого существование полюса для $\omega(\tau)$ исключается.

Пусть теперь τ_0 - нуль $\omega(\tau)$. Это возможно, если $t'(\tau_0) = 0$, и кратность нуля $t'(\tau_0)$ в τ_0 должна быть на единицу больше, чем у $N(x(\tau), y(\tau))$. Пусть $N(x(\tau), y(\tau))$ имеет в τ_0

нуль кратности m_0, $t'(\tau)$ имеет нуль кратности $m+S$, $S \geq 1$. Следовательно, $t(\tau) - t'(\tau_0)$ имеет в τ_0 нуль кратности $m+S+1$. Одновременно $Q(x(\tau), y(\tau))$, $P(x(\tau), y(\tau))$ в τ_0 в нуль обращаться не могут. Пусть $Q(x(\tau_0), y(\tau_0)) \neq 0$, а $P(x(\tau), y(\tau))$ имеет в τ_0 нуль кратности τ_0. Тогда в силу (3) $\varphi(\tau) - \varphi(\tau_0)$ имеет в τ_0 нуль кратности $S+r+1$, $\Psi(\tau) - \Psi(\tau_0)$ имеет нуль кратности $S+1$. Таким образом,

$$t(\tau_0) = (\tau - \tau_0)^{m+s+1} t_1(\tau), \quad t_1(\tau_0) \neq 0,$$

$$\varphi(\tau) - \varphi_0 = (\tau - \tau_0)^{\tau+s+1} \varphi_1(\tau), \quad \varphi_1(\tau_0) \neq 0,$$

$$\Psi(\tau) - \Psi_0 = (\tau - \tau_0)^{s+1} \Psi_1(\tau), \quad \Psi_1(\tau_0) \neq 0.$$

Если $r = 0$, окрестность точки τ_0 в K не является взаимооднозначным образом окрестностью точки $M(\varphi(\tau_0), \Psi(\tau_0))$ на интегральной кривой решения $y = y(x)$. Если же $r \neq 0$, то M алгебраическая особая точка и r непременно кратен $S+1$, то есть опять нет взаимооднозначного соответствия между указанными выше окрестностями. Это противоречит условиям униформизации $y = y(x)$.

Следовательно, $\omega(\tau) \neq 0$ для всех $\tau \in K$. Теорема доказана.

Введем обозначение:

$$\int_0^\tau \frac{t'(\tau)}{N(\varphi, \Psi)} d\tau = z \quad (6)$$

Поскольку $\frac{dz}{d\tau} = \omega \neq 0$, то (6) можно разрешить относительно τ в окрестности каждой точки $z \in K'$, где K' - область значений интеграла в (6), когда τ пробегает K. Очевидно, функция $\tau = \tau(z)$, полученная разрешением (6) относительно τ, является регулярной функцией в малой окрестности любой точки из K', но она в целом в K' может быть многозначной функцией.

Выберем ветвь $\tau = \tau(z)$, полагая $\tau(0) = 0$.

Рассмотрим функции $\varphi(\tau(z))$ и $\Psi(\tau(z))$ в $\mathbb{K}'$. Эта пара является решением системы

$$\frac{dx}{dz} = P(x, y), \quad (1)$$

$$\frac{dy}{dz} = Q(x, y). \quad (2)$$

Для определенности обозначим

$$\varphi(\tau(z)) = x_1(z), \Psi(\tau(z)) = y_1(z),$$
$$x_1(0) = \varphi(0),\ y_1(0) = \Psi(0).$$

где

Решение $x_1(z)$ и $y_1(z)$ определено в K', но оно может быть и продолжено за K' как решение системы (1) и (2).

Лемма 5. Любая точка z_0, до которой может быть продолжено решение $x_1(z)$ и $y_1(z)$, принадлежит K'.

Доказательство. В силу свойств функций $\varphi(\tau)$ и $\Psi(\tau)$ нередко существует $\tau_1 \in \mathbb{K}$ такая, что $\varphi(\tau_1) = x_1(z_0)$ и $\Psi(\tau_1) = y_1(z_0)$.

Обозначим

$$\int_0^{\tau_1} \omega(\tau) d\tau = z_1.$$

Тогда $\tau(z_1) = \tau_1$, и мы получаем

$$x_1(z_1) = \varphi(\tau(z_1)) = \varphi(\tau_1) = x_1(z_0),$$

$$y_1(z_1) = \Psi(\tau(z_1)) = \Psi(\tau_1) = y_1(z_0).$$

Это означает, что путь γ, соединяющий точки z_0 и z_1 на плоскости z, соответствует циклу на решении $y = y(x)$, а $z_0 - z_1$ является периодом функций $x_1(z)$ и $y_1(z)$. Пусть $A \in \mathbb{G}$ соответствует этому циклу. Тогда $z(A\tau_1) = z(\tau_1) + z_0 - z_1 = z_0$, т.е. $z_0 \in \mathbb{K}'$.

Лемма доказана.

Предположим, что $\tau(z)$ многозначная функция в $\mathbb{K}'$. Тогда существует замкнутый путь $Z \in \mathbb{K}'$ такой, что после его обхода, начиная с точки z_1, мы придём к значению $\tau_2^0(z_1)$ отличному от начального значения $\tau_1^0 = \tau_1(z_1)$. Это означало бы существование точек τ_1^0 и τ_2^0 в $\mathbb{K}$ таких, что $z(\tau_1^0) = z(\tau_2^0) = z_1$.

Рассмотрим уравнение

$$\int_0^{\tau} \omega(\tau)d\tau = \int_0^{\tau_2} \omega(\tau)d\tau.$$

Лемма 6. Уравнение (2) разрешимо относительно τ_2, и τ_2 является голоморфной функцией от τ_1^0.

Доказательство.

Разрешимость (2) следует из аналитичности интегралов, входящих в уравнение, и из равенства

$$\int_0^{\tau_1^0} \frac{t'(\tau)}{N}d\tau = \int_0^{\tau_2^0} \frac{\tau'(\tau)}{N}d\tau.$$

Продифференцируем обе стороны равенства 7:

$$\frac{t'(\tau)d\tau}{N(\varphi(\tau), \Psi(\tau))} = \frac{t'(\tau_2)d\tau_2}{N(\varphi(\tau_2), \Psi(\tau_2))}.$$

Отсюда получаем

$$\frac{d\tau}{d\tau_2} = \frac{t'(\tau_2)}{N(\varphi(\tau_2), \Psi(\tau_2))} \cdot \frac{N(\varphi(\tau), \Psi(\tau))}{t'(\tau)}.$$

Следовательно,

$$\frac{d\tau}{d\tau_2} \neq 0, \quad \frac{d\tau_2}{d\tau} \neq 0.$$

В силу этого уравнение (2) можно разрешить относительно τ_2 в малой окрестности τ_1^0:

$$\tau_2 = \Theta(\tau),$$

и $\Theta(\tau)$ является голоморфной функцией τ в окрестности τ_1^0. В силу конечности интегралов, входящих в (2), $\Theta(\tau)$ будет аналитической функцией во всей области $\mathbb{K}$.

Лемма доказана.

Пусть $x_1(z)$ и $y_1(z)$ - определенное выше решение (1) и (2). Тогда

$$x_1(z(\tau)) = \varphi(\tau), \quad y_1(z(\tau)) = \Psi(\tau), \quad \forall \tau \subset \mathbb{K}.$$

По определению $\Theta(\tau)$ имеем

$$z(\tau) = z(\Theta(\tau)).$$

В то же время

$$x_1(z(\Theta(\tau))) = \varphi(\Theta(\tau)), \quad y_1(z(\Theta(\tau))) = \Psi(\Theta(\tau)).$$

Следовательно,

$$x_1(z(\tau)) = \varphi(\Theta(\tau)), \quad y_1(z(\tau)) = \Psi(\Theta(\tau)),$$

или

$$\varphi(\tau) = \varphi(\Theta(\tau)), \quad \Psi(\tau) = \Psi(\Theta(\tau)).$$

Это означает, что на решении $y = y(x)$ имеется цикл. Если он гомотопен нулю, то, неизбежно, $\Theta(\tau) \equiv \tau$, т.е. $\tau = \tau(z)$ не может быть многозначной. Если же он совпадает с одним из не гомотопных нулю циклов, лежащих на $y = y(x)$, то в силу $z(\tau) = z(\Theta(\tau))$ и представления решения $y = y(x)$ как $x = x_1(z)$, $y = y_1(z)$, этот цикл проектируется на плоскость z в замкнутую кривую, чего не может быть по нашему предположению.

Таким образом, $\tau = \tau(z)$ есть однозначная функция в $\mathbb{K}'$. Тогда пара

$$x_1(z) = \varphi(\tau(z)), \quad y_1(z) = \Psi(\tau(z)),$$

является однозначными в $\mathbb{K}'$ и является решением (1) и (2). Следовательно, верна

Теорема 2. Для всякого решения $y = y(x)$ уравнения (1) и (2), не являющегося сепаратрисным решением ни одной из особых точек этого уравнения, компоненты соответствующего решения $x = x(z)$, $y = y(z)$ системы (1) и (2) являются однозначными функциями во всей области $\mathbb{K}'$.

3. Свойства особых точек

Пусть $M_0(x_0, y_0)$ - особая точка (1) и (2). В наших предположениях относительно P и Q, в окрестности M_0 система (1) и (2) приводима к виду

$$\frac{du}{dz} = \lambda_1 u, \tag{3}$$

$$\frac{d\theta}{dz} = \lambda_2 \theta, \tag{4}$$

где u, θ и x, y связаны соотношениями

$$x - x_0 = \sum_{i+j=1}^{\infty} a_{ij} u^i \theta^j, \tag{5}$$

$$y - y_0 = \sum_{i+j=1}^{\infty} b_{ij} u^i \theta^j, \tag{6}$$

ряды стоящие в правых частях сходятся при $|u| < \delta$, $|v| < \delta$, δ - положительное число.

Решим (3) и подставим значения u и θ в (5) и (6):

$$x - x_0 = \sum_{i+j=1}^{\infty} a_{ij} (c_1 e^{\lambda_1 (z - z_1)})^i (c_2 e^{\lambda_2 (z - z_1)})^j, \tag{7}$$

$$y - y_0 = \sum_{i+j=1}^{\infty} b_{ij} (c_1 e^{\lambda_1 (z - z_1)})^i (c_2 e^{\lambda_2 (z - z_1)})^j, \tag{8}$$

Рассмотрим на плоскости z сектор T с вершиной в z_1:

$$\operatorname{Re} \lambda_1 (z - z_1) \leq 0, \quad \operatorname{Re} \lambda_2 (z - z_1) \leq 0.$$

В этом секторе

$$|e^{\lambda_1 (z - z_1)}| < 1, \quad |e^{\lambda_2 (z - z_1)}| < 1,$$

и $e^{\lambda_1 (z - z_1)} \to 0$, $e^{\lambda_2 (z - z_1)} \to 0$ при $z \to \infty$ в этом секторе.

Пусть теперь $x_1(z)$ и $y_1(z)$ однозначные в $\mathbb{K}'$ решение (1) и (2). Рассмотрим z_0 такое, что точка $x_1(z_0), y_1(z_0)$ попадает в малую окрестность M_0, где справедливо разложение (7) и (8). Тогда это решение при $z \to \infty$ по сектору T будет стремиться к M_0. Если, кроме того, решение $x_1(z), y_1(z)$ периодическое, то этот сектор превратится в полуплоскость.

Особых точек системы (3) и (4) - n^2. Следовательно, найдется пара различных особых точек, секторы которых пересекаются, что невозможно.

Противоречие возникло из-за предположения, что решение $y = y(x)$ на котором лежит не гомотопный нулю цикл не является сепаратрисным решением. Отсюда верна

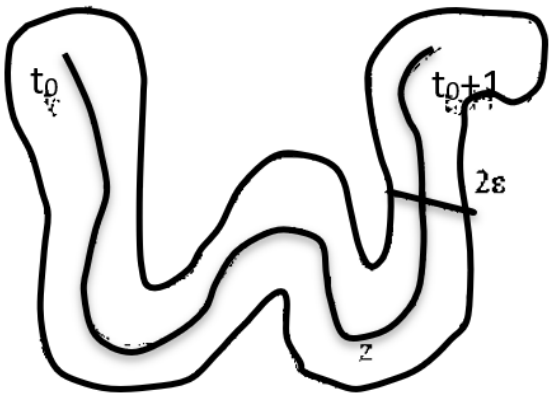
Теорема 3. Всякое решение (1) и (2), на котором лежит не гомотопный нулю цикл, является сепаратрисным решением.

Исследуем теперь сепаратрисные решения (1) и (2) и попробуем оценить число не гомотопных нулю и не гомотопных между собой циклов с несоизмеримыми периодами, лежащих на них.

4. О числе циклов, лежащих на одном решении $y = y(x)$ уравнения (1)

Согласно сказанному в начале работы, эта задача эквивалентна определению числа независимых периодов псевдопериодических решений (1) и (2). Пусть $x(t), y(t)$ - решение (1) и (2), имеющее 3 независимых периода. Тогда, как известно [3], это решение имеет всюду плотное множество периодов и само является всюду плотным в пространстве (x, y, t).

Пусть T_1 - один из периодов. Возьмем на плоскости t точку $t = t_0$, и пусть $x(t_0), y(t_0)$ - одно из значений решения $x(t), y(t)$. В силу определения T_1 существует путь L на плоскости t, начинающийся в точке t_0 и заканчивающийся в точке $t_0 + T_1$, такой, что значения $x(t), y(t)$ после обхода пути L, снова будут равны $x(t_0), y(t_0)$. Можно считать, что функции $x(t), y(t)$ являются регулярными в некоторой полосе шириной 2ϵ, окружающей линию L при движении от точки t_0 до $t_0 + T_1$ по нужной ветви.



Возьмем в малой окрестности t_0 точку t_0^* и точку $(x(t_0^*), y(t_0^*))$ принадлежащую решению $x(t), y(t)$ и такую, что $x(t_0^*) = x(t_0)$, $y(t_0^*) = y(t_0)$; такая точка существует в силу всюду плотности периодов. Продолжим ветвь, проходящую через точку $t = t_0^*$, $(x(t_0^*), y(t_0^*))$ до точки t_0. Получим новую точку $\tilde{x}(t_0), \tilde{y}(t_0)$. Возьмем путь L^* начинающийся в t_0^* и полученный из L параллельным переносом. Конец пути L^* будет в точке $t_0^* + l$. Точки t_0 и t_0^* берутся настолько близкими, что путь L лежит внутри соответствующей L^* полосы шириной 2ϵ. Это означает, что если мы начиная с точки $\tilde{x}(t_0), \tilde{y}(t_0)$, решения $x(t), y(t)$ будем двигаться от t_0 вдоль пути L до $t_0 + 1$, мы придем опять к значениям $\tilde{x}(t_0), \tilde{y}(t_0)$.

Точек $\tilde{x}(t_0), \tilde{y}(t_0)$, соответствующих различным значениям t_0^* с такими же свойствами, не менее чем счётно. Следовательно, в силу аналитичности уравнения, все решения системы, начинающиеся в малой окрестности точки $(t_0, x(t_0), y(t_0))$, будут псевдопериодическими с периодом 1. Другими словами, цикл лежащий на $y = y(x)$ и соответствующий этому периоду, будет тождественным циклом. Однако соответствующее $x(t), y(t)$ решение уравнения $y = y(x)$ является сепаратрисным решением в окрестности хотя бы одной особой точки уравнения. Тем самым, на $y = y(x)$ лежит цикл.

Таким образом, одновременно лежащий в малой окрестности точки M_0. Этот цикл очевидно не является тождественным. Мы получили противоречие с предложением о существовании трёх не гомотопных циклов на $y = y(x)$. Следовательно, верна

Теорема 3. В условиях общности P, Q, N на одном решении $y = y(x)$ могут находиться не более двух не гомотопных нулю и между собой циклов. При этом один из них лежит в малой окрестности одной из особых точек.

Из теоремы 2 и 3 вытекает основная

Теорема 4. Почти все уравнения вида $y' = P/Q$, где P, Q полиномы n-ой степени, обладают фундаментальными свойствами:

(1) Не гомотопные нулю циклы лежат на сепаратрисных решениях.
(2) Их число на каждом решении не более двух.
(3) Один из циклов на каждом решении непременно лежит в окрестности одной из особых точек уравнения.
(4) Число сепаратрисных решений равно $2n^2 + n + 1$.

Теорема 5. Число не гомотопных нулю и между собой циклов уравнения (**??**) не превышает $(2n^2 + n + 1)^2$.

Доказательство вытекает из пунктов b) и d) теоремы 4.

Как это вытекает из леммы [1], действительный предельный цикл, действительного уравнения, лежащей в малой окрестности особой точки уравнения на сепаратрисе этой особой точки, есть алгебраическая кривая.

Следовательно, верна

Теорема 6. Число действительных предельных циклов уравнения (**??**) с действительными коэффициентами не превышает $2(2n^2 + n + 1)$. При этом число не алгебраических предельных циклов не превышает $2n^2 + n + 1$.

Доказательство

Как было показано в [1], число действительных предельных циклов не больше числа комплексных циклов, лежащих на решениях (**??**) рассматриваемого в комплексной области. При этом можно считать, что циклы обладают свойствами, перечисленными в Теореме 3.

Отсюда вытекает доказательство.

Таким образом, в настоящей работе дана оценка числа предельных циклов уравнения $y' = P/Q$, где P и Q - полиномы. Кроме того, что не менее важно, указан путь, где их надо искать. А именно, доказано, что они лежат на сепаратрисных решениях. Важно также то, что элемент сепаратрисного решения явно вписывается в малую окрестность особой точки.

Список литературы

[1] Петровский И. Г., Ландис Е. М. О числе предельных циклов уравнения $y' = P/Q$, где P и Q - полиномы второй степени // Матем. сб. 1955. Т.37. Вып.2.
[2] Ландис Е. М., Петровский И.Г. О числе предельных циклов уравнения $y' = P/Q$, где P и Q - полиномы // Матем. сб. 1959. Т.48. Вып.2.
[3] Худай-Веренов М. Г. О функциях, определяемых дифференциальными уравнениями. Ашхабад, 1990 г.
[4] Л. Р. Форд. Автоморфные функции. Москва, 1936 г.
[5] А. Гурвиц, Р. Курант. Теория функций. Москва, 1968.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕРДІ ШЕШУДЕ ГРАФТАР ӘДІСІН ҚОЛДАНУ

НУРГУЛЬ ШАЖДЕКЕЕВА[1], АЙНАГУЛЬ АДИЕВА[2]

[1,2]Х. Досмұхамедов атындағы Атырау университеті , Атырау, Қазақстан
e-mail: [1]n.shazhdekeeva@mail.ru, [2]a.aina70@mail.ru

Әр түрлі қолданбаларда кездесетін дифференциалдық теңдеулерді графтардағы теңдеулер ретінде түсіндіруге болады. Графтардың қарапайым қасиеттерін пайдалана отырып, дифференциалдық теңдеулердің шешімдерінің әрекетін зерттеуге болады. Графтарда анықталатын функциялар кеңістігі мен дифференциалдық жүйелер арқылы графтардағы шеттік есептер анықталады. Бұл мақалада графтағы шеттік есепті дифференциалдық жүйе үшін шеттік есепке түрлендіру қарастырылады; графтың әрбір қырын (0, 1) интервалына түрлендіру арқылы графтағы дифференциалдық теңдеу қайта анықталады. Кейін шекаралық шарттар да интервалға сәйкес түрлендіріліп, бастапқы шеттік есеп пен жаңадан алынған шеттік есеп арасында байланыс орнатылады.

G қырларының ұзындығы ақырлы және олардың саны ақырлы K болатын граф болсын. Қырларды $e_i,\ i = 1, ..., K$ және олардың ұзындықтарын сәйкесінше $l_i,\ i = 1, ..., K$ деп белгілейік. G графында екінші ретті дифференциалдық операторды қарастырамыз:

$$ly := -\frac{d^2y}{dx^2} + q(x)y = \lambda y, \tag{1}$$

мұндағы $q(x)$ (алдағы уақытта да) G-да анықталған нақты функция.

G графындағы (1) дифференциалдық теңдеуін

$$-\frac{d^2y_i}{dx^2} + q_i(x)y_i = \lambda y_i, \quad x \in (0, l_i),\ i = 1, ..., K, \tag{2}$$

жүйесі түрінде қарастыруға болады, мұндағы $q_i = q\mid_{e_i}, y_i = y\mid_{e_i}$.

Келесі түрдегі шекаралық шарттарды қарастырамыз:

$$\sum_{j=1}^{K} \left[\alpha_{ij}y_j + \beta_{ij}y'_j\right](0) + \sum_{j=1}^{K} \left[\gamma_{ij}y_j + \delta_{ij}y'_j\right](l_j) = 0,\ i = 1, ..., 2K,$$

мұнда сызықтық тәуелсіз шекаралық шарттар саны $2K$-ға тең. Шекаралық шарттардың саны $2K$ болуы G-да шеттік есептің өз-өзіне түйіндес болуы үшін қажет (бірақ жеткілікті емес). Бұл мақалада графтағы шеттік есепті дифференциалдық жүйе үшін шеттік есепке түрлендіруді қарастырамыз.

ν төбесіндегі шекаралық шарттар осы төбеге енетін және шығатын қырлардағы y және y' шамаларының мәндері арқылы беріледі. Егер басы ν төбесі болатын қырларды $e_i,\ i \in \Lambda_s(\nu)$ және ұшы ν төбесі болатын қырларды $e_i,\ i \in \Lambda_s(\nu)$ деп белгілесек, онда ν төбесіндегі шекаралық шарттар келесі түрде өрнектеледі:

$$\sum_{j\in\Lambda_s(\nu)} \left[\alpha_{ij}y_j + \beta_{ij}y'_j\right](0) + \sum_{j\in\Lambda_e(\nu)} \left[\gamma_{ij}y_j + \delta_{ij}y'_j\right](l_j) = 0,\ i = 1, ..., N(\nu), \tag{3}$$

мұндағы $N(\nu)$ - ν төбесіндегі сызықтық тәуелсіз шекаралық шарттар саны.

$i = 1, ..., N(\nu)$ мен $j \notin \Lambda_s(\nu)$ үшін $\alpha_{ij} = 0 = \beta_{ij}$ және сол сияқты $i = 1, ..., N(\nu)$ мен $j \notin \Lambda_e(\nu)$ үшін $\gamma_{ij} = 0 = \delta_{ij}$ болсын. Онда (3) шекаралық шарттарды барлық төбелер үшін келесі түрде жазуға болады:

$$\sum_{j=1}^{K} \left[\alpha_{ij} y_j + \beta_{ij} y_j'\right](0) + \sum_{j=1}^{K} \left[\gamma_{ij} y_j + \delta_{ij} y_j'\right](l_j) = 0,\ i = 1, ..., 2K, \tag{4}$$

мұндағы $2K$- барлық сызықтық тәуелсіз шекаралық шарттардың саны. G графының толық геометриясы (қырларының саны мен ұзындықтарынан басқа) шекаралық шарттарда қамтылған.

Өз-өзіне түйіндес жағдай орын алу үшін (3) шартты қанағаттандыратын барлық $f, g \in C^2(G)$ үшін $(lf, g) - (f, lg)$ Лагранж тұрпаты нөлге айналуын талап етеміз. Өз-өзіне түйіндес шекаралық шарттар үшін $N\nu = \#\big(\Lambda_s(\nu)\big) + \#\big(\Lambda_e(\nu)\big)$ және $\sum_\nu N(\nu) = 2K$. Графтардағы өз-өзіне түйіндес есептердің қойылымы [2]-да қарастырылған және өз-өзіне түйіндес шекаралық шарттардың класы [3] пен [4]-де сипатталған.

G графындағы (2)-(3) шеттік есебін анықталу облысы

$$D(L) = \{f | f,\ f' \in AC,\ Lf \in L^2(G),\ (4) \text{ шартын қанағаттандырады}\} \tag{5}$$

немесе бұған эквивалент

$$D(L) = \{f | f,\ H^2(G),\ (3) \text{ шартын қанағаттандырады}\}$$

болатын тұйық әрі тығыз анықталған

$$Lf := -f'' + qf \tag{6}$$

операторы үшін $L^2(G)$ кеңістігіндегі оператордың меншікті мәнін табу есебі ретінде қарастыруға болады [1].

(2)-(3) есептің өз-өзіне түйіндес болуы L операторының $L^2(G)$ кеңістігінде тұйық әрі тығыз анықталған өз-өзіне түйіндес оператор болуын қамтамасыз етеді [5]. Графтағы шеттік есепті $(0, 1)$ интервалында жүйе үшін шеттік есеп ретінде қарастыруға болатынын көрсетейік.

Ұзындығы l_i болатын e_i қырын қарастырайық:

$$-y_i''(x) + q_i(x) y_i(x) = \lambda y_i(x), \quad x \in (0, l_i).$$

$t = \frac{x}{l_i}$ және $\tilde{y}_i(t) = y_i(l_i t)$ болсын. Онда

$$-\frac{d^2}{dt^2}\big[\tilde{y}_i(t)\big] = -l_i^2 y_i''(l, t) = l_i^2\big(\lambda y_i(l_i t) - q_i(l_i t) y_i(l_i t)\big) = l_i^2\big(\lambda - Q_i(t)\big)\tilde{y}_i(t),$$

мұндағы $Q_i(t) = q_i(l_i t)$.

Осылайша әрбір $i = 1, ..., K$ үшін біздің түрлендірген теңдеуіміз мына түрде болады:

$$y_i'' + l_i^2\big(Q_i - \lambda\big)\tilde{y}_i = 0,\ t \in (0, 1).$$

Алынған теңдеулерді келесі жүйе ретінде жазамыз:

$$\tilde{L}\tilde{Y} := -W\tilde{Y}'' + Q\tilde{Y} = \lambda\tilde{Y} \tag{7}$$

мұндағы $W = diag\Big[\frac{1}{l_1^2}, ..., \frac{1}{l_K^2}\Big],\ \tilde{Y} = \begin{bmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_K \end{bmatrix}$ және $Q = diag\Big[Q_1, ..., Q_K\Big]$.

Енді шекаралық шарттарды қарастырайық. Жоғарыдағы түрлендірулерден кейін барлық қырлардың ұзындықтары 1-ге тең болды және олардың шеткі нүктелері 0 мен 1 болды. Демек, шекаралық шарттарды матрицалық түрде жазуға болады:

$$\tilde{A}\tilde{Y}(0) + \tilde{B}\tilde{Y}''(0) + \tilde{C}\tilde{Y}(1) + \tilde{D}\tilde{Y}''(1) = 0 \tag{8}$$

мұндағы $\tilde{A} = \big[\alpha_{ij}\big],\ \tilde{B} = \Big[\frac{\beta_{ij}}{l_j}\Big],\ \tilde{C} = \big[\gamma_{ij}\big],\ \tilde{D} = \Big[\frac{\delta_{ij}}{l_j}\Big]$.

Осылайша G графындағы шеттік есеп (7) дифференциалдық теңдеу мен (8) шекаралық шарттар арқылы жүйе түріндегі шеттік есепке эквивалент болды.

Өлшемді L^2 векторлық кеңістігін L^2_K арқылы белгілейік:

$$L^2_K = \big\{F :\ (0,1) \to C^K \big| F_i \in L^2(0,1),\ i = 1, ..., K\big\}.$$

Ал скаляр көбейтінді келесі түрде болады:

$$\langle F, G\rangle_W = \sum_{i=1}^{K} l_i \int\limits_0^1 F_i \bar{G}_i dt = \int\limits_0^1 F^T W^{-\frac{1}{2}} \bar{G} dt \tag{9}$$

L^2_K кеңістігі $L^2(G)$ кеңістігіне изометриялы изоморфты болады. $L^2(G) \to L^2_K$ қатынасы келесі түрде анықталады:

$$f(x) \to \begin{bmatrix} f\big|_{e_1}(l_1 t) \\ \vdots \\ f\big|_{e_K}(l_K t) \end{bmatrix}$$

мұндағы $x \in G$ және $t \in (0,1)$.

(7)-(8) шеттік есебін оператордың меншікті мәнін табу есебі ретінде қарастыруға болады [6]:

$$\tilde{L}F = -WF'' + QF$$
$$D(\tilde{L}) = \big\{F \big| F, F' \in AC,\ \tilde{L}F \in L^2(G),\ (8)\ \text{қанағаттандырады}\big\}.$$

Лемма 1. (7)-(8) есебі L^2_K кеңістігінде өз-өзіне түйіндес болады сонда, тек сонда ғана, егер (2)-(3) шеттік есебі L^2_K кеңістігінде өз-өзіне түйіндес болса.

Дәлелдеу. $F, G \in C^2(0,1)$ және $F, G : (0,1) \to \mathbb{C}^K$ болсын және G графында f пен g функциялары $i = 1, ..., K,\ t \in (0,1)$ үшін $f\big|_{e_i}(l_i, t) = F_i(t)$ және $g\big|_{e_i}(l_i, t) = G_i(t)$ қатынастарымен анықталсын. Онда

$$\langle \tilde{L}F, G\rangle_W - \langle F, \tilde{L}G\rangle_W = -\sum_{i=1}^{K} l_i^{-1} \int\limits_0^1 \big[F_i'' \bar{G}_i - F_i \bar{G}_i''\big] dt = -\sum_{i=1}^{K} l_i^{-1} \big[F_i' \bar{G}_i - F_i \bar{G}_i'\big]_0^1 =$$

$$= \sum_{i=1}^{K} \Big[\big(f'\bar{g} - f\bar{g}'\big)\Big|_{e_i}\Big]_0^{l_i} = \big(Lf, g\big) - \big(f, Lg\big).$$

Демек, (3) орындалады сонда, тек сонда ғана егер (8) орындалса. Лемма дәлелденді.

Бұл жағдайда (7)-(8) есебінің өз-өзіне түйіндес болуы $\tilde{L}$ операторының L^2_K кеңістігінде тұйық әрі тығыз анықталған өз-өзіне түйіндес оператор болуын қамтамасыз етеді. Осылайша (2)-(3) есебінің өз-өзіне түйіндес болуы $\tilde{L}$ операторының L^2_K кеңістігінде тұйық әрі тығыз анықталған өз-өзіне түйіндес оператор болуын қамтамасыз етеді [5].

Әдетте өз-өзіне түйіндес болу тұрақтылықты қамтамасыз етпейді.

Жалпылықты бұзбай, шекаралық шарттар нормаланған деп есептеуге болады:

$$U_1(Y) = U_{10}(Y) - U_{11}(Y) = 0,$$

$$U_2(Y) = U_{20}(Y) - U_{21}(Y) = 0,$$

мұндағы

$$U_{10}(Y) = A_1Y'(0) + A_{10}Y(0),$$
$$U_{20}(Y) = A_2Y'(0) + A_{20}Y(0),$$
$$U_{11}(Y) = B_1Y'(0) + B_{10}Y(0),$$
$$U_{21}(Y) = B_2Y'(0) + B_{20}Y(0)$$

және әрбір $i = 1, 2$ үшін A_i мен B_i матрицаларының кемінде біреуі нөлден өзгеше.
Егер $A_i = 0$ болса, онда нормаландыру процесі бойынша A_{i0} матрицасы A_i болатынын көреміз және $B_i = 0$ үшін де осылай болады [6].

[6] сүйеніп, шекаралық шарттардың тұрақтылығын келесі түрде анықтаймыз.

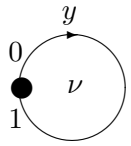
Анықтама. Жоғарыда нормаланған шекаралық щарттар тұрақты деп аталады, егер

$$\chi_- = i^{2n} det \begin{bmatrix} W^{-\frac{1}{2}}B_1 & -A_1 \\ \\ W^{-\frac{1}{2}}B_2 & -A_2 \end{bmatrix},$$

$$\chi_+ = i^{2n} det \begin{bmatrix} A_1 & -W^{\frac{1}{2}}B_1 \\ \\ A_2 & -W^{\frac{1}{2}}B_2 \end{bmatrix}$$

түрде анықталған χ_- пен χ_+ сандары нөлге тең болмаса, мұндағы W - (7) тұрақты оң диагонал матрица. Графтағы өз-өзіне түйіндес қарапайым шеттік есеп тұрақты болмауы мүмкін екендігіне мысал келтірейік.

Тек бір ғана ν төбесінен тұратын



және

$$-\frac{d^2y}{dx^2} + q(x)y = \lambda y$$

екінші ретті операторы мен

$$y(0) = y(1),\ y'(0) = y'(1)$$

шекаралық шарттарын қарастырайық. Онда

$$\chi_+ = -det \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} = 0.$$

Демек, тұрақтылық жоқ. Бұл мысалдан графтағы шеттік есептердің көпшілігінің тұрақты емес екендігін көруге болады.

Енді (7)-(8) шеттік есебін $(0, 1)$ интервалында шекаралық шарттары ажыратылған есеп ретінде қарастыруға болатынын көрсетейік. Ол үшін әрбір e_i қырының ортасы болатын m_i төбесін енгіземіз және бұл төбеде келесі шекаралық шарттар қарастырамыз:

$$y\big(m_i^-\big) = y\big(m_i^+\big)$$
$$y'\big(m_i^-\big) = y'\big(m_i^+\big)$$

m_i төбесіндегі шекаралық шарттар өз-өзіне түйіндес және G графының әрбір ν төбесіндегі шекаралық шарттар да өз-өзіне түйіндес болғандықтан, жаңадан алынған шеттік есеп те өз-өзіне түйіндес болып табылады.

Лемма 2. (9) скаляр көбейтіндісі бойынша өз-өзіне түйіндес (7)-(8) есебі $\langle F, G\rangle_M$ скаляр көбейтіндісі бойынша өз-өзіне түйіндес

$$\tau_P Y := -MY'' + PY = \lambda Y \tag{10}$$

$$A^*Y(0) - B^*Y'(0) = 0 \tag{11}$$

$$\Gamma^*Y(1) - \Delta^*Y'(1) = 0 \tag{12}$$

есебіне эквивалент, мұндағы $M = 4\begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix}$, $P = \begin{bmatrix} Q\left(\frac{t+1}{2}\right) & 0 \\ 0 & Q\left(\frac{1-t}{2}\right) \end{bmatrix}$, $A^* = \begin{bmatrix} I & -I \\ 0 & 0 \end{bmatrix}$, $-B^* = \begin{bmatrix} 0 & 0 \\ I & I \end{bmatrix}$, $\Gamma^* = \begin{bmatrix} \tilde{C} & \tilde{A} \end{bmatrix}$ және $\Delta^* = 2\begin{bmatrix} \tilde{D} & -\tilde{B} \end{bmatrix}$. Эквивалент деп меншікті мәндер мен олардың геометриялық еселіктері сақталып, ал меншікті функциялар

$$\tilde{Y}(t) \mapsto \tilde{Y}(t) := \begin{bmatrix} \tilde{Y}\left(\frac{t+1}{2}\right) \\ \tilde{Y}\left(\frac{1-t}{2}\right) \end{bmatrix}$$

түрде кескінделетінін айтамыз.

Дәлелдеу. λ (10)-(12) есебінің j-еселі меншікті мәні болсын. Онда (10)-(12) есептің Y шешімін λ меншікті мәніне сәйкес j сызықтық тәуелсіз меншікті векторлардан тұратын матрица түрінде жазуға болады.

Y_1 мен Y_2

$$Y := \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

түрдегі $K \times j$ матрицалар және

$$\tilde{Y}(s) = \begin{cases} Y_1(2s-1), \ \dfrac{1}{2} \le s \le 1 \\ Y_2(1-2s), \ 0 \le s \le \dfrac{1}{2} \end{cases} \tag{13}$$

болсын. (11) теңдеу бойынша $s = \frac{1}{2}$ нүктесінде $\tilde{Y}$ пен $\tilde{Y}'$ үзіліссіз болғандықтан, олар (0,1) интервалында үзіліссіз.

(12) теңдеуден келесі теңдеуді аламыз:

$$\begin{bmatrix} \tilde{C} & \tilde{A} \end{bmatrix} \begin{bmatrix} Y_1(1) \\ Y_2(1) \end{bmatrix} + 2\begin{bmatrix} \tilde{D} & -\tilde{B} \end{bmatrix} \begin{bmatrix} Y_1'(1) \\ Y_2'(1) \end{bmatrix} = 0.$$

Бұдан

$$\tilde{C}Y_1(1) + \tilde{A}Y_2(1) + 2\tilde{D}Y_1'(1) - \tilde{B}Y_2'(1) = 0.$$

Соңғы теңдеуден (13) бойынша (8) шығады.

Бұған қоса

$$-M\begin{bmatrix} Y_1'' \\ Y_2'' \end{bmatrix} + P\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \lambda \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}.$$

Онда

$$-4WY_1'' + Q\Big(\frac{t+1}{2}\Big)Y_1 = \lambda Y_1$$

және

$$-4WY_2'' + Q\Big(\frac{1-t}{2}\Big)Y_2 = \lambda Y_2.$$

Бұдан $\tilde{Y}$-ке қатысты (7) теңдеуді аламыз.

$\tilde{Y}$ матрицасының бағандары сызықты тәуелсіз болғандықтан, (7), (8) меншікті кеңістіктігінің өлшемі кем дегенде j-ға тең деп айта аламыз, яғни $n \geq j$.

Керісініше, егер λ (7), (8) есебінің n-еселі меншікті мәні болса, онда (7), (8) есебінің $\tilde{Y}$ шешімін λ меншікті мәніне сәйкес (7), (8) есебінің меншікті векторларынан құралған n сызықты тәуелсіз бағандардан тұратын матрица түрінде жазуға болады.

Теореманың қойылымында берілген түрлендіру бойынша

$$Y(0) = \begin{bmatrix} \tilde{Y}\left(\frac{1}{2}^{+}\right) \\ \\ \tilde{Y}\left(\frac{1}{2}^{-}\right) \end{bmatrix}$$

және

$$Y(1) = \begin{bmatrix} \tilde{Y}(1) \\ \\ \tilde{Y}(0) \end{bmatrix}.$$

Оған қоса

$$Y'(t) = \begin{bmatrix} \frac{1}{2}\tilde{Y}'\left(\frac{t+1}{2}\right) \\ \\ -\frac{1}{2}\tilde{Y}'\left(\frac{1-t}{2}\right) \end{bmatrix},$$

бұдан

$$Y'(0) = \frac{1}{2}\begin{bmatrix} \tilde{Y}'\left(\frac{1}{2}^{+}\right) \\ \\ \tilde{Y}'\left(\frac{1}{2}^{-}\right) \end{bmatrix}.$$

Осылайша $\begin{bmatrix} I & -I \\ 0 & 0 \end{bmatrix} Y(0) = 0$ және $\begin{bmatrix} 0 & 0 \\ -I & I \end{bmatrix} Y'(0) = 0.$

Бұдан $A^*Y(0) - B^*Y'(0) = 0$, яғни (11) орындалады. Оған қоса

$$Y'(1) = \frac{1}{2}\begin{bmatrix} \tilde{Y}'(1) \\ \\ -\tilde{Y}'(0) \end{bmatrix}$$

және (8) теңдеуден

$$\begin{bmatrix} \tilde{C} & \tilde{A} \end{bmatrix} Y(1) + 2\begin{bmatrix} \tilde{D} & -\tilde{B} \end{bmatrix} Y'(1) = 0,$$

бұдан $\Gamma^*Y(1) - \Delta^*Y'(1) = 0$, яғни (12) теңдеу орындалады.

$$Y''(t) = \frac{1}{4}\begin{bmatrix} \tilde{Y}''\left(\frac{t+1}{2}\right) \\ \\ \tilde{Y}''\left(\frac{1-t}{2}\right) \end{bmatrix}$$

болғандықтан,

$$-4\begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} Y''(t) = \begin{bmatrix} -W\tilde{Y}''\left(\frac{t+1}{2}\right) \\ \\ -W\tilde{Y}''\left(\frac{1-t}{2}\right) \end{bmatrix} =$$

$$= \begin{bmatrix} \left(\lambda - Q\left(\frac{t+1}{2}\right)\right)\tilde{Y}\left(\frac{t+1}{2}\right) \\ \left(\lambda - Q\left(\frac{1-t}{2}\right)\right)\tilde{Y}\left(\frac{1-t}{2}\right) \end{bmatrix} =$$

$$= \lambda Y(t) - \begin{bmatrix} Q\left(\frac{t+1}{2}\right) & 0 \\ 0 & Q\left(\frac{1-t}{2}\right) \end{bmatrix} Y(t).$$

Осылайша (10) теңдеу орындалады және (10)-(12) меншікті кеңістігінің өлшемі кем дегенде n-ға тең, яғни $j \geq n$.

Басқаша айтқанда, егер λ (10)-(12) есебінің меншікті мәні болса, онда ол (7), (8) есебінің де еселігі бірдей меншікті мәні болады және керісінше.

Бұл мақалада графтағы шеттік есеп дифференциалдық жүйе үшін берілген шеттік есеп ретінде көрсетілді. Графтағы шеттік есеп графтағы дифференциалдық операторлың меншікті мәнін табу есебі ретінде қарастырылып, оның спектрлік құрылымын зертеуге мүмкіндік береді.

Кілттік сөздер: граф, дифференциалдық теңдеу, шеттік есеп, дифференциалдық оператор, өз-өзіне түйіндес оператор, меншікті мән.

AMS пәндік классификация: 39

Әдебиеттер тізімі

[1] Sonja C., Spectral theory of differential operators on graphs, *University of the Witwatersrand,South Africa*, 2005, 130 p.

[2] Carlson R., Adjoint and self-adjoint differential operators on graphs, *Electronic J. Differential Equations*, Vol.1998, No.6, 1998, pp.1-10.

[3] Harmer M., Hermitian symplectic geometry and extension theory, *Journal of Physics A: Mathematical and General* , Vol.33, No.50, 2000, pp.9193-9203.

[4] Kostrykin V., Schrader R., Kirchhoff's rule for quantum wires, *Journal of Physics A: Mathematical and General* , Vol.32, No.4, 1999, pp.595-630.

[5] Yoshida K., Functional Analysis, *Springer-Verlag, Berlin Heidelberg New York*, 1980.

[6] Weidmann J., Linear Operators in Hilbert Spaces, *Springer-Verlag*, 1980.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ТРЕТЬЯ КРАЕВАЯ ЗАДАЧА ДЛЯ УРАВНЕНИЯ ТРЕТЬЕГО ПОРЯДКА С ПЕРЕМЕННЫМИ КОЭФФИЦИЕНТАМИ

РАХМАТИЛЛА УМАРОВ[1]

[1]Наманганский инженерно-строительный институт, Наманган, Узбекистан
e-mail: r.umarov1975@mail.ru

Аннотация. В работе рассматривается третья краевая задача в прямоугольной области для неоднородного уравнения третьего порядка с младшими членами, переменными коэффициентами . Единственность решения поставленной задачи доказана методом интегралов энергии. Решение выписано через построенную функцию Грина.

Ключевые слова:Дифференциальное уравнение, третий порядок, кратные характеристики, регулярное решение, единственность, существование, функция Грина.

Предметная классификация AMS: 35G15

1. Введение

Дифференциальные уравнения в частных производных третьего порядка рассматриваются при решении задач теории нелинейной акустики и в гидродинамической теории космической плазмы, фильтрации жидкости в пористых средах [1].

В совокупности, всех уравнений третьего порядка особое место по специфическому характеру занимают, уравнения с кратными характеристиками.

В работе [2], учитывая свойства вязкости и теплопроводности газа, из системы Навье-Стокса было получено уравнение третьего порядка с кратными характеристиками, содержащее вторую производную по времени

$$u_{xxx} + u_{yy} - \frac{\nu}{y}u_y = u_x u_{xx}, \quad \nu = const.$$

Это уравнение при описывает осесимметричный поток, а при описывает плоско - параллельный поток [3].

Первые результаты по уравнению третьего порядка с кратными характеристиками были получены в работах H. Block [4], E. DelVecehio[5].

L. Catabriga в работе [6] для уравнения $D_x^{2n+1}u - D_y^2 u = 0$ построил фундаментальное решение в виде двойного несобственного интеграла и изучил свойства потенциала, решил краевые задачи.

В работах [7-8] построены фундаментальные решения уравнения третьего порядка с кратными характеристиками, содержащие вторые производные по времени, выраженные через вырожденные гипергеометрические функции, изучены их свойства, найдена оценки при $|t| \to \infty$.

В работах [9-13], рассмотрены краевые задачи для уравнений третьего порядка с кратными характеристиками, при помощи построения функции Грина. Также, отметим работы

[14-19], в которых, рассмотрены краевые задачи для уравнений третьего порядка. В работах [20-23] решение поставленной задачи для уравнения третьего порядка с постоянными коэффициентами было найдено с другими краевыми условиями.

2. Постановка задачи

В области $D = \{(x,y): \ 0 < x < p, \ 0 < y < q\}$ рассмотрим следующее уравнение третьего порядка в вида

$$L(u) = U_{xxx} - U_{yy} + A_1(x)\,U_{xx} + A_2(x)\,U_x + A_3(x)\,U + A_4 U_y = g_1(x,y), \tag{1}$$

где $p, \ q, A_4 \in R, \ A_i(x), \ i = \overline{1,3}, \ g_1(x,y)$ заданная, достаточно гладкая функция.

Заменой

$$U(x,y) = \exp\left(-\frac{1}{3}\int\limits_0^x A_1(\xi)\,d\xi + \frac{A_4}{2}y\right) u(x,y),$$

уравнение (1) можно привести к виду

$$u_{xxx} - u_{yy} + a_1(x)\,u_x + a_2(x)\,u = g(x,y), \tag{2}$$

где

$$a_1(x) = -A_1'(x) - \frac{1}{3}A_1{}^2(x) + A_2(x),$$

$$a_2(x) = -\frac{1}{3}A_1''(x) + \frac{2}{27}A_1{}^3(x) - \frac{1}{3}A_1(x)\,A_2(x) + A_3(x) + \frac{A_4^2}{4},$$

$$g(x,y) = \exp\left(\frac{1}{3}\int\limits_0^x A_1(\xi)\,d\xi - \frac{A_4}{2}y\right) g_1(x,y).$$

Задача B_3**.** Найти функцию $u(x,y)$ из класса $C^{3,2}_{x,y}(D) \cap C^{2,1}_{x,y}\left(\overline{D}\right)$, удовлетворяющую уравнению (1) и следующим краевым условиям:

$$\begin{cases} \alpha u(x,0) + \beta\, u_y(x,0) = 0, \\ \gamma u(x,q) + \delta\, u_y(x,q) = 0, \end{cases} \qquad 0 \le x \le p, \tag{3}$$

$$u(0,y) = \psi_1(y), \quad u_x(p,y) = \psi_2(y), \quad u_{xx}(p,y) = \psi_3(y), \quad 0 \le y \le q, \tag{4}$$

где $\psi_i(y), \ i = \overline{1,3}$ заданные функции.

3. Единственность решения

Теорема 1. Если задача B_3 имеет решение, то при выполнении условий $a_1(x) \ge 0, \ \ a_2(x) - \frac{1}{2}a_1'(x) \ge 0, \ \alpha\beta \le 0, \ \gamma\delta \ge 0$ оно единственно.

Доказательство. Предположим, обратное. Пусть задача B_3 имеет два решения $u_1(x,y)$ и $u_2(x,y)$. Тогда функция $u(x,y) = u_1(x,y) - u_2(x,y)$ удовлетворяет однородному уравнению (2) с однородными краевыми условиями. Докажем, что $u(x,y) \equiv 0$ в $\bar{D}$.

В области D справедливо тождество

$$uL[u] = uu - uu + a_1(x)\,uu_x + a_2(x)\,u^2 = 0,$$

или

$$\frac{\partial}{\partial x}\left(uu_{xx} - \frac{1}{2}u_x^2 + \frac{1}{2}a_1(x)\,u^2\right) - \frac{\partial}{\partial y}(uu_y) + u_y^2 + \left(a_2(x) - \frac{1}{2}a_1'(x)\right)u^2 = 0.$$

Интегрируя тождество (5) по области D, требуя $\alpha \ne 0, \ \gamma \ne 0$ и учитывая однородные краевые условия, получим

$$\begin{gathered} \tfrac{1}{2}\int\limits_0^q a_1(x)\,u^2(p,y)\,dy + \tfrac{1}{2}\int\limits_0^q u_x^2(0,y)\,dy + \tfrac{\delta}{\gamma}\int\limits_0^p u_y^2(x,q)\,dx - \\ -\tfrac{\beta}{\alpha}\int\limits_0^p u_y^2(x,0)\,dx + \iint\limits_D u_y^2 dxdy + \iint\limits_D \left(a_2(x) - \tfrac{1}{2}a_1'(x)\right)u^2 dxdy = 0. \end{gathered} \tag{5}$$

Если $a_2(x) - \frac{1}{2}a_1'(x) \neq 0$, то из последнего слагаемого получим $u(x,y) \equiv 0$. Если то $a_2(x) - \frac{1}{2}a_1'(x) = 0,\ a_1(x) \neq 0$, тогда из первого и пятьего слагаемого $u(p,y) = 0,\ u_y(x,y) = 0$. Отсюда $u(x,y) = f(x),\ f(p) = 0$. Поставив уравнения (2) и учитывая краевые условие (4), получим задачу

$$\begin{cases} f'''(x) + a_1(x) f'(x) + a_2(x) f(x) = 0, \\ f(p) = f'(p) = f''(p) = 0. \end{cases}$$

Линейное однородное уравнение имеет фундаментальную систему решения в виде

$$f(x) = C_1 X_1(x) + C_2 X_2(x) + C_3 X_3(x),$$

Для нахождения $C_1,\ C_2,\ C_3$ воспользуемся краевыми условиями:

$$\begin{cases} C_1 X_1(p) + C_2 X_2(p) + C_3 X_3(p) = 0, \\ C_1 X'_1(p) + C_2 X'_2(p) + C_3 X'_3(p) = 0, \\ C_1 X''_1(p) + C_2 X''_2(p) + C_3 X''_3(p) = 0, \end{cases}$$

Определитель этой системы равен определителю Вронского и отлично от нуля. Тогда $C_1 = C_2 = C_3 = 0$, Решением этой задачи $f(x) \equiv 0$, тогда $u(x,y) \equiv 0$. Если, $a_2(x) - \frac{1}{2}a_1'(x) = 0,\ a_1(x) = 0$ получим задачу

$$\begin{cases} f'''(x) = 0, \\ f(0) = f'(p) = f''(p) = 0. \end{cases}$$

Решением этой задачи $f(x) \equiv 0$. В случаях $\beta \neq 0,\ \delta \neq 0; \quad \alpha \neq 0,\ \delta \neq 0; \quad \beta \neq 0,\ \gamma \neq 0$ аналогично получаем тождество $u(x,y) \equiv 0, \quad (x,y) \in \bar{D}$. Теорема 1 доказана.

Замечание. Отметим, что при нарушении условия теоремы 1 однородная задача для однородного уравнения (2) может иметь нетривиальные решения. Например, можно легко убедится, что когда $p = 1,\ q = \pi,\ a_1(x) = 0,\ a_2 = -\left(\mu^3 + 1\right) < 0$ задача

$$\begin{cases} u_{xxx}(x,y) - \left(\mu^3 + 1\right) u(x,y) - u_{yy}(x,y) = 0, \\ u_y(x,0) = u_y(x,\pi) = 0, \quad u(0,y) = u_x(1,y) = u_{xx}(1,y) = 0, \end{cases}$$

где μ решения уравнение

$$e^{-\frac{3}{2}\mu} - 2\sin\left(\frac{\sqrt{3}}{2}\mu - \frac{\pi}{2}\right) = 0,$$

имеет нетривиальные решения вида

$$u(x,y) = \left(e^{\mu x}\sin\left(\frac{\sqrt{3}}{2}\mu - \frac{\pi}{3}\right) - e^{-\frac{1}{2}\mu x}\sin\left(\frac{\sqrt{3}}{2}\mu(1-x) - \frac{\pi}{3}\right) + e^{\frac{1}{2}\mu(3-x)}\sin\left(\frac{\sqrt{3}}{2}\mu x\right)\right) Y_n(y),$$

где λ_n собственные значения, $Y_n(y)$ соответствующие собственные функции задачи

$$\begin{cases} Y''(y) + \lambda_n^3 Y(y) = 0, \\ \alpha Y(0) + \beta Y'(0) = 0, \\ \gamma Y(q) + \delta Y'(q) = 0. \end{cases} \tag{6}$$

Отсюда, следует, что если является параметром разделения, тогда при $a_2 \geq 0$ задача корректно поставлена, при $a_2 < 0$ задача не корректна, т.е. существует спектр.

4. Существование решения

Теорема 2. Если выполняются следующие условия:

1) $\psi_i(y) \in C^4[0,q]$, $i=\overline{1,3}$,

$\alpha\psi_i(0)+\beta\psi_i'(0)=0$, $\gamma\psi_i(q)+\delta\psi_i'(q)=0$,

$\alpha\psi''_i(0)+\beta\psi'''_i(0)$, $\gamma\psi''_i(q)+\delta\psi'''_i(q)=0$,

2) $\dfrac{\partial^3 g(x,y)}{\partial x\partial y^2}\in C[0,q]$, $0\le x\le p$; $\alpha g(x,0)+\beta g'(x,0)=0$, $\gamma g(x,q)+\delta g'(x,q)=0$;

3) $0\le C<\dfrac{\lambda_1^2}{Kp(1+\lambda_1)}$,

4) $a_1(p)=0$,

то решение задачи B_3 существует. Здесь $C=\max\{|a_1(x)|,|a_1'(x)-a_2(x)|, x\in[0,p]\}$, $\lambda_1=\sqrt[3]{\left(\frac{\pi}{q}\right)^2}$,$K=\frac{16}{3}\left(1-\exp\left(-\frac{2\sqrt{3}\pi}{3}\right)\right)^{-1}$.

Доказательство. Рассмотрим задачу Штурма-Лиувилля (6). Известно (см [10]), что для задачи (6) собственные значения λ_n существует только при $\lambda_n>0$ и соответствующие общее решение имеет вид

$$Y(y)=C_1\cos\sqrt{\lambda_n^3}y+C_2\sin\sqrt{\lambda_n^3}y,$$

где C_1, C_2- произвольные постоянные. Удовлетворив эту функцию краевым условиям задачи (6) получим:

$$\alpha_1+\beta\sqrt{\lambda^3}C_2=0,\quad C_1=-\frac{\beta}{\alpha}\sqrt{\lambda^3}C_2,$$

$$\cos\sqrt{\lambda_n^3}q\left(\gamma C_1+\delta\sqrt{\lambda_n^3}C_2\right)+\sin\sqrt{\lambda_n^3}q\left(\gamma C_1-\delta\sqrt{\lambda_n^3}C_2\right)=0,$$

отсюда, исключив постоянные C_1, C_2, приходим к трансцендентному уравнению для определения

$$ctg\sqrt{\lambda_n^3}q=\frac{\alpha\gamma+\delta\beta\lambda_n^3}{\sqrt{\lambda_n^3}(\gamma\beta-\alpha\delta)}. \tag{7}$$

Положив $\xi=\sqrt{\lambda_n^3}q>0$, получим $ctg\xi=\frac{a+b\xi^2}{c\xi}$, где $a=\alpha\gamma q^2$, $b=\delta\beta$, $c=q(\gamma\beta-\alpha\delta)$. С целью нахождения решения последнего уравнения, рассмотрим следующую систему кривых:

$$\begin{cases}\eta=ctg\xi,\\ \eta=\dfrac{a+b\xi^2}{c\xi}=\dfrac{1}{c}\left(\dfrac{a}{\xi}+b\xi\right).\end{cases} \tag{8}$$

Точки пересечения двух кривых, участвующих в (8) являются решением уравнения (7), т.е. собственным значением $\lambda_n=\sqrt[3]{\left(\frac{\xi_n}{q}\right)^2}$ нашей задачи, здесь

$$\lambda_n^3=O\left(n^2\right). \tag{9}$$

собственные функции имеют вид:

$$Y_n(y)=\alpha\sin\sqrt{\lambda_n^3}y-\beta\sqrt{\lambda_n^3}\cos\sqrt{\lambda_n^3}y,\ n\in N, \tag{10}$$

Известно [10], что функции (7) образует в $L_2[0,q]$ полную ортогональную систему и при выполнения условии теоремы 2 на заданных функции, можно разложит по системе $\{Y_n(y)\}$.

Разложим $g(x,y)$ в ряд Фурье по $\{Y_n(y)\}$:

$$g(x,y)=\sum_{n=1}^{\infty}g_n(x)\frac{Y_n(y)}{\|Y_n\|^2},$$

где $\|Y_n\|^2 = \int\limits_0^q Y_n^2(y)\,dy = 1, \quad g_n(x) = \int\limits_0^q g(x,\eta)\,Y_n(\eta)\,d\eta.$

Интегрируем функцию $g_n(x)$ по частям два раза и учитывая условия теоремы 2 имеем оценку

$$|g_n(x)| \le M\frac{|F_n(x)|}{\lambda_n^3}, \tag{11}$$

здесь $F_n(x) = \int\limits_0^q \frac{\partial^2 g(x,\eta)}{\partial\eta^2} Y_n(\eta)\,d\eta \le \sqrt{\int\limits_0^q \left(\frac{\partial^2 g(x,\eta)}{\partial\eta^2}\right)^2 d\eta}\sqrt{\int\limits_0^q (Y_n(\eta))^2 d\eta} \le M.$

Решение задачи B_3 ищем в виде

$$u(x,y) = \sum_{i=1}^{\infty} X_n(x)\,Y_n(y), \tag{12}$$

(12) поставляя в уравнение (2) получим следующую задачу:

$$\begin{cases} X'''_n + a_1 X'_n + a_2 X_n + \lambda_n^3 X_n = g_n(x), \\ X_n(0) = \psi_{2n},\ X'_n(p) = \psi_{3n},\ X''_n(p) = \psi_{1n}, \end{cases} \tag{13}$$

где $\psi_{in} = \frac{1}{\|Y_n\|}\int\limits_0^q \psi_i(\eta)\,Y_n(\eta)\,d\eta,\ i = \overline{1,3}.$

Учитывая условия теоремы 2, интегрируем по частям ψ_{in} четыре раза находим оценку

$$|\psi_{in}| \le \frac{1}{\|Y_n\|}\frac{|\Psi_{in}|}{\lambda^6},\ i = \overline{1,3}. \tag{14}$$

где $\Psi_{in} = \frac{1}{\|Y_n\|}\int\limits_0^q {\psi_i}^{IV}(\eta)\,Y(\eta)\,d\eta, \quad i = \overline{1,3}.$

Введём новую функцию $V(x)$, по формуле

$$V_n(x) = X_n(x) - \rho_n(x), \tag{15}$$

где $\rho_n(x)$ имеет вид:

$$\rho_n(x) = \psi_{1n} + x\psi_{2n} + \frac{x^2 - 2xp}{2}\psi_{3n}. \tag{16}$$

Подставляя (16), (15) в (13) получим задачу

$$\begin{cases} V_n''' + \lambda_n^3 V_n = \lambda_n^3 f_n(x) - a_1(x)\,V_n' - a_2(x)\,V_n, \\ V_n(0) = V_n'(p) = V_n''(p) = 0, \end{cases} \tag{17}$$

здесь

$$f_n(x) = -\left(\frac{a_2(x)}{\lambda_n^3} + 1\right)\psi_{1n} - \left(\frac{xa_2(x)+a_1(x)}{\lambda_n^3} + x\right)\psi_{2n} -$$
$$-\left(\frac{xa_1(x) - pa_1(x) - xa_2(x)p}{\lambda_n^3} + \frac{a_2(x)}{\lambda_n^3}x^2 + \frac{x^2-2xp}{2}\right)\psi_{3n} + \frac{g_n(x)}{\lambda_n^3}$$

Отсюда, учитывая оценки (11), (14) имеем

$$|f_n(x)| \le M|\psi_{1n}| + M|\psi_{2n}| + M|\psi_{3n}| + M\frac{|F_n(x)|}{\lambda_n^6} \le M\left(\frac{|\Psi_{1n}|}{\lambda_n^6} + \frac{|\Psi_{2n}|}{\lambda_n^6} + \frac{|\Psi_{3n}|}{\lambda_n^6} + \frac{F_n(x)}{\lambda_n^6}\right).$$

Учитывая это, получим следующие оценки

$$\begin{aligned} |f_n(x)| &\le \frac{M}{\lambda_n^6}\left(|\Psi_{1n}| + |\Psi_{2n}| + |\Psi_{3n}| + |F_n(x)|\right), \\ \left|f'_n(x)\right| &\le \frac{M}{\lambda_n^6}\left(|\Psi_{1n}| + |\Psi_{2n}| + |\Psi_{3n}| + \left|F'_n(x)\right|\right). \end{aligned} \tag{18}$$

Согласно теореме Гильберта, решение задачи (17) ищем следующим образом:

$$V_n(x) = \lambda_n^3 \int_0^p G_n(x,\xi) f_n(\xi)\, d\xi - \int_0^p G_n(x,\xi)\, a_1(\xi)\, V_n{}'(\xi)\, d\xi - \int_0^p G_n(x,\xi)\, a_2(\xi)\, V_n(\xi)\, d\xi \quad (19)$$

где $G_n(x,\xi)$ функция Грина задачи (17), которая имеет вид

$$G_n(x,\xi) = \begin{cases} G_{1n}(x,\xi),\ 0 \le x \le \xi, \\ G_{2n}(x,\xi),\ \xi \le x \le p. \end{cases}$$

Здесь

$$G_{1n}(x,\xi) = \frac{1}{\sqrt{3}\lambda_n^2\Delta}\left(-e^{\lambda_n\left(p-x-\frac{\xi}{2}\right)}\sin\left(\frac{\sqrt{3}}{2}\lambda_n\xi+\frac{\pi}{6}\right) - e^{\lambda_n\left(\xi-x-\frac{p}{2}\right)}\cos\left(\frac{\sqrt{3}}{2}\lambda_n p\right) + e^{\lambda_n\left(\xi+\frac{x}{2}-\frac{p}{2}\right)}\cos\left(\frac{\sqrt{3}}{2}\lambda_n(x-p)\right) + e^{\lambda_n\left(p+\frac{x}{2}-\frac{\xi}{2}\right)}\sin\left(\frac{\sqrt{3}}{2}\lambda_n(\xi-x)+\frac{\pi}{6}\right) - 2e^{-\lambda_n\left(\frac{\xi}{2}+\frac{p}{2}-\frac{x}{2}\right)}\sin\left(\frac{\sqrt{3}}{2}\lambda_n x\right)\cos\left(\frac{\sqrt{3}}{2}\lambda_n(\xi-p)+\frac{\pi}{6}\right)\right)$$

$$G_{2n}(x,\xi) = \frac{1}{\sqrt{3}\lambda_n^2\Delta}\left(1-2e^{-\frac{3\lambda_n}{2}\xi}\sin\left(\frac{\sqrt{3}}{2}\lambda_n\xi+\frac{\pi}{6}\right)\right)\cdot \left(\frac{1}{2}e^{\lambda_n(\xi+p-x)} + e^{\lambda_n\left(\xi-\frac{p}{2}+\frac{x}{2}\right)}\cos\left(\frac{\sqrt{3}}{2}\lambda_n(x-p)\right)\right).$$

Оценка $G_n(x,\xi)$имеет вид

$$|G_n(x,\xi)| \le \frac{K}{\lambda_n^2}, \quad \left|\frac{\partial G_n(x,\xi)}{\partial\xi}\right| \le \frac{K}{\lambda_n}. \quad (20)$$

Интегрируя по частям второй интеграл в (19), имеем

$$V_n(x) = V_{0n}(x) + \int_0^p \bar{G}_n(x,\xi)\, V_n(\xi)\, d\xi. \quad (21)$$

Здесь

$$V_{0n}(x) = \lambda_n^3 \int_0^p G_n(x,\xi) f_n(\xi)\, d\xi,$$

$$\bar{G}_n(x,\xi) = \frac{\partial G_n(x,\xi)}{\partial\xi} a_1(\xi) + G_n(x,\xi)\left(a_1{}'(\xi) - a_2(\xi)\right), \quad (22)$$

Используя равенство (20), имеем оценку для $\bar{G}_n(x,\xi)$ в виде

$$\left|\bar{G}_n\right| \le |a_1(\xi)|\left|\frac{\partial G_n(x,\xi)}{\partial\xi}\right| + \left|a_1{}'(\xi) - a_2(\xi)\right||G_n| \le \left(\frac{1}{\lambda_n} + \frac{1}{\lambda_n^2}\right)KC.$$

Уравнения (21) является интегральным уравнением Фредгольма второго рода. Запишем решение (21) с помощью резольвенты в виде

$$V_n(x) = V_{0n}(x) + \int_0^p R_n(x,\xi)\, V_{0n}(\xi)\, d\xi,$$

где

$$R_n(x,\xi) = \bar{G}_n(x,\xi) + \sum_{m=1}^{\infty} \bar{G}_{mn}(x,\xi). \quad (23)$$

Здесь

$$\bar{G}_{mn}(x,\xi) = \int_0^p \bar{G}_n(x,s)\, \bar{G}_{(m-1)n}(s,\xi)\, ds,\ m = 1,2,\dots,\ \bar{G}_{0n}(x,\xi) = \bar{G}_n(x,s).$$

Оценим резольвента (23). Из

$$R_n(x,\xi) = \bar{G}_{0n}(x,\xi) + \bar{G}_{1n}(x,\xi) + \bar{G}_{2n}(x,\xi) + ... + \bar{G}_{mn}(x,\xi) + ...$$

найдём оценку

$$|R_n(x,\xi)| \le \left|\bar{G}_{0n}(x,\xi)\right| + \left|\bar{G}_{1n}(x,\xi)\right| + \left|\bar{G}_{2n}(x,\xi)\right| + ... + \left|\bar{G}_{mn}(x,\xi)\right| + ...$$

Для правой части этого неравенство составим мажорирующий ряд. Введя обозначение

$$J = \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_1^2}\right) KC$$

, находим

$$\left|\bar{G}_{1n}(x,\xi)\right| \le \left|\bar{G}_n(x,\xi)\right| \le KC\left(\frac{1}{\lambda_n} + \frac{1}{\lambda_n^2}\right) \le J,$$

$$\left|\bar{G}_{2n}(x,\xi)\right| \le \int_0^p \left|\bar{G}_{1n}(x,s)\right| \left|\bar{G}_{1n}(s,\xi)\right| ds \le J^2 p,$$

. . . . . .

$$\left|\bar{G}_{mn}(x,\xi)\right| \le \int_0^p \left|\bar{G}_{1n}(x,s)\right| \left|\bar{G}_{(m-1)n}(s,\xi)\right| ds \le J^m p^{m-1},\ i = 2,3,...,$$

. . . . . .

Тогда мажорирующий ряд имеет вид

$$\frac{1}{p}\sum_{m=1}^{\infty} (Jp)^m. \tag{24}$$

Из условия теоремы 2 имеем $C < \frac{\lambda_1^2}{Kp(1+\lambda_1)}$. Эту неравенство можно записать в виде

$$\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_1^2}\right) KC < \frac{1}{p},$$

отсюда

$$Jp < 1,$$

тогда мажорирующий ряд (24) является бесконечно убывающей геометрической прогрессией. В этом случае резольвента равномерно сходиться, и его оценка имеет вид

$$|R(x,\xi)| \le \frac{J}{1-Jp} \le M. \tag{25}$$

Подставляя $G_n(x,\xi) = \frac{1}{\lambda_n^3} G_{n\xi\xi\xi}(x,\xi)$ в $V_{0n}(x)$ и интегрируя, имеем

$$V_{0n}(x) = f_n(x) - G_{n\xi\xi}(x,0) f_n(0) + G_{n\xi\xi}(x,p) f_n(p) - \int_0^p G_{n\xi\xi}(x,\xi) f_n{}'(\xi)\, d\xi$$

.

Учитывая оценки (20) и

$$|G_{2n\xi\xi}(x,0)| \le K,\ |G_{1n\xi\xi}(x,p)| \le K,\ |G_{1n\xi\xi}(x,\xi)| \le K,$$

получим

$$|V_{0n}(x)| \le \frac{M}{\lambda_n^6}\left(|\Psi_{1n}| + |\Psi_{2n}| + |\Psi_{3n}| + |F_n(x)| + |F_n(0)| + |F_n(p)| + \left|F'_n(x)\right|\right). \tag{26}$$

Из (25) и (26) имеем оценку

$$|V_n(x)| \le |V_{0n}(x)| + \int_0^p |R(x,\xi)|\,|V_{0n}(\xi)|\,d\xi \le \frac{M}{\lambda_n^6}\left(\sum_{i=1}^{3}|\Psi_{in}| + |F_n(x)| + |F_n(0)| + |F_n(p)| + \left|F'_n(x)\right|\right).$$

В силу (15) и (12) решение задачи (13) имеет вид

$$u(x,y) = \sum_{n=1}^{\infty}\left(V_n(x) + \rho_n(x)\right)Y_n(y)$$

. Проверим это решение на сходимость. Учитывая оценку

$$|\rho_n(x)| \le \frac{M}{\lambda_n^6}\left(|\Psi_{1n}| + |\Psi_{2n}| + |\Psi_{3n}|\right),$$

имеем

$$|u(x,y)| \le \sum_{n=1}^{\infty}\left(|V_n(x)| + |\rho_n(x)|\right) \le M\sum_{n=1}^{\infty}\frac{1}{\lambda_n^6}\left(\sum_{i=1}^{3}|\Psi_{in}| + |F_n(x)| + |F_n(0)| + |F_n(p)| + \left|F'_n(x)\right|\right) < \infty.$$

Покажем сходимост $u_{xxx}(x,y)$. Имеем $u_{xxx}(x,y) = \sum\limits_{n=1}^{\infty}\left(V'''_n(x) + \rho'''_n(x)\right)Y_n(y)$. После некоторых вычислений, находим $V'''_n(x)$ в следующем виде.

$$V'''_n(x) = \lambda_n^3 f_n(x) - a_1\left(V'_{0n}(x) + \int_0^p R_{nx}(x,\xi)\,V_{0n}(\xi)\,d\xi\right) -$$
$$-a_2\left(V_{0n}(x) + \int_0^p R_n(x,\xi)\,V_{0n}(\xi)\,d\xi\right) - \lambda_n^3\left(V_{0n}(x) + \int_0^p R_n(x,\xi)\,V_{0n}(\xi)\,d\xi\right).$$

Здесь

$$V_{0n}(x) = f_n(x) - G_{n\xi\xi}(x,0)\,f_n(0) + G_{n\xi\xi}(x,p)\,f_n(p) - \int_0^p G_{n\xi\xi}(x,\xi)\,f_n{}'(\xi)\,d\xi,$$
$$V'_{0n}(x) = -G_{n\xi\xi x}(x,0)\,f_n(0) + G_{n\xi\xi x}(x,p)\,f_n(p) - \int_0^p G_{n\xi\xi x}(x,\xi)\,f_n{}'(\xi)\,d\xi.$$

Используя оценки (11), (14) и (20) находим

$$\left|V'_{0n}(x)\right| \le \frac{M}{\lambda_n^4}\left(|\Psi_{1n}| + |\Psi_{2n}| + |\Psi_{3n}| + |F_n(x)| + |F_n(0)| + |F_n(p)| + \left|F'_n(x)\right|\right),$$

$$|R_{nx}(x,\xi)| \le \lambda_n M,$$

отсюда

$$\left|V'''_n(x)\right| \le \frac{M}{\lambda_n^3}\left(\sum_{i=1}^{3}|\Psi_{in}| + |F_n(x)| + |F_n(0)| + |F_n(p)| + \left|F'_n(x)\right|\right).$$

Тогда имеем

$$|u_{xxx}(x,y)| \le M\sum_{n=1}^{\infty}\frac{1}{\lambda_n^3}\left(|\Psi_{1n}| + |\Psi_{2n}| + |\Psi_{3n}| + |F_n(x)| + |F_n(0)| + |F_n(p)| + \left|F'_n(x)\right|\right).$$

Учитывая (10), $u_{xxx}(x,y)$ сходятся абсолютно и равномерно. Из неравенство

$$|u_{yy}(x,y)| \le |u_{xxx}(x,y)| + |a_1|\,|u_x(x,y)| + |a_2|\,|u(x,y)|,$$

можно заключить что и тоже сходится. Тогда решение задачи B_3 можно представить в следующем явном виде:

$$u(x,y)=\sum_{n=1}^{\infty}\lambda_n^3 Y_n(y)\int_0^p G_n(x,\xi)\,f_n(\xi)\,d\xi+$$
$$+\sum_{n=1}^{\infty}\lambda_n^3 Y_n(y)\left(\int_0^p R_n(x,\xi)\int_0^p G_n(x,s)\,f_n(s)\,dsd\xi\right)+\sum_{n=1}^{\infty}\left(\psi_{1n}+x\psi_{2n}+\tfrac{x^2-2xp}{2}\psi_{3n}\right)Y_n(y).$$

Таким образом, доказана теорема 2.

Список литературы

[1] Юлдашев Т.К., Обратная задача для одного интегро-дифференциального уравнения Фредгольма в частных производных третьего порядка, *Вестн. Сам. гос. техн. ун-та. Сер. Физ.-мат. науки, - Самара*,Vol.34, No.1, 2014 year, pp.56–65.

[2] Рыжов О.С., Асимптотическая картина обтекания тел вращения со звуковым потоком вязкого и теплопроводящего газа, *Прикл. Матем. и механ., - Москва*, Vol.29, No.6 1965 year, pp.1004-1014.

[3] Диесперов В.Н. О функции Грина линеаризованного вязкого трансзвукового уравнения, *Журнал вычисл. матем. и матем. физики. - Москва*, Vol.12, No.1, 1972 year, pp 1265-1279.

[4] Block H. Sur les equations lineaires aux derives parielles a carateristiques multiples, *Ark. Mat. Astron. Fus.*, Note 1, Vol. 13, No.7, 1912 year, pp. 1-34; Note 2, Vol.21, No.7, 1912 year, pp.1–30; Note 3, Vol.23, No.8 1912 - 1913 year,pp.1-51.

[5] Del Vicchio E. Sulleequazioni , *Memorie R. Accad. Sci.*, Vol.66, No.2, 1915 year, pp.1–41.

[6] Cattabriga L. Potenziali di linea e di dominio per equazioni non paraboliche in due variabilia caratteristiche multiple, *Rendiconti del seminario matimatico della univ. di Padava.* No.31, 1961 year, pp.1–45.

[7] Джураев Т.Д, Апаков Ю.П. Об автомодельном решении одного уравнения третьего порядка с кратными характеристиками, *Вестник Сам. гос. техн. ун-та. Сер. Физ.-мат. науки, - Самара*,Vol.25, No.2, 2007 year, pp.18–26.

[8] Джураев Т.Д, Апаков Ю.П. К теории уравнения третьего порядка с кратными характеристиками, содержащего вторую производную по времени, *Украинский математический журнал. – Киев*, Vol.62, No.1, 2010 year, С. 40–51

[9] Apakov Y.P., Rutkauskas S. On a boundary problem to third order PDE with multiple characteristics, *Nonlinear Analysis: Modeling and Control. -Vilnius*, Vol.16, No.3, 2011 year, pp.255–269.

[10] Апаков Ю.П. О решении краевой задачи для уравнения третьего порядка с кратными характеристиками, *Украинский математический журнал. -Киев.* Vol.64, No.1, 2012. С. 1–11.

[11] Апаков Ю.П., Жураев А.Х., О решении краевой задачи для уравнения третьего порядка с помощью функции Грина, *Узбекский математический журнал*, No.3, 2011 year, С.36–42

[12] Yuldashev T.K., Apakov Y.P., Zhuraev A.Kh., Boundary value problem for third order partial integro-differential equation with a degenerate kernel. *Lobachevski Journal of Mathematics*, Vol.42, No.6, 2021 year, pp.1316–1326.

[13] Иргашев Б. Ю. Краевая задача для одного вырождающего уравнения высокого порядка с младшими членами, *Bulletin of the Institute of Mathematics*, No.6, 2019 year, С.23–29.

[14] Сабитов К. Б. Задача Дирихле для уравнение смешанного типа третьего порядка, *ДАН России. – Москва*, Vol.427, No.5, 2009 year, С.593–596.

[15] АБалкизов Ж.А., Кадзаков А.Х. О представлении решения краевой задачи для неоднородного уравнения третьего порядка с кратными характеристиками, *Известия Кабардино - Балкарского научного центра РАН. - Нальчик*, No.4, 2010 year, С. 64–69.

[16] Лукина Г.А. Краевые задачи с интегральными граничными условиями для линеаризованного уравнения Кортевега - де Фриза, *Вестник Южно-Уральского государственного университета. Серия: Матем. модел. и програм. - Челябинск*, Vol.234, No.17, 2011 year,С. 52–61.

[17] Шубин В.В. Краевые задачи для уравнений третьего порядка с разрывным коэффициентом, *Вестник НГУ. Сер. Матем., мех., информ. - Новосибирск*, Vol.12, No.1, 2012 year, С.126–138.

[18] Ashyralyev A., Belakroum K., Guezane-Lakoud A. Stability of boundary-value problems for third-order partial differential equations, *Electronic Journal of Differential Equations*, Vol.2017, No.53, 2017 year, C.1–11.

[19] Кожанов А.И., Дюжева А.В. Нелокальные задачи с интегральным условием для дифференциальных уравнений третьего порядка, *Вестн. Сам. гос. техн. ун-та. Сер. Физ.- мат. науки*, Vol.24, No.4, 2020 year, С.607—620.

[20] Apakov Y. P., Umarov R. A. Solution of the Boundary Value Problem for a Third Order Equation with Little Terms. Construction of the Green's Function, *Lobachevskii Journal of Mathematics*,Vol.43, No.3, 2022 year, pp.738—748.

[21] Apakov Y.P., Umarov R.A. Construction of a solution of the boundary-value problem for the third-order equation with lower terms by using Green's function, *Nonlinear Oscillations*, Vol.25, No.2, 2022 year, pp.161–173.

[22] Apakov Yu.P., Umarov R.A., On the third boundary problem for a nonhomogeneous third order equation with multiple characteristics, *Uzbek Mathematical Journal*, Vol.67, No.1, 2023 year, pp.4–14.

[23] Апаков Ю.П., Умаров Р.А. Построение решения краевой задачи для уравнения третьего порядка с переменными коэффициентами, *Доклады Академии наук Республики Узбекистан*, No.2, 2023 year, С.3–10

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

О РАЗРЕШИМОСТИ ТРЕТЬЕЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ НЕОДНОРОДНОГО УРАВНЕНИЯ ЧЕТВЕРТОГО ПОРЯДКА С КРАТНЫМИ ХАРАКТЕРИСТИКАМИ

ЮСУПЖОН АПАКОВ[1], САНЖАРБЕК МАМАЖОНОВ[2]

[1,2]Институт математики им. В.И. Романовского АН РУз, Ташкент, Узбекистан
[1]Наманганский инженерно-строительный институт, Наманган, Узбекистан
e-mail: [1]yusupjonapakov@gmail.com, [2]sanjarbekmamajonov@gmail.com

Изучение многих задач газовой динамике, теории упругости, теории пластин и оболочек приводится к рассмотрению дифференциальных уравнений в частных производных высоких порядков. С точки зрения физических приложений представляют большой интерес и дифференциальные уравнения четвертого порядка (см. [1]-[4]).

В данной работе в области $\Omega = \{(x,y):\ 0<x<p,\ 0<y<q\}$ рассматривается уравнение четвертого порядка в виде

$$U_{xxxx} - U_{yy} + A_1(x)\,U_{xxx} + A_2(x)\,U_{xx} + A_3(x)\,U_x + A_4(x)\,U + A_5 U_y = F(x,y),$$

где $p, q, A_5 \in R$, $A_i(x)$, $i=\overline{1,4}$, $F(x,y)$ – заданные достаточно гладкие функции. Заменой

$$U(x,y) = u(x,y)\exp\left(-\frac{1}{4}\int\limits_0^x A_1(\xi)\,d\xi + \frac{A_5}{2}y\right),$$

это уравнение можно привести к уравнению

$$L[u] = u_{xxxx}(x,y) + a_1(x)\,u_{xx}(x,y) + a_2(x)\,u_x(x,y) + a_3(x)\,u(x,y) - u_{yy}(x,y) = f(x,y), \quad (1)$$

где

$$a_1(x) = A_2(x) - \frac{3A_1'(x)}{2} - \frac{3A_1^2(x)}{8}, \quad a_2(x) = \frac{A_1^3(x)}{8} - \frac{A_1(x)\,A_2(x)}{2} - A_1''(x) + A_3(x),$$
$$a_3(x) = \frac{3}{32}A_1^2(x)\,A_1'(x) - \frac{1}{4}A_1'''(x) + \frac{3}{16}{A_1'}^2(x) - \frac{3}{256}A_1^4(x) - \frac{1}{4}A_1'(x)\,A_2(x) +$$
$$+\frac{1}{16}A_1^2(x)\,A_2(x) - \frac{1}{4}A_1(x)\,A_3(x) + A_4(x) + \frac{1}{4}A_5^2,$$
$$f(x,y) = F(x,y)\exp\left[\frac{1}{4}\int\limits_0^x A_1(\xi)\,d\xi - \frac{A_5}{2}y\right].$$

Уравнение (1), обычно называется уравнением четвертого порядка с кратными характеристиками, содержащее вторую производную по времени.

Монография Джураева Т.Д., Сопуева А. [5] посвящена классификаций дифференциальных уравнений в частных производных четвертого порядка и решению краевых задач для таких уравнений. Согласно лемме 1.3 из [5], уравнение (1) относится к параболическому типу.

В работах [6]-[11] изучены ряд корректных краевых задач для уравнений третьего порядка с кратными характеристиками, разработанную в этих работах методику применяем для уравнения четвертого порядка.

В работе [12] рассмотрена задача с краевыми условиями для неоднородного уравнения четвертого порядка с кратными характеристиками. В работе [13] решена задача с начальными и граничными условиями для вынужденных колебаний консольной балки. В работе [14] изучена краевая задача для уравнения высокого порядка с кратными характеристиками методом построения функции Грина. В работе [15] исследована задача для уравнения четвертого порядка с неизвестной правой частью.

Задача, изучаемая в данной статье, во многим отличается от упомянутых выше работ. Во-первых, в уравнение включены младшие члены. Во-вторых, в вышеупомянутых работах собственные функции найдены по переменной x, по y делают остальные выкладки. В нашем работе собственные функции находим по переменной y, а по x решаем задачу с построением функции Грина и получаем интегральное уравнение Фредгольма второго рода, решая это уравнение находим решение поставленной задачи.

Для уравнения (1) в области Ω изучим следующую задачу.

Задача A**.** Найти функцию $u(x,y)$ из класса $C^{4,2}_{x,y}(\Omega)\cap C^{3,1}_{x,y}\left(\bar{\Omega}\right)$, удовлетворяющую уравнению (1) в области Ω и следующим краевым условиям:

$$\alpha u(x,0)+\beta u_y(x,0)=0,\quad \gamma u(x,q)+\delta u_y(x,q)=0,\quad 0\le x\le p, \tag{2}$$

$$u(0,y)=\psi_1(y),\ u(p,y)=\psi_2(y),\ u_{xx}(0,y)=\psi_3(y),\ u_{xx}(p,y)=\psi_4(y),\ 0\le y\le q, \tag{3}$$

где $\alpha,\beta,\gamma,\delta\in R\backslash\{0\}$, и $\psi_i(y)$, $i=\overline{1,4}$, $f(x,y)$ – заданные достаточно гладкие функции.

Отметим, что в работе [12] рассмотрен случай $a_1=a_2=0$, $a_3\neq 0$, а в работах [13], [15], [16], [17] исследованы краевые задачи для модельных уравнений четвертого порядка спектральном методом. В работе [18], метод Фурье используется для решения краевой задачи для модельного уравнения произвольного четного порядка. В работах [19], [20], исследованы краевые задачи для уравнения (1) когда все коэффициенты постоянные. В работах [21], [22] исследованы другие краевые задачи для уравнения (1). В работе [23] исследована третья краевая задача для модельного уравнение третьего порядка с кратными характеристиками.

Теорема 1. *Если задача* A *имеет решение, то при выполнении условий* $a_1(x)\le 0$, $a_1''(x)-a_2'(x)+2a_3(x)\ge 0$, $\alpha\beta<0$, $\gamma\delta>0$ *оно единственно.*

Доказательство. Предположим, обратное пусть задача A имеет два решения $u_1(x,y)$ и $u_2(x,y)$. Тогда функция $u(x,y)=u_1(x,y)-u_2(x,y)$ удовлетворяет однородному уравнению (1) и однородным краевым условиям. Докажем, что $u(x,y)\equiv 0$ в $\bar{\Omega}$.

В области Ω справедливо тождество

$$\begin{gathered} uL[u]\equiv\frac{\partial}{\partial x}\left(uu_{xxx}-u_xu_{xx}+a_1(x)uu_x-\frac{1}{2}a_1'(x)u^2+\frac{1}{2}a_2(x)u^2\right)-\\ -\frac{\partial}{\partial y}(uu_y)+u_{xx}^2-a_1(x)u_x^2+\frac{1}{2}\left(a_1''(x)-a_2'(x)+2a_3(x)\right)u^2+u_y^2=0. \end{gathered} \tag{4}$$

Интегрируя тождество (4) по области Ω и учитывая однородные краевые условия, получим

$$\begin{gathered} \int_0^p\int_0^q u_{xx}^2(x,y)\,dxdy+\int_0^p\int_0^q u_y^2(x,y)\,dxdy+\frac{1}{2}\int_0^p\int_0^q\left(a_1''(x)-a_2'(x)+2a_3(x)\right)u^2(x,y)\,dxdy-\\ -\int_0^p\int_0^q a_1(x)u_x^2(x,y)\,dxdy-\frac{\alpha}{\beta}\int_0^p\left(u(x,0)\right)^2dx+\frac{\gamma}{\delta}\int_0^p\left(u(x,q)\right)^2dx=0. \end{gathered}$$

При $a_1''(x)-a_2'(x)+2a_3(x)>0$ из третьей интеграл имеем $u(x,y)\equiv 0$, $(x,y)\in\bar{\Omega}$.

Если $a_1''(x)-a_2'(x)+2a_3(x)=0$, $a_1(x)=0$ тогда из второй интеграл получим

$$u_y(x,y)=0\quad\Rightarrow\quad u(x,y)=h(x).$$

Учитывая условие (2), имеем

$$\alpha u(x,0)+\beta u_y(x,0)=\alpha h(x)=0\quad\Rightarrow\quad h(x)=0.$$

отсюда $u(x,y)\equiv 0$, $(x,y)\in\bar{\Omega}$. Теорема 1 доказана.

Замечание. Отметим, что при нарушении условий теоремы 1, однородная задача A для однородного уравнение (1) может иметь нетривиальные решения. Например, легко можно убедится, когда $a_1 = \dfrac{\pi k}{p} > 0$, $a_2 = 0$, $a_3 = -\lambda_n < 0$, задача

$$\begin{cases} u_{xxxx}(x,y) + \left(\dfrac{\pi k}{p}\right)^2 u_{xx}(x,y) - \lambda_n u(x,y) - u_{yy} = 0, \\ \alpha u(x,0) + \beta u_y(x,0) = 0, \quad \gamma u(x,q) + \delta u_y(x,q) = 0, \\ u(0,y) = u(p,y) = u_{xx}(0,y) = u_{xx}(p,y) = 0, \end{cases}$$

имеет нетривиальные решения вида

$$u_{n,k}(x,y) = \sin\left(\frac{\pi k}{p}x\right) Y_n(y), \quad n,k \in N,$$

где $Y_n(y)$ решение задачи

$$\begin{cases} Y''(y) + \lambda Y(y) = 0, \\ \alpha Y(0) + \beta Y'(0) = 0, \\ \gamma Y(q) + \delta Y'(q) = 0. \end{cases} \tag{5}$$

Теорема 2. *Если выполняются следующие условия:*

1. $C < \dfrac{\mu_1^3\left(1 - e^{-2\mu_1 p}\right)^2}{p\left(2\mu_1^2 + 3\mu_1\left(1 + e^{-4\mu_1 p}\right) + 3\right)}$;
2. $\alpha\psi_i^{(j)}(0) + \beta\psi_i^{(j+1)}(0) = 0$, $\gamma\psi_i^{(j)}(q) + \delta\psi_i^{(j+1)}(q) = 0$, $\psi_i(y) \in C^3[0,q]$, $i = \overline{1,4}$, $j = 0,2$;
3. $\alpha f(x,0) + \beta f_y(x,0) = 0$, $\gamma f(x,q) + \delta f_y(x,q) = 0$, $f_{xyy}(x,y) \in C\left(\bar{\Omega}\right)$, $0 \le x \le p$, *то решение задачи A существует. Здесь*

$$C = \max_{\xi \in [0,p]} \left\{ \left| a_i^{(j)}(\xi) \right|, \left| a_1''(\xi) \right|, \ i = \overline{1,3}, \ j = 0,1 \right\}, \quad \mu_1 = \sqrt{\frac{\pi}{2q}}.$$

Доказательство. Рассмотрим задачу (5). Из [23], для задачи (5) собственные значения λ существует только при $\lambda = \lambda_n > 0$ и соответствующие собственные функции имеет вид

$$Y_n(y) = \frac{H_n(y)}{\|H_n(y)\|}, \quad H_n(y) = \alpha \sin\sqrt{\lambda_n} y - \beta\sqrt{\lambda_n}\cos\sqrt{\lambda_n} y, \quad n \in N,$$

и эти функции образует в $L_2[0,q]$ полную ортонормированную систему, где

$$\lambda_1 < \lambda_2 < ... < \lambda_n < ..., \quad \lambda_n = O\left(n^2\right), \ n \to \infty.$$

Решение задачи A ищем в виде

$$u(x,y) = \sum_{n=1}^{\infty} X_n(x) Y_n(y). \tag{6}$$

Формально считая, что (6) удовлетворяет уравнение (1), и подставляя (6) в (1), учитывая граничные условия (3), получим задачу:

$$\begin{cases} X_n^{(4)}(x) + a_1(x) X_n''(x) + a_2(x) X_n'(x) + a_3(x) X_n(x) + \lambda_n X_n(x) = f_n(x), \\ X_n(0) = \psi_{1n}, \ X_n(p) = \psi_{2n}, \ X_n''(0) = \psi_{3n}, \ X_n''(p) = \psi_{4n}, \end{cases} \tag{7}$$

где $\psi_{in} = \int\limits_0^q \psi_i(\eta) Y_n(\eta)\, d\eta$, $i = \overline{1,4}$, $f_n(x) = \int\limits_0^q f(x,\eta) Y_n(\eta)\, d\eta$.

Решаем задачу (7) с помощью функцию Грина. Для построения функции Грина сначала приводим краевые условия к нулю. Введем обозначение

$$V_n(x) = X_n(x) - \rho_n(x), \tag{8}$$

где

$$\rho_n(x) = c_1(x)\psi_{1n} + c_2(x)\psi_{2n} + c_3(x)\psi_{3n} + c_4(x)\psi_{4n}, \tag{9}$$

$$c_1(x) = \frac{p-x}{p}, \quad c_2(x) = \frac{x}{p}, \quad c_3(x) = \frac{3x^2p - x^3 - 2xp^2}{6p}, \quad c_4(x) = \frac{x^3 - xp^2}{6p}.$$

Учитывая (9), вставим (8) в (7). Тогда получим задачу

$$\begin{cases} V_n^{(4)}(x) + \lambda_n V_n(x) = g_n(x) - a_1(x)V_n''(x) - a_2(x)V_n'(x) - a_3(x)V_n(x), \\ V_n(0) = V_n(p) = V_n''(0) = V_n''(p) = 0, \end{cases} \tag{10}$$

здесь

$$g_n(x) = f_n(x) + d_{1n}(x)\psi_{1n} + d_{2n}(x)\psi_{2n} + d_{3n}(x)\psi_{3n} + d_{4n}(x)\psi_{4n},$$

$$d_{1n}(x) = -\frac{(p-x)a_3(x) - a_2(x)}{p} - \lambda_n c_1(x),$$

$$d_{2n}(x) = -\frac{a_2(x) + xa_3(x)}{p} - \lambda_n c_2(x),$$

$$d_{3n}(x) = -\frac{(6p-6x)a_1(x) + \left(6xp - 3x^2 - 2p^2\right)a_2(x)}{6p} + \frac{\left(3x^2p - x^3 - 2xp^2\right)a_3(x)}{6p} - \lambda_n c_3(x),$$

$$d_{4n}(x) = -\frac{6xa_1(x) + \left(3x^2 - p^2\right)a_2(x) + \left(x^3 - xp^2\right)a_3(x)}{6p} - \lambda_n c_4(x).$$

Ищем решение задачи (10) в следующем виде:

$$V_n(x) = \int_0^p G_n(x,\xi)g_n(\xi)\,d\xi - \\ - \int_0^p G_n(x,\xi)\left(a_1(\xi)V_n''(\xi) + a_2(\xi)V_n'(\xi) + a_3(\xi)V_n(\xi)\right)d\xi, \tag{11}$$

где $G_n(x,\xi)$ функция Грина для задачи

$$\begin{cases} V_n^{(4)}(x) + \lambda_n V_n(x) = 0, \\ V_n(0) = V_n(p) = V_n''(0) = V_n''(p) = 0, \end{cases}$$

которая имеет вид

$$G_n(x,\xi) = \begin{cases} G_n^1(x,\xi),\ 0 \le x < \xi \le p, \\ G_n^2(x,\xi),\ 0 \le \xi < x \le p, \end{cases}$$

где

$$G_n^1(x,\xi) = \frac{1}{2\mu_n^3\Delta}\{[\cos(\mu_n p)\,sh(\mu_n p)(\cos(\mu_n(\xi-p))\,sh(\mu_n(\xi-p)) - \\ - \sin(\mu_n(\xi-p))\,ch(\mu_n(\xi-p))) + \sin(\mu_n p)\,ch(\mu_n p)\times \\ \times(\cos(\mu_n(\xi-p))\,sh(\mu_n(\xi-p)) + \sin(\mu_n(\xi-p))\,ch(\mu_n(\xi-p)))]\times \\ \times\cos(\mu_n x)\,sh(\mu_n x) + [\sin(\mu_n p)\,ch(\mu_n p)(\cos(\mu_n(\xi-p))\,sh(\mu_n(\xi-p)) - \\ - \sin(\mu_n(\xi-p))\,ch(\mu_n(\xi-p))) - \cos(\mu_n p)\,sh(\mu_n p)\times \\ \times(\cos(\mu_n(\xi-p))\,sh(\mu_n(\xi-p)) - \sin(\mu_n(\xi-p))\,ch(\mu_n(\xi-p)))]\sin(\mu_n x)\,ch(\mu_n x)\},$$

$$G_n^2(x,\xi) = \frac{1}{2\mu_n^3\Delta}\{[\cos(\mu_n p)\,sh(\mu_n p)(\cos(\mu_n\xi)\,sh(\mu_n\xi) - \sin(\mu_n\xi)\,ch(\mu_n\xi)) + \\ + \sin(\mu_n p)\,ch(\mu_n p)(\cos(\mu_n\xi)\,sh(\mu_n\xi) + \sin(\mu_n\xi)\,ch(\mu_n\xi))]\times \\ \times\cos(\mu_n(x-p))\,sh(\mu_n(x-p)) + [\sin(\mu_n p)\,ch(\mu_n p)(\cos(\mu_n\xi)\,sh(\mu_n\xi) - \\ - \sin(\mu_n\xi)\,ch(\mu_n\xi)) - \cos(\mu_n p)\,sh(\mu_n p)(\cos(\mu_n\xi)\,sh(\mu_n\xi) + \\ + \sin(\mu_n\xi)\,ch(\mu_n\xi))]\sin(\mu_n(x-p))\,ch(\mu_n(x-p))\},$$

$$\Delta = -e^{2\mu_n p}\bar{\Delta}, \quad \bar{\Delta} = \frac{1 + e^{-4\mu_n p}}{2} - \cos(2\mu_n p)\,e^{-2\mu_n p}, \quad \mu_n = \sqrt[4]{\frac{\lambda_n}{4}} > 0.$$

Предположим, $\bar{\Delta} = 0$, тогда

$$2\cos(2\mu_n p) = e^{2\mu_n p} + e^{-2\mu_n p}.$$

Учитывая $\mu_n > 0$, имеем систему неравенств

$$\begin{cases} 2\cos(2\mu_n p) \le 2, \\ e^{2\mu_n p} + e^{-2\mu_n p} > 2, \end{cases}$$

который не имеет решение. Значить, $\bar{\Delta} > 0$.

Чтобы оценит $\bar{\Delta}$ снизу будем считать, что $\cos(2\mu_n p) = 1$, тогда имеем

$$\bar{\Delta} \ge \frac{1}{2}\left(1 - e^{-2\mu_n p}\right)^2.$$

Правая сторона этого неравенство при дискретных значениях n возрастает, поэтому при $n = 1$ принимает минимальное значение и получим

$$\bar{\Delta} \ge \frac{1}{2}\left(1 - e^{-2\mu_1 p}\right)^2 > 0.$$

Это доказывает отсутствие "малого знаменателя". И так $\Delta \neq 0$.

Интегрируя по частям второе слагаемое в равенстве (11), и введя обозначения

$$\beta_n(x) = \int_0^p G_n(x,\xi) g_n(\xi)\, d\xi,$$

$$\bar{G}_n(x,\xi) = \left(a_2'(\xi) - a_1''(\xi) - a_3(\xi)\right) G_n + \left(a_2(\xi) - 2a_1'(\xi)\right) G_{n\xi} - a_1(\xi) G_{n\xi\xi},$$

получим

$$V_n(x) = \beta_n(x) + \int_0^p \bar{G}_n(x,\xi) V_n(\xi)\, d\xi, \tag{12}$$

являющееся интегральным уравнением Фредгольма второго рода. Запишем решение (12) с помощью резольвенты

$$V_n(x) = \beta_n(x) + \int_0^p R_n(x,\xi)\beta_n(\xi)\, d\xi, \tag{13}$$

где

$$R_n(x,\xi) = \bar{G}_{0n}(x,\xi) + \sum_{m=1}^{\infty} \bar{G}_{mn}(x,\xi), \quad \bar{G}_{0n}(x,\xi) = \bar{G}_n(x,\xi),$$

$$\bar{G}_{mn}(x,\xi) = \int_0^p \bar{G}_n(x,s)\bar{G}_{(m-1)n}(s,\xi)\, ds, \ m = 1, 2, \ldots .$$

В силу (6), (8), (9), (13) решение задачи A имеет вид

$$u(x,y) = \sum_{n=1}^{\infty} \left(V_n(x) + \rho_n(x)\right) Y_n(y). \tag{14}$$

Проверим (14) на сходимость. В дальнейшем максимальное значение всех найденных положительных известных чисел в оценках будем обозначать через M.

Для оценки резольвенты мажорирующий ряд имеет вид

$$\frac{1}{p}\sum_{m=1}^{\infty} (Jp)^m, \tag{15}$$

где

$$J = C\left(\frac{2}{\mu_1^3} + \frac{3}{\mu_1^2}\left(1 + e^{-4\mu_1 p}\right) + \frac{3}{\mu_1}\right)\left(1 - e^{-2\mu_1 p}\right)^{-2}.$$

Условие 1, теоремы 2 можно записать в виде

$$\left(\frac{2}{\mu_1}+\frac{3}{\mu_1^2}\left(1+e^{-4\mu_1 p}\right)+\frac{3}{\mu_1^3}\right)\left(1-e^{-2\mu_1 p}\right)^{-2}Cp<1,$$

отсюда

$$Jp<1,$$

тогда мажорирующий ряд (15) является бесконечно убывающей геометрической прогрессией. В этом случае резольвента равномерно сходится, и его оценка имеет вид

$$|R_n(x,\xi)|\le\frac{J}{1-Jp}\le M. \tag{16}$$

Теперь оценим $\beta_n(x)$. Сначала, учитывая 2-3 условия теоремы 2, интегрируем по частям ψ_{in} четыре раза, а $f_n(x)$ два раза и получим оценки

$$|\psi_{in}|\le\frac{1}{\lambda_n^2}|\Psi_{in}|,\quad i=\overline{1,4},\quad |f_n(x)|\le\frac{1}{\lambda_n}|\Phi_n(x)|,\quad n\in N. \tag{17}$$

где $\Psi_{in}=\int\limits_0^q\psi_i^{(4)}(\eta)\,Y_n(\eta)\,d\eta,\ i=\overline{1,4},\ \Phi_n(x)=\int\limits_0^q f_{\eta\eta}(x,\eta)\,Y_n(\eta)\,d\eta,\ n\in N.$

В каждом из интервалов $0\le\xi<x$ и $x<\xi\le p$ функция $G_n(x,\xi)$, рассматриваемая как функция от ξ, является решением уравнения

$$G_{n\xi\xi\xi\xi}(x,\xi)+\lambda_n G_n(x,\xi)=0.$$

Тогда подставляя

$$G_n(x,\xi)=-\frac{1}{\lambda_n}G_{n\xi\xi\xi\xi}(x,\xi),$$

в $\beta_n(x)$, имеем

$$\beta_n(x)=-\frac{1}{\lambda_n}\left(\int\limits_0^x G^2_{n\xi\xi\xi\xi}(x,\xi)g_n(\xi)d\xi+\int\limits_x^p G^1_{n\xi\xi\xi\xi}(x,\xi)g_n(\xi)d\xi\right).$$

Учитывая свойство функции Грина $G_{n\xi\xi\xi}(x,x+0)-G_{n\xi\xi\xi}(x,x-0)=1$, интегрируем $\beta_n(x)$ по частям один раз и находим

$$\beta_n(x)=\frac{1}{\lambda_n}\left(g_n(x)+\int\limits_0^p G_{n\xi\xi\xi}(x,\xi)\,g_n'(\xi)\,d\xi\right).$$

Учитывая (17) и следующие оценки:

$$|g_n(x)|\le\frac{M}{\lambda_n}\left(\sum_{i=1}^4|\Psi_{in}|+|\Phi_n(x)|+\frac{1}{\lambda_n}\sum_{i=1}^4|\Psi_{in}|\right),$$
$$|g'_n(x)|\le\frac{M}{\lambda_n}\left(\sum_{i=1}^4|\Psi_{in}|+|\Phi'_n(x)|+\frac{1}{\lambda_n}\sum_{i=1}^4|\Psi_{in}|\right),\quad |G_{n\xi\xi\xi}(x,\xi)|\le M,$$

находим

$$|\beta_n(x)|\le\frac{M}{\lambda_n^2}\left(\sum_{i=1}^4|\Psi_{in}|+1+|\Phi_n(x)|+\frac{1}{\lambda_n}\sum_{i=1}^4|\Psi_{in}|\right). \tag{18}$$

Согласно (16) и (18), имеем

$$|V_n(x)|\le\frac{M}{\lambda_n^2}\left(\sum_{i=1}^4|\Psi_{in}|+1+|\Phi_n(x)|+\frac{1}{\lambda_n}\sum_{i=1}^4|\Psi_{in}|\right). \tag{19}$$

После некоторых вычислений находим оценку

$$|\rho_n(x)| \le \frac{M}{\lambda_n^2}\sum_{i=1}^{4}|\Psi_{in}|,$$

учитывая это и (19) для решения задачи (1)-(3) получим оценку

$$|u(x,y)| \le \sum_{n=1}^{\infty}\frac{M}{\lambda_n^2}\left(\sum_{i=1}^{4}|\Psi_{in}| + 1 + |\Phi_n(x)| + \frac{1}{\lambda_n}\sum_{i=1}^{4}|\Psi_{in}|\right) < \infty.$$

Отсюда следует, что решение задачи (1)-(3) сходится абсолютно и равномерно в $\bar{\Omega}$. Теперь, покажем равномерную сходимость $u_{xxxx}(x,y)$.

Функция Грина имеет разрывы в 3-производных. Учитывая это, находим производное 4-порядка от $V_n(x)$:

$$V_n^{(4)}(x) = g_n(x) - \lambda_n\left(\beta_n(x) + \int_0^p R_n(x,\xi)\,\beta_n(\xi)\,d\xi\right) -$$
$$-a_1\left(\beta_n''(x) + \int_0^p R_{nxx}(x,\xi)\,\beta_n(\xi)\,d\xi\right) - a_2\left(\beta_n'(x) + \int_0^p R_{nx}(x,\xi)\,\beta_n(\xi)\,d\xi\right) -$$
$$-a_3\left(\beta_n(x) + \int_0^p R_n(x,\xi)\,\beta_n(\xi)\,d\xi\right).$$

Используя следующие оценки:

$$|\beta_n'(x)| \le \frac{M}{\sqrt[4]{\lambda_n^7}}\left(\sum_{i=1}^{4}|\Psi_{in}| + 1 + \frac{1}{\lambda_n}\sum_{i=1}^{4}|\Psi_{in}|\right), |\beta_n''(x)| \le \frac{M}{\sqrt{\lambda_n^3}}\left(\sum_{i=1}^{4}|\Psi_{in}| + 1 + \frac{1}{\lambda_n}\sum_{i=1}^{4}|\Psi_{in}|\right),$$
$$|R_{nx}(x,\xi)| \le \mu_n M, \qquad |R_{nxx}(x,\xi)| \le \mu_n^2 M,$$

находим оценку

$$\left|V_n^{(4)}(x)\right| \le \frac{M}{\lambda_n}\left(\sum_{i=1}^{4}|\Psi_{in}| + 1 + |\Phi_n(x)|\right) + \varphi_n(x),$$

где

$$\varphi_n(x) = \frac{M}{\sqrt{\lambda_n^3}}\left(\sum_{i=1}^{4}|\Psi_{in}| + 1 + \frac{1}{\lambda_n}\sum_{i=1}^{4}|\Psi_{in}|\right) + \frac{M}{\sqrt[4]{\lambda_n^7}}\left(\sum_{i=1}^{4}|\Psi_{in}| + 1\right) + \frac{M}{\sqrt[4]{\lambda_n^{11}}}\sum_{i=1}^{4}|\Psi_{in}| +$$
$$+\frac{M}{\lambda_n^2}\left(\sum_{i=1}^{4}|\Psi_{in}| + 1 + |\Phi_n(x)| + \frac{1}{\lambda_n}\sum_{i=1}^{4}|\Psi_{in}|\right).$$

Тогда имеем

$$|u_{xxxx}(x,y)| \le M\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left(\sum_{i=1}^{4}|\Psi_{in}| + 1 + |\Phi_n(x)|\right) + \sum_{n=1}^{\infty}\varphi_n(x).$$

Учитывая $\lambda_n = O\left(n^2\right)$, этот ряд сходится абсолютно и равномерно. Из уравнение (1) заключаем, что u_{yy} тоже сходится. Теорема 2 доказана.

Ключевые слова:Уравнение четвертого порядка, кратные характеристики, младшие члены, краевая задача, единственность, существование, функция Грина.

Предметная классификация AMS: 35C10, 35G15

Список литературы

[1] Турбин М.В. Исследование начально-краевой задачи для модели движения жидкости Гершел - Балкли // Вест. Воронеж. Гос. Ун-та. Сер. Физ. Мат. 2013. 2. -С. 246–257.

[2] Уизем Дж. Линейние и нелинейные волны. -М.: Мир, 1977.

[3] Шабров С.А. Об оценках функций влияния одной математической модели четвертого порядка // Вест. Воронеж. Гос. Ун-та. Сер. Физ. Мат. 2015. №2. -С. 168–179.

[4] Benney D.J., Luke J.C. Interactions of permanent waves of finite amplitude // J. Math. Phys. 1964. 43. P.309–313.

[5] Джураев Т.Д., Сопуев А. К теории дифференциальных уравнений в частных производных четвертого порядка. -Ташкент: Фан, 2000. -144 с.

[6] Джураев Т.Д, Апаков Ю.П. К теории уравнения третьего порядка с кратными характеристиками, содержащего вторую производную по времени // Украинский математический журнал. -Киев, 2010, том 62. № 1. -С. 40–51.

[7] Apakov Yu.P., Rutkauskas S. On a boundary problem to third order PDE with multiple characteristics // Nonlinear Analysis: Modeling and Control. -Vilnius, 2011. -Vol. 16. №3. -pp. 255–269.

[8] Apakov Yu.P. On the solution of a boundary-value problem for a third- order equation with multiple characteristics // Ukrainian Mathematical Journal. Springer, New York, June, 2012 -Vol. 64, №1. -P. 1–11.

[9] Apakov Yu.P. Irgashev B.Yu. Boundary-value problem for a degenerate high-odd-order equation // Ukrainian Mathematical Journal. Springer, New York, June, 2015 -Vol. 66, № 10. -P. 1475–1488.

[10] Apakov Yu.P., A. Kh. Zhuraev. Third boundary-value problem for a third-order differential equation with multiple characteristics // Ukrainian Mathematical Journal. Springer, New York, February, 2019 -Vol. 70, № 9. -P. 1467–1476.

[11] Apakov Yu.P. On Unique Solvability of Boundary-Value Problem for a Viscous Transonic Equation // Lobachevski Journal of Mathematics.2020 Vol, 41, № 9, -P. 1754–1761.

[12] Аманов Д., Мурзамбетова М. Б. Краевая задача для уравнения четвертого порядка с младшим членом, Вестн. Удмуртск. ун-та. Матем. Мех. Компьют. науки, 2013, выпуск 1, 3–10.

[13] Сабитов К. Б., Фадеева О. В. Начально-граничная задача для уравнения вынужденных колебаний консольной балки // Вестн. Сам. гос. техн. ун-та. Сер. Физ.-мат. науки, 25:1 (2021), 51–66.

[14] Иргашев Б. Ю. Краевая задача для одного вырождающегося уравнения высокого порядка с младшими членами // Бюллетень Института математики 2019, №6, -С.23–30.

[15] Urinov A.K., Azizov M.S. Boundary Value Problems for a Fourth Order partial Equation with an Unknown Right – hand Part // Lobachevskii Journal of Mathematics, 2021, Vol. 42 №3 pp.632–640.

[16] Аманов Д., Бекиев А.Б., Отарова Ж.А. Об одной краевой задаче для уравнения четвертого порядка // УзМЖ. 2015. №4. -С.11–18.

[17] Сабитов К. Б. Обратные задачи для уравнения колебаний балки по определению правой части и начальных условий // Дифференциальные уравнения, 2020, том 56, №6, -С. 761–774.

[18] Иргашев Б. Ю. Краевая задача для уравнения высокого четного порядка, Вестн. Волгогр. гос. ун-та. Сер. 1, Мат. Физ., 2016, выпуск 3(33), 6–18.

[19] Y.P.Apakov and S.M.Mamajonov: A boundary-value problem for the fourthorder equation with multiple characteristics in a rectangular domain. Nonlinear Oscillations. Published in vol. 24 (2021), No. 3, pp. 291–305.

[20] Mamajonov S.M. The third boundary problem for a fourth-order non-homogeneous equation with constant coefficients. Bull. Inst. Math., 2022, Vol. 5, No. 6, pp. 100–109.

[21] Ю.П.Апаков, С.М.Мамажонов. Краевая задача для неоднородного уравнения четвертого порядка с переменными коэффициентами. Доклады Академии наук Республики Узбекистан (ДАН), 2022, -No. 4, -С. 7–13.

[22] Ю.П.Апаков, С.М.Мамажонов. Краевая задача для неоднородного уравнения четвертого порядка с младшими членами. Дифференциальные уравнения, 2023, том 59, № 2, -С.183–192.

[23] Апаков Ю.П. Решение краевых задач для уравнения третьего порядка с кратными характеристиками методом разделение переменных // УзМЖ. 2007. №1, -С. 14-23.

Функциялар теориясы және функционалдық талдау

Theory of functions and functional analysis

Теория функций и функциональный анализ

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

THE INFINITE SUMMATION FORMULAS OF THE MULTIPLE LAURICELLA'S HYPERGEOMETRIC FUNCTIONS

ANVAR HASANOV[1,3], TUHTASIN ERGASHEV[2,3]

[1]Institute of Mathematics, University Street, Tashkent 700174, Uzbekistan
[2]Department of Mathematics, Analysis, Logic and Discrete Mathematics Ghent University, Belgium
[3] National Research University "TIIAME", 39, Kari-Niyazi Street, Tashkent, 100000, Uzbekistan
e-mail:anvarhasanov@yahoo.com, ergashev.tukhtasin@gmail.com

Abstract. Inspired by the work by Wang, who gave some infinite summation formulas of Appell functions (in two variables) with the help of some famous summation theorems, we, introducing into consideration new inverse pair of symbolic operators with the multidimensional analogues and studying their properties, establish the infinite summation formulas for the four multiple Lauricella's functions.

Keywords: Appell functions; multiple Lauricella's hypergeometric functions; the inverse pair symbolic operators; Poole's formula; infinity summation formulas.

AMS Subject Classification: Primary 33C20, 33C65

1. Introduction and definitions

Hypergeometric functions in one and more variables occur naturally in a wide variety of problems in applied mathematics, statistics, operations research, theoretical physics, and engineering sciences. For instance, Srivastava and Kashyap [25] presented a number of interesting applications of hypergeometric functions in one and more variables in queuing theory and related stochastic processes. The work of Niukkanen [23] on the multiple hypergeometric functions is motivated by various physical and quantum chemical applications of such functions. Especially, many problems in gas dynamics lead to solutions of degenerate second-order partial differential equations which are then solvable in terms of multiple hypergeometric functions. Among examples, we can cite the problem of adiabatic flat-parallel gas flow without whirlwind, the flow problem of supersonic current from vessel with flat walls, and a number of other problems connected with gas flow [15].

The success of the theory of hypergeometric functions in one variable has stimulated the development of a corresponding theory in two and more variables. Appell [1] has defined, in 1880, four functions, F_1 to F_4 which are all analogous to Gauss' $F(a,b;c;z)$. A great merit in the further development of the theory of the hypergeometric series in two variables belongs to Horn [20], who gave a general definition and order classification of double hypergeometric series. He has investigated the convergence of hypergeometric series of two variables and established the systems of partial differential equations which they satisfy. Horn investigated in particular hypergeometric series of order two and found that, apart from certain series which are either expressible in terms of one variable or are products of two hypergeometric series, each in one variable, there are essentially 34 (14 complete and 20 confluent) convergent series of order two.

Lauricella [21, p. 114] further generalized the four Applell series F_1, F_2, F_3, F_4 to series in n variables and defined his multiple hypergeometric series denoted by $F_A^{(n)}$, $F_B^{(n)}$,$F_C^{(n)}$ and $F_D^{(n)}$

which have important applications. For instance, explicit fundamental solutions of a multidimensional singular elliptic equation are expressed in terms of the Lauricella's hypergeometric function $F_A^{(n)}$ [11, 12]. For a given multiple hypergeometric function, it is useful to fund a decomposition formula which would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions involving fewer variables. The familiar operator method of Burcnall and Chaundy [8, 9] has been used by them rather extensively for finding decomposition formulas for hypergeometric functions of two variables in terms of the classical Gauss hypergeometric function of one variable. In the papers [3, 4, 5, 6] interesting results were obtained on the study of the double hypergeometric functions. Recently, in the work [19], for the first time, formulas of analytic continuation for the Lauricella's hypergeometric functions in three variables.

Following the works [8, 9], Hasanov and Srivastava [17, 18] introduced operators generalizing the Burcnall-Chaundy operators and found expansion formulas for many triple hypergeometric functions, and they proved recurrent formulas when the dimension of hypergeometric function exceeds three. However, due to the recurrence, additional difficulties may arise in the applications of those decomposition formulas. Recently, in the works [13, 14, 16] new non-recurrence decomposition formulas for n-variable Lauricella functions $F_A^{(n)}$ and $F_B^{(n)}$ are obtained which are directly applied to the solving boundary value problems for multidimensional singular elliptic equation.

In this work, inspired by the work Wang [7] we establish the infinite summation formulas for Lauricella functions in n variables $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$.

The plan of this paper is as follows. In Section 2 we briefly give some preliminary information, which will be used later. In Section 3, we define inverse pair symbolic operators $H_x(a,b)$, $\bar{H}_x(a,b)$ and their multidimensional analogues. In Sections 4 and 5, we write the infinity summation formulas associated with one- and multi-dimensional inverse pair operators, respectively.

2. Preliminaries

Throughout this work it is convenient to employ the Pochhammer symbol $(\lambda)_n$ defined by

$$(\lambda)_n = \lambda(\lambda+1)...(\lambda+n-1),\ n=1,2,...;\quad (\lambda)_0 \equiv 1$$

for which the following equalities are true:

$$(1)_n = n!,\ \ (\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda),\ \ (\lambda)_{n+k} = (\lambda)_n(\lambda+n)_k,$$

where $\Gamma(z)$ is a famous Euler's gamma-function.

It is known that the Euler gamma-function $\Gamma(a)$ has property [10, p.17, Eq. (2)]

$$\Gamma(a+m) = \Gamma(a)(a)_m.$$

Here $(a)_m$ is a Pochhammer symbol, for which the equality $(a)_{m+n} = (a)_m(a+m)_n$ and its particular case $(a)_{2m} = (a)_m(a+m)_m$ are true [10, p.67, Eq. (5)].

A function

$$F(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k,\ c \neq 0,-1,-2,... \tag{1}$$

is known as the Gaussian hypergeometric function.

If $Rec > Reb > 0$, we have Euler's formula [10, p.114, Eq.(1)]

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} dt. \tag{2}$$

Here the right-hand side is a one-valued analytic function of z within the domain $|arg(1-z)| < \pi$; therefore (2) gives also the analytic continuation of $F(a,b;c;z)$.

The integral representation (2) allows to derive the Boltz formula [10, p. 105, Eq. (3)]

$$F(a,b;c;z)=(1-z)^{-b}F\left(c-a,b;c;\frac{z}{z-1}\right) \tag{3}$$

and get the value of the Gaussian function in unity (the summation formula) [10, p.73, Eq. (73)]

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ Re(c-a-b)>0,\ c\neq 0,\ -1,\ -2,\ ... \tag{4}$$

When studying the properties of a hypergeometric function, infinite sums of this function are very important. For example, the following formula shows that the infinite sum of a function can give an elementary function [22, p. 413, Eq. 6.7.1(8)]

$$\sum_{k=0}^{\infty}\frac{(a)_k(c-b)_k}{k!(c)_k}z^kF(a+k,b;c+k;z)=(1-z)^{-a}. \tag{5}$$

The double Appell hypergeometric functions are defined as following [1]:

$$F_1\left(a,b,b';c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m\left(b'\right)_n}{(c)_{m+n}m!n!}x^my^n, \tag{6}$$

$$F_2\left(a,b,b';c,c';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m\left(b'\right)_n}{(c)_m\left(c'\right)_n m!n!}x^my^n, \tag{7}$$

$$F_3\left(a,a',b,b';c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m\left(a'\right)_n(b)_m\left(b'\right)_n}{(c)_{m+n}m!n!}x^my^n, \tag{8}$$

$$F_4\left(a,b;c,c';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m\left(c'\right)_n m!n!}x^my^n. \tag{9}$$

We introduce a following notations:

$$\mathbf{a}:=(a_1,...,a_n)\,,\ \mathbf{b}:=(b_1,...,b_n)\,,\ \mathbf{c}:=(c_1,...,c_n)\,;\ \ \mathbf{x}:=(x_1,...,x_n)\,;$$

$$(\mathbf{a})_{\mathbf{k}}:=\prod_{j=1}^{n}\left(a_j\right)_{k_j}\,;\ \ \mathbf{x}^{\mathbf{k}}:=\prod_{j=1}^{n}x_j^{k_j};$$

$$\mathbf{k}:=(k_1,...,k_n)\,,\quad |\mathbf{k}|:=k_1+...+k_n,\ k_1\geq 0,...,k_n\geq 0;K!:=k_1!k_2!...k_n!;$$

$$\mathbf{l}:=(l_1,...,l_n)\,,\quad |\mathbf{l}|:=l_1+...+l_n,\ l_1\geq 0,...,l_n\geq 0;L!:=l_1!l_2!...l_n!.$$

The Lauricella hypergeometric functions in three (real or complex) variables are defined as following [2, p.114]

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(a)_{|\mathbf{k}|}\left(\mathbf{b}\right)_{\mathbf{k}}}{K!\left(\mathbf{c}\right)_{\mathbf{k}}}\mathbf{x}^{\mathbf{k}}\ \ \left(|x_1|+...+|x_n|<1\right), \tag{10}$$

$$F_B^{(n)}\left(\mathbf{a},\mathbf{b};c;\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{\left(\mathbf{a}\right)_{\mathbf{k}}\left(\mathbf{b}\right)_{\mathbf{k}}}{K!(c)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}},\ \ \left(|x_1|<1,\ ...,\ |x_n|<1\right). \tag{11}$$

$$F_C^{(n)}\left(a,b;\mathbf{c};\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(a)_{|\mathbf{k}|}(b)_{|\mathbf{k}|}}{K!\left(\mathbf{c}\right)_{\mathbf{k}}}\mathbf{x}^{\mathbf{k}}\ \ \left(\sqrt{|x_1|}+...+\sqrt{|x_n|}<1\right), \tag{12}$$

$$F_D^{(n)}\left(a,\mathbf{b};c;\mathbf{x}\right)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(a)_{|\mathbf{k}|}\left(\mathbf{b}\right)_{\mathbf{k}}}{K!(c)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}},\ \ \left(|x_1|<1,\ ...,\ |x_n|<1\right). \tag{13}$$

In all definitions (10) – (13), as usual, the denominator parameters c, c_1, ..., c_n are neither zero nor a negative integer.

Clearly, we have

$$F_A^{(2)} = F_2, \;\; F_B^{(2)} = F_3, \;\; F_C^{(2)} = F_4, \;\; F_D^{(2)} = F_1,$$
$$F_A^{(1)} = F_B^{(1)} = F_C^{(1)} = F_D^{(1)} \equiv F,$$

where F_1, ..., F_4 are the Appell series defined from (6) to (9) and F is Gaussian hypergeometric function defined in (1).

Really that in all definitions (10) – (13) a number of variables is natural: $n = 1, 2, ...$, however, in our further studies, if $n = 0$, then we accept that

$$F_A^{(0)} = F_B^{(0)} = F_C^{(0)} = F_D^{(0)} \equiv 1.$$

The following summation formula [2, p. 117]

$$F_D^{(n)}(a, \mathbf{b}; c; 1, ..., 1) = \frac{\Gamma(c)\Gamma(c-a-B)}{\Gamma(c-a)\Gamma(c-B)}, \;\; Re(c-a-B) > 0, \tag{14}$$

is valid, where $B := b_1 + ... + b_n$.

It is easy to see that the formula (14) generalizes the famous Gaussian summation formula (4).

3. The inverse pair operators $H_x(a,b)$ and $\bar{H}_x(a,b)$

Burchnall and Chaundy [8, 9] systematically presented a number of expansions for some double hypergeometric functions in series of simpler hypergeometric functions. However, the Burchnall-Chaundy method is limited to functions of two variables.

Developing the idea of Burchnall and Chaundy, we introduce the following mutually inverse symbolic operators for n variables:

$$H_{\mathbf{x}}(\alpha, \beta) := \frac{\Gamma(\alpha + \delta_1 + ... + \delta_n)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta + \delta_1 + ... + \delta_n)} \tag{15}$$

and

$$\bar{H}_{\mathbf{x}}(\alpha, \beta) := \frac{\Gamma(\alpha)\Gamma(\beta + \delta_1 + ... + \delta_n)}{\Gamma(\beta)\Gamma(\alpha + \delta_1 + ... + \delta_n)}, \tag{16}$$

where

$$\mathbf{x} := (x_1, ..., x_n); \;\; \delta_j = x_j \frac{\partial}{\partial x_j}, \;\; j = 1, ..., n.$$

In the one-dimensional case, these operators look like

$$H_{x_j}(\alpha, \beta) := \frac{\Gamma(\alpha + \delta_j)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta + \delta_j)}, \;\; j = 1, ..., n \tag{17}$$

and

$$\bar{H}_{x_j}(\alpha, \beta) := \frac{\Gamma(\alpha)\Gamma(\beta + \delta_j)}{\Gamma(\beta)\Gamma(\alpha + \delta_j)}, \;\; j = 1, ..., n. \tag{18}$$

Using the Gauss' formula (4) for $F(a, b; c; 1)$, we get

$$H_{x_j}(\alpha, \beta) = \sum_{r=0}^{\infty} \frac{(\beta - \alpha)_r (-\delta_j)_r}{(\beta)_r \, r!}, \;\; j = 1, ..., n \tag{19}$$

and

$$\bar{H}_{x_j}(\alpha, \beta) = \sum_{r=0}^{\infty} \frac{(\beta - \alpha)_r (-\delta_j)_r}{(1 - \alpha - \delta_j)_r \, r!}, \;\; j = 1, ..., n. \tag{20}$$

Similarly, using the summation formula (14) for $F_D^{(n)}(a, \mathbf{b}; c; 1, ..., 1)$, we get

$$H_{\mathbf{x}}(\alpha,\beta)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(\beta-\alpha)_{|\mathbf{k}|}}{K!(\beta)_{|\mathbf{k}|}}\prod_{j=1}^{n}(-\delta_j)_{k_j} \tag{21}$$

and

$$\bar{H}_{\mathbf{x}}(\alpha,\beta)=\sum_{|\mathbf{k}|=0}^{\infty}\frac{(\beta-\alpha)_{|\mathbf{k}|}}{K!(1-\alpha-\delta)_{|\mathbf{k}|}}\prod_{j=1}^{n}(-\delta_j)_{k_j}. \tag{22}$$

Note, that for every analytic function $f(z)$ the following Poole's formula [24, p. 26, Eq. (33)]

$$\left(-z\frac{\partial}{\partial z}\right)_r\{f(z)\}=(-1)^r z^r\frac{d^r}{dz^r}\{f(z)\} \tag{23}$$

is valid.

The companion to the Poole's formula (23), the following operator identity [24, p.93, Eq. (7)]

$$\left(a+z\frac{\partial}{\partial z}\right)_r\{f(z)\}=z^{1-a}\frac{d^r}{dz^r}\left\{z^{a+r-1}f(z)\right\} \tag{24}$$

is valid for every analytic function $f(z)$.

The operator identities (23) and (24) can be proved by the mathematical induction method.

4. The infinity summation formulas associated with one-dimensional inverse pair operators

Consider the inverse pair operators $H_{x_j}(\alpha,\beta)$ and $\bar{H}_{x_j}(\alpha,\beta)$, defined in (17) and (18), respectively $(1\le j\le n)$. A composition of these operators r times will be denoted by

$$H^r_{\mathbf{x}_r}(\mathbf{a}_r,\,\mathbf{b}_r):=H_{x_1}(a_1,b_1)\,H_{x_2}(a_2,b_2)\,...H_{x_r}(a_r,b_r),$$
$$\bar{H}^r_{\mathbf{x}_r}(\mathbf{a}_r,\,\mathbf{b}_r):=\bar{H}_{x_1}(a_1,b_1)\,\bar{H}_{x_2}(a_2,b_2)\,...\bar{H}_{x_r}(a_r,b_r),$$

where $\mathbf{a}_r:=(a_1,...,a_r)$, $\mathbf{b}_r:=(b_1,...,b_r)$ and $\mathbf{x}_r:=(x_1,...,x_r)$ are vectors with r components $(1\le r\le n)$.

Theorem 1. Let n be a number of the variables of Lauricella's functions defined (10) – (13). If r is a natural number and $1\le r\le n$, then the following symbolic forms

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x})=H^r_{\mathbf{x}_r}(\mathbf{b}_r,\mathbf{d}_r)\,F_A^{(n)}(a,\mathbf{d}_r,\mathbf{b}_{r+1,n};\mathbf{c};\mathbf{x}), \tag{25}$$

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x})=\bar{H}^r_{\mathbf{x}_r}(\mathbf{d}_r,\mathbf{b}_r)\,F_A^{(n)}(a,\mathbf{d}_r,\mathbf{b}_{r+1,n};\mathbf{c};\mathbf{x}), \tag{26}$$

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x})=H^r_{\mathbf{x}_r}(\mathbf{b}_r,\mathbf{c}_r)\,(1-X_r)^{-a}\times \tag{27}$$

$$\times F_A^{(n-r)}\left(a,\mathbf{b}_{r+1,n};\mathbf{c}_{r+1,n};\frac{x_{r+1}}{1-X_r},...,\frac{x_n}{1-X_r}\right), \tag{28}$$

$$F_A^{(n-r)}\left(a,\mathbf{b}_{r+1,n};\mathbf{c}_{r+1,n};\frac{x_{r+1}}{1-X_r},\right.$$
$$\left....,\frac{x_n}{1-X_r}\right)=(1-X_r)^a\,\bar{H}^r_{\mathbf{x}_r}(\mathbf{b}_r,\mathbf{c}_r)\,F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}), \tag{29}$$

$$F_B^{(n)}(\mathbf{a},\mathbf{b};c;\mathbf{x})=H^r_{\mathbf{x}_r}(\mathbf{a}_r,\mathbf{d}_r)\,F_B^{(n)}(\mathbf{d}_r,\mathbf{a}_{r+1,n};c;\mathbf{x}), \tag{30}$$

$$F_B^{(n)}(\mathbf{a},\mathbf{b};c;\mathbf{x})=\bar{H}^r_{\mathbf{x}_r}(\mathbf{d}_r,\mathbf{a}_r)\,F_B^{(n)}(\mathbf{d}_r,\mathbf{a}_{r+1,n};c;\mathbf{x}), \tag{31}$$

$$F_C^{(n)}(a,b;\mathbf{c};\mathbf{x})=H^r_{\mathbf{x}_r}(\mathbf{d}_r,\mathbf{c}_r)\,F_C^{(n)}(a,b;\mathbf{d}_r,\mathbf{c}_{r+1,n};\mathbf{x}), \tag{32}$$

$$F_D^{(n)}(a,\mathbf{b};c;\mathbf{x})=H^r_{\mathbf{x}_r}(\mathbf{b}_r,\mathbf{d}_r)\,F_D^{(n)}(a,\mathbf{d}_r,\mathbf{b}_{r+1,n};c;\mathbf{x}), \tag{33}$$

$$F_D^{(n)}(a,\mathbf{b};c;\mathbf{x}) = \bar{H}_{\mathbf{x}_r}^{r}(\mathbf{d}_r,\mathbf{b}_r)\, F_D^{(n)}(a,\mathbf{d}_r,\mathbf{b}_{r+1,n};c;\mathbf{x}) \tag{34}$$

are valid, where

$$\mathbf{b}_{r+1,n} := (b_{r+1},...,b_n)\,,\ \ \mathbf{b}_{n+1,n} = \emptyset,\ \ \mathbf{c}_{r+1,n} := (c_{r+1},...,c_n)\,,\ \ \mathbf{c}_{n+1,n} = \emptyset,$$

$$\mathbf{d}_r := (d_1,...,d_r)\,,\ \ X_r := x_1 + ... + x_r,\ 1 \le r \le n.$$

The symbolic forms (25) – (34) are used to obtain a large number of infinite summation formulas of the multiple Lauricella's hypergeometric functions. Namely, using the formulas (19), (20) and applying them many times, by virtue of Poole's formulas (23) and (24), we get

$$\begin{aligned} F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) &= \sum_{|\mathbf{k}_r|=0}^{\infty}(a)_{|\mathbf{k}_r|}\prod_{j=1}^{r}\left[\frac{(d_j-b_j)_{k_j}}{k_j!\,(c_j)_{k_j}}(-x_j)^{k_j}\right]\times \\ &\quad\times F_A^{(n)}(a+|\mathbf{k}_r|,\mathbf{d}_r+\mathbf{k}_r,\mathbf{b}_{r+1,n};\mathbf{c}_r+\mathbf{k}_r,\mathbf{c}_{r+1,n};\mathbf{x}) \qquad (35) \\ &= \sum_{|\mathbf{k}_r|=0}^{\infty}(a)_{|\mathbf{k}_r|}\prod_{j=1}^{r}\left[\frac{(b_j-d_j)_{k_j}}{k_j!\,(c_j)_{k_j}}x_j^{k_j}\right]\times \\ &\quad\times F_A^{(n)}(a+|\mathbf{k}_r|,\mathbf{d}_r,\mathbf{b}_{r+1,n};\mathbf{c}_r+\mathbf{k}_r,\mathbf{c}_{r+1,n};\mathbf{x}) \qquad (36) \\ &= (1-X_r)^{-a}\sum_{|\mathbf{k}_r|=0}^{\infty}(a)_{|\mathbf{k}_r|}\prod_{j=1}^{r}\left[\frac{(c_j-b_j)_{k_j}}{k_j!\,(c_j)_{k_j}}\left(\frac{-x_j}{1-X_r}\right)^{k_j}\right]\times \\ &\quad\times F_A^{(n-r)}\left(a+|\mathbf{k}_r|,\mathbf{b}_{r+1,n};\mathbf{c}_{r+1,n};\frac{x_{r+1}}{1-X_r},...,\frac{x_n}{1-X_r}\right), \qquad (37) \end{aligned}$$

$$\begin{aligned} F_A^{(n-r)}\left(a,\mathbf{b}_{r+1,n};\mathbf{c}_{r+1,n};\frac{x_{r+1}}{1-X_r},...,\frac{x_n}{1-X_r}\right) &= (1-X_r)^a\times \\ \times\sum_{|\mathbf{k}_r|=0}^{\infty}(a)_{|\mathbf{k}_r|}\prod_{j=1}^{r}\left[\frac{(c_j-b_j)_{k_j}}{k_j!\,(c_j)_{k_j}}x_j^{k_j}\right]&\cdot F_A^{(n)}(a+|\mathbf{k}_r|,\mathbf{b};\mathbf{c}_r+\mathbf{k}_r,\mathbf{c}_{r+1,n};\mathbf{x})\,, \qquad (38) \end{aligned}$$

$$\begin{aligned} F_B^{(n)}(\mathbf{a},\mathbf{b};c;\mathbf{x}) &= \sum_{|\mathbf{k}_r|=0}^{\infty}\frac{1}{(c)_{|\mathbf{k}_r|}}\prod_{j=1}^{r}\left[\frac{(d_j-a_j)_{k_j}(b_j)_{k_j}}{k_j!}(-x_j)^{k_j}\right]\times \\ &\quad\times F_B^{(n)}(\mathbf{d}_r+\mathbf{k}_r,\mathbf{a}_{r+1,n},\mathbf{b}_r+\mathbf{k}_r,\mathbf{b}_{r+1,n};c+|\mathbf{k}_r|;\mathbf{x}) \qquad (39) \\ &= \sum_{|\mathbf{k}_r|=0}^{\infty}\frac{1}{(c)_{|\mathbf{k}_r|}}\prod_{j=1}^{r}\left[\frac{(a_j-d_j)_{k_j}(b_j)_{k_j}}{k_j!}x_j^{k_j}\right]\times \\ &\quad\times F_B^{(n)}(\mathbf{d}_r,\mathbf{a}_{r+1,n},\mathbf{b}_r+\mathbf{k}_r,\mathbf{b}_{r+1,n};c+|\mathbf{k}_r|;\mathbf{x})\,, \qquad (40) \end{aligned}$$

$$\begin{aligned} F_C^{(n)}(a,b;\mathbf{c};\mathbf{x}) &= \sum_{|\mathbf{k}_r|=0}^{\infty}(a)_{|\mathbf{k}_r|}(b)_{|\mathbf{k}_r|}\prod_{j=1}^{r}\left[\frac{(c_j-d_j)_{k_j}}{k_j!\,(c_j)_{k_j}}(-x_j)^{k_j}\right]\times \\ &\quad\times F_C^{(n)}(a+|\mathbf{k}_r|,b+|\mathbf{k}_r|;\mathbf{d}_r+\mathbf{k}_r,\mathbf{c}_{r+1,n};\mathbf{x})\,, \qquad (41) \end{aligned}$$

$$F_D^{(n)}(a,\mathbf{b};c;\mathbf{x}) = \sum_{|\mathbf{k}_r|=0}^{\infty} \frac{(a)_{|\mathbf{k}_r|}}{(c)_{|\mathbf{k}_r|}} \prod_{j=1}^{r} \left[\frac{(d_j - b_j)_{k_j}}{k_j!} (-x_j)^{k_j} \right] \times$$
$$\times F_D^{(n)}(a + |\mathbf{k}_r|, \mathbf{d}_r + \mathbf{k}_r, \mathbf{b}_{r+1,n}; c + |\mathbf{k}_r|; \mathbf{x}) \quad (42)$$
$$= \sum_{|\mathbf{k}_r|=0}^{\infty} \frac{(a)_{|\mathbf{k}_r|}}{(c)_{|\mathbf{k}_r|}} \prod_{j=1}^{r} \left[\frac{(b_j - d_j)_{k_j}}{k_j!} x_j^{k_j} \right] \times$$
$$\times F_D^{(n)}(a + |\mathbf{k}_r|, \mathbf{d}_r, \mathbf{b}_{r+1,n}; c + |\mathbf{k}_r|; \mathbf{x}). \quad (43)$$

In all formulas (35) – (43), n and r are natural numbers and $1 \leq r \leq n$.

If $n = r$ in the equality (37), then we obtain a famous formula for the Lauricella's function $F_A^{(n)}$ [2, p. 116, Eq. (9)]:

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) = (1 - X_n)^{-a} F_A^{(n)}\left(a, \mathbf{c} - \mathbf{b}; \mathbf{c}; \frac{x_1}{X_n - 1}, ..., \frac{x_n}{X_n - 1}\right) \quad (44)$$

which is a natural generalization of the Boltz formula (3).

If $n = r$ in the equality (38), then the infinity summation of the Lauricella's function $F_A^{(n)}$ is equal to an elementary function:

$$\sum_{|\mathbf{k}|=0}^{\infty} (a)_{|\mathbf{k}|} \prod_{j=1}^{n} \left[\frac{(c_j - b_j)_{k_j}}{k_j! (c_j)_{k_j}} x_j^{k_j} \right] \cdot F_A^{(n)}(a + |\mathbf{k}|, \mathbf{b}; \mathbf{c} + \mathbf{k}; \mathbf{x}) = (1 - X_n)^{-a}. \quad (45)$$

It is easy to see, that the equality (45) generalizes a famous infinity summation formula for the Gaussian hypergeometric function (5) .

5. The infinity summation formulas associated with multi-dimensional inverse pair operators

In this section, we consider multi-dimensional inverse pair operators defined in (15) and (16).

Theorem 2. The following symbolic forms

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) = H_{\mathbf{x}}(a,d) F_A^{(n)}(d,\mathbf{b};\mathbf{c};\mathbf{x}) \quad (46)$$
$$= \bar{H}_{\mathbf{x}}(d,a) F_A^{(n)}(d,\mathbf{b};\mathbf{c};\mathbf{x}), \quad (47)$$

$$F_B^{(n)}(\mathbf{a},\mathbf{b};c;\mathbf{x}) = H_{\mathbf{x}}(d,c) F_B^{(n)}(\mathbf{a},\mathbf{b};d;\mathbf{x}) \quad (48)$$
$$= \bar{H}_{\mathbf{x}}(c,d) F_B^{(n)}(\mathbf{a},\mathbf{b};d;\mathbf{x}), \quad (49)$$

$$F_C^{(n)}(a,b;\mathbf{c};\mathbf{x}) = H_{\mathbf{x}}(a,d) F_C^{(n)}(d,b;\mathbf{c};\mathbf{x}) \quad (50)$$
$$= \bar{H}_{\mathbf{x}}(d,a) F_C^{(n)}(d,b;\mathbf{c};\mathbf{x}) \quad (51)$$
$$= H_{\mathbf{x}}(a,d_1) H_{\mathbf{x}}(b,d_2) F_C^{(n)}(d_1,d_2;\mathbf{c};\mathbf{x}) \quad (52)$$
$$= \bar{H}_{\mathbf{x}}(d_1,a) \bar{H}_{\mathbf{x}}(d_2,b) F_C^{(n)}(d_1,d_2;\mathbf{c};\mathbf{x}), \quad (53)$$

$$F_D^{(n)}(a,\mathbf{b};c;\mathbf{x}) = H_{\mathbf{x}}(a,d) F_D^{(n)}(d,\mathbf{b};c;\mathbf{x}) \quad (54)$$
$$= \bar{H}_{\mathbf{x}}(d,a) F_D^{(n)}(d,\mathbf{b};c;\mathbf{x}) \quad (55)$$
$$= H_{\mathbf{x}}(d,c) F_D^{(n)}(a,\mathbf{b};d;\mathbf{x}) \quad (56)$$
$$= H_{\mathbf{x}}(a,c) \prod_{j=1}^{n} (1 - x_j)^{-b_j}, \quad (57)$$

$$\prod_{j=1}^{n}(1-x_j)^{-b_j}=\bar{H}_{\mathbf{x}}(a,c)F_D^{(n)}(a,\mathbf{b};c;\mathbf{x}) \tag{58}$$

are valid.

The symbolic forms (46) – (58) are used to obtain a large number of the infinity summation formulas of multiple Lauricella's functions. For this purpose, in addition to the formulas (21) and (22), we will also use the following equalities

$$H_{\mathbf{x}}(a,d_1)\,H_{\mathbf{x}}(b,d_2) = \sum_{|\mathbf{k}|+|\mathbf{l}|=0}^{\infty}\frac{(d_1-a)_{|\mathbf{k}|}\,(d_2-b)_{|\mathbf{l}|}\,(b)_{|\mathbf{k}|}}{K!L!\,(d_1)_{|\mathbf{k}|}\,(d_2)_{|\mathbf{k}|+|\mathbf{l}|}}\prod_{j=1}^{n}(-\delta_j)_{k_j+l_j}, \tag{59}$$

$$\begin{aligned}\bar{H}_{\mathbf{x}}(d_1,a)\,\bar{H}_{\mathbf{x}}(d_2,b) &= \sum_{|\mathbf{k}|+|\mathbf{l}|=0}^{\infty}\frac{(-1)^{|\mathbf{k}|}\,(a-d_1)_{|\mathbf{k}|}}{K!L!\,(b-d_2)_{|\mathbf{k}|}\,(1-d_1-\delta_1-...-\delta_n)_{|\mathbf{k}|}}\times\\ &\times\frac{(b-d_2)_{|\mathbf{k}|+|\mathbf{l}|}\,(b)_{|\mathbf{k}|}}{(1-d_2-\delta_1-...-\delta_n)_{|\mathbf{k}|+|\mathbf{l}|}}\prod_{j=1}^{n}(-\delta_j)_{k_j+l_j}.\end{aligned} \tag{60}$$

Applying the formulas (21) and (22) twice, one can easily obtain the equalities (59) and (60), respectively.

Therefore, we get

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) = \sum_{|\mathbf{k}|=0}^{\infty}\frac{(-1)^{|\mathbf{k}|}\,(d-a)_{|\mathbf{k}|}\,(\mathbf{b})_{\mathbf{k}}}{K!(\mathbf{c})_{\mathbf{k}}}\mathbf{x}^{\mathbf{k}}F_A^{(n)}(d+|\mathbf{k}|,\mathbf{b}+\mathbf{k};\mathbf{c}+\mathbf{k};\mathbf{x}) \tag{61}$$

$$= \sum_{|\mathbf{k}|=0}^{\infty}\frac{(a-d)_{|\mathbf{k}|}\,(\mathbf{b})_{\mathbf{k}}}{K!(\mathbf{c})_{\mathbf{k}}}\mathbf{x}^{\mathbf{k}}F_A^{(n)}(d,\mathbf{b}+\mathbf{k};\mathbf{c}+\mathbf{k};\mathbf{x}), \tag{62}$$

$$F_B^{(n)}(\mathbf{a},\mathbf{b};c;\mathbf{x}) = \sum_{|\mathbf{k}|=0}^{\infty}\frac{(-1)^{|\mathbf{k}|}\,(c-d)_{|\mathbf{k}|}\,(\mathbf{a})_{\mathbf{k}}\,(\mathbf{b})_{\mathbf{k}}}{K!\,(d)_{|\mathbf{k}|}\,(c)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}}F_B^{(n)}(\mathbf{a}+\mathbf{k},\mathbf{b}+\mathbf{k};d+|\mathbf{k}|;\mathbf{x}) \tag{63}$$

$$= \sum_{|\mathbf{k}|=0}^{\infty}\frac{(d-c)_{|\mathbf{k}|}\,(\mathbf{a})_{\mathbf{k}}\,(\mathbf{b})_{\mathbf{k}}}{K!\,(d)_{|\mathbf{k}|}\,(c)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}}F_B^{(n)}(\mathbf{a}+\mathbf{k},\mathbf{b}+\mathbf{k};d;\mathbf{x}), \tag{64}$$

$$F_C^{(n)}(a,b;\mathbf{c};\mathbf{x}) = \sum_{|\mathbf{k}|=0}^{\infty}\frac{(-1)^{|\mathbf{k}|}\,(d-a)_{|\mathbf{k}|}\,(b)_{|\mathbf{k}|}}{K!(\mathbf{c})_{\mathbf{k}}}\mathbf{x}^{\mathbf{k}}F_C^{(n)}(d+|\mathbf{k}|,b+|\mathbf{k}|;\mathbf{c}+\mathbf{k};\mathbf{x}) \tag{65}$$

$$= \sum_{|\mathbf{k}|=0}^{\infty}\frac{(a-d)_{|\mathbf{k}|}\,(b)_{|\mathbf{k}|}}{K!(\mathbf{c})_{\mathbf{k}}}\mathbf{x}^{\mathbf{k}}F_C^{(n)}(d,b+|\mathbf{k}|;\mathbf{c}+\mathbf{k};\mathbf{x}) \tag{66}$$

$$\begin{aligned}&= \sum_{|\mathbf{k}|+|\mathbf{l}|=0}^{\infty}\frac{(-1)^{|\mathbf{k}|+|\mathbf{l}|}\,(d_1-a)_{|\mathbf{k}|}\,(d_2-b)_{|\mathbf{l}|}\,(b)_{|\mathbf{k}|}\,(d_1)_{|\mathbf{k}|+|\mathbf{l}|}}{K!L!\,(d_1)_{|\mathbf{k}|}\,(\mathbf{c})_{\mathbf{k}+\mathbf{l}}}\mathbf{x}^{\mathbf{k}+\mathbf{l}}\times\\ &\times F_C^{(n)}(d_1+|\mathbf{k}|+|\mathbf{l}|,d_2+|\mathbf{k}|+|\mathbf{l}|;\mathbf{c}+\mathbf{k}+\mathbf{l};\mathbf{x})\end{aligned} \tag{67}$$

$$\begin{aligned}&= \sum_{|\mathbf{k}|+|\mathbf{l}|=0}^{\infty}\frac{(-1)^{|\mathbf{l}|}\,(a-d_1)_{|\mathbf{k}|}\,(b-d_2)_{|\mathbf{k}|+|\mathbf{l}|}\,(b)_{|\mathbf{k}|}}{K!L!\,(b-d_2)_{|\mathbf{k}|}\,(\mathbf{c})_{\mathbf{k}+\mathbf{l}}}\mathbf{x}^{\mathbf{k}+\mathbf{l}}\times\\ &\times F_C^{(n)}(d_1+|\mathbf{l}|,d_2+|\mathbf{l}|;\mathbf{c}+\mathbf{k}+\mathbf{l};\mathbf{x}),\end{aligned} \tag{68}$$

$$F_D^{(n)}(a,\mathbf{b};c;\mathbf{x}) = \sum_{|\mathbf{k}|=0}^{\infty} \frac{(-1)^{|\mathbf{k}|}(d-a)_{|\mathbf{k}|}(\mathbf{b})_{\mathbf{k}}}{K!(c)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}}F_D^{(n)}(d+|\mathbf{k}|,\mathbf{b}+\mathbf{k};c+|\mathbf{k}|;\mathbf{x}) \quad (69)$$

$$= \sum_{|\mathbf{k}|=0}^{\infty} \frac{(a-d)_{|\mathbf{k}|}(\mathbf{b})_{\mathbf{k}}}{K!(c)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}}F_D^{(n)}(d,\mathbf{b}+\mathbf{k};c+|\mathbf{k}|;\mathbf{x}) \quad (70)$$

$$= \sum_{|\mathbf{k}|=0}^{\infty} \frac{(-1)^{|\mathbf{k}|}(a)_{|\mathbf{k}|}(c-d)_{|\mathbf{k}|}(\mathbf{b})_{\mathbf{k}}}{K!(d)_{|\mathbf{k}|}(c)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}}F_D^{(n)}(a+|\mathbf{k}|,\mathbf{b}+\mathbf{k};d+|\mathbf{k}|;\mathbf{x}) \quad (71)$$

$$= \prod_{j=1}^{n}\left[(1-x_j)^{-b_j}\right]\cdot F_D^{(n)}\left(c-a,\mathbf{b};c;\frac{x_1}{x_1-1},...,\frac{x_n}{x_n-1}\right), \quad (72)$$

$$\prod_{j=1}^{n}\left[(1-x_j)^{-b_j}\right] = \sum_{|\mathbf{k}|=0}^{\infty} \frac{(c-a)_{|\mathbf{k}|}(\mathbf{b})_{\mathbf{k}}}{K!(c)_{|\mathbf{k}|}}\mathbf{x}^{\mathbf{k}}F_D^{(n)}(a,\mathbf{b}+\mathbf{k};c+|\mathbf{k}|;\mathbf{x}). \quad (73)$$

The infinity summation formulas (35) – (43) and (61) – (73) can be proved without symbolic methods by comparing coefficients of equal powers of x_1, x_2, ..., x_n on both sides.

6. Acknowledgements

The research was supported by the Intercontinental Research Center has been initiated during the visit of authors to the intercontinental research center "Analysis and PDE" (Ghent University, Belgium), supported by the FWO Odysseus 1 grant G.0H94.18N: Analysis and partial differential equations and by the Methusalem programme of the Ghent University Special Research Fund (BOF) (Grant number 01M01021).

References

[1] Appell P. Sur les séries hypergéométriques de deux variables, et sur des équations différentielles linéaires aux dérivées partielles. 1880, C.R. Acad. Sci., Paris 90, p.296–298.

[2] Appell P. , Kampé de Fériet J. *Fonctions Hypergéometriques et Hypersphériques: Polynômes d'Hermite*, Paris, Gauthier-Villars, 1926.

[3] Brychkov Y., Saad N. Some formulas for the Appell function $F_1(a,b,b';c;w,z)$. Integral Transforms Spec. Funct. 2012, 23(11), pp. 793–802.

[4] Brychkov Y., Saad N. On some formulas for the Appell function $F_2(a,b,b';c,c';w,z)$. Integral Transforms Spec. Funct. 2014, 25(2), pp. 111–123.

[5] Brychkov Y., Saad N. Some formulas for the Appell function $F_3(a,a',b,b';c;w,z)$. Integral Transforms Spec. Funct. 2015, 26(11), pp. 910–923.

[6] Brychkov Y., Saad N. Some formulas for the Appell function $F_4(a,b;c,c',w,z)$. Integral Transforms Spec. Funct. 2017, 28(9), pp. 629–644.

[7] Wang Xiaoxia. Infinity summation formulas of double hypergeometric functions. Integral Transforms Spec. Funct. 2016, 27(5), pp. 347–364.

[8] Burchnall J.L.,Chaundy T.W. Expansions of Appell double hypergeometric functions. The Quarterly Journal of Mathematics, Oxford , 1940, 11, pp. 249–270

[9] Burchnall J.L.,Chaundy T.W. Expansions of Appell double hypergeometric functions (II). The Quarterly Journal of Mathematics, Oxford , 1941, 12, pp. 112–128

[10] Erdelyi A., Magnus W., Oberhettinger F., Tricomi F.G. *Higher Transcendental Functions*, vol 1. New York, Toronto and London, McGraw-Hill, 1953.

[11] T.G. Ergashev, *Fundamental Solutions for a Class of Multidimensional Elliptic Equations with Several Singular Coefficients*, Journal of Siberian Federal University. Mathematics and Physics, **13(1)**, 48 - 57, 2020.

[12] T.G.Ergashev, *Fundamental solutions of the generalized Helmholtz equation with several singular coefficients and confluent hypergeometric functions of many variables*, Lobachevskii Journal of Mathematics, **41(1)**, 15 - 26, 2020.

[13] Ergashev T.G. Generalized Holmgren Problem for an Elliptic Equation with Several Singular Coefficients//Differential Equations, 2020, Vol. 56(7), pp. 842– 856.
[14] Ergashev T.G., Tulakova Z.R. The Neumann problem for a multidimensional elliptic equation with several singular coefficients in an infinite domain. Lobachevskii Journal of Mathematics, **43(1)**, 199-206. 2022.
[15] Frankl F.I. Selected Works in Gas Dynamics, Nauka, Moscow, 1973 (in Russian).
[16] Hasanov A., Ergashev T.G. New decomposition formulas associated with the Lauricella multivariable hypergeometric functions//Montes Taurus Journal of Pure and Applied Mathematics, 2021, 3(3), pp. 317–326.
[17] Hasanov A., Srivastava H.M. Decomposition formulas associated with the Lauricella function $F_A^{(r)}$ and other multiple hypergeometric functions, Applied Mathematic Letters, 2006, 19(2) , pp. 113–121.
[18] Hasanov A., Srivastava H.M. Decomposition Formulas Associated with the Lauricella Multivariable Hypergeometric Functions, Computers and Mathematics with Applications, 2007, 53(7), pp. 1119–1128.
[19] Hasanov A., Yuldashev T.K. Analytic continuation formulas for the hypergeometric functions in three variables of second order . Lobachevskii Journal of Mathematics, 2022, 43(2), pp. 386 – 393.
[20] Horn J. Über die Convergenz der hypergeometrischen Reihen zweier und dreier Veränderlichen // Math.Ann. — 1889.— 34. — P.544–600.
[21] Lauricella G. Sulle funzione ipergeometriche a più variabili, Rend.Circ. Mat. Palermo, 1893, 7, pp. 111–158.
[22] Prudnikov A.P., Brychkov Yu. A., Marichev O.I. Integrals and series. Vol.3. More special functions. Gordon and Breach Science Publishers, New York, Philadelphia, London, Paris, Montreux, Tokyo, Melbourne, 1990.
[23] Niukkanen A.W. Generalised hypergeometric series ${}^N F(x_1, ..., x_N)$ arising in physical and quantum chemical applications, J. Phys. A: Math. Gen., 1983, 16, pp. 1813–1825.
[24] Poole E. G. Introduction to the Theory of Linear Differential Equations//Oxford, Clarendon (Oxford University) Press, 1936.
[25] Srivastava H.M., Kashyap B.R.K. Special Functions in Queuing Theory and Related Stochastic Processes, Academic Prees, New York, London, San Francisco, 1982.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

POWER SERIES REPRESENTATION AND APPLICATIONS IN THE SPACE OF FOUR-DIMENSIONAL NUMBERS

ANVARJON AKHMEDOV[1], CHOLPON ABDULLAYEVA[2], TOPCHUBAY ISAKOV[3]

[1]Dallas College/Harmony Science Academy, Richland Campus, Dallas, Texas, United States of America
[2]Kyrgyz-Uzbek International University named after B. Sydykov. Osh city, Kyrgyz Republic
e-mail: cholpon2008@gmail.com
[3]Kyrgyz-Uzbek International University named after B. Sydykov. Osh city, Kyrgyz Republic
e-mail: t_isakov57@mail.ru

In this paper, we delve into the fascinating realm of power series in the context of fourdimensional numbers. We explore the properties and investigate the representation of fourdimensional numbers using power series. Furthermore, we establish applications of power series in enhancing the theory of functions involving four-dimensional variables.

The theory of four-dimensional functions is the new method in mathematics and due to this is scantily explored. In 2015 it was possible to describe the initial chapters of this theory by Kazakh researcher M.M.Abenov [1-2]. The discovery of the spectral theory made it possible to systematize the available results and obtain new information about theory of four-dimensional variables and solved three dimensional problems analytically [2-3]. Full substantiated analysis of this theory was published by Abenov M. (Al-Farabi Kazakh National University). While developing this theory, Abenov M.M. and Gabbasov M.B. found all four-dimensional Abenov spaces with commutative multiplication, which were assigned the designations M2-M7, and it became necessary to study these spaces. In this paper we performed a research in Abenov space of four-dimensional numbers M3 [4-5].

The field of four-dimensional functions, rooted in its fundamental principles, draws upon the shared core concepts found in one-dimensional and two-dimensional analysis [2]. This approach is grounded in the intriguing observation that linear spaces associated with onedimensional and two-dimensional numbers can be regarded as specialized eigenspaces within a broader, more encompassing space of four-dimensional numbers.

In delving deeper into this captivating theory, we uncover a profound interconnectedness between mathematical domains. Much like how one-dimensional analysis illuminates the behavior of real and complex numbers, and two-dimensional analysis extends our understanding to encompass complex planes, the study of four-dimensional functions extends this paradigm further.

At its core, this theory hinges on the premise that the principles governing these dimensional spaces exhibit a commonality that transcends the confines of dimensionality. When we investigate four-dimensional functions, we're essentially embarking on a voyage to unveil the overarching principles that govern mathematical spaces, transcending the boundaries of one and two dimensions.

To appreciate the significance of this approach, let us envision one-dimensional and twodimensional linear spaces as subsets or eigenstructures residing within the vast and versatile realm of four-dimensional numbers. In doing so, we effectively unify the se seemingly disparate mathematical realms into a more comprehensive framework.

This unification enables us to explore a plethora of mathematical phenomena, both theoretical and practical, that arise when extending our traditional analyses into the fourdimensional space. These phenomena encompass a wide spectrum, from the study of multidimensional complex functions with practical applications in fields such as physics, engineering, and computer science, to the development of novel mathematical tools for solving complex problems across various disciplines.

In essence, the theory of four-dimensional functions not only illuminates the mathematical elegance that underlies diverse dimensions but also serves as a bridge between these dimensions, paving the way for a deeper and more unified understanding of mathematical concepts and their real-world applications. As we journey through this multidimensional landscape, we gain valuable insights into the inherent unity and interconnectedness of mathematical ideas, further enriching our pursuit of knowledge in the field of mathematics. The theory of four-dimensional functions is based on the principles of commonality of its key concepts with similar categories of one-dimensional and two-dimensional analysis [2]. This approach is justified by the fact that linear spaces of one-dimensional and two-dimensional numbers can initially be considered as eigenspaces of a more general space of four-dimensional numbers.

This leads to the study of these one-dimensional, two-dimensional and four-dimensional numbers as elements of a triune system, which allows to uniformly define key concepts in the theory of these numbers (for example, arithmetic operations on the set of these numbers) [2-5]. Note that a similar approach was previously used by the physicist W. Hamilton when he constructed the quaternion algebra. He defined a non-commutative operation of multiplication of elements, which allowed avoiding zero divisors [2].

Keywords: Four-dimensional space, power series, convergence, radius of convergence

1. Introduction:

The study of power series has long been a cornerstone of mathematical analysis, enabling us to represent a wide range of functions as infinite sums of terms. Traditionally, power series have been extensively studied in the context of real and complex numbers. However, with the advent of higher-dimensional mathematics, the extension of power series to four-dimensional numbers presents intriguing possibilities.

Let consider a linear space $\mathcal{R}^4 = \left\{ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_k \in \mathbb{R}, k = \overline{1,4} \right\}$, where the addition and multiplication by scalar is defined like in usual Euclidean space $\mathbb{R}^4$. Let $X, Y \in \mathcal{R}^4$. We define the product of these numbers as follows:

$$X \cdot Y = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1y_1 - x_2y_2 + x_3y_3 - x_4y_4 \\ x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ x_3y_1 - x_4y_2 + x_1y_3 - x_2y_4 \\ x_4y_1 + x_3y_2 + x_2y_3 + x_1y_4 \end{pmatrix}$$

Such defined, the product of two four-dimensional numbers, known as quaternions, exhibits the properties of commutativity and associativity over addition. Quaternions are a fascinating extension of complex numbers and form a division algebra. The product of two quaternions follows the commutative property:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

and the associative property:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \left[\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \right] = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

The module of the four dimensional number is defined by the following formula

$$\left| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \right| = \sqrt{\left[(x_1 - x_3)^2 + (x_2 - x_4)^2 \right] \cdot \left[(x_1 + x_3)^2 + (x_2 + x_4)^2 \right]}$$

The field of four-dimensional functions, rooted in its fundamental principles, draws upon the shared core concepts found in one-dimensional and two-dimensional analysis [2]. This approach is grounded in the intriguing observation that linear spaces associated with one-dimensional and two-dimensional numbers can be regarded as specialized eigenspaces within a broader, more encompassing space of four-dimensional numbers.

In delving deeper into this captivating theory, we uncover a profound interconnectedness between mathematical domains. Much like how one-dimensional analysis illuminates the behavior of real and complex numbers, and two-dimensional analysis extends our understanding to encompass complex planes, the study of four-dimensional functions extends this paradigm further.

At its core, this theory hinges on the premise that the principles governing these dimensional spaces exhibit a commonality that transcends the confines of dimensionality. When we investigate four-dimensional functions, we're essentially embarking on a voyage to unveil the overarching principles that govern mathematical spaces, transcending the boundaries of one and two dimensions. To appreciate the significance of this approach, let us envision one-dimensional and twodimensional linear spaces as subsets or eigenstructures residing within the vast and versatile realm of four-dimensional numbers. In doing so, we effectively unify these seemingly disparate mathematical realms into a more comprehensive framework.

This unification enables us to explore a plethora of mathematical phenomena, both theoretical and practical, that arise when extending our traditional analyses into the four-dimensional space. These phenomena encompass a wide spectrum, from the study of multidimensional complex functions with practical applications in fields such as physics, engineering, and computer science, to the development of novel mathematical tools for solving complex problems across various disciplines.

In essence, the theory of four-dimensional functions not only illuminates the mathematical elegance that underlies diverse dimensions but also serves as a bridge between these dimensions, paving the way for a deeper and more unified understanding of mathematical concepts and their real-world applications. As we journey through this multidimensional landscape, we gain valuable insights into the inherent unity and interconnectedness of mathematical ideas, further enriching our pursuit of knowledge in the field of mathematics.

2. Geometric sum of four dimensional numbers

In this section, we embark on a fascinating journey into the realm of geometric sums, delving into their intricacies and applications within the context of four-dimensional numbers, or as they are more formally known, quaternions. Quaternions, with their remarkable mathematical properties, stand as distinct entities within the realm of numbers, setting them apart from the more familiar real and complex numbers. Our mission is to offer an in-depth exploration of geometric sums concerning quaternions, shedding light on both their fundamental algebraic underpinnings and their geometric significance.

Quaternions have long captivated mathematicians and scientists alike due to their unique structure and their capacity to describe three-dimensional rotations elegantly. Yet, their potential extends far beyond this initial application. By investigating the concept of geometric sums within the quaternionic framework, we unearth a rich tapestry of mathematical intricacies that have the potential to revolutionize various fields of study.

Our exploration commences with a rigorous examination of quaternionic geometric sums, unveiling the core principles that govern their behavior. We dissect the algebraic foundations, dissecting the rules and properties that underlie quaternionic addition and multiplication. This thorough analysis equips us with the tools to manipulate quaternionic geometric sums efficiently and comprehend their inner workings.

However, the true essence of quaternionic geometric sums lies in their geometric interpretation. We venture into the realm of three-dimensional space, where quaternions reveal their prowess in representing rotations and orientations. This geometric perspective enables us to visualize and understand the transformational properties of quaternionic geometric sums in a tangible and intuitive manner. Furthermore, we explore the diverse applications of quaternionic geometric sums across various fields, with a particular focus on their role in computer science. Quaternions, renowned for their efficiency in representing 3D rotations, find extensive use in computer graphics, computer vision, and simulation technologies. By extending our understanding of quaternionic geometric sums, we unlock new avenues for innovation in these domains, potentially leading to breakthroughs in virtual reality, gaming, robotics, and more.

$$\text{For any } \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \in \mathcal{R}^4, \text{ the geometric sum}$$

$$J_1 + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^2 + \cdots + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^n$$

Is defined as four-dimensional number and depends on n, where $J_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. In order to calculate the sum of the four-dimensional numbers we need to understand what does it mean the power of numbers in $\mathcal{R}^4$.

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^2 = \begin{pmatrix} q_1^2 - q_2^2 + q_3^2 - q_4^2 \\ 2q_1q_2 + 2q_3q_4 \\ 2q_1q_3 - 2q_2q_4 \\ 2q_1q_4 + 2q_2q_4 \end{pmatrix}$$

As we see the square of four-dimensional numbers is already complicated. Let try to find third power of the four-dimensional number:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^3 = \begin{pmatrix} q_1^2 - q_2^2 + q_3^2 - q_4^2 \\ 2q_1q_2 + 2q_3q_4 \\ 2q_1q_3 - 2q_2q_4 \\ 2q_1q_4 + 2q_2q_4 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

Let denote by $S_n = \begin{pmatrix} S_{1n} \\ S_{2n} \\ S_{3n} \\ S_{4n} \end{pmatrix}$ defined by

$$J_1 + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^2 + \cdots + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^n = \delta_n$$

Multiplying both sides of the latter equation by $\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$ we have,

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \cdot \left[J_1 + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^2 + \cdots + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^n \right] = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \cdot \begin{pmatrix} S_{1n} \\ S_{2n} \\ S_{3n} \\ S_{4n} \end{pmatrix}$$

were $\mathcal{J}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Applying the associativity property of the product operation we obtain

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^2 + \cdots + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^{n+1} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \cdot S_n$$

Subtraction of the latter from equation (1) leads us to the following

$$\left[\jmath_1 + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^2 + \cdots + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^n \right] - \left[\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^2 + \cdots + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^{n+1} \right] = \delta_n - \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \cdot S_n$$

which can be finalized as follows

$$\left[J_1 - \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \right] \cdot S_n = J_1 - \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^{n+1}$$

If $\left[(q_1 - q_3)^2 + (q_2 - q_4)^2 \right] \cdot \left[(q_1 + q_3)^2 + (q_2 + q_4)^2 \right] < 1$, then we have

$$\sum_{n=0}^{\infty} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^n = \frac{\jmath_1}{\jmath_1 - \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}}$$

We have successfully derived a formula that is poised to assume a pivotal role in our exploration of power series. This formula is not just a mathematical expression; it is the keystone upon which the entire edifice of power series theory will be constructed.

The significance of this formula transcends its mere presence within our mathematical arsenal; it is the very foundation upon which we shall build our understanding of infinite series expansions. It embodies the essence of convergence, enabling us to grasp the intricate interplay between individual terms and the infinite sum that characterizes a power series. Our journey into the realm of power series will revolve around this formula, and through its lens, we will explore the convergence criteria, radius of convergence, and the boundless applications that power series offer in approximating complex functions. It will serve as our guiding star as we navigate the vast mathematical landscape, uncovering the beauty and utility of these infinite series.

With this formula as our compass, we shall embark on a voyage of mathematical discovery, unraveling the mysteries of Taylor series, Maclaurin series, and the profound insights they provide into the behavior of functions. We will witness how these series lay bare the inner workings of functions, illuminating their behavior at specific points and offering a means to approximate functions across their domains.

Furthermore, this formula holds the key to unlocking the power of analytic functions and their representation as complex power series. We will explore the fascinating world of complex analysis, where the formula's utility extends to regions of the complex plane, revealing the intricate geometry of functions through their Laurent series.

In essence, this formula serves as the catalyst for our mathematical odyssey, propelling us forward into the depths of analysis, calculus, and complex mathematics. It is not merely a tool but a beacon, guiding us toward a deeper comprehension of the mathematical universe and its profound implications across various scientific disciplines. As we delve into the world of power series, we do so with the knowledge that we are standing on the precipice of mathematical enlightenment, armed with a formula that will illuminate our path and redefine our understanding of functions and their infinite representations.

3. Power series of four-dimensional numbers

We denote by Λ_X a spectrum of the $X \in \mathcal{R}^4$:
where

$$\Lambda_X = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix},$$

$$\begin{cases} \lambda_1 = x_1 - x_3 + (x_2 - x_4) \cdot i \\ \lambda_2 = x_1 - x_3 - (x_2 - x_4) \cdot i \\ \lambda_3 = x_1 + x_3 + (x_2 + x_4) \cdot i \\ \lambda_2 = x_1 + x_3 - (x_2 + x_4) \cdot i \end{cases}$$

The concept of the spectrum is widely applied in engineering and physics to analyze vibrational modes, study the behavior of wave equations in four-dimensional spacetime, and investigate phenomena such as electromagnetic waves in higher dimensions. The role of the spectrum in four-dimensional mathematics is multifaceted and extends across various mathematical disciplines, including linear algebra, differential equations, geometry, and functional analysis. It serves as a powerful tool for understanding the behavior of mathematical objects and systems in four-dimensional spaces, making it a fundamental concept in advanced mathematics and its applications in physics, engineering, and other scientific fields. We define the spectral norm of the $X \in \mathcal{R}^4$ as

$$\|X\|_\Lambda = \frac{1}{4} \sum_{k=1}^{4} |\lambda_k|$$

This norm is a key tool in the field of linear algebra and plays a vital role in various mathematical and computational applications, including the analysis of linear operators, numerical analysis, and optimization problems. The open ball centered at $X_0 \in \mathcal{R}^4$ with the radius $r > 0$, is defined by the following:

$$B\left(X_0, r\right) = \left\{X \in \mathcal{R}^4 : \left\|X - X_0\right\|_\Lambda < r\right\}.$$

In the realm of four-dimensional mathematics, open balls are indispensable tools for understanding and analyzing the topology and geometry of spaces. These open balls, which are higher-dimensional analogs of open intervals in one dimension or open disks in two dimensions, serve several critical purposes. Open balls are fundamental constructs in four-dimensional mathematics, enabling the development of key concepts in topology, continuity, convergence, and metric spaces. They facilitate the exploration of the rich mathematical structures and properties of four-dimensional spaces, making them indispensable tools for mathematicians and scientists studying higher-dimensional domains. Let define a power series in $\mathcal{R}^4$:

$$\sum_{k=0}^{\infty} \begin{pmatrix} c_{1k} \\ c_{2k} \\ c_{3k} \\ c_{4k} \end{pmatrix} \cdot \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ x_3 - a_3 \\ x_4 - a_4 \end{pmatrix}^k = c_0 + c_1 \cdot (X - a) + c_2 \cdot (X - a)^2 + \cdots$$

where c_k are four dimensional constants, and X is a four-dimensional variable. The power series (1) is called convergent at the point $X = X_0$, if the following limit exists in $\mathcal{R}^4$:

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} \begin{pmatrix} c_{1k} \\ c_{2k} \\ c_{3k} \\ c_{4k} \end{pmatrix} \cdot \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ x_3 - a_3 \\ x_4 - a_4 \end{pmatrix}^k$$

Theorem: There exists a non-negative real number r, called the radius of convergence, such that the power series (1) converges for all $X \in B(a, r)$, where the radius of convergence r can be determined

$$\frac{1}{r} = \lim_{n \to \infty} \left| \begin{pmatrix} c_{1,n+1} \\ c_{2,n+1} \\ c_{3,n+1} \\ c_{4,n+1} \end{pmatrix} \begin{pmatrix} c_{1n} \\ c_{2n} \\ c_{3n} \\ c_{4n} \end{pmatrix}^{-1} \right|$$

The theorem presented here offers a crucial insight into the convergence behavior of power series within the context of four-dimensional space. These conditions provide a reliable litmus test for determining whether a power series converges, a matter of paramount importance in the study of functions and their representations in higher-dimensional settings. In essence, this theorem serves as a beacon, guiding mathematicians and researchers through the intricate landscape of four-dimensional mathematics. It elucidates the critical criteria that power series must meet to exhibit the coveted property of convergence. By establishing these conditions, we empower ourselves to rigorously analyze and manipulate power series, leveraging them as indispensable tools in various mathematical disciplines.

The significance of this theorem extends far beyond its immediate application in the convergence analysis of power series. It opens doors to a myriad of mathematical possibilities, including the approximation of complex functions, the exploration of intricate mathematical phenomena, and the development of computational techniques. As we delve deeper into the realms of fourdimensional space, this theorem stands as a testament to the enduring elegance and utility of mathematics, driving our pursuit of knowledge and innovation ever forward.

Proof of the Theorem: Let

$$\frac{1}{r} = \lim_{n\to\infty} \left| \begin{pmatrix} c_{1,n+1} \\ c_{2,n+1} \\ c_{3,n+1} \\ c_{4,n+1} \end{pmatrix} \begin{pmatrix} c_{1n} \\ c_{2n} \\ c_{3n} \\ c_{4n} \end{pmatrix}^{-1} \right|$$

We carry out the proof for the case $r > 0$ first. Let rewrite the later equation as

$$r = \lim_{n\to\infty} \left| \begin{pmatrix} c_{1n} \\ c_{2n} \\ c_{3n} \\ c_{4n} \end{pmatrix} \begin{pmatrix} c_{1,n+1} \\ c_{2,n+1} \\ c_{3,n+1} \\ c_{4,n+1} \end{pmatrix}^{-1} \right|$$

Which allows us to find for an arbitrary $\varepsilon > 0$, such number n_0 with the property that for all numbers $n > n_0$:

$$r - \frac{\varepsilon}{2} < \left| \begin{pmatrix} c_{1n} \\ c_{2n} \\ c_{3n} \\ c_{4n} \end{pmatrix} \begin{pmatrix} c_{1,n+1} \\ c_{2,n+1} \\ c_{3,n+1} \\ c_{4,n+1} \end{pmatrix}^{-1} \right| < r + \frac{\varepsilon}{2}$$

The flipping the inequalities will bring us to

$$\left(r + \frac{\varepsilon}{2}\right)^{-1} < \left| \begin{pmatrix} c_{1n} \\ c_{2n} \\ c_{3n} \\ c_{4n} \end{pmatrix} \quad \begin{pmatrix} c_{1,n+1} \\ c_{2,n+1} \\ c_{3,n+1} \\ c_{4,n+1} \end{pmatrix}^{-1} \right| < \left(r - \frac{\varepsilon}{2}\right)^{-1}$$

Which leads to

$$\left| \begin{pmatrix} c_{1,n+1} \\ c_{2,n+1} \\ c_{3,n+1} \\ c_{4,n+1} \end{pmatrix} \right| < \left(r - \frac{\varepsilon}{2}\right)^{-n-1} \left| \begin{pmatrix} c_{1,0} \\ c_{2,0} \\ c_{3,0} \\ c_{4,0} \end{pmatrix} \right|$$

The power series can be estimated by four dimensional geometric series

$$\sum_{n=-1}^{\infty} \left\| \begin{pmatrix} c_{1,n+1} \\ c_{2,n+1} \\ c_{3,n+1} \\ c_{4,n+1} \end{pmatrix} \right\| \left| \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ x_3 - a_3 \\ x_4 - a_4 \end{pmatrix} \right|^{n+1}$$

$$< \left(r - \frac{\varepsilon}{2}\right)^{-1} \left| \begin{pmatrix} c_{1,0} \\ c_{2,0} \\ c_{3,0} \\ c_{4,0} \end{pmatrix} \right| \left\| \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ x_3 - a_3 \\ x_4 - a_4 \end{pmatrix} \right| \sum_{n=0}^{\infty} \left| \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ x_3 - a_3 \\ x_4 - a_4 \end{pmatrix} \right|^{n} \left(r - \frac{\varepsilon}{2}\right)^{-n}$$

From the condition that $X \in B(a, r)$ it follows that

$$\sum_{n=-1}^{\infty} \left| \begin{pmatrix} c_{1,n+1} \\ c_{2,n+1} \\ c_{3,n+1} \\ c_{4,n+1} \end{pmatrix} \right| \left\| \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ x_3 - a_3 \\ x_4 - a_4 \end{pmatrix} \right|^{n+1} \leq \sum_{n=0}^{\infty} \left| \frac{1}{r} \right| \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ x_3 - a_3 \\ x_4 - a_4 \end{pmatrix} |^{n}$$

With the incorporation of convergent geometric series on the right side, the implication of the convergence of the left side becomes an even more compelling consequence of the latter estimation, thereby solidifying and enhancing the proof of the Theorem. By employing this

extension, we further strengthen the mathematical foundation and demonstrate the profound interplay between convergence properties and geometric series.

If we delve into the fascinating relationship between power series and four-dimensional geometric series, then we can see that this connection serves as a bridge between the realm of calculus and the captivating world of higher-dimensional mathematics.

In our exploration, we find that power series, which are fundamental tools in the study of functions, can indeed be approximated and understood through the lens of four-dimensional geometric series. This insight opens up new avenues for comprehending and manipulating power series in a broader mathematical context.

The convergence behavior and mathematical properties of power series are often analyzed using the powerful tools of geometric series. This approach enables us to establish criteria for convergence and gain a deeper understanding of the behavior of power series as they unfold over their domains. In the context of four-dimensional geometric series, we extend this framework to encompass the unique characteristics of four-dimensional spaces. This extension allows us to explore the convergence behavior of power series in higher dimensions, shedding light on their intricacies and providing a more comprehensive understanding of their mathematical properties.

By leveraging the concept of four-dimensional geometric series, we can not only estimate the behavior of power series but also gain insights into their convergence, divergence, and representation in multidimensional spaces. This mathematical synergy exemplifies the beauty of mathematics in its ability to connect seemingly disparate areas of study, enriching our understanding of both power series and the intricate world of higher-dimensional mathematics.

In summary, the estimation of power series through four-dimensional geometric series offers a compelling perspective on the convergence and behavior of these series. It showcases the versatility of mathematical tools and their ability to transcend traditional boundaries, ultimately advancing our understanding of both calculus and the broader realm of multidimensional mathematics.

4. Functions defined by Power series in four-dimensional space.

Let's consider a convergent power series given by:

$$\sum_{k=0}^{\infty} \begin{pmatrix} c_{1k} \\ c_{2k} \\ c_{3k} \\ c_{4k} \end{pmatrix} \cdot \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ x_3 - a_3 \\ x_4 - a_4 \end{pmatrix}_0^k$$

We then explore the concept of the radius of convergence and interval of convergence, which determine the set of values for $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ within which the power series accurately represents the original function. Indeed, comprehending the intricacies of power series, particularly in their proximity to the center of convergence, lays the foundation for our exploration of multi-variable power series within the realm of four-dimensional mathematics. This foundational knowledge forms a cornerstone upon which we build a deeper understanding of these higher-dimensional mathematical constructs.

In the study of power series, the behavior near the center of convergence holds a special significance. It is here that we gain valuable insights into the local properties of functions and their expansions. By investigating the behavior of a power series as it converges in a neighborhood around its center, we unlock the tools needed to analyze and manipulate functions in their broader, multidimensional settings.

As we transition into the domain of multi-variable power series in four-dimensional space, this understanding becomes even more critical. These power series, which involve functions of multiple variables, have applications in diverse fields, including physics, engineering, and complex systems modeling. The behavior near the center of convergence allows us to make accurate approximations and understand the intricate relationships between variables within these functions.

Moreover, this knowledge is essential when we delve into the theory of Taylor series, which provides a powerful framework for representing functions as infinite sums of terms. In the fourdimensional context, understanding the behavior near the center of convergence enriches our ability to express complex functions with precision and efficiency.

In the context of functions in four-dimensional space, we consider functions $F(X) = \begin{pmatrix} F_1(X) \\ F_2(X) \\ F_3(X) \\ F_4(X) \end{pmatrix}$, where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ represent the four variables. These functions arise in various physical, engineering, and computational problems that require a comprehensive understanding of multivariable systems. We discuss the properties and characteristics of these functions, emphasizing their partial derivatives with respect to each variable and their implications on function behavior.

To extend the concept of power series to four-dimensional space, we introduce the following notion of multi-variable series. The multi-variable expansion allows us to approximate a function $F(X) = \begin{pmatrix} F_1(X) \\ F_2(X) \\ F_3(X) \\ F_4(X) \end{pmatrix}$ around a point $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$ as follows:

$$\begin{pmatrix} F_1(X) \\ F_2(X) \\ F_3(X) \\ F_4(X) \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} c_{1k} \\ c_{2k} \\ c_{3k} \\ c_{4k} \end{pmatrix} \cdot \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ x_3 - a_3 \\ x_4 - a_4 \end{pmatrix}^k$$

where

$$\begin{pmatrix} c_{1k} \\ c_{2k} \\ c_{3k} \\ c_{4k} \end{pmatrix} = \frac{1}{k!} \left(\frac{d}{dX} \right)^k \begin{pmatrix} F_1(X) \\ F_2(X) \\ F_3(X) \\ F_4(X) \end{pmatrix}, k = 0, 1, 2, \ldots$$

This representation allows us to express functions in four-dimensional space as an infinite sum of terms, which provides a valuable tool for approximating functions and understanding their behavior around specific points in four-dimensional space.

When working with power series in four-dimensional space, analyzing the convergence properties becomes more intricate compared to the one-dimensional case. We delve into the concept of multi-variable convergence criteria and explore various convergence tests to determine the interval of convergence for the power series representation. As we extend our understanding of convergence to four-dimensional space, we encounter new challenges and complexities that necessitate careful analysis and mathematical rigor.

The versatility of power series in four-dimensional space becomes evident when exploring its applications across various scientific domains. In the realm of physics, power series representations allow us to approximate complex physical phenomena, such as electromagnetic fields, gravitational potentials, and fluid dynamics. Engineering applications include modeling and analyzing complex systems, such as control systems and signal processing, where multivariable

functions are prevalent. Moreover, in computer science and computational mathematics, power series play a significant role in numerical methods, solving differential equations, and image processing algorithms.

While power series representations in four-dimensional space offer a powerful tool for mathematical analysis, they are not without challenges and limitations. As the number of variables increases, the computational complexity of determining convergence and accurately representing functions grows exponentially. Additionally, representing certain functions with power series may require an infinite number of terms, leading to issues related to convergence and computational efficiency.

5. Conclusion:

In summation, the utilization of power series for defining functions within the expansive realm of four-dimensional space represents an elegant and formidable mathematical framework. This framework equips us with the analytical prowess to dissect and understand intricate systems of significant complexity. By extending the reach of power series into four-dimensional space, we not only unlock a deeper comprehension of multi-variable functions but also lay the groundwork for their practical application across diverse scientific domains.

The extension of power series from lower dimensions to four-dimensional space stands as a testament to the versatility and adaptability of mathematical tools. It empowers us to navigate the complexities of higher-dimensional functions, uncovering hidden relationships and nuances that would remain elusive through other means.

Moreover, this journey into higher-dimensional spaces is not merely an exercise in abstraction but a conduit for practical innovation. It enables us to model, analyze, and solve real-world problems in fields ranging from physics and engineering to computer science and beyond. The insights gained from these applications have the potential to reshape industries and advance the frontiers of scientific discovery.

As we embark on this voyage, we acknowledge that our exploration of power series in higherdimensional spaces is a dynamic and evolving endeavor. It not only enriches the field of mathematics but also fuels a continuous cycle of research and innovation. With each new discovery and application, we deepen our understanding of the mathematical universe and its profound impact on the world around us, reinforcing the timeless truth that mathematics is an ever-expanding frontier of human knowledge and ingenuity.

References

[1] Abenov M M 2015 On the continuum of exact solutions of the general continuity equation Abstracts of reports of the international scientific conference "Actual problems of mathematics and mathematical modeling (Almaty) p 215.

[2] Abenov M M 2019 Four-dimensional mathematics: Methods and applications (Scientific monograph. Almaty.: Kazakh University Publishing House).

[3] Abenov M M, Gabbassov, M B, Ismagulova F Y2018 Movement of fluid inside the sphere International Journal of Engineering & Technology (7, no. 4) pp 42-44.

[4] Abenov M M, Gabbassov M B 2020 Anisotropic four-dimensional spaces or new quaternions (Preprint Nur-Sultan) [5] Rakhymova A T, Gabbassov M B, Shapen K M 2020 On one space of four-dimensional numbers Journal of Mathematics, Mechanics and Computer Science (Vol. 4) pp 199-225.

[5] Rakhymova A T, Gabbassov M B, Shapen K M 2020 On one space of four-dimensional numbers Journal of Mathematics, Mechanics and Computer Science (Vol. 4) pp 199-225.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ОПИСАНИЕ ОПЕРАТОРА ВЗЯТИЯ ПОЛЯРЫ

АЛИ ЯГУБОВ

Национальная Академия Авиации, Баку, Азербайджан,
e-mail: alihaqverdi@mail.ru

В работе изучаются условия, при которых оператор, переводящий совокупность всех выпуклых замкнутых, содержащих нуль подмножеств гильбертова пространства в себя, является оператором взятия поляры.

Рассмотрим вещественное гильбертово пространство H. Через *conv* обозначим совокупность всех выпуклых замкнутых подмножеств этого пространства, содержащих нуль. Поляру $U^0 = \{x \in H \ : \ (u,x) \leq 1, \ \forall u \in U\}$ множества $U \in conv$ обозначим через $\pi(U)$. Оператор π взаимно однозначно отображает множество *conv* на себя.

Теми же свойствами, наряду с оператором π обладает оператор – π, где
$(-\pi)(U) = -(\pi(U)) = \{y \in H \ : < y, \ x > \geq -1$ для всех $x \in U\}$.

Рассмотрим оператор $\mathrm{P} : conv \to conv$ со следующими свойствами:

a) $P(B) = B$,

b) $P(AU) = (A^*)^{-1}P(U)$, (1)

для любого $U \in conv$ и следующих операторов:

b_1) $A = cI$, где I тождественный оператор, c– число,

b_2) $A = A_v$, где A_v – оператор ортогонального проектирования на одномерное подпространство, проходящее через вектор v. Если $\|v\| = 1$ то $A_v x =< x, v > v$

b_3) $A = \widetilde{A}_v$, где $\widetilde{A}_v$ – оператор ортогонального проектирования на гиперплоскость, ортогональную вектору v. Если $\|v\| = 1$, то $\widetilde{A}_v x = x - < x, v > v$,

b_4) $A = A_{u,v,\alpha}$, где $A_{u,v,\alpha}$ оператор "двумерного поворота": $A_{u,v,\alpha} = G_\alpha \cdot \mathrm{Pr}_{u,v}$; здесь $\mathrm{Pr}_{u,v}$-the оператор ортогонального проектирования на плоскость, натянутую на векторы u, v; G_α-оператор поворота на угол α, определенный на плоскости.

c) Если $U_\lambda \in conv \ (\lambda \in \Lambda)$, то

c_1) $P\left(\overline{co} \bigcup\limits_{\lambda\in\Lambda} V_\lambda\right) = \bigcap\limits_{\lambda\in\Lambda} P(V_\lambda)$ для любого множества Λ.

c_2) $P\left(\bigcap\limits_{\lambda\in\Lambda} V_\lambda\right) = \overline{co} \bigcup\limits_{\lambda\in\Lambda} P(V_\lambda)$ для конечного множества индексов Λ.

Замечание. Так как операторы A, фигурирующие в $b_1), b_2), b_3)$ самосопряжены, то для них равенство (1) записывается в виде $P(AU) = A^{-1}P(U)$

Теорема 1. Пусть оператор $\mathrm{P} : conv \to conv$ обладает свойствами $a), b), c)$. Тогда либо $\mathrm{P} = \pi$, либо $\mathrm{P} = -\pi$.

Доказательство теоремы опирается на ряд лемм.

Лемма 1. Пусть $v \in H, v \neq 0, \ \{x \in B :< x, v >= 0\}$. Тогда
$P(C) = B + R_v$ (где $R_v = \{\lambda v : \lambda \in (-\infty, +\infty)\}$)

Доказательство. Не умаляя общности, считаем, что $\|v\| = 1$. Рассмотрим гиперплоскость H_v, ортогональному вектору v, и вектор $\tilde{A}_v$ проектирования на эту гиперплоскость $\tilde{A}_v x = x - < x, v > v$. Понятно, что $\tilde{A}_v B = c$. Используя самосопряжённость оператора $\tilde{A}_v$, получим, что $P(C) = P\left(\tilde{A}_v\right)^{-1}(B)$. Так как $\tilde{A}_v H = H_v$ и $B \cap H_v = C$, то

$\left(\tilde{A}_v\right)^{-1}(B) = \left(\tilde{A}_v\right)^{-1}(B \cap H_v) = \left(\tilde{A}_v\right)^{-1}(C)$. Осталось проверить равенство $\left(\tilde{A}_v\right)^{-1}(C) = C + R_v = B + R_v$.

Пусть $h \in \left(\tilde{A}_v\right)^{-1}(C)$. Тогда $\tilde{A}_v h \in C \subset C + R_v$. Значит существует $x \in C$ такой, что $h- < h, v > v = x$. Отсюда получим $h = x+ < h, v > v \in x + R_v \subset C + R_v$ т.е. $\left(\tilde{A}_v\right)^{-1}(C) \subset C + R_v$.

Пусть наоборот $h \in C + R_v$. Тогда $h = x + \lambda v$, где $x \in C$, $\lambda v \in R_v$. Умножая это равенство скалярно на v, получим $\lambda =< h, v >$ или $x = h- < h, v > v = \tilde{A}_v h$. Так как $x \in C$, то $h \in \left(\tilde{A}_v\right)^{-1}(C)$ т.е. $C + R_v \subset \left(\tilde{A}_v\right)^{-1}(C)$ и значит $\left(\tilde{A}_v\right)^{-1}(C) = C + R_v = B + R_v$.

Лемма 2. Пусть $v \in H, \|v\| = 1$ и $W = \{x \in B :< v, x >\geq 0\}$ - один их двух единичных полушаров, определяемых вектором v. Тогда либо $P(W) = \pi(W)$ либо $P(W) = -\pi(W)$.

Доказательство. Обозначим $P(W) = Z$. Рассмотрим множество - $(-W)$- второй полушар, определенный элементом v. Так как $-W = (-I)W$, то в силу b_1) $P(-W) = -P(W) = -Z$. Так как $W \cup (-W) = B$, $W \cap (-W) = C$, где C множество, определенное в лемме 1, то используя свойства $a), c)$ и лемму 1, получим следующую систему уравнений относительно множества Z :

$Z \cap (-Z) = B$; $Z \cup (-Z) = B + R_v$ (2)

Покажем, что решением этой системы (в пространстве $conv$) могут служить лишь множества $Z_+ = B + R_{+v}$ и $Z_- = B + R_{-v}$ (здесь $R_{+v} = \{\lambda v, \lambda \geq 0\}$). Отметим что $Z_+ = -Z_-$. Второе из равенств (2) показывает, что $R_v \subset Z \cup (-Z)$. В частности, точки nv , где $n \in N = \{0, 1, 2, ...\}$ содержатся в этом объединении.

Пусть $N_+ = \{n \in N : nv \in Z\}$, $N_- = \{n \in N : nv \in -Z\}$.

Хоть одно из множеств N_+, N_-, - бесконечно. Пусть, например, N_+ бесконечно и n_0 наименьший элемент N_+ . Из выпуклости множества Z в этом случае следует, что луч $[\, n_0, +\infty)v$ содержится в Z . Если $n_0 > \infty$, то тоска $0 = 0 \cdot v$ не входит в Z. Понятно, что одновременно $0 \notin -Z$ и , следовательно, $0 \notin Z \cup (-Z)$. Это однако, невозможно, так как $0 \in R_v$. Таким образом $n_0 = 0$. Снова привлекая выпуклость Z получим, что рассматриваемом случае $N_+ = N, N_- = \varnothing$ и $R_{+v} \subset Z$. Первое из равенств (2) показывает, $B \subset Z$. Из выпуклости и замкнутости Z вытекает, что $\overline{co}(B \cup R_{+v}) \subset Z$. Так как R_{+v}- луч, исходящий из нуля, то $\overline{co}(B \cup R_{+v}) = B + R_{+v}$ и, следовательно, $Z_+ \subset Z$. Таким же образом предположение о том, что множество N_-- бесконечно приводит к включению $Z_- \subset Z$.

Покажем теперь, что соотношение $Z_+ \subset Z$ равносильно равенству $Z_+ = Z$. Допустим противное, тогда пусть существует элемент такой $x \in Z$, что $x \notin Z_+ = B + R_{+v}$. Второе из равенств (2) показывает, что $x \in B + R_v = C + R_v$. Это означает, что при некоторых $c \in C$ и $\mu \in R$ справедливо равенство $x = c + \mu v$. Если $\mu \geq 0$ то $x \in C + R_{+v} \subset Z_+$, что невозможно. Таким образом $\mu < 0$ и поэтому $-x = -c + (-\mu)v \in C + R_{+v} \subset Z_+ \subset Z$. Последние соотношения показывают, что $x \in -Z$. Следовательно, $x \in Z \cap (-Z) = B$. В то же время, так как $x \notin Z_+ = B + R_{+v}$, то $x \notin B$. Полученное противоречие и показывает, что $Z_+ = Z$. Таким же образом соотношение $Z_- \subset Z$ приводит к равенству $Z_- = Z$. Итак, показано, что если $Z = P(W)$, то либо $Z = Z_+$, либо $Z = Z_-$. Для завершения доказательства осталось заметить, что $\pi(W) = Z_-, -\pi(W) = Z_+$. Действительно, так как $W = B \cap H_+, \pi(H_+) = R_{-v}, \pi(B) = B$, где $H_+ = \{x \in H :< v, x >\geq 0\}$, то $\pi(W) = \pi(B \cap H_+) = \overline{co}(B \cup R_{-v}) = B + R_{-v}$. Второе равенство проверяется аналогично.

Лемма 3. Пусть $v \in N, \|v\| = 1, [0, v] = \{\lambda v : \lambda \in [0, 1]\}$ Тогда либо $P([0, v]) = \pi([0, v])$, либо $P([0, v]) = -\pi([0, v])$.

Доказательство. Рассмотрим оператор проектирования на прямую $R_v : A_v x =< x, v > v$. Так как $[0, v] = A_v(W)$, то привлекая b_2) получим, что $P([0, v]) = A_v^{-1}(P(W))$ (Здесь используется самосопряжённость оператора A_v). В то же время $\pi([0, v]) = A_v^{-1}\pi(W)$. Поэтому если $P(W) = \pi(W)$, то $P([0, v]) = \pi([0, v])$, если же $P(W) = -\pi(W)$ то $P([0, v]) = -\pi([0, v])$.

Доказательство теоремы. Выберем произвольный вектор $v, \|v\| = 1$ и найдем $P([0, v])$. Если $P([0, v]) = \pi([0, v])$, то используя операторы "двумерного поворота" из b_4) легко убедимся в том, что $P([0, w]) = \pi([0, w])$, для любого $w \in H, \|w\| = 1$. Если $w \neq 0$, то привлекая b_1) и используя оператор $A = \|w\| I$, можно проверить, что и в этом случае $P([0, w]) = \pi([0, w])$. Наконец, если $w = 0$, то $[0, w] = (0 \cdot I)v$, $(\|v\| = 1)$ и снова используя b_1) получим, что $P([0, v]) = \pi([0, v]) = H$. Пусть теперь $U \in conv$. Тогда $U = \bigcup\limits_{w \in U} [0, w]$ и поэтому, привлекая c_1) имеем $P(U) = \bigcap\limits_{w \in U} P([0, w]) = \bigcap\limits_{w \in U} \pi([0, w]) = \pi(v)$.

Подобным же образом, если $P([0, v]) = -\pi([0, v])$, то для любого $U \in conv$ выполняется $P(U) = -\pi(U)$. Теорема доказана.

Ключевые слова: гильбертово пространство, полярный оператор, двумерный оператор вращения, гиперплоскостной ортогональный.
Предметная классификация AMS: 46C07, 46B70, 46M35

Список литературы

[1] Рубинов А.М., *Суперлинейные Многозначные Отображения и Их Приложения Экономико-Математическим Задачам*, Ленинград, «Наука», 1980, 170 р.
[2] Иоффе А.Д., Иоффе А.Д., Тихомиров В.М., *Теория Экстремальных Задач*, Москва, «Наука», 1974, 481 р.

Алгебра, математикалық логика және геометрия

Algebra, mathematical logic and geometry

Алгебра, математическая логика и геометрия

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON THE GEOMETRY OF HAMILTONIAN SYSTEMS

ABDIGAPPAR NARMANOV[1], SHOHIDA ERGASHOVA[1]

[1]National university of Uzbekistan, Tashkent, Uzbekistan
e-mail: shohida.ergashova@mail.ru

Abstract. The paper studies the geometry of integrable Hamiltonian system. The basic concept of a Hamiltonian system of differential equations forms the basis of much of the more advanced work in classical mechanics, including motions of rigid bodies, celestial mechanics, quantization theory and so on. More recently, Hamiltonian methods have become increasingly important in the study of the equations of continuum mechanics, including fluids, plasmas and elastic media. It is shown that the level surfaces are three-dimensional surfaces with the positive or negative Gauss curvature at the point, where it is depend on which canonical form it has. All surfaces can be characterized by their Gaussian curvature, which describes the geometry of the surface: Euclidean, elliptic, or hyperbolic. Furthermore, curvature dictates how a vector may be transported across the surface or within the space.

Keywords: Poisson bracket, Hamiltonian function, completely integrable Hamiltonian system in the sense of Liouville, hyper space, 3-dimensional surfaces on 4-dimensional space, Gauss curvature.

AMS Subject Classification: 57R30, 57R50, 57R25.

1. Introduction and preliminaries

Let M be a smooth manifold of dimension m.

Definition 1 [3, 4, 5]**.** A *Poisson bracket* on a smooth manifold M is an operation that assigns a smooth real-valued function {F,H} on M to each pair F, H of smooth, real-valued functions, with the basic properties:

(a) *Bilineality*:

$$\{cF + c^{'}P, H\} = c\{F, H\} + c^{'}\{P, H\}, \{F, cH + c^{'}P\} = c\{F, H\} + c^{'}\{F, P\}.c, c^{'} \in \mathbb{R}$$

(b) *Skew-Symmetry*:

$$\{F, H\} = -\{H, F\}$$

(c) *Jacobi Identity*:

$$\{\{F, H\}, P\} + \{\{P, F\}, H\} + \{\{H, P\}, F\} = 0$$

(d) *Leibniz' Rule*:

$$\{F, H \cdot P\} = \{F, H\} \cdot P + H \cdot \{F, P\}$$

A manifold M with a Poisson bracket is called *Poisson manifold*, the bracket defining *Poisson structure* on M.

Example 1 [3, 4]**.** Let M be the Euclidean space $\mathbb{R}^m$, $m = 2n + l$ with coordinates

$(p,q,z) = (p^1,...,p^n,q^1,...,q^n,z^1,...,z^l)$. If $F(p,q,z)$ and $H(p,q,z)$ are smooth functions, we define their Poisson bracket to be the function:

$$\{F,H\} = \sum_{i=1}^{n}\{\frac{\partial H}{\partial p^i}\cdot\frac{\partial F}{\partial q^i} - \frac{\partial H}{\partial q^i}\cdot\frac{\partial F}{\partial p^i}\} \tag{1}$$

We note the particular bracket identities:

$$\{p^i,p^j\} = 0, \{q^i,q^j\} = 0, \{q^i,p^j\} = \delta^i_j.$$

$$\{p^i,z^k\} = \{q^i,z^k\} = \{z^t,z^k\} = 0.$$

in which i and j run from 1 to n, when t and k run from 1 to l. δ^i_j is the Kronecker symbol, which is 1 if $i = j$ and 0 otherwise.

Definition 2 [4, 5]**.** Let M be a Poisson manifold and $H : M \to \mathbb{R}$ a smooth function. The *Hamiltonian vector field* associated with H is the unique smooth vector field $sgradH$ on M satisfying

$$sgradH(F) = \{F,H\} = -\{H,F\} \tag{2}$$

for every smooth function $F : M \to \mathbb{R}$.

The equations governing the flow of $sgradH$ are referred to as *Hamilton's equations* for the *Hamiltonian function* H.

In the case of the Poisson bracket on $\mathbb{R}^m$ $m = 2n + l$, the Hamiltonian vector field corresponding to $H(p,q,z)$ is clearly

$$sgradH = \sum_{i=1}^{n}(\frac{\partial H}{\partial p^i}\cdot\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\cdot\frac{\partial}{\partial p^i}) \tag{3}$$

The corresponding flow is obtained by integrating the system of ordinary differential equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p^i}, \frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i}, i = 1,...,n \tag{4}$$

$$\frac{dz^j}{dt} = 0, j = 1,...,l \tag{5}$$

Which are *Hamiltonian systems* in this case.

Proposition 1 [1, 4, 6]**.** Let M be a Poisson manifold and $F, H : M \to \mathbb{R}$ be smooth functions with corresponding Hamiltonian vector fields $sgradF$, $sgradH$. The Hamiltonian vector field associated with the Poisson bracket of F and H is, up to sign, the Lie bracket of the two Hamiltonian vector fields:

$$sgrad\{F,H\} = [sgradF, sgradH]$$

Definition 3 [2, 4]**.** Let M^{2n} be a Poisson manifold and $sgradH$ Hamiltonian vector field with a smooth Hamiltonian function H.

Hamiltonian system $sgradH$ is called *completely integrable in the sense of Liouville or completely integrable*, if exists set of smooth functions $f_1,...,f_n$ as:

1) $f_1,...,f_n$ are first integrals of $sgradH$ Hamiltonian vector field,

2) they are functionally independent on M, that is, almost everywhere on M their gradients are linearly independent,

3) $\{f_i,f_j\} = 0$ for any i and j,

4) the vector fields $sgradf_i$ are complete, that is natural parameter on their integral trajectories is defined on the whole number line.

Denote by $\mathcal{Q}$ the Riemannian ambient manifold. Let $dim\mathcal{Q} = N$. For its metric is $g^{\mathcal{Q}}$. Denote by M the n-dimensional manifold $(n < N)$, the embedding $f : M \to \mathcal{Q}$.

There are three vector bundles:

1) $TM = (P,X)$, $P \in M$, $X \in T_PM$,

2) Normal bundle: NM, (P,ξ), $P \in M$, $\xi \in N_P M$,

$$m = dim N_P M = N - n$$

3) $\mathcal{Q}$-bundle: (P,ξ), $P \in M$, $\xi \in N_P \mathcal{Q}$.

$$T_P \mathcal{Q} = T_P M \oplus N_P M$$

Definition 4 [2]**.** An n-dimensional surface M in an N-dimensional Euclidean space is called a *hyper surface* if $N = n + 1$.

If $\mathcal{Q} = \mathbb{R}^N$ and $dim M = n$, $N = n + 1$, then $T_P M$ is a hyper surface.

Definition 5 [2]**.** The *normal section of the submanifold M at the point P*, corresponding to existing pair $X \in T_P M$ and $\xi \in N_P M$ is the intersection of M with the 2-plane passing through X and ξ.

The normal section in a neighborhood of the point P is smooth regular curve on M.

Theorem 1 [2]**.** For any point $P \in M$ in the space $T_P M$ there is a basis $X_1, ..., X_n$ in which the matrix of the first quadratic form is identity and matrix of the second quadratic form is diagonal:

$$\begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{pmatrix}$$

Definition 6 [2]**.** The numbers λ_j are called the *principal curvatures* of the manifold M at the point P, and the lines passing through the vectors X_j are the *principal directions*.

The *Gaussian curvature* of the manifold M at the point P is:

$$K = \prod_{j=1}^{n} \lambda_j. \tag{6}$$

Let $\mathcal{Q} = \mathbb{R}^4$ and $M \in \mathbb{R}^4$ and $dim M = 3$.

Definition 7. At each point p of a 3-dimensional differentiable surface in 4-dimensional Euclidean space one may choose a unit normal vector. A normal plane at p is one that contains the normal vector, and will therefore also contain a unique direction tangent to the surface and cut the surface in a plane curve, called normal section. This curve will in general have different curvatures for different normal planes at p. The *principal curvatures at p*, denoted k_1, k_2 and k_3, are the extreme values of this curvature and that the corresponding directions called *principal directions at p* which are orthogonal.

Definition 8. The *Gauss curvature* of the 3-dimensional differentiable surface in 4-dimensional Euclidean space at a point is defined at the product of the principal curvatures where denoted by $K = k_1 \cdot k_2 \cdot k_3$.

2. Results and Discussion

Let given 3-dimensional differentiable surface with the parametric equation as:

$$\begin{cases} p_1 = f^1(u,v,w) \\ p_2 = f^2(u,v,w) \\ q_1 = f^3(u,v,w) \\ q_2 = f^4(u,v,w) \end{cases} \tag{7}$$

At each point of this surface has tangent plane with the basic vectors and a normal vectors as:

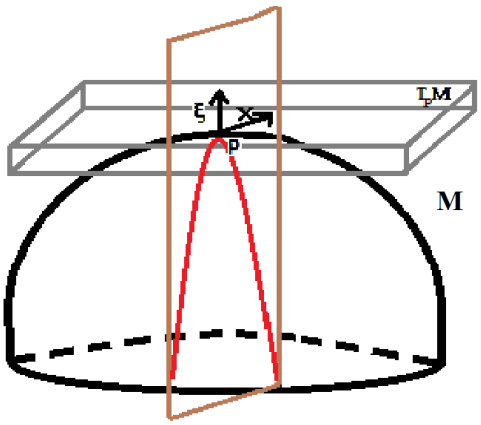


FIGURE 1. The normal section of the M at the point P on 4-dimensional Euclidean space.

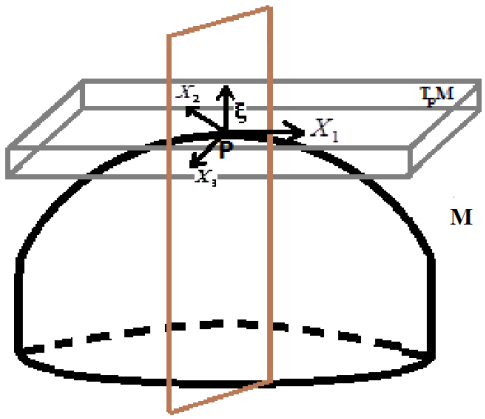


FIGURE 2. Principal directions X_1, X_2 and X_3 at the point P on 4-dimensional Euclidean space.

$$
\begin{gathered}
r_1 = \{f_u^1, f_u^2, f_u^3, f_u^4\}, \\
r_2 = \{f_v^1, f_v^2, f_v^3, f_v^4\}, \\
r_3 = \{f_w^1, f_w^2, f_w^3, f_w^4\}, \\
\xi = \{\xi_1, \xi_2, \xi_3, \xi_4\}.
\end{gathered}
$$

And vectors

$$
\begin{gathered}
r_{11} = \{f_{uu}^1, f_{uu}^2, f_{uu}^3, f_{uu}^4\}, \\
r_{12} = \{f_{uv}^1, f_{uv}^2, f_{uv}^3, f_{uv}^4\}, \\
r_{13} = \{f_{uw}^1, f_{uw}^2, f_{uw}^3, f_{uw}^4\}, \\
r_{22} = \{f_{vv}^1, f_{vv}^2, f_{vv}^3, f_{vv}^4\}, \\
r_{23} = \{f_{vw}^1, f_{vw}^2, f_{vw}^3, f_{vw}^4\}, \\
r_{33} = \{f_{ww}^1, f_{ww}^2, f_{ww}^3, f_{ww}^4\}.
\end{gathered}
$$

where f_j^i is partial derivative of f with respect to u, v, w, f_{jk}^i is the mixed derivative of the second order and $i = \overline{1,4}$.

These vectors are enough to write matrix of first quadratic form A and matrix of second quadratic form B.

$$A = \begin{pmatrix} \langle r_1, r_1 \rangle & \langle r_1, r_2 \rangle & \langle r_1, r_3 \rangle \\ \langle r_1, r_2 \rangle & \langle r_2, r_2 \rangle & \langle r_2, r_3 \rangle \\ \langle r_1, r_3 \rangle & \langle r_2, r_3 \rangle & \langle r_3, r_3 \rangle \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$$

$$B = \begin{pmatrix} \langle r_{11}, \xi \rangle & \langle r_{12}, \xi \rangle & \langle r_{13}, \xi \rangle \\ \langle r_{12}, \xi \rangle & \langle r_{22}, \xi \rangle & \langle r_{23}, \xi \rangle \\ \langle r_{13}, \xi \rangle & \langle r_{23}, \xi \rangle & \langle r_{33}, \xi \rangle \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}$$

where bracket $\langle \cdot, \cdot \rangle$ is inner product.

According to Theorem 1 we have following determinant:

$$\begin{vmatrix} b_{11} - kg_{11} & b_{12} - kg_{12} & b_{13} - kg_{13} \\ b_{12} - kg_{12} & b_{22} - kg_{22} & b_{23} - kg_{23} \\ b_{13} - kg_{13} & b_{23} - kg_{23} & b_{33} - kg_{33} \end{vmatrix} = 0 \tag{8}$$

By calculating the determinant of the matrix we obtain for us a third-order equation required function k, where we can find formula for Gauss curvature of 3-dimensional surface on 4-dimensional space:

$$K = \frac{b_{11}(b_{22}b_{33} - b_{23}^2) + b_{12}(b_{13}b_{23} - b_{12}b_{33}) + b_{13}(b_{12}b_{23} - b_{13}b_{22})}{g_{11}(g_{23}^2 - g_{22}g_{33}) + g_{12}(g_{12}g_{33} - g_{13}g_{23}) + g_{13}(g_{13}g_{22} - g_{12}g_{23})} \tag{9}$$

Let us consider the Hamiltonian functions $H : M^4 \to \mathbb{R}$ on the Poisson manifold M^4 which are given by the formulas

$$H(p_1, p_2, q_1, q_2) = (-1)^i p_1^2 + (-1)^j p_2^2 + (-1)^k q_1^2 + (-1)^l q_2^2. \tag{10}$$

$$i, j, k, l = 1, 2.$$

There generates 3-dimensional surfaces on 4-dimensional space in each value of Hamiltonian function with the equations as

$$(-1)^i p_1^2 + (-1)^j p_2^2 + (-1)^k q_1^2 + (-1)^l q_2^2 = c. \tag{11}$$

$$i, j, k, l = 1, 2.$$

Theorem 2. Let surface can be written in canonical form as

$$(-1)^i p_1^2 + (-1)^j p_2^2 + (-1)^k q_1^2 + (-1)^l q_2^2 = 1. \tag{12}$$

If the value of all or one of i,j,k,l is equal to 2, then 3-dimensional surface has the positive Gauss curvature. In other cases, if the value of two or three of i,j,k,l is equal to 2, then 3-dimensional surface is the surface with the negative Gauss curvature.

Proof. First case: If all values of i,j,k,l are equal to 2, then

$$p_1^2 + p_2^2 + q_1^2 + q_2^2 = 1$$

the surface is the standart 3-dimensional sphere, so it is obvious that it's Gauss curvature is always positive.

Second case: If the one of values of i,j,k,l is equal to 2, then the equation of the surface can be as:

$$p_1^2 - p_2^2 - q_1^2 - q_2^2 = 1$$

Now we find A and B matrix for this surface.

Gauss curvature is not depend on choosing parameterization method, so we will parameterize it as:

$$\begin{cases} p_1 = chu \\ p_2 = shusinv \\ q_1 = shucosvcosw \\ q_2 = shucosvsinw \end{cases}$$

where vectors are

$$r_1 = \{shu, chusinv, chucosvcosw, chucosvsinw\},$$
$$r_2 = \{0, shucosv, -shusinvcosw, -shusinvsinw\},$$
$$r_3 = \{0, 0, -shucosvsinw, shucosvcosw\},$$
$$\xi = \{-chu, shusinv, shucosvcosw, shucosvsinw\},$$
$$r_{11} = \{chu, shusinv, shucosvcosw, shucosvsinw\},$$
$$r_{12} = \{0, chucosv, -chusinvcosw, -chusinvsinw\},$$
$$r_{13} = \{0, 0, -chucosvsinw, chucosvcosw\},$$
$$r_{22} = \{0, -shusinv, -shucosvcosw, -shucosvsinw\},$$
$$r_{23} = \{0, 0, shusinvsinw, -shusinvcosw\},$$
$$r_{33} = \{0, 0, -shucosvcosw, -shucosvsinw\}.$$

We have coeffitcients

$$\begin{array}{lll} g_{11} = ch2u, & g_{12} = 0, & g_{13} = 0, \\ g_{22} = sh^2u, & g_{23} = 0, & g_{33} = sh^2ucos^2v, \\ b_{11} = -1, & b_{12} = 0, & b_{13} = 0, \\ b_{22} = -sh^2u, & b_{23} = 0, & b_{33} = -sh^2ucos^2v. \end{array}$$

Now we ready to write the formula for the Gauss curvature

$$K = -\frac{b_{11}b_{22}b_{33}}{g_{11}g_{22}g_{33}} = \frac{shu^4cos^2v}{ch2ushu^4cos^2v} = \frac{1}{ch2u}$$

So, we proved that the surface has positive Gauss curvature.

If the surface is given with the equation as:

$$\begin{array}{ll} p_2^2 - p_1^2 - q_1^2 - q_2^2 = 1 & or \\ q_1^2 - p_1^2 - p_2^2 - q_2^2 = 1 & or \\ q_2^2 - p_1^2 - p_2^2 - q_1^2 = 1 & \end{array}$$

We can choose a parameterization since the coefficients will be the same as the coefficients of the given surface.

Third case: If the two of values of i,j,k,l is equal to 2, then the equation of the surface can be as:

$$p_1^2 + p_2^2 - q_1^2 - q_2^2 = 1$$

Now we find A and B matrix for this surface.

Gauss curvature is not depend on choosing parameterization method, so we will parameterize it as:

$$\begin{cases} p_1 = chuchvcosw \\ p_2 = chuchvsinw \\ q_1 = chushv \\ q_2 = shu \end{cases}$$

where vectors are

$$\begin{gathered}
r_1 = \{shuchvcosw, shuchvsinw, shushv, chu\},\\
r_2 = \{chushvcosw, chushvsinw, chuchv, 0\},\\
r_3 = \{chuchvsinw, chuchvsinw, 0, 0\},\\
\xi = \{-chuchvcosw, -chuchvsinw, chushv, shu\},\\
r_{11} = \{chuchvcosw, chuchvsinw, chushv, shu\},\\
r_{12} = \{shushvcosw, shushvsinw, shuchv, 0\},\\
r_{13} = \{-shuchvsinw, shuchvcosw, 0, 0\},\\
r_{22} = \{chuchvcosw, chuchvsinw, chushv, 0\},\\
r_{23} = \{-chushvsinw, chushvcosw, 0, 0\},\\
r_{33} = \{-chuchvcosw, -chuchvsinw, 0, 0\}.
\end{gathered}$$

We have coeffitcients

$$\begin{array}{ccc}
g_{11} = sh^2uch2v + ch^2u, & g_{12} = 2chuchvshushv, & g_{13} = 0,\\
g_{22} = ch^2uch2v, & g_{23} = 0, & g_{33} = ch^2uch^2v,\\
b_{11} = -1, & b_{12} = 0, & b_{13} = 0,\\
b_{22} = -ch^2u, & b_{23} = 0, & b_{33} = -ch^2uch^2v.
\end{array}$$

Now we ready to write the formula for the Gauss curvature

$$K = -\frac{b_{11}b_{22}b_{33}}{g_{33}(g_{12}^2 - g_{11}g_{22})} = \frac{1}{4ch^2vsh^2ush^2v - sh^2uch^22v - ch^2uch2v} = -\frac{1}{sh^2u + ch^2uch2v}$$

So, we proved that the surface has negative Gauss curvature.

If the surface is given with the equation as:

$$\begin{array}{llll}
p_1^2 - p_2^2 - q_1^2 + q_2^2 = 1 & or & p_1^2 - p_2^2 + q_1^2 - q_2^2 = 1 & or\\
p_2^2 - p_1^2 - q_1^2 + q_2^2 = 1 & or & p_2^2 - p_1^2 + q_1^2 - q_2^2 = 1 & or\\
q_1^2 + q_2^2 - p_1^2 - p_2^2 = 1 & or & q_1^2 - q_2^2 - p_1^2 + p_2^2 = 1 & or\\
q_2^2 - p_1^2 - p_2^2 - q_1^2 = 1 & & &
\end{array}$$

We can choose a parameterization since the coefficients will be the same as the coefficients of the given surface.

In the last case: If the three of values of i,j,k,l is equal to 2, then the equation of the surface can be as:

$$p_1^2 + p_2^2 - q_1^2 + q_2^2 = 1$$

Now we find A and B matrix for this surface.

Gauss curvature is not depend on choosing parameterization method, so we will parameterize it as:

$$\begin{cases}
p_1 = chucosvcosw\\
p_2 = chucosvsinw\\
q_1 = shu\\
q_2 = chusinv
\end{cases}$$

where vectors are

$$r_1 = \{shucosvcosw, shucosvsinw, chu, shusinv\},$$
$$r_2 = \{-chusinvcosw, -chusinvsinw, 0, chucosv\},$$
$$r_3 = \{-chucosvsinw, chucosvcosw, 0, 0\},$$
$$\xi = \{chucosvcosw, chucosvsinw, -shu, chusinv\},$$
$$r_{11} = \{chucosvcosw, chucosvsinw, shu, chusinv\},$$
$$r_{12} = \{-shusinvcosw, -shusinvsinw, 0, shucosv\},$$
$$r_{13} = \{-shucosvsinw, shucosvcosw, 0, 0\},$$
$$r_{22} = \{-chucosvcosw, -chucosvsinw, 0, -chusinv\},$$
$$r_{23} = \{chusinvsinw, -chusinvcosw, 0, 0\},$$
$$r_{33} = \{-chucosvcosw, -chucosvsinw, 0, 0\}.$$

We have coeffitcients

$$\begin{array}{lll} g_{11} = ch2u, & g_{12} = 0, & g_{13} = 0, \\ g_{22} = ch^2u, & g_{23} = 0, & g_{33} = ch^2ucos^2v, \\ b_{11} = 1, & b_{12} = 0, & b_{13} = 0, \\ b_{22} = -ch^2u, & b_{23} = 0, & b_{33} = -ch^2ucos^2v. \end{array}$$

Now we ready to write the formula for the Gauss curvature

$$K = -\frac{b_{11}b_{22}b_{33}}{g_{11}g_{22}g_{33}} = \frac{ch^4ucos^2v}{ch^4uch2ucos^2v} = -\frac{1}{ch2u}$$

So, we proved that the surface has negative Gauss curvature.

If the surface is given with the equation as:

$$\begin{array}{l} p_2^2 - p_1^2 + q_1^2 + q_2^2 = 1 \quad or \\ p_1^2 - p_2^2 + q_1^2 + q_2^2 = 1 \quad or \\ p_1^2 + p_2^2 + q_1^2 - q_2^2 = 1 \end{array}$$

We can choose a parameterization since the coefficients will be the same as the coefficients of the given surface.

So, proof is finished, we have proved theorem 2.

References

[1] Bolsinov A., Fomenko A., *Integrable Hamiltonian systems 1*, Udmurtskiy universitet, Izhevsk, 1999, 443 p.

[2] Shafarevich A.I., *Klassicheskaya differensialnaya geometriya* MFTI, Moskow, 2010.

[3] Olver P., *Applications of Lie Groups to Differential Equations*, Springer, New York, 1993, 544 p.

[4] Narmanov A.Ya., Ergashova Sh.R., On the geometry of Liouville foliations, *E3S Web of Conferences*, Vol.402, No.03032(2023). TransSiberia 2023, pp.1-9.

[5] Narmanov A.Ya., Rajabov E.O., The Geometry of Vector Fields and Two Dimensional Heat Equation, *International Electronic Journal of Geometry*, Vol.16, No.1, 2023, pp.73-80.

[6] Narmanov A.Ya., Rajabov E.O., Vector fields and differential equations, *Journal of Physics: conference Series*, Vol.2388(012041), 2022, pp.1-8.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON THE GEOMETRY OF VECTOR FIELDS

ABDIGAPPAR Y. NARMANOV, BEKZOD R. DIAYAROV

National University of Uzbekistan, Tashkent, Uzbekistan
e-mail: narmanov@yandex.ru

Abstract. It is known that vector fields play an important role in many areas of mathematics and technology, in particular, in the theory of dynamical systems. Hamiltonian dynamical systems are generated by Hamiltonian vector fields. The paper studies the geometry of a singular foliation of a four-dimensional Euclidean space generated by the orbits of two Hamiltonian vector fields. It is proved that the regular leaf of this foliation is a surface of positive normal curvature and nonzero normal torsion.

Keywords: Singular foliation,curvature,torsion,vector field, orbit

AMS Subject Classification: 37C10, 37C27, 57R25

1. Introduction

Let M be a Riemannian (smooth) manifold of dimension n.

Definition 1. A partition F of a riemannian manifold M by path-connected immersed submanifolds L_α is called a singular foliation of M if it verifies condition: for each leaf L_α and each vector $v \in T_pL_\alpha$ at the point p, there is $X \in XF$ such that $X(p) = v$, where T_pL_α is the tangent space of the leaf L_α at the point p , XF is the module of smooth vector fields on M tangent to leaves (XF acts transitively on each leaf) (see [6, 7, 8]). If the dimension of L is maximal, it is called regular, otherwise L is called singular. It is known that orbits of vector fields generate singular foliation (see [6, 7, 8]).

Let us consider a set of vector fields $D \subset V(M)$of the Lie algebra of all smooth (class C^∞) vector fields $V(M)$ and the smallest Lie subalgebra containing D by $A(D)$. Let $t \to X^t(x)$ be an integral curve of the vector field X with the initial point x for $t = 0$, which is defined in some region $I(x)$ of real line.

Definition 2. The orbit $L(x)$ of a system D of vector fields through a point x is the set of points y in M such that there exist $t_1, t_2, ..., t_k \in R$ and vector fields $X_{i_1}, X_{i_2}, ..., X_{i_k} \in D$ such that

$$y = X_{i_k}^{t_k}(X_{i_{k-1}}^{t_{k-1}}(...(X_{i_1}^{t_1})...)),$$

where k is an arbitrary positive integer.

The fundamental result in study of orbits is Sussmann theorem [8], which asserts that every orbit is an immersed submanifold of M.

By setting $A_x(D) = \{X(x) : X \in A(D)\}$, we obtain an involutive distribution $P_D : x \to Ax(D)$. If $dimA_x(D) = const$ for every x, then the distribution $P_D : x \to Ax(D)$ is completely integrable by the Frobenius theorem. If the dimension $dimA_x(D)$ depends on x, then we can use Hermann theorem for distributions of variable dimension, which gives condition for the complete integrability of distributions which is finitely generated [9].

Definition 3. A system of vector fields

$$D = \{X_1, X_2, ..., X_k\}$$

on M is in involution if there exist smooth real-valued functions $f_{ij}^l(x), x \in M, i, j, l = 1, ..., k$ such that for each (i, j) it takes

$$[X_i, X_j] = \sum_{l=1}^{k} f_{ij}^l(x) X_l.$$

Hermann Theorem [9]. In order a system of smooth vector fields

$$D = \{X_1, X_2, ..., X_k\}$$

on M to generate completely integrable distribution, it is necessary and sufficient that it be involutive.

Let us recall some notions from geometry of Hamilton systems [9],[2],. Given a smooth manifold M a Poisson bracket on M assigns to each pair of smooth, real-valued functions $\{F, H\} : M \to R$ another smooth, real-valued function, which we denote by $\{F, H\}$ There are certain basic properties that such a bracket operation must satisfy in order to qualify as a Poisson bracket.

Definition 1.1. *A Poisson bracket on a smooth manifold M is an operation that assigns a smooth real-valued function F, H on M to each pair F, H of smooth, real-valued functions, with the basic properties:*

(1) Bilinearity: $\{\lambda F + \mu P, H\} = \lambda\{F, H\} + \mu\{P, H\}$,
$\{F, \lambda H + \mu P\} = \lambda\{F, H\} + \mu\{F, P\}$ *for constants* $\lambda, \mu \in R$,
(2) Skew-Symmetry: $\{F, H\} = -\{H, F\}$,
(3) Jacobi Identity: $\{\{F, H\}, P\} + \{\{P, F\}, H\} + \{\{H, P\}, F\} = 0$
(4) Leibniz' Rule: $\{F, H \cdot P\} = \{F, H\} \cdot P + H \cdot \{F, P\}$.

(Here $\cdot$ denotes the ordinary multiplication of real-valued functions.)

A manifold M with a Poisson bracket is called a Poisson manifold, the bracket defining a Poisson structure on M. The notion of a Poisson manifold is slightly more general than that of a symplectic manifold, or manifold with Hamiltonian structure; in particular, the underlying manifold M need not be even-dimensional. This is borne out by the standard examples from classical mechanics.

Let M be the Euclidean space R^m with coordinates

$$(p, q, z) = (p^1, p^2..., p^n, q^1, q^2..., q^n, z^1, z^2..., z^l),$$

where $2n + l = m$. If $F(p, q, z), H(p, q, z)$ are smooth functions, we define their Poisson bracket to be the function

$$\{F, H\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p^i} - \sum_{i=1}^{n} \frac{\partial F}{\partial p^i}\frac{\partial H}{\partial q^i}. \tag{1}$$

This bracket is clearly bilinear and skew-symmetric; the verifications of the Jacobi identity and the Leibniz rule are simple exercises.

Definition 1.2. *Let M be a Poisson manifold and $H : M \to R$ a smooth function. The Hamiltonian vector field associated with H is the unique smooth vector field $sgradH$ on M satisfying $sgradH = \{F, H\} = -\{H, F\}$ for every smooth function $F : M \to R$.*

In the case of the canonical Poisson bracket on R^m, $m = 2n + l$, the Hamiltonian vector field corresponding to $H(p, q, z)$ is clearly

$$sgradH = \sum_{i=1}^{n} \{\frac{\partial H}{\partial p^i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p^i}\}. \tag{2}$$

The corresponding flow is obtained by integrating the system of ordinary differential equations

$$\frac{dp^i}{dt} = -\frac{\partial H}{dq^i}, \frac{dq^i}{dt} = \frac{\partial H}{dp^i}, \frac{dz^l}{dt} = 0. \tag{3}$$

There is a fundamental connection between the Poisson bracket of two functions and the Lie bracket of their associated Hamiltonian vector fields, which forms the basis of much of the theory of Hamiltonian systems. It is well known following theorem:

Theorem 1.1. *Let M be a Poisson manifold and let $\{F, H\} : M \to R$ be smooth functions with corresponding Hamiltonian vector fields $sgradF, sgradH$. The Hamiltonian vector field associated with the Poisson bracket of F and H is the Lie bracket of the two Hamiltonian vector fields:*

$$sgrad\{F, H\} = [sgradF, sgradH]$$

Definition 1.3. *Let M^{2n} be a symplectic manifold and $sgradH$ Hamiltonian vector field with a smooth Hamiltonian function H.*

Hamiltonian system $sgradH$ is called completely integrable in the sense of Liouville, if exists set of smooth functions $f_1, ..., f_n$ as:

1) $f_1, ..., f_n$ are first integrals of $sgradH$ Hamiltonian vector field,

2) they are functionally independent on M, that is, almost everywhere on M their gradients are linearly independent,

3) $\{f_i, f_j\} = 0$ for any i and j,

4) the vector fields $sgradf_i$ are complete, that is natural parameter on their integral trajectories is defined on the whole number line.

Definition 1.4. *Partition of the manifold M^{2n} into connected components of joint level surfaces of the integrals $f_1, ..., f_n$ is called The Liouville foliation corresponding to the completely integrated system* [9].

Since $f_1, ..., f_n$ is preserved by $sgradH$, each leaf of the Liouville foliation is invariant surface. Liouville foliation is consists of regular leaves (which fill almost all M) and special leaves (a subset of zero measure).

2. On the geometry of Hamiltonian vector fields

Let us consider a family of $D = \{X_1, X_2\}$ vector fields on four-dimensional Euclidean space E^4 with Cartesian coordinates p_1, p_2, q_1, q_2, where

$$X_1 = q_1\frac{\partial}{\partial p_1} + q_2\frac{\partial}{\partial p_2} + p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2}, X_2 = q_1\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial q_1}. \tag{4}$$

Theorem 1. The family of orbits of the vector fields (2.1) is a singular foliation regular leaf of which is a surface with a positive normal curvature and with non zero normal torsion.

Proof. First of all, we note that for the Lie bracket of the vector fields X_1, X_2 it holds $[X_1, X_2] = 0$. It follows from Hermann theorem this system generates completely integrable distribution

Vector field

$$X_1 = q_1\frac{\partial}{\partial p_1} + q_2\frac{\partial}{\partial p_2} + p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2} \tag{5}$$

generates following one parametrical group of transformations

$$(p_1, p_2, q_1, q_2) \to (p_1cht + q_1sht, p_2cht + q_2sht, p_1sht + q_1cht, p_2sht + q_2cht), \tag{6}$$

where $t \in R$.

The vector field

$$X_2 = q_1\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial q_1}, \tag{7}$$

generates following one parametrical group of transformations

$$(p_1, p_2, q_1, q_2) \to (p_1 chs + q_1 shs, p_2, p_1 shs + q_1 chs, q_2), s \in R. \tag{8}$$

We need to find the invariant functions of the groups generated by vector fields (2.1). It is easy to check the following equalities

$$X_1(I_1) = 0, X_1(I_2) = 0, X_2(I_1) = 0, X_2(I_2) = 0 \tag{9}$$

for the functions

$$I_1(p_1, p_2, q_1, q_2) = p_1^2 - q_1^2, I_2(p_1, p_2, q_1, q_2) = p_2^2 - q_2^2. \tag{10}$$

It follows the functions I_1, I_2 are invariant functions [9].

This invariant functions give us a family of two dimensional surfaces

$$p_1^2 - q_1^2 = C_1, p_2^2 - q_2^2 = C_2, \tag{11}$$

where C_1, C_2 are constants.

Let L is a regular connected surface defined by equatons (2.10). For a point $p \in L$ the orbit $L(p)$ of the family of vector fields (2.1) contained in L. According to Sussmann theorem [8] the orbit $L(p)$ is a two-dimensional manifold and therefore it is an open subset of L. This implies that the surface L is a union of orbits. Due to the connectedness of L , we get $L(p) = L$.

Now we can check metric characteristics of the surface L.

Let us recall some notions on the geometry of two dimensional surface in a four-dimensional Euclidean space E^4.

Consider on a two-dimensional surface F a four-dimensional Euclidean space E^4 at the point x some direction given by non-zero vector t. The vector t and the normal plane N of the surface at the point x define a hyper plane E^3 (x, t, N) in E^4 that intersects over the validity of F along some curve γ. The curve γ is called the normal section we eat the surface F at the point x in the direction t. According to its construction the curve γ is a three-dimensional curve. Curvature $k_N(x, t)$ and torsion $\chi_N(x, t)$ of the curve γ at the point x is called, respectively, the normal curvature and normal torsion of the surface at the point x in the direction t. Geometry of two dimensional surfaces in the four dimensional Euclidean space is studied by many authors [1]

We will parameterize the surface (2.10) by the following equations

$$\begin{cases} p_1 = \sqrt{|c_1|} chu \\ p_2 = \sqrt{|c_2|} chv \\ q_1 = \sqrt{|c_1|} shu \\ q_2 = \sqrt{|c_2|} shv \end{cases}$$

Now we find

$$\frac{\partial r}{\partial u} = r_1 = \{\sqrt{|c_1|} shu; 0; \sqrt{|c_1|} chu; 0\},$$
$$\frac{\partial r}{\partial v} = r_2 = \{0; \sqrt{|c_2|} shv; 0; \sqrt{|c_2|} chv\}$$

and coefficients of first quadratic form

$$g_{11} = |c_1| ch2u, g_{12} = g_{21} = 0, g_{22} = \langle r_2, r_2 \rangle = |c_2| ch2v,$$

where $\langle \cdot, \cdot \rangle$ is the inner product.

We need two normal vectors to find coefficients of second quadratic forms

$$n_1 = \{-\frac{chu}{ch2u}; 0; \frac{shu}{ch2u}; 0\}$$

and

$$n_2 = \{0; -\frac{chv}{ch2v}; 0; \frac{shv}{ch2v}\}.$$

Now we can find two second quadratic forms of the regular surface. Coefficients of the first of them is calculated by the formula

$$b_{ij} = -\frac{1}{|n_1|}\langle \partial_i r, \partial_j n_1 \rangle.$$

By using these equation we find that

$$b_{11} = -\frac{\sqrt{|c_1|}}{ch2u}, b_{12} = b_{21} = 0, b_{22} = 0$$

Coefficients of the second of them are calculated by the formula

$$c_{ij} = -\frac{1}{|n_2|}\langle \partial_i r, \partial_j n_2 \rangle.$$

It follows from here

$$c_{11} = 0, c_{12} = c_{21} = 0, c_{22} = -\frac{\sqrt{|c_2|}}{ch2v}.$$

Let given $x(x_1, x_2)$ point and $t(t_1, t_2)$ vector in (u, v) parametrization. Then normal curvature at the point $x(x_1, x_2)$ in a direction of the vector $t(t_1, t_2)$ is calculated by the formula:

$$k_N(x,t) = \left(\left(\frac{\sum\limits_{i=1}^{2} b_{ij}t_i t_j}{\sum\limits_{i=1}^{2} g_{ij}t_i t_j} \right)^2 + \left(\frac{\sum\limits_{i=1}^{2} c_{ij}t_i t_j}{\sum\limits_{i=1}^{2} g_{ij}t_i t_j} \right)^2 \right)^{\frac{1}{2}}$$

From this formula we have that normal curvature at the any point $x(x_1, x_2)$ in any direction $t(t_1, t_2)$ is a positive and equal to:

$$k_N(x,t) = \left(\frac{\frac{|c_1|t_1^4}{ch^2 2u} + \frac{|c_2|t_2^4}{ch^2 2v}}{|c_1|ch2ut_1^2 + |c_2|ch2vt_2^2} \right)^{\frac{1}{2}}$$

Now we calculate normal torsion at the point $x(x_1, x_2)$ in the direction $t(t_1, t_2)$ by the formula from [1]and we have following:

$$\sigma_N(x,t) = (g_{ij}t^i t^j)^{\frac{1}{2}} \left(\frac{\frac{4\sqrt{|c_2|}sh2v}{ch^2 2v}t_1^2 t_2^3 + \frac{4\sqrt{|c_1|}sh2u}{ch^2 2u}t_1^3 t_2^2}{\frac{\sqrt{|c_1|}}{ch^2 2u}t_1^2 + \frac{\sqrt{|c_2|}}{ch^2 2v}t_2^2} \right).$$

It is to check normal torsion is not equal zero. **Theorem 1 is proved.**

Remark 1 It is known that Gauss curvature of two-dimensional surface in $E^4$is calculated by the formula [10, 3, 4]:

$$K_G = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2} + \frac{c_{11}c_{22} - c_{12}^2}{g_{11}g_{22} - g_{12}^2}.$$

It follows that $K_G = 0$.

Remark 2 It is known that Gauss torsion of two-dimensional surface in $E^4$is calculated by the formula [10, 3, 4]:

$$\sigma_G = \frac{\langle n_{1u}, n_{2v} \rangle - \langle n_{2u}, n_{1v} \rangle}{\sqrt{g_{11}g_{22} - g_{12}^2}}$$

From this formula we have $\sigma_G = 0$.

Remark 3 The vector field X_1 is a Hamiltonian vector field with the Hamiltonian $H : M^4 \to \mathbb{R}$ which is given by the formula

$$H_1(p_1, p_2, q_1, q_2) = \frac{1}{2}(p_1^2 + p_2^2 - q_1^2 - q_2^2), \tag{12}$$

the vector field X_2 is a Hamiltonian vector field with the Hamiltonian $H : M^4 \to \mathbb{R}$ which is given by the formula

$$H_2(p_1, p_2, q_1, q_2) = \frac{1}{2}(p_1^2 - q_1^2). \tag{13}$$

In fact the vector field X_1 generates Hamiltonian system which is completely integrable in the sense of Liouville.

Invariant functions are functionally independent first integrals of this hamiltonian system and following equations

$$F_1 = p_1^2 - q_1^2 \tag{14}$$

$$F_2 = p_2^2 - q_2^2$$

generate Liouville foliation.

3. Conclusion

The paper studies the foliation generated by the family of orbits of vector fields. It is shown that regular leaves are two-dimensional manifolds of positive normal curvature and non-zero normal torsion.

References

[1] Yu. A. Aminov, M. G. Shayevska, *Krucheniye Gaussa 2-mernoy poverxnosti, zadannoy v neyavnom vide v 4-mernom yevklidovom prostranstve*, Matematicheskiy sbornik. 2004; 195(11), 1-12.

[2] A.V.Bolsinov, A.T.Fomenko, Integrable Hamiltonian systems, Ud- murtskiy universitet, Izhevsk, 1999.

[3] V.Fomenko, *Classification of Two-Dimensional Surfaces with Zero Normal Torsion in Four-Dimensional Spaces of Constant Curvature*, Math. Notes. 2004; 75(5), 690-701.

[4] V.Fomenko, *Some properties of two-dimensional surfaces with zero normal torsion in* E^4, Sb. Math. 1979; 35(2), 251-265.

[5] S.Kadomcev, A study of certain properties of normal torsion of a two-dimensional surface in four-dimensional space, VINITI, Problems in geometry, 1975, 7,267-278.

[6] A.Ya. Narmanov and S. Saitova, On the geometry of the reachability set of vector fields,Differential Equations, 2017,53 .

[7] A.Ya. Narmanov and O. Yu. Qosimov, On the Geometry of the Set of Orbits of Killing Vector Fields on Euclidean Space, J. Geom. Symmetry Phys, 2020, 55, 39-49.

[8] H.Sussman, Orbits of families of vector fields and integrability of distributions, Transactions of the AMS, 1973, 180, 171-188.

[9] P.Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, 1993.

[10] K. Ramazanova, *The theory of curvature of* X^2 *in* E^4, Izv. Vyssh. Uchebn. Zaved. Mat, Vol.6 (1966), 137-143.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON THE MINIMIZATION OF k-VALUED LOGIC FUNCTIONS IN THE CLASS OF DISJUNCTIVE NORMAL FORMS

ANVAR KABULOV* [1,3,4,5], ABDUSSATTAR BAIZHUMANOV [2], MANSUR BERDIMURODOV[1]

[1]National University of Uzbekistan named after Mirzo Ulugbek, Tashkent, Uzbekistan
[2]South Kazakhstan State Pedagogical University, Shymkent, Kazakhstan
[3]Tashkent State Technical University, Tashkent, Uzbekistan
[4]Gulistan State University, Sirdaryo Region, Uzbekistan
[5]Engineering Federation of Uzbekistan, Tashkent, Uzbekistan

*e-mail: islambeksaymanov@gmail.com

Abstract. The paper considers the representation of k-valued logical functions in the class of disjunctive normal forms. Various classes of monotone functions of k-valued logic are investigated. Theorems are proved on the coincidence of reduced and shortest disjunctive nominal forms of k-valued functions. For a certain class of k-valued monotone functions, we prove an estimate for the number of functions in this class. we prove criteria for the absorption of elementary conjunctions by a first-order neighborhood of disjunctive normal forms of k-valued functions.

Keywords: k-valued, minimization, disjunctive normal form, rank, abbreviated d.n.f., monotone function.

AMS Subject Classification: 03B50

1. Introduction

The methods of k-valued logics are generally necessary for the study of a number of important problems from the most diverse fields: biology, medicine, military affairs, automation, control, planning of experiments etc., everywhere where not only the quantitative relationships between the quantities characterizing the processes under consideration are significant, but also the logical dependences connecting them [1, 2, 3, 4]. A multi-valued logical function can be represented as a disjunction (multi-place function "or") $K_1 \vee K_2 \vee ... \vee K_m$, where each term is a conjunction (multi-place function "and") of certain variables from the set $\{x_1, \ldots, x_n\}$, taken with or without negation. k-valued function gives a description functioning of the control system, and the formula realizing it, in particular, the disjunctive normal form (d.n.f.), describes the scheme of this system, so that the nodes and elements of the scheme correspond to the terms and letters of the d.n.f. as a rule, k-valued function has many essentially different d.n.f. [5, 6, 7, 8, 9, 10, 11, 12]. In mathematical logic they are considered from the qualitative side. With the development of cybernetics, the terms and letters of the d.n.f. began to reflect equipment costs in circuits and this drew attention to the quantitative side. Therefore, one of the problems of k-valued logics dictated by practice is the problem of minimizing multi-valued functions. The results of research in some areas in this area, in particular, minimization in certain systems multi-valued functions are quite widely displayed in the literature [13, 14, 15]. Therefore, it should immediately be noted that we will only discuss the minimization of multi-valued functions in the class of d.n.f..

2. Problem statement

Consider the set of multivalued logic functions depending on n variables, etc. the set of functions defined on the set of all vertices of the n-dimensional k-lattice E_n^k and taking values from the set $E_k = \{0, 1, \ldots, k-1\}$. With this interpretation, there is a one-to-one correspondence between multivalued logic functions depending on n arguments and subsets of $N_f \subseteq E_n^k$. The function $f(x_1, \ldots, x_n)$ and the subset $N_f \subseteq E_n^k$ correspond to each other in the case $f(x) = \begin{cases} \gamma, & if\ x \in N_f \\ 0, & if\ x \in E_n^k \backslash N_f \end{cases}$, where

$$\gamma \in E_f \subseteq \{0, 1, \ldots, k-1\} \tag{1}$$

We can assume that the set E_f divides the function f into a number of subfunctions $f_{\gamma_1}(\widetilde{x}), \ldots, f_{\gamma_m}(\widetilde{x})$, and the set N_f into pairwise disjoint subsets $N_{f_{\gamma_1}}, \ldots, N_{f_{\gamma m}}$, where $m = |E_f|$,

$$f_{\gamma_i}(\widetilde{x}) = \begin{cases} \gamma_i, & if\ f(x) = \gamma_i \\ 0, & if\ f(x) \neq \gamma_i \end{cases}$$

$$N_{f_i} = \left\{ \widetilde{\alpha} : \left(\widetilde{\alpha} \in E_n^k \right) \wedge \left(f(\widetilde{\alpha}) = \gamma_i \right), \left(i = \overline{1, m} \right). \right\}$$

It is easy to see that

$$f(\widetilde{\alpha}) = \max_{\gamma_i} \{ f_{\gamma_1}(\widetilde{\alpha}), \ldots, f_{\gamma_m}(\widetilde{\alpha}) \} \tag{2}$$

The function $f_\gamma(\widetilde{x})$, which takes only two values (0 *and* γ), will be called quasi-Boolean, and the representation of the function $f(x_1, \ldots, x_n)$ in the form of a (2)-quasi-Boolean representation of the multi-valued logical function $f(x_1, \ldots, x_n)$.

We introduce the function
$J_M(x) = \begin{cases} k-1, & if\ x \in M \\ 0, & if\ x \notin M \end{cases}$, where

$$M \subseteq E_k = \{0, 1, 2, \ldots, k-1\} \tag{3}$$

An elementary conjunction (e.c.) is an expression
$\mathfrak{A} = \min [J_{M_1}(x_1), \ldots, J_{M_n}(x_n), \gamma]$, where

$$\varnothing \neq M_j \subseteq E_k, \left(j = \overline{1, n} \right) \tag{4}$$

Further, for brevity, formulas $\max[\mathfrak{A}_1, \ldots, \mathfrak{A}_m]$ will be conventionally denoted as $\mathfrak{A}_1 \vee \ldots \vee \mathfrak{A}_m = \bigvee\limits_{i=1}^{m} \mathfrak{A}_i$: if $\mathfrak{A}_i$ is an analog of e.c., then this formula will be called disjunctive normal form (d.n.f.).

The area of truth of e.c. let's call $\mathfrak{A}$ the region $N_{\mathfrak{A}}$ in which $\mathfrak{A}$ takes the value γ. It is easy to see that the domain $N_{\mathfrak{A}} = \prod\limits_{j=1}^{n} M_j$ is a sub lattice (a subset of the set E_n^k) of the lattice E_n^k. With such a geometric consideration, the e.c. the sub lattice $N_{\mathfrak{A}}$ corresponds to the lattice E_n^k.

Rank e.c. let's $\mathfrak{A}$ call a number $r(\mathfrak{A}) = \sum\limits_{j=1}^{n} (k - |M_j|) = kn - \sum\limits_{j=1}^{n} |M_j|$.

The formula $\mathfrak{M} = \bigvee\limits_{i=1}^{t} \mathfrak{A}_i$ where all $\mathfrak{A}_i$, $\left(i = \overline{1, t} \right)$ are e.c. will be called the disjunctive normal form.

Note that each set-valued logic function $f(x_1, \ldots, x_n)$ corresponds to a non-empty class of d.n.f. realizing the given function. The set of all intervals corresponding to e.c. a certain d.n.f. from this class determines the covering of N_f by sub lattices of the lattice E_n^k. Hence it follows that the subsets $M \subseteq E_n^k$ can be defined using the d.n.f.

Let $I = \{N_{\mathfrak{A}}\}$, be some subset of sub lattices from E_n^k.

A sub lattice $N_B \in I$ is said to be maximal with respect to M if there is no sub lattice $N_{\mathfrak{A}}$ in I such that $N_{\mathfrak{A}} \neq N_B$ and $N_{\mathfrak{A}} \supseteq N_B$.

To represent the function $f(x_1, \ldots, x_n)$ in the form of a d.n.f. we considered its quasi-Boolean representation: $f = f_{\gamma_1} \vee \ldots \vee f_{\gamma_m}$ and $\gamma_1 < \gamma_2 < \ldots < \gamma_m$.

Note that for the same function $f(\widetilde{x})$ there can be several equivalent quasi-Boolean representations. Indeed, $f = f_{\gamma_1} \vee \ldots \vee f_{\gamma_m} = f^* = f^*_{\gamma_1} \vee \ldots \vee f^*_{\gamma_m}$ where $N_{f^*_{\gamma_i}} = N_{f_{\gamma_i}} \cup Q_i$, $Q_i \subseteq \bigcup\limits_{j>i} N_{\gamma_j}, \left(i = \overline{1,m}\right)$.

We will consider only one "maximum" representation $f' = {f'}_{\gamma_1} \vee \ldots \vee {f'}_{\gamma_m}$ where $N_{{f'}_{\gamma_i}} = \bigcup\limits_{j=1}^{n} N_{f_{\gamma_j}}$, $\left(i = \overline{1,m}\right)$.

Select all maximal sub lattices $N_{B^i_j}, \left(i = \overline{1,m}\right)$ contained in $N_{{f'}_{\gamma_i}}$ that have non-empty intersection with $N_{f_{\gamma_i}}$ and such that the value of B^i_j is equal to γ_i in $N_{B^i_j}, \left(i = \overline{1,m}\right)$. D.n.f. $\mathfrak{M} = \bigvee\limits_{i=1}^{m} \bigvee\limits_{j=1}^{I_i} B^i_j$ is called the reduced disjunctive normal form of the function $f(\widetilde{x})$.

A covering of a set N_f by maximal sub lattices is said to be irreducible if, after the removal of any of its sub lattices, it ceases to be a covering. A d.n.f. realizing a function f is called dead-end if it corresponds to an irreducible cover of the set N_f.

Consider a multivalued logic function $F(x_1, \ldots, x_n)$ defined at $M \subseteq E^k_n : F(\widetilde{x}) = \gamma_j$, if $\widetilde{x} \in M_j, \left(j = \overline{1,m}\right)$, $m < k$, $\gamma_j \in E_k$, $M = \bigcup\limits_{i=0}^{m} M_i$ and $M_i \bigcap M_j = \varnothing$ at $\left(i \neq j,\ i,\ j = \overline{0,m}\right)$. And $\gamma_1 < \ldots < \gamma_m$, $\gamma_0 = 0$.

Thus, $F(x_1, \ldots, x_n)$ is defined by specifying pairwise disjoint sets $M_0, \ldots, M_m$. The function $F(\widetilde{x})$ is defined, generally speaking, not on the entire set E^k_n. There are different before the definition in the class of functions $F(\widetilde{x})$, multi-valued logic, not equivalent to each other.

Our task is to find the simplest ones, in a certain sense, before definitions.

For $F(\widetilde{x})$, select all maximal intervals $N_{B^i_j}, \left(i = \overline{1,m},\ j = \overline{0,I_i}\right)$ contained in $E^k_n \setminus \bigcup\limits_{v=0}^{i-1} M_v$ that have non-empty intersection with M_i such that the value of B^i_j is equal to γ_i.

D.n.f. $\mathfrak{M} = \bigvee\limits_{i=1}^{m} \bigvee\limits_{j=1}^{I_i} B^i_j$ is called the reduced normal form for $F(\widetilde{x})$. It is easy to see that d.n.f. $\eta_{\Sigma TF}$ is uniquely determined by the function F.

Let us now indicate the points at which, when the values of the function F change (transition to F'), the values of $\eta_{\Sigma TF}$ change (transition to $\eta_{\Sigma TF'}$).

3. Monotone functions of k-valued logic

Let's consider some order on the set ε_k. For two sets $\Im = (\alpha_1, \alpha_2, \ldots \alpha_n)$ and $\widetilde{\beta} = (\beta_1, \beta_2, \ldots \beta_n)$ the precedence relation $\Im \leq \widetilde{\beta}$ is satisfied if $\alpha_i \leq \beta_i$ in this order for any $i = \overline{1,n}$.

Definition 1. A k-valued logic function $f(x_1, x_2, \ldots, x_n)$ is said to be monotonic with respect to a given order if for any tuples α and β such that $\left(\Im \leq \widetilde{\beta}\right)$ we have $f(\alpha) \leq f(\beta)$.

If $0 < 1 < 2 < \ldots < k-1$, then the set of functions that are monotone in this order constitutes the class of monotone functions of k-valued logic.

Theorem 1. Abbreviated d.n.f. monotone function f_γ of k-valued logic in n variables

a) consists of e.c. K^A, and only elementary formulas of the form $J_{[a,k-1]}(x), 0 \leq a \leq k-1$ are used;

b) Is the only minimal (shortest) d.n.f. of the function f.

Proof.

a) Let $K = J_{T_1}(x_1) \cdot J_{T_2}(x_2) \cdot \ldots \cdot J_{T_n}(x_n) \cdot \gamma$, where $T_j = [a_j, k-1]$, $j \neq i$, $0 \leq a_j \leq k-1$, $T_i = [b_i] \bigcup [a_i, k-1]$, $b_i \leq a_i$.

Then the conjunction K, and hence the function f_γ, takes the value γ on the set $\widetilde{a} = (a_1, a_2, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n)$.

b) It follows from the monotonicity condition for the function f_γ that for any set $\widetilde{b}$ such that $\widetilde{b} \geq \widetilde{a}$, $f_\gamma\left(\widetilde{b}\right) = \gamma$ therefore, there exists an e.c. $K' = J_{T'_1}(x_1) \cdot J_{T'_2}(x_2) \cdot \ldots \cdot J_{T'_n}(x_n) \cdot \gamma$, where $T'_j = T_j$, $j \neq i$, $T'_i = [b_i, k-1]$, for which $U_K \subset U_{K'} \subseteq U_{f_\gamma}$, and e.c. K is not maximal for U_{f_γ}.

c) As proved, each maximum e.c. K functions has the form $K = J_{[a_1,k-1]}(x_1) \cdot J_{[a_2,k-1]}(x_2) \cdot \ldots \cdot J_{[a_n,k-1]}(x_n) \cdot \gamma$, $0 \leq a_j \leq k-1$, $j = \overline{1,n}$.

Let us show that the set $\widetilde{a} = (a_1, a_2, \ldots, a_n)$ is core for the function f_γ, etc. in the abbreviated d.n.f. the function f_γ has no e.c., except for K, which takes the value γ on this set.

Indeed, if in the reduced d.n.f. function f_γ was an e.c. K', which takes the values γ on the tuple $\widetilde{a}$, then it would follow from the monotonicity condition for the function f_γ that the e.c. K' takes the value γ on all tuples $\widetilde{b}$ such that $\widetilde{b} \geq \widetilde{a}$, then $U_K \subseteq U_{K'}$, which contradicts the maximum e.c. K.

The theorem is proved.

Corollary. Abbreviated d.n.f. monotone function f of k-valued logic in n variables consists of e.c. K^A and is the only minimal d.n.f. functions f.

Proof. A monotone function f has the following obvious properties: for any comparable collections $\widetilde{a} \in N_{f_\gamma}$ and $\widetilde{b} \in N_{f_\theta}$ we have $\widetilde{b} > \widetilde{a}$ for $\gamma < \theta$. Therefore, e.c. K, is included in the abbreviated d.n.f. functions $f_\gamma\,(\gamma \in \{\varepsilon_k/0\})$ do not contain elementary formulas of type $J_T(x)$, where the set T is a disconnected set of points from ε_k.

The second assertion is proved by the method of theorem 1, and the core sets for the function f are the sets $\widetilde{a} = (a_1, a_2, \ldots, a_n)$ in the e.c. $K = J_{[a_1,b_1]}(x_1) \cdot J_{[a_2,b_2]}(x_2) \cdot \ldots \cdot J_{[a_n,b_n]}(x_n) \cdot \gamma$.

Corollary proven. On the set ε_k we introduce a partial order $0 < 1$, $0 < 2$, $\ldots$, $0 < k-1$ where i is incomparable with j if $i, j \in \{\varepsilon_k/0\}$.

The set of functions of k-valued logic f in n variables, monotone in a given order, is combined into the class S. Let us estimate the cardinality of the class S.

The set ε_k is associated with a basic graph - a directed graph with K vertices corresponding to the elements of the set ε_k, in which there is an arc (i, j) if and only if $i > j$.

Let us introduce the Z axis on the plane. Associate each point A of the plane with the number Z_A, the projection of A onto the Z axis. In particular, if the basis graph is drawn on the plane, then each vertex corresponds to the number Z_i. An image of a basic graph is called admissible if for any arc (i, j) of the graph $Z_i - Z_j \geq 1$.

Consider a random variable $\xi = Z_A$, where the point A can fall into any vertex of the graph with probability $\frac{1}{k}$ then the expectation is $M\xi = Z_{cp} = \frac{Z_1 + \ldots + Z_k}{k}$ and the variance is $D\xi = \frac{1}{k} \sum\limits_{i=1}^{k} (Z_i - Z_{cp})^2$.

We will consider the image of the graph shifted so that $M\xi = 0$.

In older articles an estimate is obtained for finding the number $\psi(n)$ of monotone functions of n variables from an arbitrary partially ordered set of k elements:

$$\psi(n) = d^{\frac{1}{\sqrt{2\pi D}} \cdot \frac{k^n}{\sqrt{n}}(1+\varepsilon(n))} \tag{5}$$

where $\varepsilon(n) \to 0$ at $n \to \infty$; $D = \inf D\xi$; $d = \max(|H_1|, \ldots, |H_s|)$, $H_0 \leq H_1 \leq \ldots \leq H_{s+1}$, all $H_i \subseteq \varepsilon_k$, $s \geq 1$, $|H_0| = |H_{s+1}| = 1$ and $H_i \neq H_j$, at $i \neq j$ ($H_i \leq H_j$ if $a \leq b$ for any $a \in H_i$, $b \in H_j$), the maximum is taken over all possible chains. This estimate is also valid for the class of functions S.

For the order in ε_k we have:

$$\begin{cases} Z_1 - Z_0 \geq 1 \\ \ldots \quad \ldots \quad \ldots \\ Z_{k-1} - Z_0 \geq 1 \end{cases} \tag{6}$$

$$Z_{cp} = \frac{Z_0 + Z_1 + \ldots + Z_{k-1}}{k} = 0,\ D\xi = \frac{1}{k}\sum_{i=0}^{k-1} {Z_i}^2 \tag{7}$$

It follows that $Z_0 \le -\frac{k-1}{k}$ and $Z_i \ge \frac{1}{k}$, $i = \overline{1, k-1}$, so $D\xi \ge \frac{k-1}{k^2}$ and $D \ge \frac{k-1}{k^2}$.

On the set ε_k for the introduced order there is only $(k-1)$ chain

$$\begin{cases} \{0\} < \{0,1\} < \{1\} \\ \cdots\cdots\cdots \\ \{0\} < \{0,k-1\} < \{k-1\} \end{cases} \tag{8}$$

It is obvious that in the considered case $d = 2$.

Consequently

$$\begin{cases} \psi(n) = d^{\frac{1}{\sqrt{2\pi(k-1)}} \cdot \frac{k^{n+1}}{\sqrt{n}}(1+\varepsilon(n))} \end{cases} \tag{9}$$

where $\varepsilon(n) \to 0$ at $n \to \infty$.

For functions of class S, e.c. consists of elementary formulas of the form $J_T(x)$, where $T \subseteq \{\varepsilon_k/0\}$.

If for monotone functions f of k-valued logic the abbreviated d.n.f. is the only minimal one, then for functions of the class S this property does not hold, which is demonstrated by the following.

Example. $k = 3$, $n = 3$.

Let the function $f(x_1, x_2, x_3)$ take the values 1 on the sets $(0,1,1), (1,1,1), (1,2,1), (2,1,1), (1,2,2)$ and 0 in other cases.

Abbreviated d.n.f. function f has the form:

$$D_C(f) = J_1(x_2) \cdot J_1(x_3) \vee J_1(x_1) \cdot J_2(x_2) \cdot J_{[1,2]}(x_3) \tag{10}$$

and the minimum:

$$D_M(f) = J_1(x_2) \cdot J_1(x_3) \vee J_1(x_1) \cdot J_{[1,2]}(x_2) \cdot J_1(x_3) \tag{11}$$

The process of transition from the abbreviated d.n.f. functions f of a k-valued logic to a dead-end one can be divided into elementary steps, each of which is a removal from the d.n.f. D obtained in the previous step, one e.c. K. Removed e.c. is such that $U_K \subseteq \bigcup_{j=1}^{m} U_{K_j}$, where K_j are some e.c. from d.s.f. D different from K.

In older articles the criterion for covering an interval by the sum of other intervals for functions f of k-valued logic is described. For functions f of class S, this criterion has a simpler form.

E.c. K_1 and K_2 are called orthogonal if $K_1 \cdot K_2 = 0$. In other words, conjunctions K_1 and K_2 are orthogonal if and only if $U_{K_1} \bigcap U_{K_2} = \varnothing$. Obviously, when studying the absorption, some sets of e.c. $\{K_j\}$, $j = \overline{1,m}$ e.c. K it suffices to consider only those e.c. that are non-orthogonal to K. To check orthogonality, it is easiest to use the following properties: two e.c. are orthogonal if and only if there exists a variable x_j for which $T_j^1 \bigcap T_j^2 = \varnothing$ is satisfied in the elementary formulas $J_{T_j^1}(x_1)$ and $J_{T_j^2}(x_2)$.

D.n.f. D realizing the function f absorbs the e.c. K if $K(\widetilde{x}) \le D(\widetilde{x})$ for any $\widetilde{x} = \varepsilon_k^n$.

So let $K = J_{T_1}(x_1) \cdot J_{T_2}(x_2) \cdot \ldots \cdot J_{T_t}(x_t) \cdot \gamma$.

Obviously, K can be absorbed only by those sets of e.c. $\{K_j\}$, which take values from $\{0, \gamma\}$, so let's consider the absorption process using the example of the quasi-Boolean function f_γ.

For each e.c. $\{K_j\}$, $j = \overline{1,m}$ construct an e.c. K_j, replacing the elementary formulas $J_T(x)$ occurring in K_j with $J_{[1,k-1]}(x)$.

It's obvious that $U_{K_j} \subseteq U_{K_j}$, $U_{K_j} \cap \left(U_D / U_{K_j}\right) = \varnothing$.

Let us introduce into consideration the set $\varepsilon_k^{n,t}$- the collection of all sets from ε_k^n, in which the t first coordinates take values from $\{\varepsilon_k/0\}$, and the rest are arbitrary.

Theorem 2. The disjunction $D = \bigvee\limits_{j=1}^{m} K_j$ absorbs the e.c. K if and only if $\bigvee\limits_{j=1}^{m} K_j = \gamma$ for any $\widetilde{x} = \varepsilon_k^{n,t}$, etc. if $\bigvee\limits_{j=1}^{m} K_j = \varepsilon_k^{n,t}$.

Proof. *Need.* Let D absorb the e.c. K. Let us prove that in this case $\bigvee\limits_{j=1}^{m} K_j = \gamma$ for any $\widetilde{x} = \varepsilon_k^{n,t}$. Let us assume that this is not the case, etc. there is a collection $\Im$ such that $\bigvee\limits_{j=1}^{m} K_j(\Im) = 0$. Denote by $x_{i_1}, \ldots, x_{i_p}$ the variables that are not included in any of the e.c. from D. Obviously, the values of the remaining variables do not affect the value of the expression $\bigvee\limits_{j=1}^{m} K_j$.

The value of the function f on the tuples $\Im$ will be denoted by $[f]\,\Im$. Then one can write

$$\left[\bigvee_{j=1}^{m} K_j\right]\{\Im\} = 0 \tag{12}$$

where $\{\Im\}$ is the set of sets whose $(x_1, x_2, \ldots, x_t)$ coordinates take all possible values from $\{\varepsilon_k/0\}$, and the remaining $(n-t)$ coordinates are such that (12) is satisfied. hence we get that $[K_j]\{\Im\} = 0$ for all $j = \overline{1,m}$, so $[D]\,\Im = 0$.

Let us determine the values of the remaining variables entering K ($K \neq 0$ on U_D, since K is not orthogonal to D) so that K turns into γ on these sets. The intersection of these two sets determines the values of all variables in such a way that D takes the value 0 on this set, and e.c. K value γ. This contradicts the condition $K(\widetilde{x}) \leq D(\widetilde{x})$ for any $\widetilde{x} = \varepsilon_k^n$, hence the assumption that $\bigvee\limits_{j=1}^{m} K_j \neq \gamma$ on $\varepsilon_k^{n,t}$ is false, and the necessity of the condition of the theorem is proved.

Adequacy. Let the condition of the theorem be satisfied etc., $\bigcup\limits_{j=1}^{m} U_{K_j} = \varepsilon_k^{n,t}$. Then $U_K \subseteq \bigcup\limits_{j=1}^{m} U_{K_j}$, but $U_K \bigcap \left(\bigcup\limits_{j=1}^{m} U_{K_j} / \bigcup\limits_{j=1}^{m} U_{K_j}\right) = \varnothing$. Consequently. $U_K \subseteq \bigcup\limits_{j=1}^{m} U_{K_j}$.

The theorem has been proven.

4. Conclusions

The paper proposes a representation of k-valued functions in the class of disjunctive normal forms. Monotone functions of k-valued logic are investigated. We prove theorems on the coincidence of abbreviated and shortest d.n.f. k-valued functions. For a certain class of k-valued monotone functions, the number of functions from this class is calculated. Criteria for the absorption of elementary conjunctions by a first-order neighborhood are proposed.

References

[1] A. Kabulov, I. Yarashov, A. Otakhonov Algorithmic Analysis of the System Based on the Functioning Table and Information Security // 2022 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Toronto, ON, Canada, 2022, pp. 1-5, doi: 10.1109/IEMTRONICS55184.2022.9795746.

[2] A. Kabulov, I. Saymanov, I. Yarashov, A. Karimov Using Algorithmic Modeling to Control User Access Based on Functioning Table // 2022 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Toronto, ON, Canada, 2022, pp. 1-5, doi: 10.1109/IEMTRONICS55184.2022.9795850.

[3] E. Navruzov, A. Kabulov Detection and analysis types of DDoS attack // 2022 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Toronto, ON, Canada, 2022, pp. 1-7, doi: 10.1109/IEMTRONICS55184.2022.9795729.

[4] A. Kabulov, I. Saymanov, I. Yarashov, F. Muxammadiev Algorithmic method of security of the Internet of Things based on steganographic coding // 2021 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Toronto, ON, Canada, 2021, pp. 1-5, doi: 10.1109/IEMTRONICS52119.2021.9422588.

[5] A. Kabulov, I. Normatov, E. Urunbaev, F. Muhammadiev Invariant Continuation of Discrete Multi-Valued Functions and Their Implementation // 2021 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Toronto, ON, Canada, 2021, pp. 1-6, doi: 10.1109/IEMTRONICS52119.2021.9422486.
[6] A. Kabulov, I. Normatov, A. Seytov, A. Kudaybergenov Optimal Management of Water Resources in Large Main Canals with Cascade Pumping Stations // 2020 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Vancouver, BC, Canada, 2020, pp. 1-4, doi: 10.1109/IEMTRONICS51293.2020.9216402.
[7] Kabulov, A. V., Normatov, I. H. (2019). About problems of decoding and searching for the maximum upper zero of discrete monotone functions // Journal of Physics: Conference Series, 1260(10), 102006, 2019. doi:10.1088/1742-6596/1260/10/102006.
[8] Kabulov, A. V., Normatov, I. H., Ashurov A.O. Computational methods of minimization of multiple functions // Journal of Physics: Conference Series, 1260(10), 10200, 2019. doi:10.1088/1742-6596/1260/10/102007.
[9] Anvar Kabulov, Alimdzhan Babadzhanov, Islambek Saymanov Correct models of families of algorithms for calculating estimates // AIP Conference Proceedings 8 June 2023; 2781 (1): 020010. https://doi.org/10.1063/5.0144830.
[10] Anvar Kabulov, Alimdzhan Babadzhanov, Islambek Saymanov Completeness of the linear closure of the voting model // AIP Conference Proceedings 8 June 2023; 2781 (1): 020020. https://doi.org/10.1063/5.0144832.
[11] Anvar Kabulov, Mansur Berdimurodov Gost R 34.12-2015 (Kuznechik) analysis of a cryptographic algorithm // AIP Conference Proceedings 8 June 2023; 2781 (1): 020046. https://doi.org/10.1063/5.0145975.
[12] Valeri Sklyarov, Iouliia Skliarova, Anvar Kabulov Hardware accelerators for solving computationally intensive problems over binary vectors and matrices // AIP Conference Proceedings 8 June 2023; 2781 (1): 020013. https://doi.org/10.1063/5.0145125.
[13] Preface: International Conference on Modern Problems of Applied Mathematics and Information Technology // AIP Conference Proceedings 8 June 2023; 2781 (1): 010001. https://doi.org/10.1063/12.0017180.
[14] Ibrokhimali Normatov Endless individual areas of logic and beginnings of arithmetics // AIP Conference Proceedings 2781, 020008 (2023). https://doi.org/10.1063/5.0144824
[15] I. Normatov, I. Yarashov, A. Otakhonov and B. Ergashev Construction of reliable well distribution functions based on the principle of invariance for convenient user access control // 2022 International Conference on Information Science and Communications Technologies (ICISCT), Tashkent, Uzbekistan, 2022, pp. 1-5, doi: 10.1109/ICISCT55600.2022.10146952.
[16] Normatov I. Principle of Independence of Continuation of Functions Multivalued Logic from Coding // Journal of Physics: Conference Series, 2019, 1210(1), 012107. doi:10.1088/1742-6596/1210/1/012107.
[17] I. Normatov, E. Kamolov Development of an algorithm for optimizing the technological process of kaolin enrichment // 2020 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Vancouver, BC, Canada, 2020, pp. 1-4, doi: 10.1109/IEMTRONICS51293.2020.9216371.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

THE STUDY OF GEOMETRIC LAWS IN THE PATTERNS OF THE KHOJA AHMED YASAWI COMPLEX

AUYEZ BAIDABEKOV[1], ELMIRA KEMELBEKOVA[1], ZHANNA SADYKOVA[1]

[1]L.N. Gumilyov Eurasian National University, Astana, Kazakhstan
e-mail: baydabekov@mail.ru, e.kemelbekova@mail.ru, sjm67@mail.ru

Abstract. The article describes the history of geometric ornaments in the Islamic world from the beginning of the VII th century to the end of the IX th century with a break of almost three centuries. The use of geometric patterns has become one of the distinctive features of Islamic art. The creation of this unique new style required significant intellectual input from Islamic mathematicians, astronomers, and scientists. Its heyday in medieval Eastern architecture was the doctrine of balance and harmony. Therefore, geometric statics becomes a part of the doctrine of geometric harmony. The unity of the architectural-spatial format of the structure and the method of coordinating its structure is visible right here. At that time, geometric methods as the basis of architecture became the leading ideological system of architecture, which had great significance in the formation and development of medieval Middle Eastern architecture. The works of medieval Eastern architects, artists engaged in ornamentation and geometric patterns, formed as a summary of the experience formulated in connection with the new classification of mathematical sciences, had a decisive influence on the direction of the worldview of many generations. At the same time, the theoretical foundations are born from the experience itself, from the experience of construction and art production. The scope of use of geometric expressions was wide. The spatial structure of the building, the balance of the architectural constructions, the stability and strength of the structure as a whole, and the geometric expression included geometric theory and practical problems. Khoja Akhmat Yaasawi was a complex that preserved such a balance of geometric patterns and architectural structures in medieval Central Asia. Like many of Amir Temir's buildings, the Khoja Akhmat Yasawi complex is distinguished by its scale and grandeur of design, but the volume-spatial interpretation of the complex is simple. The building is an elongated parallelepiped with two portals and two domes rising along the longitudinal axis. At the same time, the problem of studying the geometric patterns on the facades of the Khoja Ahmet Yasawi mausoleum based on this theory and other geometric laws is considered.

Keywords: al-Farabi, al-Khorezmi, division of geometric bodies into equal parts, mausoleum of ruler Amir Temir, Khoja Ahmet Yasawi, dimensions of the mausoleum, geometric expressions.

1. Introduction

The use of geometric patterns in various buildings, including mosques and mausoleums, was one of the features of Islamic art in the Arab Caliphate from the beginning of the VII th century to the end of the XIV th century. Mathematicians and other scientists who lived in Islamic countries made a great contribution to the creation of this unique new style [15]. The development of architecture and its flourishing period in medieval countries, including Central Asia, was associated with the rapid development of the doctrine of balance. The doctrine of equilibrium becomes a part of the doctrine of geometric harmony. Here, the architectural-spatial format of the structure and the method of coordinating its structure are used together in one application. Islamic scholars, including the great encyclopedists of the Middle Ages, considered harmony and balance as an aesthetic principle and turned it into a practical art. At this time, geometric

methods as the basis of architecture became the leading ideological system of architecture, which had a great importance in the formation and development of medieval Central Asian architecture [13]. The works of medieval architects, artists engaged in ornamentation and geometric patterns, formed as the core of the practice formulated in connection with the new classification of mathematical sciences, had a decisive influence on the direction of the worldview of many generations. At the same time, theoretical foundations emerged from the experience itself, from the experience of construction and artistic production. Spatial structure of the building, balance of architectural structures, stability and strength of the structure as a whole, geometric ornament and geometric theory and practical problems are included. The works and researches of medieval mathematicians, who looked at the lens of the geometric technique, which was the basis of the architecture of their time, had a wide scope of application of geometric expressions in mathematics. He looks for support in mathematics, and the most important part of mathematics is algebra, which defines numerical and geometric unknowns. Al-Farabi appreciated many practical arts, including architecture. In this regard, al-Farabi writes: "Also, many masters use geometrical methods, including the art of construction management, project implementation, and architectural construction" [17, 18]. In another treatise, al-Farabi writes that knowledge is the geometric basis of a master, and one should start learning it from geometric methods. Al-Farabi emphasizes that experimental art represents the foundation of geometric methods and their relation to bodies, figures, order, position and intention.

Among the following works of the mathematician Al-Khwarizmi dedicated to our research topic, "A small book on algebra and muqabala problems" [4] is of interest. The emergence of algebraic calculations was connected with the necessity of life: the economic life of the caliphate, including internal and external trade, agriculture and irrigation construction (irrigation, construction of cities, architecture), geometric methods of construction of architectural objects, etc. was in demand. The preface of the treatise reads: "I have compiled a small book on algebra and calculation, including simple and complex questions of arithmetic, as it is necessary for people in inheritances, wills, distributions. Litigation, trade, as well as land, canal construction, geometry (engineering art, construction of architectural objects) and other types of such situations" - transactions, as well as land, canal construction, geometry (engineering art, architecture, architectural construction object) and other types of such situations" [7]. Al-Khwarizmi said that there are three types of numbers using algebra to calculate such balls. These numbers are divided into roots, squares and prime numbers. Al-Khorezmi in his scientific work considered the square as a unit of measurement of the surface. It provides information about whole right triangles with sides, citing the rules for calculating the areas of squares, triangles and rhombuses [14].

In the Middle Ages, the leading architects in Central Asia belonged to the cultural environment of the society, intelligent workers, mastered mathematical sciences, which made it possible to use the geometric methods of masters in the construction of architectural objects [9]. In the Middle Ages in the East, the formation of a highly qualified architect was achieved through practical training, that is, the transfer of experience and skills on the construction site and learning from books intended for people in the construction profession.

Based on such experiences and theories, a geometric study of a large medieval architectural object built by the ruler Amir Temir at the end of the XIV th century in order to modernize the mausoleum of Khoja Ahmet Yasawi, the spreader of Sufi teachings in Central Asia. This building is an amazing multifunctional modern architecture and is the largest brick complex in Central Asia. At the same time, Amir Temir built a building based on the Holy Quran in order to show the glory and wealth of the Iron Country and promote the Muslim religion. During the construction of the building, he made a great contribution to the use of new methods of new architecture, apart from the previous modern construction technologies of that time. Amir Temir attached great importance to the main dimensions and decoration of the building. Examples include various cast candles and cauldrons, candlesticks and large and small doors decorated

with gold and silver-plated wood carvings, as well as carvings and inscriptions on the facades and exteriors of buildings, ceramic labels painted in various brick colors. In addition, in some doors and architectural details of the building, elements of traditional Kazakh ornament can be found [11]. In the construction of this building, Amir Temir used some features of the architecture of the Karakhan period before him. At the same time, the article deals with the problem of studying the geometric patterns of the geometric patterns on the facades of the mausoleum of Khoja Ahmet Yasawi.

Among the theoretical calculations made by Ì.Ñ. Bulatov [6] (1907-2004) as a result of research work on old projects, two main rules are fully preserved. First, the theoretical account of the irrational relations $1 : \sqrt{2}$, $1 : \sqrt{3}$, $1 : \sqrt{5}$ and the exact correspondence notes between the rectangles drawn on the radial grid of coordinates. Secondly, the method of translating drawings into nature is based on different proportionality acting at the same time. In addition, the analysis of architectural ornaments of Central Asia conducted by N. Baklanov [5], G. Gaganov [8] and L. Rempel [9] was a scientific innovation. They posed and partially solved the problem of the theory of construction of architectural ornaments of Central Asia connected with the success of applied mathematics. Some of the researchers underestimated the mathematical knowledge of ornament masters, while others, in particular N. Bandanov, on the contrary, recognized their higher education [10].

2. The use of mathematical methods by the masters and architects of Central Asia

The idea of harmony in the aesthetics of the Muslim Middle Ages has attracted the attention of orientalists in recent years. Researches of scientists Thanks to the publication of philosophical works, musical-mathematical treatises of the countries of the Near and Middle East, our understanding of the aesthetics of the Muslim Middle Ages has expanded significantly. Craftsmen and architects recognized that mathematical methods were the basis of architecture and used them in the drawing and construction of medieval projects. Abu-Nasr Muhammad ben Tarchan ben-Ausalag al-Farabi (870-950) "Mathematical treatise" [2] presents two methods of constructing simple geometric constructions using circle and line, parabola, multiplication and angle trisection. Al-Farabi's sixth book is entirely about equilateral quadrilaterals (squares). The book shows how to draw an octagon with equal sides into a square (Figure 1).

First he took the square ABCD. He drew the diagonal lines of the square and marked the point of intersection as E. Point G was determined by measuring half the side of the square from point E along the diagonal AD. From this point we draw lines perpendicular to the square from point G. These lines intersect the sides of square ABCD and give points HF, XN, ML and KJ. These found points divide the given square ABCD into an equilateral octagon (Figure 1). In addition, this method divides square ABCD into four equilateral triangles.

According to al-Farabi, arithmetic and geometry are included in all sciences and arts. As a result of the activities of mathematical sciences, the application of geometry to the construction of architectural forms, the technique of geometric masters in architecture gives rise to the science of geometric harmony.

According to al-Farabi, the concepts created as a generalization of the experience formulated in connection with the classification of sciences had a decisive influence on the orientation of the worldview of many generations of medieval Eastern architects and decorative artists. The theory arising from the practice itself, the active experience of collective construction and artistic production had a great impact on the experience and creativity of architects who considered the basis of architecture at that time through the prism of the geometric technique of the masters. The scope of their application was much wider and included problems of architectural theory and practice, such as the spatial structure of the structure, the balance of architectural masses, fragments, architectural ornaments [1].

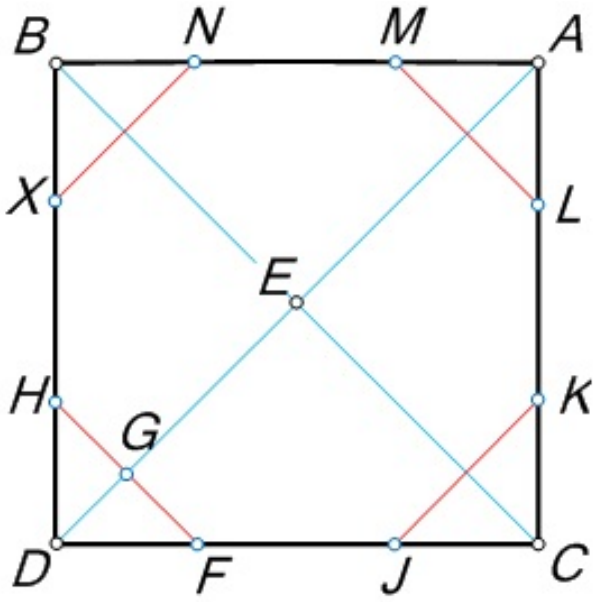


FIGURE 1. Al-Farabi's division of a square into an equilateral octagon

His creative biography of al-Khwarizmi [4] (813-847), who lived in the northern edge of the Arab caliphate, is given thanks to some preserved scientific works. Al-Khwarizmi, whose creative origins date back to the ancient Babylonian tradition, was well versed in ancient Vidian mathematics. Of all the works of al-Khwarizmi on the topic of our study, "A short book on algebra and algebraic calculations" is of interest, the appearance of which was, of course, due to a vital need. The growing economic life of the Caliphate, internal and external trade, agriculture, irrigation construction (irrigation, construction of cities, architecture), geometric methods of architectural construction, etc. demanded popularization of primary mathematics education. Al-Khwarizmi said that there are three types of numbers using algebra to calculate such balls. These numbers are divided into roots, squares and prime numbers. A root is anything that can be multiplied by itself, whether it is one or a large number or a small fraction. And it is obtained by multiplying the square root, and a prime number means any number that is not related to the root or the square. At the same time, translating what has been said into the language of geometry, the root is considered to be the side of the square. According to al-Khorezmi [7], he considered the square as a unit of measurement of the surface. Gives information about whole right triangles with sides 3:4:5, 5:12:13, 9:12:15 and gives rules for calculating areas of squares, triangles, rhombuses. In addition, the diagram shows a special case of proving the stated theorem (the sum of the squares of the legs is equal to the square of the hypotenuse) (Figure 2).

Interestingly, he proposed to define irrational roots (for example: $\sqrt{50}$ and $\sqrt{75}$) as prime numbers and approximate their values.

It can be assumed that the "Dimensions" section of the book is intended for architects, because there we find phrases such as "such a column has a round base" or "pyramidal column" (these terms belong to the art of architecture), but mainly the problem solving examples given are architectural production and architectural objects. was characteristic of the construction.

The essence of the above methods is that they act together geometrically and arithmetically. The Greeks defined geometrical volume, mass, and articulation in the same way as the relation caps of body parts in art; Arithmetic - geometry translated into the language of indicators necessary during construction works (calculated bricks, casings, curved parts, etc., needed to cover domes, drums of domes in other parts of the building); translated modular geometric

proportionality as a general principle (expressed in building plans and three-dimensional models) into the language of the builder who needed working drawings.

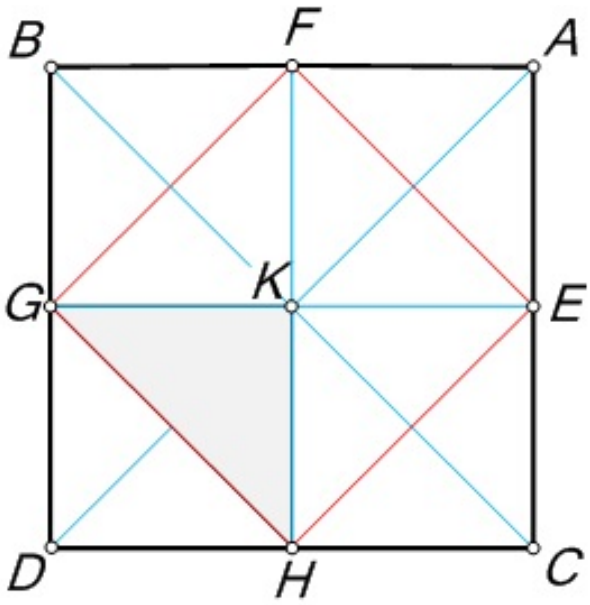


FIGURE 2. Taken from Al-Khwarizmi's treatise, square surface unit

But it seems that he had no technical documents other than mathematical problems and models. That's why the module was needed to help architects and builders. At that time, length was not a model of the standard. The size of the bricks varied from region to region. It was a dimensionless module, brick by name, because in all cases it must contain an integer number of bricks, it lost the role of modules. The modular mesh corresponds to the brick standard adopted for this structure (taking into account the thickness of the seam). Since the layers are not stable (the work is done by hand), the arithmetic calculation and the modular grid are broken. During the work, these violations were corrected with the help of ropes, poles and rulers, that is, in a geometric way. Hence the idea of inconsistencies, errors, and strange applications of geometrical or modular or arithmetical relations.

In the X th-XV th centuries, the conformity of the proportions of medieval buildings and constructions to one or another module belongs to the geometric methods of coordinating spatial architectural objects of this era. This can be seen when considering a number of architectural monuments and theoretical treatises of encyclopedist scientists of that era.

3. Geometrical secrets of the facade of Khoja Ahmet Yasawi

In order to increase the ideological policy of the state he founded in the steppes of the nomads led by Amir Temir, Khoja Ahmet Yasawi, the propagator and spreader of Islam, built a new complex in order to modernize the mausoleum in accordance with modern requirements. Like many buildings built by Amir Temir, the complex of Khoja Ahmad Yasawi is distinguished by its scale and grandeur of design, although the volume-spatial interpretation of the complex is simple. It is an elongated parallelepiped with a portal and two domes raised along the longitudinal axis [3].

The facades of the Khoja Ahmet Yasawi mausoleum are not only beautiful, but also rich in geometric patterns that have important symbolic and spiritual meaning in Islamic art and architecture. Islamic geometric patterns are based on a set of mathematical principles that reflect the fundamental order of the universe. These patterns are often used in Islamic art and architecture as a way to convey spiritual symbols and create a sense of harmony and balance.

The facade of the mausoleum of Khoja Ahmet Yasawi is decorated with complex geometric patterns based on sacred geometry, for example, the use of circles, squares, and polygons.

The repetition of geometric patterns on the facade of the mausoleum creates a sense of unity and continuity. Patterns are often based on a central motif, which is then repeated and expanded to create larger and more complex patterns.

Let's look at the facades of the mausoleum of Khoja Ahmet Yasawi, that is, a skyscraper, from a geometric point of view. First, let's look at the front view of the building. This main facade faces south-east. In ancient times, Kazakhs built their yurts facing south-east like this. The main facade of the building consists of a main portal and two towers, three entrance doors and a large entrance arch.

The facades of the mausoleum of Khoja Ahmet Yasawi are decorated with many interesting geometric patterns. The side facade of the mausoleum has a series of stars and polygons used to create complex geometric patterns. These patterns are often based on a central motif, which is then repeated and expanded to create larger images [16].

The side facades of the building (west and east) are decorated with decorative burnt bricks of different colors. Geometric patterns (gyrih) woven with blue plates are very clear, the harmony of delicate lines around them harmonizes the whole composition without scattering.

The side facade on the right side is oriented to the northeast. This side facade consists of a square. There is also a 2,20-meter-high polygonal plinth mosaic and 2,50-meter Bashkir images with Arabic letters. The facade wall consists of a puzzle-like mosaic made of dark blue bricks, consisting of two horizontal and two vertical rows of two rhombus-shaped octagonal geometric elements. Such a dark blue consists of 45° smaller square bricks. Among these geometrical elements are the words "Allah", "Muhammad", "O Lord", "Power belongs to God", "Allah is my ruler" written in "kufa" script from light blue bricks [15].

Comparing the complex of Khoja Ahmad Yasawi (XIV century) with other geometric ornaments of this era, if we consider the laws of creation of this gerich, it allows us to assume that it is built on the products of a square and an equilateral triangle. However, here the side of the equilateral triangle is the side of the square, and the side of the octagon inscribed in the square. The method of drawing a square is very simple. For this, first we draw the initial square ABCD. We connect the vertices AC and CD of the square, i.e. the rectangle, draw the diagonals of the square and determine the point O. We draw arcs of radius R from point O and determine points E and G from the points that intersect their walls. These points divide the sides of the square ABCD into three segments. Ariphetically, these sections are written as follows (Figure 3) [5]:

$$DG = \frac{2-\sqrt{2}}{2}; \quad GE = \sqrt{2}-1; \quad EC = \frac{2-\sqrt{2}}{2}$$

If we connect the points E and G of these sides, we find four lines parallel to the diagonal of the square. These lines divide the square into four lines at an angle of 450 degrees to the sides, and these lines divide the square into four equilateral triangles. These lines cross each other and form a small square whose sides are 450 degrees to the square of ABCD (Figure 3).

Let's try to draw the base of the mosaic on the side facade using this drawn geometric pattern. We divide the walls into squares. We extend the sides of the small square in the geometric model to the point E on the side of the square ABCD. Thus, if we collect a small square in this geometric model, the mosaic base of the facade appears on the right.

The geometric patterns on the facade of the mausoleum of Khoja Ahmet Yasawi are rich in symbolism. For example, the octagonal star used in the design is associated with the eight gates in the Islamic tradition. The use of the octagonal shape is also important because it represents the balance between heaven and earth.

Calligraphy is an important form of art in Islamic culture and is often used in conjunction with geometric patterns to create beautiful and expressive images. The facade of Khoja Ahmet

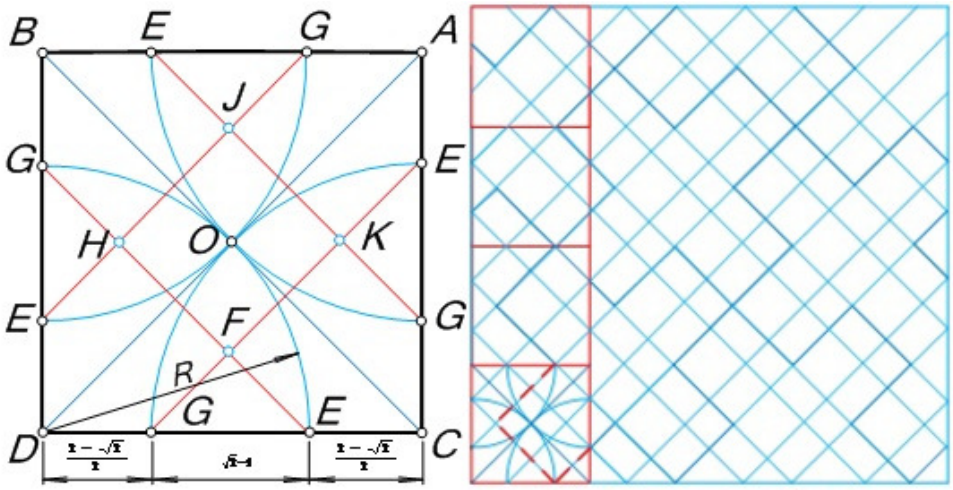


FIGURE 3. Drawing of the side facade of the building

Yasawi mausoleum is decorated with thoughtful calligraphy combined with geometric patterns, which gives a sense of the unity of the two art forms.

The geometric pattern in Figure 3 is used on the side facade of the Khoja Ahmet Yasawi mausoleum, where calligraphy is used to create a border around the geometric patterns, as well as to add text to the overall design.

Symmetry is an important aspect of Islamic art and is often used to create balance and harmony in design. On the side facade of the mausoleum, the geometric patterns are symmetrical, the same patterns and shapes are repeated on both sides of the central axis.

The left facade of the building faces southwest. It has a square side facade on the right side (Fig. 4). The facade has a plinth and baskuri and an entrance door. On the left wall, two rows of twelve-pointed cross-shaped geometric elements are placed on a slope made of dark blue bricks. If we consider the outline of this cross as a line, then the sides of the cross-shaped polygon give the swastika. In addition, a swastika with wings is woven into a cross-shaped polygon with dark blue bricks. And inside this geometric element, the words "Allah", "Muhammad", "O Creator", "Power belongs to God", "Allah is my ruler" are written in the "kufa" script of light blue bricks (Fig 4).

Using the geometric pattern on the left side of Figure 4, let's create a mosaic of the left facade. For this, we first connect the point G located on the AC side of the square ABCD and the point DC E of the square. Next, we connect this point E to the side FH of the small square. We connect the vertex H of the side FH and the side HJ of the smaller square and the point G on the side AB of the larger square. The resulting half-square can be symmetrical to side AB to produce a second square.

Use of light and shadow The intricate geometric patterns on the side facade of the Khoja Ahmet Yasawi mausoleum create a play of light and shadow that changes throughout the day. This effect adds overall beauty and complexity to the design and creates a sense of movement and dynamism in a static facade.

Let's look at the geometric regularity of the geometric pattern of the foundation mosaic of the mausoleum of Khoja Ahmet Yasawi (Fig. 5). This plinth mosaic features a twelve-pointed square geometric figure. A feature of a polygon is that eight sides are the same length, and the other four angles are approximately four times shorter. Inside these twelve corners, that is, in the center, there is a star-shaped octagon. The four acute angles of this polygon are equal to 72° degrees. To construct this foundation mosaic, we construct a geometric model using one of al-Faradi's four methods of dividing the equilateral rectangle in Figure 1 above into an equilateral octagon.

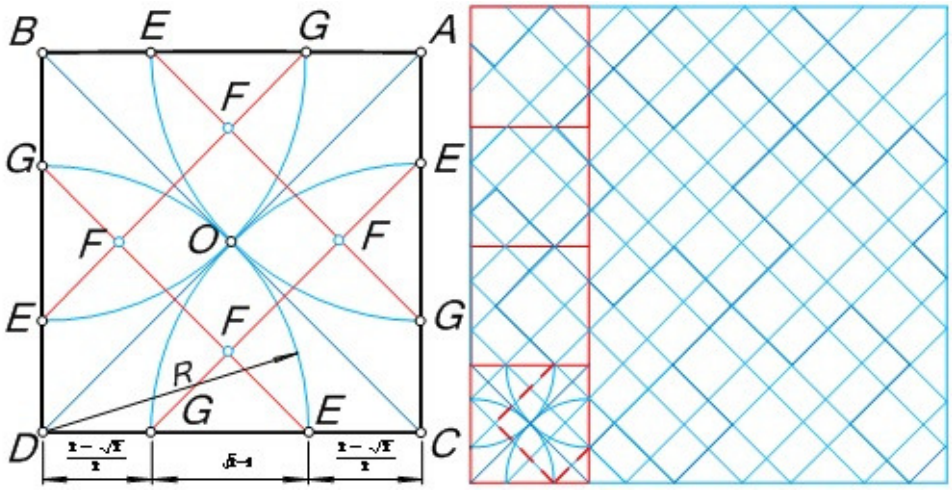


FIGURE 4. Mosaic on the left fañade

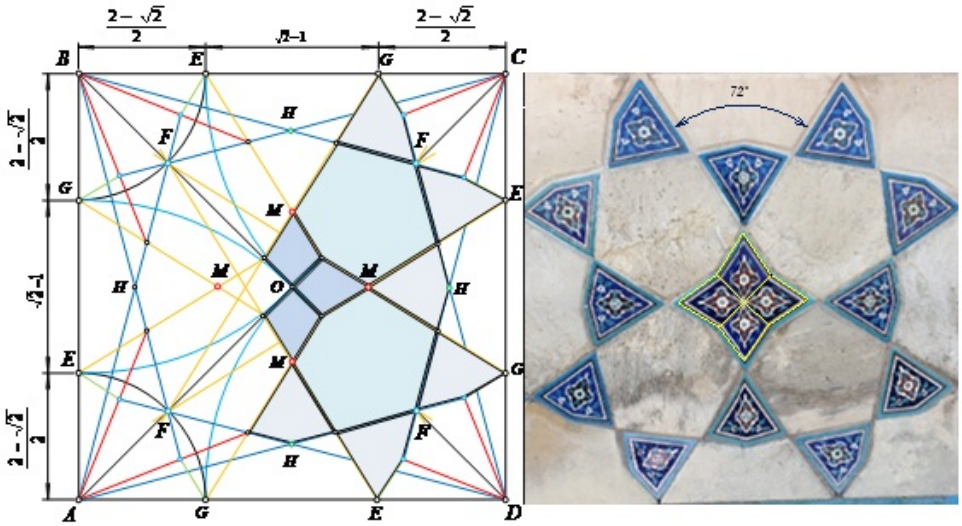


FIGURE 5. An equilateral octagon on the foundation

If we connect the points E and G of these sides with an arc, then the diagonals of the square intersect at four points F. If we connect these found points with the vertices of the square ABCD, the lines AF and BF intersect and point H. Thus we find three more points H by connecting the lines BF and FC, CF and DF, DF and AF. By connecting four points H and four points F, we get an equilateral octagon from the square ABCD.

From the four points F of this equilateral octagon, we connect the points M with the points E and G located on opposite sides of the outer square ABCD. If we connect these three points, we will find four equilateral triangles inside the square. We connect the found M points with the K points along the diagonal and find four stars (Fig. 5). This star divides an equilateral octagon into four equal hexagons and four quadrilaterals. We find a way to draw eight rectangles outside the equilateral octagon as shown in Figure 5.

First, we draw the initial square ABCD. We connect the vertices AC and BD of the square, draw the diagonals and determine the point O from the point of intersection. From the point O, we draw arcs of radius R cutting the sides of the square ABCD and determine four points E and four G.

These points divide the sides of the square ABCD into three sections:

$$DG = \frac{2-\sqrt{2}}{2}; \quad GE = \sqrt{2}-1; \quad EC = \frac{2-\sqrt{2}}{2}$$

If we connect the points E and G of these sides with an arc, then the diagonals of the square intersect at four points F. If we connect these found points with the vertices of the square ABCD, the lines AF and BF intersect and point H. Thus we find three more points H by connecting the lines BF and FC, CF and DF, DF and AF. By connecting four points H and four points F, we get an equilateral octagon from the square ABCD.

From the four points F of this equilateral octagon, we connect the points M with the points E and G located on opposite sides of the outer square ABCD. If we connect these three points, we will find four equilateral triangles inside the square. We connect the found M points with the K points along the diagonal and find four stars (Fig. 5). This star divides an equilateral octagon into four equal hexagons and four quadrilaterals. We find a way to draw eight rectangles outside the equilateral octagon as shown in Figure 5.

4. Conclusion

The great medieval encyclopedists considered harmony, balance, sobriety as an aesthetic principle and spread it to practical art. The masterly geometrical methods declared by al-Farabi to be the basis of architecture have become a system of leading ideas of architecture, which are of great importance in the formation and development of Eastern architecture. Correspondence of the proportions of the buildings with one or another module belongs to the geometric methods of coordination of spatial architectural objects of this era. Therefore, in this era, the geometric concept won in the theory and practice of architects. This can be seen when considering the proportionality of a number of architectural monuments and theoretical treatises of the encyclopedist scientists of that era.

The facade of Khoja Ahmet Yasawi mausoleum is an excellent example of Islamic geometric design and architectural ornamentation. Its use of complex patterns, symmetrical plans and color inlays, as well as the incorporation of natural elements, light and shadow, reflect the skill of the craftsmen who created it and reflect the rich cultural and artistic heritage of Islam in Central Asia.

At the same time, it was found that the masters who built the mausoleum of Khoja Ahmet Yasawi drew the visible dimensions of the building based on geometric figures. It was found that the geometric model of octagons and twelve corners and rhombuses was built on the side walls by different methods of dividing the square into polygons. Thanks to horizontal and vertical laying, bricks are laid with great precision in building geometric shapes. The main part of the building is designed to draw polygons on the foundation using mathematical methods. This shows that carpenters who built mausoleums in the Middle Ages knew well the laws of mathematics and the ways of constructing polygons.

References

[1] D. Reisman, Al-Farabi and the Philosophical.Adamson P., Taylor R. Cambridge: University press. 2005.

[2] Al-Farabi, Matematicheskij traktaty [A mathematical treatise].Almaty: Nauka. 1972. (In Russian).

[3] Kemelbekova E., Baidabekov A., Yeziyeva Nehir M., *Study of the Basic Geometric Dimensions of the Mausoleum of Hodja Ahmed Yassawi in Central Asia.* ICGG 2022 - Proceedings of the 20th International Conference on Geometry and Graphics. San Paulo: Brazil. (2023). P. 986-993.

[4] Al-Khorozmi Muhammad, *Matematicheskij traktaty [A mathematical treatise].* Tashkent: 1983. (In Russian).

[5] Baklanov N.B., *Geometricheskij ornament Srednej Azii i sposoby ego postroeniya [Geometric ornament of Central Asia and methods of its construction].* Moscow: (1947). Vol. 9. P. 101-121. (In Russian)

[6] Bulatov M.S. , *On Geometricheskaya garmoniya v arhitekture Srednej Azii IX-XV vv. [Geometric harmony in the architecture of Central Asia in the IX-XV centuries].* Moscow: Nauka. 1978. (In Russian).

[7] Bulgakov P.G., Rosenfeld B.A., Akhmedov A.A., *Muhammad al'-Horezmi, ok. 783 - ok. 850. [Muhammad al-Khwarizmi, c. 783 - c. 850.]* Moscow: Nauka. 1983. (In Russian).

[8] Gaganov G.I, *Geometricheskij ornament Srednej Azii [Geometric ornament of Central Asia]. Architectural heritage.* Moscow. (1958). Sat. 11. P. 181-208. (In Russian).
[9] Kubesov A.K., *Matematicheskoe nasledie al'-Farabi [Mathematical legacy of al-Farabi].* Alma-Ata: 1974. (In Russian).
[10] Mankovskaya L.Yu., *Issledovanie arhitekturnogo kompleksa-mavzoleya Ahmada YAsavi v gorode Turkestan i voprosy ego restavracii [Study of the architectural complex-mausoleum of Ahmad Yasavi in the city of Turkestan and questions of its restoration].* Tashkent: 1963. (In Russian).
[11] Margulan A.H., *Iz istorii gorodov i arhitekturnogo iskusstva drevnego Kazahstana [From the history of cities and architectural art of ancient Kazakhstan].* Alma-Ata: AN KazSSR. 1950. (In Russian).
[12] Rempel LI., *Arhitekturnyj ornament Uzbekistan. Istoriya razvitiya i teoriya stroitel'stva [Architectural ornament Uzbekistan. History of development and theory of construction].* Tashkent. 1961.(In Russian).
[13] Resenfeld B.A., Yushkevich A.P. , *Matematika Blizhnego i Srednego Vostoka v srednie veka [Mathematics of the Middle and Middle East in the Middle Ages].* Sverdlovsk: 1958. No. 3. (In Russian).
[14] Sirajdinov S.H., Matvievskaya G.P. , *Al'-Horezmi vydayushchijsya matematik i astronom srednevekov'ya [Al Khwarizmi is an outstanding mathematician and astronomer of the Middle Ages].* Moscow: Enlightenment. 1983. (In Russian).

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON GENERALIZED QUASI-TYPE RECTIFYING CURVES

BAHAR DOĞAN YAZICI[1], MURAT TOSUN[2]

[1]Department of Mathematics, Faculty of Science, Bilecik Şeyh Edebali University, Bilecik, Turkey
[2]Department of Mathematics, Faculty of Science, Sakarya University, Sakarya, Turkey
e-mail: bahar.dogan@bilecik.edu.tr

Abstract. In this paper, we introduce quasi-type rectifying curves in Myller configuration. Firstly, we give the necessary and sufficient conditions for a curve in the Myller configuration to be a quasi- type rectifying curve. We obtain relationships between quasi-curvatures of quasi-type rectifying curves. Next, we state and prove some theorems according to special cases of quasi-curvatures by using the quasi-type frame. Finally, we express special cases corresponding to some selections of invariants given in Myller configuration.

Keywords: Quasi-type rectifying curves, Myller configuration, quasi-type frame, versor field.

AMS Subject Classification: 53A04

1. Introduction

In geometry, it is quite common to examine special types of curves according to different frames in various spaces. Basically, the Frenet frame has a large working area [6]. In recent years, new frames have been defined where the Frenet frame has a disadvantage. One of them is the quasi frame. Quasi frame or q-frame can be installed even where the curvature is zero [5]. In differential geometry, special types of curves called rectifying curves, osculating curves and normal curves are important [2, 9, 10]. Rectifying curves are introduced by Chen as curves whose position vector always lies in the rectifying plane [2]. Rectifying curves are studied in various spaces according to the Frenet frame, according to the Bishop frame, and finally according to the quasi frame [3, 8].

In Euclidean space, T is a unit tangent vector field, N is a unit normal vector field, and B is a unit binormal vector field. The plane formed by T and N is called the osculating plane, the plane by N and B is called the normal plane, and the plane by T and B is called the rectifying plane. The Myller configuration structure is a structure that generalizes these concepts. In Myller configuration, it is most commonly supplied with a versor field (C, Ψ) and a plane field (C, π). If $\Psi \in \pi$, the mate $\{(C, \Psi), (C, \pi)\}$ is called a Myller configuration and denoted by $\mathcal{M}(C, \overline{\Psi}, \pi)$. If the planes π are tangent to C, it is called a tangent Myller configuration $\mathcal{M}_t(C, \overline{\Psi}, \pi)$ [16]. On the other hand, by using the normal of the plane π, the normal versor field was defined and its geometry was examined. It is known in differential geometry that the first derivative of the unit speed curve is related to the tangent of the curve. In the Myller configuration, the position vector of the curve C is $\overline{r}(s)$, while the structure $\overline{r}'(s)$ is a linear combination of versor fields [16, 4, 17]. Therefore, in some special cases, the case in Euclidean space is obtained. In the Myller configuration, the Frenet-type frame is established and the fundamental theorem of curvature is expressed using curvatures and invariants. In [14, 11, 12, 15], there are studies such as rectifying curves, osculating curves, Bertrand and Mannheim curves in 3-dimensional and 4-dimensional space in Myller configuration. In [7], quasi-type frame is

defined in Myller configuration and quasi-type osculating curves are investigated.

In this study, we introduce quasi-type rectifying curves in Myller configuration. One of the best results obtained is that quasi-type rectifying curves are a general form of rectifying curves studied according to different frames in various spaces. Firstly, we define general characterizations of quasi-type rectifying curves. Then, we explain some theorems according to the special cases of quasi-curvatures.

2. Preliminaries

In this section, we give some basic definitions and theorems of Frenet-type frame and quasi-type frame in Myller configuration.
Let us consider a versor field $(C, \overline{\Psi})$. If $\overline{\Psi} = \overline{\Psi}_1$, the Frenet-type frame is denoted by

$$\begin{pmatrix} \overline{\Psi}_1'(s) \\ \overline{\Psi}_2'(s) \\ \overline{\Psi}_3'(s) \end{pmatrix} = \begin{pmatrix} 0 & K_1(s) & 0 \\ -K_1(s) & 0 & K_2(s) \\ 0 & -K_2(s) & 0 \end{pmatrix} \begin{pmatrix} \overline{\Psi}_1(s) \\ \overline{\Psi}_2(s) \\ \overline{\Psi}_3(s) \end{pmatrix} \tag{1}$$

and

$$\frac{d\overline{r}(s)}{ds} = a_1(s)\overline{\Psi}_1(s) + a_2(s)\overline{\Psi}_2(s) + a_3(s)\overline{\Psi}_3(s) \tag{2}$$

where $\overline{r}(s)$ is a position vector of C in Myller configuration. It is clear that, if $a_1 = 1$, $a_2 = 0$ and $a_3 = 0$, the classic Frenet frame is obtained in Euclidean space [16].
Let us assume that $\overline{\Psi}_1 = \Psi_1^* = \overline{\Psi}$. Consequently, we define versor fields as

$$\Psi_1^* = \overline{\Psi}_1, \quad \Psi_2^* = \frac{\Psi_1^* \wedge \sigma}{\|\Psi_1^* \wedge \sigma\|}, \quad \Psi_3^* = \Psi_1^* \wedge \Psi_2^* \tag{3}$$

where σ is the projection vector of the form $\sigma = (1,0,0)$, $\sigma = (0,1,0)$ or $\sigma = (0,0,1)$. By according to (3), if Ψ_1^* and σ are parallel, the quasi-type frame becomes singular. Then, the projection vector can be chosen to eliminate this situation. Now, we give relationships between Frenet-type frame and quasi-type frame in Myller configuration:

Theorem 2.1. [7] *Let $(C, \overline{\Psi})$ be a versor field. By considering $\overline{\Psi}_1 = \Psi_1^* = \overline{\Psi}$, the derivative formulas for the quasi-type frame $\{\Psi_1^*, \Psi_2^*, \Psi_3^*\}$ are*

$$\begin{pmatrix} \Psi_1^{*\prime}(s) \\ \Psi_2^{*\prime}(s) \\ \Psi_3^{*\prime}(s) \end{pmatrix} = \begin{pmatrix} 0 & K_1^*(s) & K_2^*(s) \\ -K_1^*(s) & 0 & K_3^*(s) \\ -K_2^*(s) & -K_3^*(s) & 0 \end{pmatrix} \begin{pmatrix} \Psi_1^*(s) \\ \Psi_2^*(s) \\ \Psi_3^*(s) \end{pmatrix} \tag{4}$$

and

$$\frac{d\overline{r}}{ds} = a_1^*(s)\Psi_1^*(s) + a_2^*(s)\Psi_2^*(s) + a_3^*(s)\Psi_3^*(s) \tag{5}$$

where

$$\begin{aligned} K_1^*(s) &= K_1(s)\cos\theta(s), \\ K_2^*(s) &= -K_1(s)\sin\theta(s), \\ K_3^*(s) &= K_2(s) + \theta'(s), \\ a_1^*(s) &= a_1(s), \\ a_2^*(s) &= a_2(s)\cos\theta(s) + a_3(s)\sin\theta(s), \\ a_3^*(s) &= -a_2(s)\sin\theta(s) + a_3(s)\cos\theta(s) \end{aligned}$$

In the Myller configuration, the quasi-type frame contains several frames. The following table describes these exceptions:

Special Cases	$a_1^*(s)$	$a_2^*(s)$	$a_3^*(s)$	$K_1^*(s)$	$K_2^*(s)$	$K_3^*(s)$
Classical quasi frame in Euclidean space	1	0	0	$K_1^*(s)$	$K_2^*(s)$	$K_3^*(s)$
Bishop-type frame in Myller configuration	$a_1^*(s)$	$a_2^*(s)$	$a_3^*(s)$	$K_1^*(s)$	$K_2^*(s)$	0
Classical Bishop frame in Euclidean space	1	0	0	$K_1^*(s)$	$K_2^*(s)$	0
Frenet-type frame in Myller configuration	$a_1^*(s)$	$a_2^*(s)$	$a_3^*(s)$	$K_1(s)$	0	$K_2(s)$
Classical Frenet frame in Euclidean space	1	0	0	$K_1(s)$	0	$K_2(s)$

3. Quasi-Type Rectifying Curves in Myller Configuration

In this section, we define quasi-type rectifying curves in Myller configuration. We examine some relations between its quasi curvatures and quasi invariants.

Definition 3.1. *Let $\overline{r}(s)$ is a curve with quasi-type frame $\{\Psi_1^*, \Psi_2^*, \Psi_3^*\}$. The $\overline{r}(s)$ is called a generalized quasi-type rectifying curves if*

$$\overline{r}(s) = \mu(s)\Psi_1^*(s) + \eta(s)\Psi_3^*(s) \tag{6}$$

where μ, η are functions.

Let us now give theorems describing the necessary and sufficient conditions for a curve given with a quasi-type frame to be a generalized quasi-type rectifying curve:

Theorem 3.1. *Let $\overline{r}(s)$ is a curve with quasi-type frame $\{\Psi_1^*, \Psi_2^*, \Psi_3^*\}$. The $\overline{r}(s)$ is generalized quasi-type rectifying curve if*

$$\mu^2(s) + \eta^2(s) = 2\int (\mu(s)a_1^*(s) + \eta(s)a_3^*(s))ds \tag{7}$$

and

$$\mu(s)K_1^*(s) - \eta(s)K_3^*(s) = a_2^*(s). \tag{8}$$

Conversely, if $\overline{r}(s)$ satisfies equations (7) and (8), $\overline{r}(s)$ is either a quasi-type rectifying curve or a curve with $a_2^* = 0$.

Proof. Assume that $\overline{r}(s)$ is generalized quasi-type rectifying curve. Then, by differentiating equation (6), we have

$$\begin{aligned} \mu^{'}(s) - \eta(s)K_2^*(s) &= a_1^*(s), & (9)\\ \eta^{'}(s) + \mu(s)K_2^*(s) &= a_3^*(s), & (10)\\ \mu(s)K_1^*(s) - \eta(s)K_3^*(s) &= a_2^*(s). & (11) \end{aligned}$$

By multiplying the equation (9) by $\mu(s)$ and by multiplying the equation (10) by $\eta(s)$ and by adding, we get

$$\mu(s)\mu^{'}(s) + \eta(s)\eta^{'}(s) = \mu(s)a_1^*(s) + \eta(s)a_3^*(s). \tag{12}$$

By integrating the equation (12) , we can write

$$\mu^2(s) + \eta^2(s) = 2\int (\mu(s)a_1^*(s) + \eta(s)a_3^*(s))ds$$

Also, from equation (11), we have

$$\mu(s)K_1^*(s) - \eta(s)K_3^*(s) = a_2^*(s).$$

Conversely, if $\overline{r}(s)$ satisfies equations (7) and (8), we have from the differential of equation (7),

$$\mu(s)\mu^{'}(s) + \eta(s)\eta^{'}(s) = \mu(s)a_1^*(s) + \eta(s)a_3^*(s). \tag{13}$$

Since

$$\mu(s) = \langle \overline{r}(s), \Psi_1^*(s)\rangle, \quad \mu^{'}(s) = a_1^*(s) + \langle \overline{r}(s), K_1^*(s)\Psi_2^*(s) + K_2^*(s)\Psi_3^*(s)\rangle \tag{14}$$

$$\eta(s) = \langle \overline{r}(s), \Psi_3^*(s)\rangle, \quad \eta^{'}(s) = a_3^*(s) + \langle \overline{r}(s), -K_2^*(s)\Psi_1^*(s) - K_3^*(s)\Psi_2^*(s)\rangle \tag{15}$$

By using equations (13) and (14), we get

$$\langle \overline{r}(s), \Psi_2^*(s)\rangle (K_1^*(s)\langle \overline{r}(s), \Psi_1^*(s)\rangle - K_3^*(s)\langle \overline{r}(s), \Psi_3^*(s)\rangle) = 0$$

Consequently, $\langle \overline{r}(s), \Psi_2^*(s)\rangle = 0$ or $K_1^*(s)\langle \overline{r}(s), \Psi_1^*(s)\rangle - K_3^*(s)\langle \overline{r}(s), \Psi_3^*(s)\rangle = 0$. If $\langle \overline{r}(s), \Psi_2^*(s)\rangle = 0$, $\overline{r}(s)$ is a generalized quasi-type rectifying curve. If $K_1^*(s)\langle \overline{r}(s), \Psi_1^*(s)\rangle - K_3^*(s)\langle \overline{r}(s), \Psi_3^*(s)\rangle = 0$, from equation (8), we get $a_2^* = 0$. □

Corollary 3.2. *Let $\overline{r}(s)$ is a curve with quasi-type frame $\{\Psi_1^*, \Psi_2^*, \Psi_3^*\}$. If $\overline{r}(s)$ is a generalized quasi-type rectifying curve, the relationship between quasi curvatures are given by*

$$K_2^*(s) = \frac{\mu^{'}(s)a_3^*(s) - \eta^{'}(s)a_1^*(s)}{\mu(s)\mu^{'}(s) + \eta(s)\eta^{'}(s)},$$

$$\mu(s)K_1^*(s) - \eta(s)K_3^*(s) = a_2^*(s).$$

Corollary 3.3. *Let us assume that $a_1^* = 1$, $a_2^* = a_3^* = 0$ where $K_1^* > 0$. Then, by using equations (9), (10) and (11), we get*

$$\overline{r}(s) = \left(\int 1 + K_2^*(s) e^{-\int \frac{K_2^*(s)K_3^*(s)}{K_1^*(s)}} \right) \Psi_1^*(s) + e^{-\int \frac{K_2^*(s)K_3^*(s)}{K_1^*(s)}} \Psi_3^*(s). \tag{16}$$

From this it can be said that equation (16) is a characterization of rectifying curves in Euclidean space according to the classical quasi frame [8].

Now, let us give the following theorems according to the special cases of quasi curvatures:

Theorem 3.4. *Let $\overline{r}(s)$ is a curve with quasi-type frame $\{\Psi_1^*, \Psi_2^*, \Psi_3^*\}$ and $K_1^* = 0, K_2^* \neq 0, K_3^* \neq 0$. $\overline{r}(s)$ is a generalized quasi-type rectifying curve if and only if*

$$\left(\frac{a_3^*(s)}{K_2^*(s)} + \frac{1}{K_2^*(s)} \left(\frac{a_2^*(s)}{K_3^*(s)} \right)^{'} \right)^{'} - \frac{a_2^*(s)K_2^*(s)}{K_3^*(s)} = a_1^*(s). \tag{17}$$

Proof. Assume that $\overline{r}(s)$ is a generalized quasi-type rectifying curve where $K_1^* = 0, K_2^* \neq 0, K_3^* \neq 0$. By using equations (10) and (11), we get

$$\mu(s) = \frac{a_3^*(s)}{K_2^*(s)} + \frac{1}{K_2^*(s)} \left(\frac{a_2^*(s)}{K_3^*(s)} \right)^{'}, \quad \eta(s) = -\frac{a_2^*(s)}{K_3^*(s)}.$$

If equation (9) is used, we have equation (17). Conversely, assume that (17) is provided where $K_1^* = 0, K_2^* \neq 0, K_3^* \neq 0$. Hence, it can be easily calculated that the equation

$$\frac{d}{ds} \left[\overline{r}(s) - \left(\frac{a_3^*(s)}{K_2^*(s)} + \frac{1}{K_2^*(s)} \left(\frac{a_2^*(s)}{K_3^*(s)} \right)^{'} \right) \Psi_1^*(s) + \frac{a_2^*(s)}{K_3^*(s)} \Psi_3^*(s) \right]$$

is equal to zero. Therefore, $\overline{r}(s)$ is a generalized quasi-type rectifying curve. □

Remark 3.1. *When $a_1^* = 1$ and $a_2^* = a_3^* = 0$, equation (17) gives the contradiction $0 = 1$. Thus, the case of K_1^* is not a characterization of rectifying curves in Euclidean space with respect to the quasi frame.*

Theorem 3.5. *Let $\overline{r}(s)$ is a curve with quasi-type frame $\{\Psi_1^*, \Psi_2^*, \Psi_3^*\}$ where $K_2^* = 0, K_1^* \neq 0, K_3^* \neq 0$. $\overline{r}(s)$ is a generalized quasi-type rectifying curve if and only if*

$$K_1^*(s) \int a_1^*(s)ds - K_3^*(s) \int a_3^*(s)ds = a_2^*(s). \tag{18}$$

Proof. Let us assume that that $\overline{r}(s)$ is a generalized quasi-type rectifying curve where $K_2^* = 0, K_1^* \neq 0, K_3^* \neq 0$. From equations (9) and (10), we get

$$\mu(s) = \int a_1^*(s)ds, \quad \eta(s) = \int a_3^*(s)ds. \tag{19}$$

By using equation (11), we obtain equation (18). Conversely, suppose that equation (18) be satisfied with $K_2^* = 0, K_1^* \neq 0, K_3^* \neq 0$. From equation (18), since

$$\frac{d}{ds}\left[\overline{r}(s) - \left(\int a_1^*(s)ds\right)\Psi_1^*(s) - \left(\int a_1^*(s)ds\right)\Psi_3^*(s)\right] = 0.$$

Consequently, $\overline{r}(s)$ is a generalized quasi-type rectifying curve. □

Corollary 3.6. *Considering $K_2^* = 0$, $K_1^*(s) = K_1(s)$ and $K_3^*(s) = K_2(s)$, the equation (18) is a representation of rectifying-type curves in Myller configuration with Frenet-type frame* [14].

Corollary 3.7. *If $a_1^* = 1$, $a_2^* = a_3^* = 0$ where $K_2^* = 0$, the equation (18) is written by*

$$\frac{K_3^*(s)}{K_1^*(s)} = \frac{\tau(s)}{\kappa(s)} = as + b$$

where a, b are constants. This corresponds to the relationship between the curvatures of classical rectifying curves in Euclidean space [2].

Theorem 3.8. *Let $\overline{r}(s)$ is a curve with quasi-type frame $\{\Psi_1^*, \Psi_2^*, \Psi_3^*\}$ where $K_3^* = 0, K_1^* \neq 0, K_2^* \neq 0$. Then, $\overline{r}(s)$ is a generalized quasi-type rectifying curve if and only if*

$$\left(-\frac{a_1^*(s)}{K_2^*(s)} + \frac{1}{K_2^*(s)}\left(\frac{a_2^*(s)}{K_1^*(s)}\right)'\right)' + \frac{a_2^*(s)K_2^*(s)}{K_1^*(s)} = a_3^*(s). \tag{20}$$

Proof. Let us suppose that $\overline{r}(s)$ is a generalized quasi-type rectifying curve with $K_3^* = 0, K_1^* \neq 0, K_2^* \neq 0$. From equations (9) and (11), we get

$$\mu(s) = \frac{a_2^*(s)}{K_1^*(s)}, \quad \eta(s) = -\frac{a_1^*(s)}{K_2^*(s)} + \frac{1}{K_2^*(s)}\left(\frac{a_2^*(s)}{K_1^*(s)}\right)'.$$

By according to equation (10), we have equation (20). Conversely, let us suppose that (20) is provided where $K_3^* = 0, K_1^* \neq 0, K_2^* \neq 0$. From equation (20), we can write

$$\frac{d}{ds}\left[\overline{r}(s) - \frac{a_2^*(s)}{K_1^*(s)}\Psi_1^*(s) - \left(-\frac{a_1^*(s)}{K_2^*(s)} + \frac{1}{K_2^*(s)}\left(\frac{a_2^*(s)}{K_1^*(s)}\right)'\right)\Psi_3^*(s)\right] = 0.$$

Then, $\overline{r}(s)$ is a generalized quasi-type rectifying curve. □

Corollary 3.9. *Since $K_3^* = 0$, $K_1^* \neq 0$ and $K_2^* \neq 0$, the equation (20) is a characterization of rectifying curves in Myller configuration with Bishop-type frame* [13].

Corollary 3.10. *If $a_1^* = 1$, $a_2^* = a_3^* = 0$ when $K_3^* = 0$, by according to equation (20), we get equation K_2^* =constant. This is a representation of rectifying curves in Euclidean space according to the classical Bishop frame* [1].

References

[1] Bishop R. L., There is more than one way to frame a curve, *Amer. Math. Monthly*, Vol.82, 1975, pp.246-251.

[2] Chen B. Y., When does the position vector of a space curve always lie in its rectifying plane?, *Amer. Math. Monthly*, Vol.110, No.2, 2003, pp.147-152.

[3] Chen B.Y., Dillen F., Rectifying curves as centrodes and extremal curves, *Bull. Inst. Math. Academia Sinica*, Vol.33, No.2, 2005, pp.77-90.

[4] Constantinescu O., Myller configurations in Finsler spaces, *Differential Geometry-Dynamical Systems*, Vol.8, 2006, pp.69-76.

[5] Dede M., Ekici C., Görgülü A., Directional q-frame along a space curve, *IJARCSSE*, Vol.5, No.12, 2015, 775-780.

[6] do Carmo P. M., *Differential geometry of curves and surfaces*, Prentice-Hall, Englewood Cliffs, New York, 1976.

[7] Doğan Yazıcı B., Tosun M., Quasi-type frame and quasi-type osculating curves in Myller configuration, 2023 (submitted).

[8] Erdoğan Doğan A., q-frame and its geometrical applications, Master Thesis, 2020.

[9] İlarslan, K., Nešović, E., Some characterizations of osculating curves in the Euclidean spaces, *Demonstratio Mathematica*, Vol.41, No.4, 2008, 931-940.

[10] İlarslan, K., Spacelike normal curves in Minkowski space E_1^3, *Turkish Journal of Mathematics*, Vol.29, No.1, 2005, 53-63.

[11] İşbilir Z., Tosun M., On generalized osculating-type curves in Myller configuration, *An. St. Univ. Ovidius Constanta, Ser. Mat.* 2023 (accepted).

[12] İşbilir Z., Tosun M., An extended framework for osculating-type curves in 4-dimensional space, 2023 (submitted).

[13] Keskin Ö., Yaylı, Y., Rectifying-type curves and rotation minimizing frame R_n, arXiv preprint, 2019, arXiv:1905.04540.

[14] Macsim, G., Mihai, A., Olteanu, A., On rectifying-type curves in a Myller configuration, *Bulletin of the Korean Mathematical Society*, Vol.56, No.2, 2019, 383-390.

[15] Macsim G. F., Mihai A., Olteanu A., Special curves in a Myller configuration, *Proceedings of the 16th Workshop on Mathematics, Computer Science and Technical Education, Department of Mathematics and Computer Science*, Vol. 2, 2019.

[16] Miron, R., *The Geometry of Myller Configurations, Applications to Theory of Surfaces and Nonholonomic Manifolds*, Romanian Academy, 2010.

[17] Miron R. Myller configurations and Vranceanu nonholonomic manifolds, *Scientific Studies and Research*, Vol.21, No.1, 2011.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON INNER AND LOCAL INNER DERIVATIONS OF FOUR-DIMENSIONAL JORDAN ALGEBRAS

FARHODJON ARZIKULOV[1], FURKAT URINBOYEV[2]

[1]V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan
[2]Namangan State University, Namangan, Uzbekistan
e-mail: arzikulovfn@rambler.ru, furqatjonforever@gmail.com

Abstract. In the present paper, properties of inner derivations and local, 2-local inner derivations of four-dimensional Jordan algebras are studied. The main goal of the work is to construct matrices of inner derivations and local inner derivations of four-dimensional Jordan algebras, and to give a characterization of inner derivations in terms of 2-local derivations.

Keywords: Jordan algebra, inner derivation, local inner derivation, 2-local inner derivation.

AMS Subject Classification: 16W25, 46L57, 47B47, 17C65

1. Introduction

Jordan algebras are classical algebras widely studied by specialists. Jordan algebras turned out to be related to other classical algebras such as associative algebras and Lie algebras. The classification of finite-dimensional Jordan algebras is one of the primary areas of modern algebra and, as is known, was first studied by P. Jordan, J. von Neumann and E. Wigner [10]. The algebraic and geometric classifications of 3- and 4-dimensional Jordan algebras were constructed by I. Kashuba and M.E. Martin in [12], M.E. Martin in [16], I. Kashuba and I. Shestakov in [13].

In this paper, we study local derivations of four-dimensional Jordan algebras. The concept of local derivations was introduced by R. Kadison [11] and D. Larson, A. Sourour [17] independently in 1990. A linear mapping ∇ of an algebra A into itself is a local derivation if, for each a in A, there exists a derivation D_a on A such that $D_a(a) = \nabla(a)$. R. Kadison proves that every continuous local derivation of a von Neumann algebra into its dual Banach the bimodule is a derivation. B. Johnson [9] generalizes the above result by proving that each a local derivation of a C*-algebra into its Banach bimodule is a derivation. The study of local derivations of non-associative algebras began with the work [8] for the case of Jordan algebras. They proved that every bounded local triple derivation on a JB*-triple is a triple derivation. By Theorem 2.4 in [8] every local derivation on a JB-algebra is a derivation. In particular, every local derivation on a finite dimensional semisimple Jordan algebra is a derivation. The study of local derivations of Lie algebras began with the work [6]. They proved that every local derivation on a semisimple finite-dimensional Lie algebra is a derivation. The inner and local inner derivations of two- and three-dimensional Jordan algebras are given in [7]. In [7] the second author proved that every local inner derivation on two- and three-dimensional Jordan algebras is an inner derivation.

In this paper, we also study 2-local derivations of four-dimensional Jordan algebras. In 1997,P. Šemrl [19] introduced and investigated so-called 2-local derivations and 2-local automorphisms on operator algebras. He described such maps on the algebra $B(H)$ of all bounded linear

operators on an infinite-dimensional separable Hilbert space H. Namely he proved that every 2-local derivation (automorphism) on $B(H)$ is a derivation (respectively an automorphism).

A similar description of 2-local derivations for the finite-dimensional case appeared later in [15]. In the paper [18] 2-local derivations have been described on matrix algebras over finite-dimensional division rings. In [4] Sh. Ayupov and K. Kudaybergenov generalized the above mentioned results of [19] and [15] for arbitrary Hilbert spaces. Namely they proved that every 2-local derivation on the algebra $B(H)$ of all linear bounded operators on an arbitrary Hilbert space H is a derivation. A similar result is obtained for automorphisms by them. In [1, 5] the authors extended the above results for 2-local derivations and give a proof of the theorem for arbitrary von Neumann algebras.

In the present paper we constructed the matrices of inner derivations and local inner derivations of four-dimensional Jordan algebras and described 2-local inner derivations of these algebras.

2. Preliminary information

In this section, we provide the necessary terminology and notations that will be needed further in this paper.

Definition 2.1. A Jordan algebra $\mathcal{J}$ is a vector space over a field $\mathbb{F}$ equipped with a bilinear operator $\cdot : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$, satisfying the following identities:

$$x \cdot y = y \cdot x, \ \ (x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x) \tag{1}$$

for any x, $y \in \mathcal{J}$. An important special case of mapping derivation is the so-called "inner derivation". In this section, we introduce the notion of inner derivations of Jordan algebras and give their properties.

Let $\mathcal{J}$ be a Jordan algebra. Recall that a linear map $D : \mathcal{J} \to \mathcal{J}$ is called a derivation if $D(xy) = D(x)y + xD(y)$ for any two elements x, $y \in \mathcal{J}$.

Given subsets B and C of the Lie algebra with bracket $[\cdot, \cdot]$, denote by $[B, C]$ the set of all finite sums of elements $[b, c]$, where $b \in B$ and $c \in C$.

Consider the Jordan algebra $\mathcal{J}$ and let $m = \{xM : x \in \mathcal{J}\}$, where xM denotes the multiplication operator defined by $(xM)y := x \cdot y$ for all x, $y \in \mathcal{J}$. Let $aut(\mathcal{J})$ denote the Lie ring of all derivations of $\Re$. The elements of the ideal $int(\mathcal{J}) := [m, m]$ of the algebra $aut(\Re)$ are called inner derivations of $\mathcal{J}$. In other words, derivation of D by $\mathcal{J}$ is called inner derivation if there exist elements a_1, a_2, ..., a_m, b_1, b_2, ..., b_m into $\mathcal{J}$ such that

$$D(x) = \sum_{k=1}^{m} [a_k(b_k x) - b_k(a_k x)], x \in \mathcal{J}.$$

A linear mapping $\nabla : \mathcal{J} \to \mathcal{J}$ is called a local inner derivation if for every $x \in \mathcal{J}$ there exists an inner derivation $D : \mathcal{J} \to \mathcal{J}$ such that $\nabla(x) = D(x)$.

A map $\Delta : \mathcal{J} \to \mathcal{J}$ is called a 2-local inner derivation, if for any two elements x, $y \in \mathcal{J}$ there exists an inner derivation $D_{x,y} : \mathcal{J} \to \mathcal{J}$ such that $\Delta(x) = D_{x,y}(x)$, $\Delta(y) = D_{x,y}(y)$.

Let $\mathcal{J}$ be a Jordan algebra of dimension four over an algebraically closed field $\mathbb{F}$ of characteristic $\neq 2$ with a basis $\{e_1, e_2, e_3, e_4\}$. Let x be an element in $\mathcal{J}$. Then $x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4$, for some elements x_1, x_2, x_3, x_4 in $\mathbb{F}$. Throughout of the paper let $\bar{x} = (x_1, x_2, x_3, x_4)^{Tr}$. Conversely, if $v = (x_1, x_2, x_3, x_4)^{Tr}$ is a column vector with x_1, x_2, x_3, x_4 in $\mathbb{F}$, then, throughout of the paper, by $\widehat{v}$ we will denote the element $x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4$, i.e.,

$$\widehat{v} = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4.$$

Let $D : \mathcal{J} \to \mathcal{J}$ be a linear operator. Then $D(x) = \widehat{A\bar{x}}, x \in \mathcal{J}$, for the matrix $A = (a_{i,j})_{i,j=1}^{4}$ of the linear operator D, where $x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4$. For example, if $\mathcal{J} = \mathcal{J}_2$ (see table 2) and D is a inner derivation on $\mathcal{J}$, then D has the following form

$$D(x) = (-\beta x_3 - \alpha x_4)e_1 + (-\beta x_3 - \alpha x_4)e_2 +$$

$$+(\alpha x_1 + \alpha x_2 + \gamma x_3)e_3 + (\beta x_1 + \beta x_2 - \gamma x_4)e_4$$

with respect to the basis $\{e_1, e_2, e_3, e_4\}$, for some α, β, γ in $\mathbb{F}$. This form of a inner derivation can be directly calculated, and we will omit the calculations of the forms of inner derivations on the Jordan algebras. Throughout this paper, let $E_{i,j}$ be a linear mapping of the algebra $\mathcal{J}$ with basis $\{e_1, \ldots, e_4\}$, which maps the basis element e_j to e_i and the rest basis elements to zero. Then, for example, the vector space of inner derivations of the algebra $\mathcal{J}_2$ admits a basis of the following form:

$$\{E_{3,1} + E_{3,2} - E_{1,4} - E_{2,4}, E_{4,1} + E_{4,2} - E_{1,3} - E_{2,3}, E_{3,3} - E_{4,4}\}.$$

Theorem 2.1. [14] *Let A be an indecomposable four-dimensional Jordan algebra. Then it is isomorphic to one of the following pairwise non-isomorphic Jordan algebras:*

Table 1
Four-dimensional Jordan algebras.

$\mathcal{J}$	Multiplication table	$\mathcal{J}$	Multiplication table
$\mathcal{J}_2$	$e_1e_4 = \frac{1}{2}(e_1 + e_2)$, $e_1e_3 = \frac{1}{2}e_3, e_1e_4 = \frac{1}{2}e_4$, $e_2e_3 = \frac{1}{2}e_3, e_2e_4 = \frac{1}{2}e_4$	$\mathcal{J}_{52}$	$e_1n_1 = \frac{1}{2}n_1$, $e_1n_2 = n_2, n_1^2 = n_3$
$\mathcal{J}_9$	$e_3^2 = e_1 + e_2$, $e_1e_3 = \frac{1}{2}e_3, e_2e_3 = \frac{1}{2}e_3$, $e_1n_1 = \frac{1}{2}n_1, e_2n_1 = \frac{1}{2}n_1$	$\mathcal{J}_{53}$	$e_1n_1 = \frac{1}{2}n_1$, $e_1n_2 = n_2, n_1^2 = n_2 + n_3$
$\mathcal{J}_{16}$	$e_1n_2 = \frac{1}{2}n_2$, $e_1n_1 = \frac{1}{2}n_1, e_2n_1 = \frac{1}{2}n_1$	$\mathcal{J}_{55}$	$e_1n_1 = n_1, e_1n_2 = n_2$, $e_1n_3 = \frac{1}{2}n_3, n_3^2 = n_2$
$\mathcal{J}_{17}$	$e_1n_2 = n_2$, $e_1n_1 = \frac{1}{2}n_1, e_2n_1 = \frac{1}{2}n_1$	$\mathcal{J}_{56}$	$e_1n_1 = n_1, e_1n_2 = n_2$, $e_1n_3 = \frac{1}{2}n_3, n_1^2 = n_2$
$\mathcal{J}_{18}$	$e_1n_1 = \frac{1}{2}n_1, e_1n_2 = \frac{1}{2}n_2$, $e_2n_1 = \frac{1}{2}n_1, e_2n_2 = \frac{1}{2}n_2$	$\mathcal{J}_{57}$	$e_1n_1 = n_1, e_1n_2 = n_2$, $e_1n_3 = \frac{1}{2}n_3, n_1^2 = n_2, n_3^2 = n_2$
$\mathcal{J}_{25}$	$n_1^2 = n_2, e_1n_2 = n_2$, $e_1n_1 = \frac{1}{2}n_1, e_2n_1 = \frac{1}{2}n_1$	$\mathcal{J}_{58}$	$e_1n_1 = n_1, e_1n_2 = \frac{1}{2}n_2$, $e_1n_3 = \frac{1}{2}n_3, n_3^2 = n_1$
$\mathcal{J}_{31}$	$e_1n_3 = \frac{1}{2}n_3$, $e_1n_1 = n_1, e_1n_2 = n_2$	$\mathcal{J}_{59}$	$e_1n_1 = n_1, e_1n_2 = \frac{1}{2}n_2$, $e_1n_3 = \frac{1}{2}n_3, n_3^2 = n_1, n_2n_3 = n_1$
$\mathcal{J}_{32}$	$e_1n_1 = \frac{1}{2}n_1$, $e_1n_2 = \frac{1}{2}n_2, e_1n_3 = n_3$	$\mathcal{J}_{60}$	$e_1n_1 = n_1, e_1n_2 = \frac{1}{2}n_2$, $e_1n_3 = \frac{1}{2}n_3, n_1n_2 = n_3$
$\mathcal{J}_{33}$	$e_1n_1 = \frac{1}{2}n_1$, $e_1n_2 = \frac{1}{2}n_2, e_1n_3 = \frac{1}{2}n_3$	$\mathcal{J}_{61}$	$n_1^2 = n_2, n_2^2 = n_4$, $n_1n_3 = n_4, n_1n_2 = n_3$
$\mathcal{J}_{36}$	$e_1n_1 = n_1$, $e_1n_2 = n_2, e_1n_3 = n_3$	$\mathcal{J}_{62}$	$n_1^2 = n_2, n_4^2 = n_2$, $n_1n_2 = n_3$
$\mathcal{J}_{39}$	$n_1^2 = n_2, n_1n_2 = n_3$, $e_1n_1 = n_1, e_1n_2 = n_2, e_1n_3 = n_3$	$\mathcal{J}_{63}$	$n_1^2 = n_2, n_4^2 = -n_2 - n_3$, $n_1n_2 = n_3, n_2n_4 = n_3$
$\mathcal{J}_{42}$	$n_1^2 = n_2, e_1n_1 = n_1$ $e_1n_2 = n_2, e_1n_3 = n_3$	$\mathcal{J}_{64}$	$n_1^2 = n_2, n_4^2 = -n_2$, $n_1n_2 = n_3, n_2n_4 = n_3$
$\mathcal{J}_{43}$	$n_1^2 = n_2, n_1n_3 = n_2$, $e_1n_1 = n_1, e_1n_2 = n_2, e_1n_3 = n_3$	$\mathcal{J}_{65}$	$n_1^2 = n_2$, $n_1n_2 = n_3, n_2n_4 = n_3$
$\mathcal{J}_{45}$	$e_1n_1 = \frac{1}{2}n_1$, $n_1^2 = n_2, n_3^2 = n_2$	$\mathcal{J}_{66}$	$n_1^2 = n_2$, $n_4^2 = n_3, n_1n_2 = n_3$
$\mathcal{J}_{48}$	$e_1n_3 = \frac{1}{2}n_3, e_1n_2 = \frac{1}{2}n_2, n_3^2 = n_1$	$\mathcal{J}_{69}$	$n_1^2 = n_2, n_1n_3 = n_4$
$\mathcal{J}_{49}$	$e_1n_2 = \frac{1}{2}n_2, e_1n_3 = \frac{1}{2}n_3, n_3^2 = n_1, n_2n_3 = n_1$	$\mathcal{J}_{70}$	$n_1^2 = n_2, n_3n_4 = n_2$
$\mathcal{J}_{50}$	$n_1n_2 = n_3, e_1n_2 = \frac{1}{2}n_2, e_1n_3 = \frac{1}{2}n_3$		

3. Inner derivations and local inner derivations on four-dimensional Jordan algebras

In this section, we apply our algorithm to describe inner derivations of four-dimensional Jordan algebras. Let $\mathcal{J}$ be a four-dimensional Jordan algebra. Then the following theorem takes place.

Theorem 3.1. *Inner derivations and local inner derivations of four-dimensional indecomposable Jordan algebras are given as follows:*

Table 2
Inner derivations and local inner derivations on four-dimensional Jordan algebras.

$\mathcal{J}$	Common form of a inner derivation	Common form of a local inner derivation	Is each inner local derivation a inner derivation?
$\mathcal{J}_2$	$\{E_{3,1}+E_{3,2}-E_{1,4}-E_{2,4}, E_{4,1}+E_{4,2}-E_{1,3}-E_{2,3}, E_{3,3}-E_{4,4}\}$	$\{E_{3,1}+E_{3,2}-E_{1,4}-E_{2,4}, E_{4,1}+E_{4,2}-E_{1,3}-E_{2,3}, E_{3,3}-E_{4,4}\}$	+
$\mathcal{J}_9$	$\{E_{3,1}+E_{3,2}-2E_{1,3}-2E_{2,3}, E_{4,1}+E_{4,2}, E_{4,3}\}$	$\{E_{3,1}+E_{3,2}-2E_{1,3}-2E_{2,3}, E_{4,1}+E_{4,2}, E_{4,3}\}$	+
$\mathcal{J}_{16}$	$\{E_{3,1}+E_{3,2}, E_{4,1}\}$	$\{E_{3,1}+E_{3,2}, E_{4,1}\}$	+
$\mathcal{J}_{17}$	$\{E_{3,1}+E_{3,2}, E_{4,1}\}$	$\{E_{3,1}+E_{3,2}, E_{4,1}\}$	+
$\mathcal{J}_{18}$	$\{E_{3,1}+E_{3,2}, E_{4,1}+E_{4,2}\}$	$\{E_{3,1}+E_{3,2}, E_{4,1}+E_{4,2}\}$	+
$\mathcal{J}_{25}$	$\{E_{3,1}+E_{3,2}+2E_{4,3}, E_{3,1}+E_{3,2}-2E_{4,3}, E_{4,1}\}$	$\{E_{3,1}, E_{3,2}, E_{4,1}, E_{4,3}\}$	-
$\mathcal{J}_{31}$	$\{E_{2,1}, E_{3,1}, E_{4,1}\}$	$\{E_{2,1}, E_{3,1}, E_{4,1}\}$	+
$\mathcal{J}_{32}$	$\{E_{2,1}, E_{3,1}, E_{4,1}\}$	$\{E_{2,1}, E_{3,1}, E_{4,1}\}$	+
$\mathcal{J}_{33}$	$\{E_{2,1}, E_{3,1}, E_{4,1}\}$	$\{E_{2,1}, E_{3,1}, E_{4,1}\}$	+
$\mathcal{J}_{36}$	$\{E_{2,1}, E_{3,1}, E_{4,1}\}$	$\{E_{2,1}, E_{3,1}, E_{4,1}\}$	+
$\mathcal{J}_{39}$	$\{E_{2,1}, E_{3,1}, E_{4,1}\}$	$\{E_{2,1}, E_{3,1}, E_{4,1}\}$	+
$\mathcal{J}_{42}$	$\{E_{2,1}, E_{3,1}, E_{4,1}\}$	$\{E_{2,1}, E_{3,1}, E_{4,1}\}$	+
$\mathcal{J}_{43}$	$\{E_{2,1}, E_{3,1}, E_{4,1}\}$	$\{E_{2,1}, E_{3,1}, E_{4,1}\}$	+
$\mathcal{J}_{45}$	$\{E_{2,1}-2E_{3,2}\}$	$\{E_{2,1}-2E_{3,2}\}$	+
$\mathcal{J}_{48}$	$\{E_{3,1}, E_{4,1}-2E_{2,4}\}$	$\{E_{3,1}, E_{4,1}-2E_{2,4}\}$	+
$\mathcal{J}_{49}$	$\{E_{3,1}-2E_{2,4}, E_{4,1}-2E_{2,3}-2E_{2,4}\}$	$\{E_{3,1}-2E_{2,4}, E_{4,1}-2E_{2,3}-2E_{2,4}\}$	+
$\mathcal{J}_{50}$	$\{E_{3,1}+2E_{4,2}, E_{4,1}\}$	$\{E_{3,1}, E_{4,1}, E_{4,2}\}$	-
$\mathcal{J}_{52}$	$\{E_{2,1}-2E_{4,2}, E_{3,1}\}$	$\{E_{2,1}-2E_{4,2}, E_{3,1}\}$	+
$\mathcal{J}_{53}$	$\{E_{2,1}+2E_{3,2}-2E_{4,2}, E_{3,1}\}$	$\{E_{2,1}+2E_{3,2}-2E_{4,2}, E_{3,1}\}$	+
$\mathcal{J}_{55}$	$\{E_{2,1}, E_{3,1}, E_{4,1}+2E_{3,4}\}$	$\{E_{2,1}, E_{3,1}, E_{4,1}, E_{3,4}\}$	-
$\mathcal{J}_{56}$	$\{E_{2,1}, E_{3,1}, E_{4,1}\}$	$\{E_{2,1}, E_{3,1}, E_{4,1}\}$	+
$\mathcal{J}_{57}$	$\{E_{2,1}, E_{3,1}, E_{4,1}+2E_{3,4}\}$	$\{E_{2,1}, E_{3,1}, E_{4,1}, E_{3,4}\}$	-
$\mathcal{J}_{58}$	$\{E_{2,1}, E_{3,1}, E_{4,1}+2E_{2,4}\}$	$\{E_{2,1}, E_{3,1}, E_{4,1}, E_{2,4}\}$	-
$\mathcal{J}_{59}$	$\{E_{2,1}, E_{3,1}+2E_{2,4}, E_{4,1}+2E_{2,3}+2E_{2,4}\}$	$\{E_{2,1}, E_{3,1}, E_{4,1}, E_{2,3}, E_{2,4}\}$	-
$\mathcal{J}_{60}$	$\{E_{2,1}, E_{3,1}-2E_{4,2}, E_{4,1}\}$	$\{E_{2,1}, E_{3,1}, E_{4,1}, E_{4,2}\}$	-
$\mathcal{J}_{61}$	0	0	+
$\mathcal{J}_{62}$	$\{E_{3,4}\}$	$\{E_{3,4}\}$	+
$\mathcal{J}_{63}$	$\{E_{3,1}+E_{3,4}\}$	$\{E_{3,1}+E_{3,4}\}$	+
$\mathcal{J}_{64}$	$\{E_{3,1}+E_{3,4}\}$	$\{E_{3,1}+E_{3,4}\}$	+
$\mathcal{J}_{65}$	$\{E_{3,1}\}$	$\{E_{3,1}\}$	+
$\mathcal{J}_{66}$	0	0	+
$\mathcal{J}_{69}$	0	0	+
$\mathcal{J}_{70}$	0	0	+

Proof. Let us prove the theorem for the algebra $\mathcal{J}_2$. The cases of the other algebras are proved similarly. The assertion of the theorem concerning local inner derivation is proven in the next section.

Let us construct the matrix of an inner derivation of the Jordan algebra $\mathcal{J}_2$. The algebra $\mathcal{J}_2$ has a basis $\{e_1, e_2, e_3, e_4\}$ with the following table of multiplication:

$$e_3e_4=\frac{1}{2}(e_1+e_2), e_1e_3=\frac{1}{2}e_3, e_1e_4=\frac{1}{2}e_4, e_2e_3=\frac{1}{2}e_3 \quad \text{and} \quad e_2e_4=\frac{1}{2}e_4.$$

The decompositions of arbitrary elements a, $b \in \mathcal{J}_2$ in the basis $\{e_1, e_2, e_3, e_4\}$ are

$$a = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 \quad \text{and} \quad b = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 + \mu_4 e_4$$

respectively.

Let D be a derivation of the algebra $\mathcal{J}_2$, A be the matrix of the derivation D and

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

Then $D(x) = \widehat{A\bar{x}}$. We have $D(e_1) = \widehat{A\overline{e_1}}$ and

$$A\overline{e_1} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{pmatrix},$$

$$D(e_1) = A\overline{e_1} = a(be_1) - b(ae_1) =$$

$$= (\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4)[(\mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 + \mu_4 e_4)e_1] -$$

$$-(\mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 + \mu_4 e_4)[(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4)e_1] =$$

$$= \frac{1}{4}(\lambda_1\mu_3 + \lambda_2\mu_3 - \lambda_3\mu_1 - \lambda_3\mu_2)e_3 + \frac{1}{4}(\lambda_1\mu_4 + \lambda_2\mu_4 - \lambda_4\mu_1 - \lambda_4\mu_2)e_4$$

Let $\alpha = \frac{1}{4}(\lambda_1\mu_3 + \lambda_2\mu_3 - \lambda_3\mu_1 - \lambda_3\mu_2)$, $\beta = \frac{1}{4}(\lambda_1\mu_4 + \lambda_2\mu_4 - \lambda_4\mu_1 - \lambda_4\mu_2)$. Then

$$a_{11} = 0, a_{21} = 0, a_{31} = \alpha \quad \text{and} \quad a_{41} = \beta.$$

Similarly, using $A\overline{e_2}$, $A\overline{e_3}$ and $A\overline{e_4}$, we have

$$a_{12} = 0, a_{22} = 0, a_{32} = \alpha, a_{42} = \beta,$$

$$a_{13} = -\beta, a_{23} = -\beta, a_{33} = \gamma, a_{41} = 0,$$

$$a_{14} = -\alpha, a_{24} = -\alpha, a_{34} = 0, a_{44} = -\gamma.$$

From these it follows that

$$A = \begin{pmatrix} 0 & 0 & -\beta & -\alpha \\ 0 & 0 & -\beta & -\alpha \\ \alpha & \alpha & \gamma & 0 \\ \beta & \beta & 0 & -\gamma \end{pmatrix} =$$

$$= \{E_{3,1} + E_{3,2} - E_{1,4} - E_{2,4}, E_{4,1} + E_{4,2} - E_{1,3} - E_{2,3}, E_{3,3} - E_{4,4}\}.$$

The proof is complete. □

4. Description of local inner derivations on some Jordan algebras of dimension four

Theorem 4.1. *Each local inner derivation of the Jordan algebra $\mathcal{J}_2$ is an inner derivation.*

Proof. Let ∇ be a local inner derivation on $\mathcal{J}_2$. Then

$$\nabla(x) = \sum_{i=1}^{4}(\sum_{j=1}^{4} a_{i,j}x_j)e_i, x \in \mathcal{J}_2$$

for the matrix $A = (a_{i,j})_{i,j=1}^4$ of the local inner derivation ∇, where $x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4$.

By the definition for any element $x \in \mathcal{J}_2$ there exist a derivation D_x such that $\nabla(x) = D_x(x)$. By table 2 the matrix of the derivation D_x is equal to

$$A_x = \begin{pmatrix} 0 & 0 & -\beta^x & -\alpha^x \\ 0 & 0 & -\beta^x & -\alpha^x \\ \alpha^x & \alpha^x & \gamma^x & 0 \\ \beta^x & \beta^x & 0 & -\gamma^x \end{pmatrix}$$

for some elements

$$\alpha^x, \beta^x, \gamma^x$$

in $\mathbb{F}$, depending on x.

If we take $x = e_1$, then

$$\nabla(e_1) = D_{e_1}(e_1) = \alpha^{e_1} e_3 + \beta^{e_1} e_4.$$

At the same time,

$$\nabla(e_1) = a_{1,1}e_1 + a_{2,1}e_2 + a_{3,1}e_3 + a_{4,1}e_4$$

Hence,

$$a_{1,1} = 0, a_{2,1} = 0, a_{3,1} = \alpha^{e_1}, a_{4,1} = \beta^{e_1}.$$

Similarly we get

$$a_{1,2} = 0, a_{2,2} = 0, a_{3,2} = \alpha^{e_2}, a_{4,2} = \beta^{e_2}.,$$

if we take $x = e_2$. Also, we similarly get

$$a_{1,3} = -\beta^{e_3}, a_{2,3} = -\beta^{e_3}, a_{3,3} = \gamma^{e_3}, a_{4,3} = 0,$$
$$a_{1,4} = -\alpha^{e_4}, a_{2,4} = -\alpha^{e_4}, a_{3,4} = 0, a_{4,4} = -\gamma^{e_4}.$$

Hence,

$$A = \begin{pmatrix} 0 & 0 & -\beta^{e_3} & -\alpha^{e_4} \\ 0 & 0 & -\beta^{e_3} & -\alpha^{e_4} \\ \alpha^{e_1} & \alpha^{e_2} & \gamma^{e_3} & 0 \\ \beta^{e_1} & \beta^{e_2} & 0 & -\gamma^{e_4} \end{pmatrix}.$$

Now, since ∇ is linear we have

$$\nabla(e_1 + e_4) = \nabla(e_1) + \nabla(e_4).$$

From this it follows that

$$\nabla(e_1 + e_4) = -\alpha^{e_1+e_4}e_1 - \alpha^{e_1+e_4}e_2 + \alpha^{e_1+e_4}e_3 + (\beta^{e_1+e_4} - \gamma^{e_1+e_4})e_4 =$$
$$= -\alpha^{e_4}e_1 - \alpha^{e_4}e_2 + \alpha^{e_1}e_3 + (\beta^{e_1} - \gamma^{e_4})e_4.$$

Hence,

$$\alpha^{e_1+e_4} = \alpha^{e_4}, \alpha^{e_1+e_4} = \alpha^{e_1}$$

and

$$\alpha^{e_4} = \alpha^{e_1}.$$

Similarly, from the equality

$$\nabla(e_2 + e_4) = \nabla(e_2) + \nabla(e_4).$$

We get

$$\nabla(e_2 + e_4) = -\alpha^{e_2+e_4}e_1 - \alpha^{e_2+e_4}e_2 + \alpha^{e_2+e_4}e_3 + (\beta^{e_2+e_4} - \gamma^{e_1+e_4})e_4 =$$
$$= -\alpha^{e_4}e_1 - \alpha^{e_4}e_2 + \alpha^{e_1}e_3 + (\beta^{e_1} - \gamma^{e_4})e_4.$$

Hence,

$$\alpha^{e_2+e_4} = \alpha^{e_4}, \alpha^{e_2+e_4} = \alpha^{e_1}$$

and

$$\alpha^{e_4} = \alpha^{e_2}.$$

Similarly, if we take $\nabla(e_1 + e_3), \nabla(e_2 + e_3), \nabla(e_3 + e_4)$ then, we get $\beta^{e_1} = \beta^{e_2} = \beta^{e_3}, \gamma^{e_3} = \gamma^{e_4}$.

Thus, we have

$$A = \begin{pmatrix} 0 & 0 & -\beta^{e_1} & -\alpha^{e_1} \\ 0 & 0 & -\beta^{e_1} & -\alpha^{e_1} \\ \alpha^{e_1} & \alpha^{e_1} & \gamma^{e_1} & 0 \\ \beta^{e_1} & \beta^{e_1} & 0 & -\gamma^{e_1} \end{pmatrix} =$$

$$= \{E_{3,1} + E_{3,2} - E_{1,4} - E_{2,4}, E_{4,1} + E_{4,2} - E_{1,3} - E_{2,3}, E_{3,3} - E_{4,4}\}.$$

Therefore, by the form of an inner derivation in table 2, ∇ is an inner derivation. The proof is complete. □

The following theorem is also proved similar to the proof of Theorem 4.1.

Theorem 4.2. *Each local inner derivation of the Jordan algebras $\mathcal{J}_{45}$, $\mathcal{J}_{48}$, $\mathcal{J}_{49}$, $\mathcal{J}_{52}$, $\mathcal{J}_{53}$ is an inner derivation.*

Theorem 4.3. *Each local inner derivation of the Jordan algebra $\mathcal{J}_{31}$ is an inner derivation.*

Proof. Let ∇ be a local inner derivation on $\mathcal{J}_{31}$. Then

$$\nabla(x) = \sum_{i=1}^{4}(\sum_{j=1}^{4} a_{i,j}x_j)f_i, x \in \mathcal{J}_{31}$$

for the matrix $A = (a_{i,j})_{i,j=1}^{4}$ of the local inner derivation ∇, where $x = x_1e_1+x_2n_1+x_3n_2+x_4n_3$.

By the definition for any element $x \in \mathcal{J}_{31}$ there exist an inner derivation D_x such that $\nabla(x) = D_x(x)$. By table 2 the matrix of the inner derivation D_x is equal to

$$A_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha^x & 0 & 0 & 0 \\ \beta^x & 0 & 0 & 0 \\ \gamma^x & 0 & 0 & 0 \end{pmatrix}$$

for some elements

$$\alpha^x, \beta^x, \gamma^x$$

in $\mathbb{F}$, depending on x.

If we take $x = e_1$, then

$$\nabla(e_1) = D_{e_1}(e_1) = \alpha^{e_1}n_1 + \beta^{e_1}n_2 + \gamma^{e_1}n_3.$$

At the same time,

$$\nabla(e_1) = a_{1,1}e_1 + a_{2,1}n_1 + a_{3,1}n_2 + a_{4,1}n_3$$

Hence,

$$a_{1,1} = 0, a_{2,1} = \alpha^{e_1}, a_{3,1} = \beta^{e_1}, a_{4,1} = \gamma^{e_1}.$$

Similarly we get

$$a_{1,2} = 0, a_{2,2} = 0, a_{3,2} = 0, a_{4,2} = 0,$$
$$a_{1,3} = 0, a_{2,3} = 0, a_{3,3} = 0, a_{4,3} = 0,$$
$$a_{1,4} = 0, a_{2,4} = 0, a_{3,4} = 0, a_{4,4} = 0.$$

Thus, we have

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha^{e_1} & 0 & 0 & 0 \\ \beta^{e_1} & 0 & 0 & 0 \\ \gamma^{e_1} & 0 & 0 & 0 \end{pmatrix} = \{E_{2,1}, E_{3,1}, E_{4,1}\}.$$

Therefore, by the form of an inner derivation in table 2, ∇ is an inner derivation. The proof is complete. □

The following theorem can be proved similar to the proof of Theorem 4.3.

Theorem 4.4. *Each local inner derivation of the Jordan algebras $\mathcal{J}_{32}$, $\mathcal{J}_{33}$, $\mathcal{J}_{36}$, $\mathcal{J}_{39}$, $\mathcal{J}_{42}$, $\mathcal{J}_{43}$, $\mathcal{J}_{56}$, $\mathcal{J}_{61}$, $\mathcal{J}_{62}$, $\mathcal{J}_{65}$, $\mathcal{J}_{66}$, $\mathcal{J}_{69}$, $\mathcal{J}_{70}$ is an inner derivation.*

Theorem 4.5. *Each local inner derivation of the Jordan algebra $\mathcal{J}_{16}$ is an inner derivation.*

Proof. Let ∇ be a local inner derivation on $\mathcal{J}_{16}$. Then

$$\nabla(x) = \sum_{i=1}^{4}(\sum_{j=1}^{4} a_{i,j}x_j)f_i, x \in \mathcal{J}_{16}$$

for the matrix $A = (a_{i,j})_{i,j=1}^{4}$ of the local inner derivation ∇, where $x = x_1e_1+x_2e_2+x_3n_1+x_4n_2$.

By the definition for any element $x \in \mathcal{J}_{16}$ there exist an inner derivation D_x such that $\nabla(x) = D_x(x)$. By table 2 the matrix of the inner derivation D_x is equal to

$$A_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha^x & \alpha^x & 0 & 0 \\ \beta^x & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

for some elements α^x, β^x in $\mathbb{F}$, depending on x.

If we take $x = e_1$, then

$$\nabla(e_1) = D_{e_1}(e_1) = \alpha^{e_1}e_3 + \beta^{e_1}e_4.$$

At the same time,

$$\nabla(e_1) = a_{1,1}e_1 + a_{2,1}e_2 + a_{3,1}e_3 + a_{4,1}e_4$$

Hence,

$$a_{1,1} = 0, a_{2,1} = 0, a_{3,1} = \alpha^{e_1}, a_{4,1} = \beta^{e_1}.$$

Similarly we get

$$a_{1,2} = 0, a_{2,2} = 0, a_{3,2} = \alpha^{e_2}, a_{4,2} = 0.,$$

if we take $x = e_2$. Also, we similarly get

$$a_{1,3} = 0, a_{2,3} = 0, a_{3,3} = \gamma^{e_3} = 0, a_{4,3} = 0,$$

$$a_{1,4} = 0, a_{2,4} = 0, a_{3,4} = 0, a_{4,4} = 0.$$

Hence,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha^{e_1} & \alpha^{e_2} & 0 & 0 \\ \beta^{e_1} & 0 & 0 & 0 \end{pmatrix}.$$

Since the local inner derivation ∇ was chosen arbitrarily, we have such matrices has following form

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta & \omega & 0 & 0 \\ \lambda & 0 & 0 & 0 \end{pmatrix}, \tag{3}$$

where δ, ω, λ, belong to $\mathbb{F}$.

Let, $\nabla(x) = \widehat{A\overline{x}}$, $x \in \mathcal{J}_{16}$. We suppose that for each $a \in \mathcal{J}_{16}$ there exists a matrix A_a of the form (2), which generates a derivation, such that

$$\nabla(a) = \widehat{A\overline{a}} = \widehat{A_a\overline{a}}.$$

We find the form of the matrix A, for which the map, defined ny the matrix A is a local inner derivation. For this propose we consider the following system of linear equations

$$\begin{cases} a_1\alpha_a + a_2\alpha_a = a_1\delta + a_2\omega \\ a_1\beta_a = a_1\lambda \end{cases} \tag{4}$$

with respect to the variables α_a, β_a for each $a \in \mathcal{J}_{16}$. Here, δ, ω, λ are elements from $\mathbb{F}$. Note, that if the left part of any equation of the system (4) is equal to zero, then the right part of this equation is also equal to zero.

Now, if $a_1 \neq 0$, $a_2 = -a_1$ then

$$\begin{cases} \delta = \omega \\ \beta_a = \lambda \end{cases}.$$

Then (3) has the following form

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta & \delta & 0 & 0 \\ \lambda & 0 & 0 & 0 \end{pmatrix} = \{E_{3,1} + E_{3,2}, E_{4,1}\}$$

Therefore, by the form of an inner derivation in table 2, ∇ is an inner derivation. The proof is complete. □

The following theorem is proved similar to the proof of Theorem 4.5.

Theorem 4.6. *Each local inner derivation of the Jordan algebras $\mathcal{J}_9$, $\mathcal{J}_{17}$, $\mathcal{J}_{18}$, $\mathcal{J}_{63}$, $\mathcal{J}_{64}$ is an inner derivation.*

Theorem 4.7. *A linear operator on the Jordan algebra $\mathcal{J}_{50}$ is a local inner derivation if and only if the matrix of this linear operator has the following matrix form:*

$$LocDer(\mathcal{J}_{50}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ \beta & \gamma & 0 & 0 \end{pmatrix} = \{E_{3,1}, E_{4,1}, E_{4,2}\}. \tag{5}$$

Proof. Let ∇ be a local inner derivation on $\mathcal{J}_{50}$. Then

$$\nabla(x) = \sum_{i=1}^{4}(\sum_{j=1}^{4} a_{i,j}x_j)n_i, x \in \mathcal{J}_{50}$$

for the matrix $A = (a_{i,j})_{i,j=1}^{4}$ of the local inner derivation ∇, where $x = x_1n_1+x_2n_2+x_3n_3+x_4n_4$. By the definition for any element $x \in \mathcal{J}_{50}$ there exist an inner derivation D_x such that $\nabla(x) = D_x x = \widehat{A\overline{x}}$. By the form in table 2 of an inner derivation

$$D_x(x) = \widehat{A_x\overline{x}}$$

for

$$A_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha^x & 0 & 0 & 0 \\ \beta^x & 2\alpha^x & 0 & 0 \end{pmatrix} \tag{6}$$

where α^x and β^x in $\mathbb{F}$, depending on x.

If we take $x = n_1$, then

$$\nabla(n_1) = D_{n_1}(n_1) = \alpha^{n_1}n_3 + \beta^{n_1}n_4.$$

At the same time,

$$\nabla(n_1) = a_{1,1}n_1 + a_{2,1}n_2 + a_{3,1}n_3 + a_{4,1}n_4.$$

Hence,

$$a_{1,1} = 0, a_{2,1} = 0, a_{3,1} = \alpha^{e_1}, a_{4,1} = \beta^{e_1}.$$

Similarly we get

$$a_{1,2} = 0, a_{2,2} = 0, a_{3,2} = 0, a_{4,2} = \alpha^{n_2},$$

if we take $x = n_2$. Also, we similarly get

$$a_{1,3} = 0, a_{2,3} = 0, a_{3,3} = 0, a_{4,3} = 0,$$

$$a_{1,4} = 0, a_{2,4} = 0, a_{3,4} = 0, a_{4,4} = 0.$$

Thus, we have

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha^{n_1} & 0 & 0 & 0 \\ \beta^{n_1} & 2\alpha^{n_2} & 0 & 0 \end{pmatrix}.$$

Since the local inner derivation ∇ was chosen arbitrarily, we have such matrices has following form

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ \beta & \gamma & 0 & 0 \end{pmatrix},$$

where α, β, γ belong to $\mathbb{F}$.

Let, $\nabla(x) = \widehat{A\overline{x}}$, $x \in \mathcal{J}_{50}$. We suppose that for each $a \in \mathcal{J}_{50}$ there exists a matrix A_a of the form (6), which generates a derivation, such that

$$\nabla(a) = \widehat{A\overline{a}} = \widehat{A_a\overline{a}}.$$

We find the form of the matrix A, for which the map, defined by the matrix A is a local inner derivation. For this propose we consider the following system of linear equations

$$\begin{cases} a_1\alpha_a = a_1\alpha \\ a_1\beta_a + 2a_2\alpha_a = a_1\beta + a_2\gamma \end{cases} \tag{7}$$

with respect to the variables

$$\alpha_a, \beta_a,$$

for each $a \in \mathcal{J}_{50}$. Here, $\alpha, \beta, \gamma, \lambda$ are elements from $\mathbb{F}$. Note, that if the left part of any equation of the system (7) is equal to zero, then the right part of this equation is also equal to zero.

Now, if $a_1 \neq 0$, $a_2 = 0$, then the system (7) has the following form

$$\begin{cases} \alpha_a = \alpha \\ \beta_a = \beta \end{cases}.$$

If $a_1 \neq 0$, $a_2 \neq 0$, then the system (7) has the following form

$$\begin{cases} \alpha_a = \alpha \\ \beta_a = \beta + \gamma\frac{a_2}{a_1} - 2\alpha\frac{a_2}{a_1} \end{cases}.$$

If $a_1 = 0$, $a_2 \neq 0$, then the system (7) has the following form

$$\alpha_a = \frac{1}{2}\gamma.$$

Thus, for each element $a \in \mathcal{J}_{50}$, the system of linear equations (7) has a solution. Hence, the linear operator, defined by the matrix A is a local inner derivation. Since the local derivation ∇ was chosen arbitrarily and ∇ is defined by derivations of the form (6) we have that the matrix of each local inner derivation on the Jordan algebra $\mathcal{J}_{50}$ has the form (5). This ends the proof. □

Similar theorems for the Jordan algebras $\mathcal{J}_{25}$, $\mathcal{J}_{55}$, $\mathcal{J}_{57}$, $\mathcal{J}_{58}$, $\mathcal{J}_{59}$, $\mathcal{J}_{60}$ can be proved by the way indicated in the proof of Theorem 4.7, where the appropriate matrices are given in Table 2.

5. Description of 2-local inner derivations on some Jordan algebras of dimension four

This section is devoted to the description of 2-local inner derivations of indecomposable four dimensional Jordan algebras.

Inner derivations and local inner derivations on Jordan algebras of dimension four are classified in Table 2. All notations, concerning the Jordan algebras below, are taken from [14] (see table 2 in [14]).

In the following, all algebras are assumed to be Jordan algebras over an algebraically closed field $\mathbb{F}$ of characteristic $\neq 2$.

Theorem 5.1. *Every 2-local inner derivation on $\mathcal{J}_2$ Jordan algebra is an inner derivation.*

Proof. We take the basis $\{e_1, e_2, e_3, e_4\}$ of $\mathcal{J}_2$ indicated in Table 1 above. Note that in the following order $\{e_1, e_2, e_4, e_3\}$ with respect to this basis every inner derivation of $\mathcal{J}_2$ is generated by a skew-symmetric matrix. So, by Theorems 2.3 and 3.1 from [3] every 2-local inner derivation on $\mathcal{J}_2$ Jordan algebra is an inner derivation. This ends the proof. □

Theorem 5.2. *Every 2-local inner derivation on $\mathcal{J}_{59}$ Jordan algebra is an inner derivation.*

Proof. Let Δ be 2-local inner derivation on $\mathcal{J}_{59}$. By the definition, for every element $x \in \mathcal{J}_{59}$,

$$x = x_1e_1 + x_2n_1 + x_3n_2 + x_4n_3,$$

there exists a matrix $A_{x,e_1} = (a^{x,e_1})_{i,j=1}^4$ such that $\Delta(x) = \widehat{A_{x,e_1}\bar{x}}$, where $\bar{x} = (x_1, x_2, x_3, x_4)^T$ is the vector corresponding to x, $\widehat{\bar{x}}$ is an operation on $\bar{x}$ such that $\widehat{\bar{x}} = x$, and

$$\Delta(e_1) = \widehat{A_{x,e_1}\overline{e_1}} = (\alpha^{x,e_1}, \widehat{\beta^{x,e_1}}, \gamma^{x,e_1})^T.$$

Since $\Delta(e_1) = L_{x,e_1}(e_1) = L_{y,e_1}(e_1)$, we have

$$(\alpha^{x,e_1}, \widehat{\beta^{x,e_1}}, \gamma^{x,e_1})^T =$$

$$= (\alpha^{y,e_1}, \widehat{\beta^{y,e_1}}, \gamma^{y,e_1})^T$$

for each pair, x, y of elements in $\mathcal{J}_{59}$. Hence, $\alpha^{x,e_1} = \alpha^{y,e_1}$, $\beta^{x,e_1} = \beta^{y,e_1}$, $\gamma^{x,e_1} = \gamma^{y,e_1}$. We have

$$a_{2,3}^{z,e_1} = 2a_{4,1}^{z,e_1}, a_{2,4}^{z,e_1} = 2(a_{3,1}^{z,e_1} + a_{4,1}^{z,e_1})$$

and the rest components $a_{i,j}^{z,e_1}$ are equal to zero, where $z \in \{x, y\}$. Therefore $a_{i,j}^{x,e_1} = a_{i,j}^{y,e_1}$, $i, j = 1, 2, 3, 4$, i.e. $A_{x,e_1} = A_{y,e_1}$, and

$$\Delta(x) = \widehat{A_{y,e_1}\bar{x}}$$

for any $x \in V$, and the matrix of $\Delta(x)$ does not depend on x. Hence Δ is a linear operator, and the matrix of L_{y,e_1} is the matrix of Δ. The proof is complete. □

The following theorem is also proved similar to the proof of Theorem 5.2.

Theorem 5.3. *Each 2-local inner derivation of the Jordan algebras $\mathcal{J}_{16}$, $\mathcal{J}_{17}$, $\mathcal{J}_{18}$, $\mathcal{J}_{31}$, $\mathcal{J}_{32}$, $\mathcal{J}_{33}$, $\mathcal{J}_{36}$, $\mathcal{J}_{39}$, $\mathcal{J}_{42}$, $\mathcal{J}_{43}$, $\mathcal{J}_{45}$, $\mathcal{J}_{48}$, $\mathcal{J}_{49}$, $\mathcal{J}_{50}$, $\mathcal{J}_{52}$, $\mathcal{J}_{53}$, $\mathcal{J}_{55}$, $\mathcal{J}_{56}$, $\mathcal{J}_{57}$, $\mathcal{J}_{58}$, $\mathcal{J}_{59}$, $\mathcal{J}_{60}$, $\mathcal{J}_{61}$, $\mathcal{J}_{62}$, $\mathcal{J}_{63}$, $\mathcal{J}_{64}$, $\mathcal{J}_{65}$, $\mathcal{J}_{66}$, $\mathcal{J}_{69}$, $\mathcal{J}_{70}$ is an inner derivation.*

Theorem 5.4. *In the Jordan algebras $\mathcal{J}_9$, $\mathcal{J}_{25}$ there exist 2-local inner derivations which are not derivations.*

Proof. We prove the theorem for the algebra $\mathcal{J}_9$; for the algebra $\mathcal{J}_{25}$ the proof is similar. Let us define a homogeneous non additive function f on $\mathbb{C}^3$ as follows

$$f(z_1, z_2, z_3) = \begin{cases} \frac{(z_1+z_2)^2}{z_3}, & \text{if} \quad z_3 \neq 0; \\ 0, & \text{if} \quad z_3 = 0. \end{cases}$$

where $(z_1, z_2, z_3) \in \mathbb{C}^3$. Consider the map $\Delta : \mathcal{J}_9 \to \mathcal{J}_9$ defined by the rule

$$\Delta(x) = f(x_1, x_2, x_3)e_4, \text{ where } x = \sum_{i=1}^{4} x_ie_i \in \mathcal{J}_9.$$

Since f is not additive, we have Δ is not a derivation. Let us show that Δ is a 2-local derivation. For the elements

$$x = \sum_{i=1}^{4} x_ie_i, y = \sum_{i=1}^{4} y_ie_i,$$

we search an inner derivation D with the matrix of the following form (by Theorem 3.1 this matrix generates an inner derivation of $\mathcal{J}_9$):

$$\begin{pmatrix} 0 & 0 & -2a & 0 \\ 0 & 0 & -2a & 0 \\ a & a & 0 & 0 \\ b & b & c & 0 \end{pmatrix},$$

such that $\Delta(x) = D(x)$ and $\Delta(y) = D(y)$. Then we obtain the following system of equations for b and c:

$$\begin{cases} x_1b + x_2b + x_3c = f(x_2, x_3), \\ y_1b + y_2b + y_3c = f(y_2, y_3). \end{cases}$$

Case 1. Let $(x_1 + x_2)y_3 - x_3(y_1 + y_2) = 0$, then the system has infinitely many solutions, because of the right-hand side of this system is homogeneous.

Case 2. Let $(x_1 + x_2)y_3 - x_3(y_1 + y_2) \neq 0$, then the system has a unique solution. The proof is complete. □

References

[1] Sh. Ayupov and F. Arzikulov, 2-Local derivations on semi-finite von Neumann algebras, *Glasg. Math. J.* 56 (2014), 9—12.

[2] Sh. Ayupov, F. Arzikulov, 2-Local derivations on associative and Jordan matrix rings over commutative rings, *Linear Algebra Appl.* 522 (2017) 28–50. DOI: http://dx.doi.org/10.1016/j.laa.2017.02.012

[3] Sh. Ayupov, F. Arzikulov, N. Umrzaqov, O. Nuriddinov, Description of 2-local derivations and automorphisms on finite-dimensional Jordan algebras, *Linear Multilinear Algebra* 70, 18 (2022), 3525–3542.

[4] Sh. Ayupov, K. Kudaybergenov, 2-local derivations and automorphisms on $B(H)$, *J. Math. Anal. Appl.* 395 (2012), 15—18.

[5] Sh. Ayupov, K. Kudaybergenov, 2-Local derivations on von Neumann algebras, *Positivity* 19 (2015), 445—455.

[6] Ayupov Sh., Kudaybergenov K., Local derivations on finite-dimensional Lie algebras, *Linear Algebra and its Applications,* 493 (2016), 381–398.

[7] Urinboyev F., On inner and local inner derivations of finite-dimensional Jordan algebras, *Bull. Inst. Math.,* 2023, Vol.6, no. 2, pp. 142-155.

[8] Burgos M., Fernandez-Polo F.J., Peralta A.M., Local triple derivations on C*-algebras and JB*-triples, *Bull. London Math. Soc.,* 46 (4) (2014), 709–724.

[9] Johnson B., Local derivations on C*-algebras are derivations, *Trans. Amer. Math. Soc.* 353 (2001), 313–325. DOI: 10.2307/221975

[10] Jordan P., von Neumann J., Wigner E., On an algebraic generalisation of the quantum mechanical formalism, *Annals Math.* 35 (1934), 29–64.

[11] Kadison R., Local derivations, *Journal of Algebra,* 130 (1990), 2, 494–509.

[12] Kashuba I., Martin M.E., The variety of three-dimensional real Jordan algebras, *Journal of Algebra and Its Applications,* Vol. 15 (2016), No. 08, 16501589. https://doi.org/10.1142/S0219498816501589

[13] Kashuba I., Shestakov I., Jordan algebras of dimension three: geometric classification and representation type, in: *Actas del XVI Coloquio Latinoamericano de Algebra,* Colonia del Sacramento, Uruguay, 2005, in: *Bibl. Rev. Mat.* Iberoamericana, 2007, pp. 295—315.

[14] I. Kashuba, M.E. Martin, Deformations of Jordan algebras of dimension four, *Journal of Algebra* 399, 277–289 (2014).

[15] S. Kim, J. Kim, Local automorphisms and derivations on M_n, *Proc. Amer. Math. Soc.* 132 (2004), 1389—1392.

[16] M.E. Martin, Four dimensional Jordan algebras, *Int. J. Math. Game Theory Algebra* 20 (2013), 41-–59.

[17] D.R. Larson, A.R. Sourour, Local derivations and local automorphisms of $B(X)$, *Proc. Sympos. Pure Math.,* 51, Part 2, Providence, Rhode Island 1990, 187–194.

[18] Y. Lin, T. Wong, *A note on 2-local maps*, *Proc. Edinb. Math. Soc.* **49** (2006), 701-–708.

[19] P. Šemrl, Local automorphisms and derivations on $B(H)$, *Proc. Amer. Math. Soc.* **125** (1997), 2677–2680.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON SOME COUNTERPARTS OF RICKART AND BAER $*$-ALGEBRAS

FARHODJON ARZIKULOV[1,2], UTKIRBEK KHAKIMOV[2]

[1] V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan
[2] Andijan State University, Andijan, Uzbekistan

e-mail:khakimov_u@inbox.ru, arzikulovfn@gmail.com

Abstract. In the present paper, we introduce and study counterparts of Rickart and Baer involutive algebras, i.e., inner Rickart algebras and inner Baer algebras. We prove that a nilpotent associative algebra, which has no nilpotent elements with nonzero square roots, is an inner Rickart algebra. A nilpotent associative algebra, which has no nilpotent elements with a square root b such that $b^3 \neq 0$, is not an inner Rickart algebra if there exists a nonzero element a such that $a^2 \neq 0$.

As a main result of the paper, we describe a finite-dimensional inner Rickart algebra $\mathcal{A}$, isomorphic to $\mathbb{R}^2 \dotplus \mathcal{N}$, with a nilradical $\mathcal{N}$. Also, we describe a finite-dimensional inner Baer algebra with a nonzero nilradical $\mathcal{N}$.

Keywords: Associative algebra; nilradical; nilpotent associative algebra; nilpotent element; idempotent.

AMS Subject Classification: 16W10, 16E50, 16N40.

1. Introduction

In the present paper, we introduce and study counterparts of Rickart and Baer $*$-algebras, that is inner Rickart algebras and inner Baer algebras. We define an inner Rickart algebra as an associative algebra which is an inner RJ-algebra with respect to the Jordan multiplication $a \cdot b = \frac{1}{2}(ab + ba)$. Similarly, an inner Baer algebra is an associative algebra which is an inner BJ-algebra with respect to the Jordan multiplication. Inner RJ-algebras and inner BJ-algebras are introduced and studied in the papers [1], [2], [3].

The chosen notions were built around an (inner) quadratic annihilator. For each nonempty subset $\mathcal{S}$ of an associative algebra $\mathcal{A}$, the (inner) quadratic annihilator of $\mathcal{S}$ is defined by

$$^{\perp_q}\mathcal{S} := \{a \in \mathcal{A} : sas = 0 \text{ for all } s \in \mathcal{S}\}.$$

Thus, following by [4], an associative algebra $\mathcal{A}$ is called an inner Rickart algebra if, for each element $x \in \mathcal{A}$, there exists an idempotent $e \in \mathcal{A}$ such that $^{\perp_q}\{x\} \cap \mathcal{A}^2 = e\mathcal{A}e \cap \mathcal{A}^2$, where $\mathcal{A}^2 := \{a^2 : a \in \mathcal{A}\}$. There exist examples of inner Rickart algebras without unit element (cf. [2, p. 32]). Note that, there exist pairwise non-isomorphic (associative) inner Rickart (Baer) algebras, the inner RJ-algebras (respectively BJ-algebras) of which coincide (see Examples 5)–8) below). This is a motivation to introduce the notions of inner Rickart and Baer algebra.

An associative algebra $\mathcal{A}$ is called an inner Baer algebra if, for each subset $\mathcal{S} \subseteq \mathcal{A}$, there exists an idempotent $e \in \mathcal{A}$ such that $^{\perp_q}\mathcal{S} \cap \mathcal{A}^2 = e\mathcal{A}e \cap \mathcal{A}^2$. Note that, an associative algebra $\mathcal{A}$ is an inner Baer algebra if, and only if, it is an inner Rickart algebra and the set of all idempotents of $\mathcal{A}$ is a complete lattice.

In the present paper, we prove that a nilpotent associative algebra, for every element a of which $a^2 = 0$, is an inner Rickart algebra. We explain that a nilpotent associative algebra that has no nilpotent elements with a square root b such that $b^3 \neq 0$ is not an inner Rickart algebra if there exists a nonzero element a such that $a^2 \neq 0$.

As a main result of the paper we describe a finite-dimensional inner Rickart algebra $\mathcal{A}$, isomorphic to $\mathbb{R}^2 \dot{+} \mathcal{N}$, with a nilradical $\mathcal{N}$ (Theorem 2.8). Also, we describe a finite-dimensional inner Baer algebra with a nonzero nilradical $\mathcal{N}$.

2. Description of finite-dimensional inner Rickart algebras

Let $\mathcal{A}$ be an associative algebra, and let

$$\mathcal{A}^2 = \{a^2 : a \in \mathcal{A}\},\ \mathcal{S}^{\perp} := \{a \in \mathcal{A} : axa = 0,\ (\forall x \in \mathcal{S})\},$$

$$^{\perp_q}\mathcal{S} := \{a \in \mathcal{A} : xax = 0, (\forall x \in \mathcal{S})\}$$

For an associative algebra $\mathcal{A}$, we consider the following condition:

(A) for every element $x \in \mathcal{A}$, there exists an idempotent $e \in \mathcal{A}$ such that

$$^{\perp_q}\{x\} \cap \mathcal{A}^2 = e\mathcal{A}e \cap \mathcal{A}^2.$$

Definition. An associative algebra satisfying condition (A) is called an inner Rickart algebra, and condition (A) itself is called an inner Rickart condition. In other words, every associative algebra, which is an inner RJ-algebra with respect to the Jordan multiplication $a \cdot b = \frac{1}{2}(ab+ba)$, is called an inner Rickart algebra.

Examples. 1) It is obvious that every associative algebra which is an inner RJ-algebra with respect to the multiplication $a \cdot b = \frac{1}{2}(ab + ba)$ is an inner Rickart algebra.

2) Let $M_n(\mathbb{R})$ be the associative algebra of $n \times n$ matrices over the field $\mathbb{R}$ of real numbers, where $\mathbf{1}$ is the unit of the algebra $M_n(\mathbb{R})$, $\{e_{i,j}\}_{i,j=1}^n$ is the system of matrix units of the algebra $M_n(\mathbb{R})$, $n = 2k$.

a) Consider the associative algebra

$$\mathcal{A} = \mathbb{R}\mathbf{1} + \sum_{i=1}^{k} \mathbb{R}e_{2i-1,2i}.$$

This algebra is an inner Rickart algebra which is not involutive [1, p. 30].

b) Similarly, the associative algebra

$$\mathcal{A} = \mathbb{R}\mathbf{1} + \sum_{i=1}^{k} \mathbb{R}e_{2i,2i-1},$$

is an inner Rickart algebra.

c) Consider the associative algebra

$$\mathcal{A} = \sum_{i=1}^{k} \mathbb{R}e_{2i-1,2i}$$

This algebra is an inner Rickart algebra without a unit element [1, Page 30, Example 3]. Indeed, since $A^2 = 0$, then

$$^{\perp}\{\mathbf{0}\} \cap \mathcal{A}^2 = \mathbf{0}(\mathcal{A})\mathbf{0} \cap \mathcal{A}^2.$$

Moreover, $\{e_{2i-1,2i}\}^{\perp} \cap \mathcal{A}^2 = \mathbf{0}(\mathcal{A})\mathbf{0} \cap \mathcal{A}^2$ for any i from $\{1, 2, \ldots, k\}$.

d) Similar to example c) the associative algebra

$$\mathcal{A} = \sum_{i=1}^{k} \mathbb{R}e_{2i,2i-1}$$

is also an inner Rickart algebra without a unit element.

3) Let $\mathcal{A}$ be an associative algebra of dimension $n+1$, and $\{e, e_1, e_2, \ldots, e_n\}$ be a basis of $\mathcal{A}$ such that

$$ee = e,\ ee_i + e_ie = 2e_i,\ e_ie_j = -e_je_i,\ i,j = 1,2,...,n.$$

Then $\mathcal{A}$ is an associative inner Rickart algebra. Indeed, with respect to the Jordan multiplication $a \cdot b = \frac{1}{2}(ab + ba)$ we have

$$e \cdot e = e,\ e \cdot e_i = e_i,\ e_i \cdot e_j = 0,\ i,j = 1,2,...,n.$$

The Jordan algebra $\mathcal{N}$, generated by the elements e_1, e_2, ..., e_n, is a nilradical of the Jordan algebra $(\mathcal{A}, \cdot)$, the quotient $(\mathcal{A}, \cdot)/\mathcal{N}$ is isomorphic to $\mathbb{F}e$, and $\mathcal{N}$ does not have nilpotent elements with a square root in $\mathcal{A}$. Therefore, by theorem 2.6 in [2] $(\mathcal{A}, \cdot)$ is an inner RJ-algebra.

4) A vector space $\mathcal{A}$ of dimension $n+1$ with a basis $\{e, e_1, e_2, \ldots, e_n\}$ such that

$$ee = e,\ ee_i = e_ie = e_i,\ e_i^2 = 0,\ e_ie_j = 0, i,j = 1,2,...,n.$$

is a commutative associative algebra and the Jordan algebra $(\mathcal{A}, \cdot)$ is the inner RJ-algebra from 3).

5) A vector space $\mathcal{A}_2^2$ of dimension 2 with a basis $\{e_1, e_2\}$ such that

$$e_1e_1 = e_1, e_1e_2 = e_2, e_2e_1 = 0, e_2e_2 = 0$$

is an associative algebra (see Table 1) and the Jordan algebra $(\mathcal{A}_2^2, \cdot)$ is an inner RJ-algebra. At the same time a vector space $\mathcal{A}_2^3$ of dimension 2 with a basis $\{e_1, e_2\}$ such that

$$e_1e_1 = e_1, e_1e_2 = 0, e_2e_1 = e_2, e_2e_2 = 0$$

is an associative algebra (Table 1) and the Jordan algebra $(\mathcal{A}_2^3, \cdot)$ coincides with the inner RJ-algebra $(\mathcal{A}_2^2, \cdot)$.

6) A vector space $\mathcal{A}_3^4$ of dimension 3 with a basis $\{e_1, e_2, e_3\}$ such that

$$e_1e_3 = e_2, e_2e_3 = e_2, e_3e_3 = e_3,$$

the omitted products vanish, is an associative algebra (see Table 2) and the Jordan algebra $(\mathcal{A}_3^4, \cdot)$ is an inner RJ-algebra by Table 2. At the same time a vector space $\mathcal{A}_3^6$ of dimension 3 with a basis $\{e_1, e_2, e_3\}$ such that

$$e_3e_1 = e_2, e_3e_2 = e_2, e_3e_3 = e_3,$$

the omitted products vanish, is an associative algebra and the Jordan algebra $(\mathcal{A}_3^6, \cdot)$ coincides with the inner RJ-algebra $(\mathcal{A}_3^4, \cdot)$.

7) A vector space $\mathcal{A}_4^8$ of dimension 4 with a basis $\{e_1, e_2, e_3, e_4\}$ such that

$$e_1e_1 = e_1, e_3e_1 = e_3, e_4e_3 = e_2, e_2e_1 = e_2,$$

the omitted products vanish, is an associative algebra (cf. Table 2.3 in [5]) and the Jordan algebra $(\mathcal{A}_4^8, \cdot)$ is an inner RJ-algebra. At the same time a vector space $\mathcal{A}_4^9$ of dimension 4 with a basis $\{e_1, e_2, e_3, e_4\}$ such that

$$e_1e_1 = e_1, e_1e_3 = e_3, e_3e_4 = e_2, e_1e_2 = e_2,$$

the omitted products vanish, is an associative algebra (Table 2.3 in [5]) and the Jordan algebra $(\mathcal{A}_4^9, \cdot)$ coincides with the inner RJ-algebra $(\mathcal{A}_4^8, \cdot)$.

8) A vector space $\mathcal{A}_4^{19}$ of dimension 4 with a basis $\{e_1, e_2, e_3, e_4\}$ such that

$$e_1e_1 = e_1, e_2e_2 = e_2, e_2e_3 = e_3, e_3e_1 = e_3, e_4e_1 = e_4,$$

the omitted products vanish, is an associative algebra (Table 2.3 in [5]) and the Jordan algebra $(\mathcal{A}_4^{19}, \cdot)$ is an inner RJ-algebra. At the same time a vector space $\mathcal{A}_4^{20}$ of dimension 4 with a basis $\{e_1, e_2, e_3, e_4\}$ such that

$$e_1e_1 = e_1, e_2e_2 = e_2, e_3e_2 = e_3, e_1e_3 = e_3, e_1e_4 = e_4,$$

the omitted products vanish, is an associative algebra (Table 2.3 in [5]) and the Jordan algebra $(\mathcal{A}_4^{20}, \cdot)$ coincides with the inner RJ-algebra $(\mathcal{A}_4^{19}, \cdot)$. Thus, two mutually non-isomorphic inner

Rickart algebras give the same inner RJ-algebra. The examples above are a motivation to introduce the notion of an inner Rickart algebra.

Lemma 2.1. *Let $\mathcal{A}$ be an inner Rickart algebra. Then the algebra $\mathcal{A}$ has no a nilpotent element a such that $a^3 \neq 0$.*

Proof. Take an element $a \in \mathcal{A}$. Suppose that $a^4 = 0$. Then $a^2 \in {}^{\perp}\{a\} \cap \mathcal{A}^2$. Since $\mathcal{A}$ is an inner Rickart algebra, there exists a nonzero idempotent $e \in \mathcal{A}$ such that ${}^{\perp}\{a\} \cap \mathcal{A}^2 = e\mathcal{A}e \cap \mathcal{A}^2$. Therefore $aea = 0$ and $ea^2 = a^2$ since $e \in e\mathcal{A}e \cap \mathcal{A}^2$. Hence, $a^3 = aa^2 = aea^2 = (aea)a = 0$.

Further, we shall apply the induction. The induction assumption is $a^k = 0$, $3 \leq k \leq n$, implies that $a^3 = 0$, where the induction step is n. If $n = 4$ then the induction assumption is valid by the previous paragraph.

Suppose that $a^{n+1} = 0$. Then, if $n = 2k$, then $n \geq 4$ and $a^{n+1}a = a^{2k+2} = (a^2)^{k+1} = 0$ and $1 \leq k+1 < n$. By the inductive assumption $a^3 = 0$. So, $U_a a = 0$ and $a \in {}^{\perp}\{a\} \cap \mathcal{A}^2$. Since $\mathcal{A}$ is an inner Rickart algebra, there exists an idempotent $f \in \mathcal{A}$ such that ${}^{\perp}\{a\} \cap \mathcal{A}^2 = f\mathcal{A}f \cap \mathcal{A}^2$. Therefore $fff = f \in f\mathcal{A}f \cap \mathcal{A}^2$, $afa = 0$ and $fa = a$. Hence, $a^2 = afa = 0$ since $f\mathcal{A}f$ is a subalgebra and $a^2 \in f\mathcal{A}f$. By the previous paragraph $b^3 = 0$.

If $n = 2k+1$, then $a^{n+1} = a^{2k+2} = (a^2)^{k+1} = 0$ and $1 \leq k+1 < n$. Again, by the inductive assumption $a^3 = 0$. Hence, by the previous paragraph $b^3 = 0$. Thus, by the induction, we obtain that for $a \in \mathcal{A}^2$, if $a = b^2$ and $a^n = 0$ for some natural number n, then $b^3 = 0$. The proof is complete. □

Theorem 2.2. *A nilpotent associative algebra that has no nilpotent elements with a nonzero square root is an inner Rickart algebra.*

Proof. Let $\mathcal{A}$ be a nilpotent associative algebra that has no nilpotent elements with a nonzero square root. Let a be a nonzero element from the algebra $\mathcal{A}$. Since $\mathcal{A}$ is nilpotent, then $a^m = 0$ for some element m. Then $a^2 = 0$. Otherwise, the algebra $\mathcal{A}$ contains an element b, which is a nilpotent element with a nonzero square root. So, $a^2 = 0$ for every non-zero element a from $\mathcal{A}$. Therefore, for any a, b from $\mathcal{A}$,

$$ab + ba = (a+b)^2 - a^2 - b^2 = 0, \;\; aba + baa = aba = 0.$$

The algebra $\mathcal{A}$ is an inner Rickart algebra. Indeed, the algebra $\mathcal{A}$ does not contain a nonzero idempotent. In particular, there is no non-zero idempotent e such that

$${}^{\perp_q}\{\mathbf{0}\} = e\mathcal{A}e.$$

However, since $\mathcal{A}^2 = \{\mathbf{0}\}$, we have

$${}^{\perp}\{\mathbf{0}\} \cap \mathcal{A}^2 = \mathbf{0}\mathcal{A}\mathbf{0} \cap \mathcal{A}^2.$$

Here ${}^{\perp}\{\mathbf{0}\} = \mathcal{A}$. Similarly, for any element a from $\mathcal{A}$, we have

$${}^{\perp}\{a\} \cap \mathcal{A}^2 = \{\mathbf{0}\} = \mathbf{0}\mathcal{A}\mathbf{0} \cap \mathcal{A}^2.$$

The proof is complete. □

Theorem 2.3. *Let $\mathcal{A}$ be a nilpotent associative algebra. Then $\mathcal{A}$ is an inner Rickart algebra if and only if, for any element a in $\mathcal{A}$, $a^2 = 0$.*

Proof. Suppose that $\mathcal{A}$ is an inner Rickart algebra. Since the algebra $\mathcal{A}$ is nilpotent, there is no nonzero idempotent in the algebra A. A has only idempotent element $\mathbf{0}$.

Let a be an arbitrary nonzero element from the algebra $\mathcal{A}$ such that $a^2 \neq 0$. Since the algebra $\mathcal{A}$ is nilpotent, a is a nilpotent element. Then, by Lemma 2.1, $a^3 = 0$. Then

$$a^2 \in^{\perp} \{a\} \cap \mathcal{A}^2.$$

Since the algebra $\mathcal{A}$ is nilpotent, there is no nonzero idempotent in the algebra A. In particular, there is no nonzero idempotent e such that

$$\{a\}^{\perp} \cap \mathcal{A}^2 = e\mathcal{A}e \cap \mathcal{A}^2.$$

So, $\mathcal{A}$ is not an inner Rickart associative algebra. This is a contradiction. So, $a^2 = 0$.

Now, suppose that for any element a in $\mathcal{A}$, $a^2 = 0$. Then, by Theorem 2.2 $\mathcal{A}$ is an inner Rickart algebra. This ends the proof. □

Theorem 2.4. *Let $\mathcal{A}$ be an inner Rickart associative algebra over a field $\mathbb{F}$, $\mathcal{N}$ be an n-dimensional commutative nilradical of the associative algebra $\mathcal{A}$ such that the codimension of $\mathcal{N}$ is 1 and $\mathcal{N}^2 = \{a^2 : a \in \mathcal{N}\} \neq \{0\}$. Then there is a nonzero idempotent $e \in \mathcal{A}$ such that $\mathcal{A} = \mathbb{F}e + \mathcal{N}$ and for any basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathcal{N}$ the following conditions are valid*

$$e_i e_j e_k = 0, \qquad i, j, k = 1, 2, \ldots, n, \tag{1.1}$$

$$e(e_i e_j) = (e_i e_j)e = e_i e_j, \qquad i, j = 1, 2, \ldots, n, \tag{1.2}$$

$$e_i e e_j + e_j e e_i = 0, \qquad i, j = 1, 2, \ldots, n, \tag{1.3}$$

$$e e_i \in \mathcal{N}, e_i e \in \mathcal{N}. \tag{1.4}$$

Conversely, any associative algebra with the basis $\{e, e_1, e_2, \ldots, e_n\}$, where e is an idempotent element, satisfying the properties (1.1)–(1.4), is an inner Rickart associative algebra.

Proof. By Lemma 2.1, the associative algebra $\mathcal{N}$ has no nilpotent elements a such that $a^3 \neq 0$. Let a be a nonzero element from the nilradical $\mathcal{N}$. Then $a^3 = 0$. Therefore, for any a, b from $\mathcal{N}$,

$$0 = (a+b)^3 - a^3 - b^3 = (a^2 + ab + ba + b^2)(a+b) - a^3 - b^3,$$
$$= a^3 + a^2 b + aba + ab^2 + ba^2 + bab + b^2 a + b^3 - a^3 - b^3 = 3a^2 b + 3b^2 a,$$
$$0 = (a-b)^3 - a^3 + b^3 = (a^2 - ab - ba + b^2)(a-b) - a^3 + b^3,$$
$$= a^3 - a^2 b - aba + ab^2 - ba^2 + bab + b^2 a - b^3 - a^3 - b^3 = -3a^2 b + 3b^2 a.$$

Hence,

$$a^2 b = 0, b^2 a = 0.$$

As well as,

$$abc = a\frac{1}{2}((b+c)^2 - b^2 - c^2) = \frac{1}{2}(a(b+c)^2 - ab^2 - ac^2) = 0.$$

So, for any elements a, b, c from $\mathcal{N}$,

$$abc = 0. \tag{1.5}$$

Let a be an arbitrary nonzero element from the radical $\mathcal{N}$ such that $a^2 \neq 0$ and $\mathcal{N}^2 = \{x^2 : x \in \mathcal{N}\}$. Then there exists a nonzero idempotent $e \in \mathcal{A}$ such that

$$a^2 \in^{\perp} \{a\} \cap \mathcal{A}^2 = e\mathcal{A}e \cap \mathcal{A}^2, \ \ aea = 0.$$

Due to (1.5) we have $\mathcal{N}^2 \subseteq^{\perp} \{a\} \cap \mathcal{A}^2 = e\mathcal{A}e \cap \mathcal{A}^2$. Hence,

$$\mathcal{N}^2 \subseteq e\mathcal{A}e \cap \mathcal{A}^2.$$

As well as,

$$a^2 \in^{\perp} \{a^2\} \cap \mathcal{A}^2 =^{\perp} \{a\} \cap \mathcal{A}^2.$$

Let b be another nonzero element from the nilradical $\mathcal{N}$ such that $b^2 \neq 0$. Then there exists a nonzero idempotent $f \in \mathcal{A}$ such that

$$b^2 \in^{\perp} \{b\} \cap \mathcal{A}^2 = f\mathcal{A}f \cap \mathcal{A}^2$$

and

$$\mathcal{N}^2 \subseteq f\mathcal{A}f \cap \mathcal{A}^2$$

Thus, $ea^2 = fa^2 = a^2$, $eb^2 = fb^2 = b^2$. Furthermore,

$$ec = fc = c, \ \ c \in \mathcal{N}^2.$$

Since codimension of $\mathcal{N}$ equals 1 we have $e = f$. If we take $a + b$ then

$$a^2, \ b^2, \ (a+b)^2 \in^{\perp} \{a\} \cap \mathcal{A}^2 = e\mathcal{A})e \cap \mathcal{A}^2,$$
$$(a+b)^2 = e(a+b)^2 = a^2 + 2e(ab) + b^2,$$

$$(a+b)^2 = (a+b)^2 e = a^2 + 2(ab)e + b^2.$$

Hence,

$$e(ab) = (ab)e = ab. \tag{1.6}$$

Let $e_1, e_2, \ldots, e_n$ be a basis in $\mathcal{N}$. Then, by (1.5) we have (1.1). Also, by (1.6), we have (1.2). At the same time,

$$e_i e e_j + e_j e e_i = 0,\ i, j = 1, 2, \ldots, n,$$

i.e. (1.3) is valid. Indeed, since $aea = 0$, $a \in \mathcal{N}$, then, for any i, j, $(e_i + e_j)e(e_i + e_j) = 0$. Hence,

$$e_i e e_j + e_j e e_i = 0.$$

So, (1.3) is valid. Since $\mathcal{N}$ is an ideal, we have (1.4) is also valid.

Now let us prove that the associative algebra $\mathcal{A}$ with a basis $\{e, e_1, e_2, \ldots, e_n\}$, satisfying the properties (1.1)–(1.4), is an inner Rickart algebra. We have

$$^{\perp}\{e\} \cap A^2 = \{\mathbf{0}\} = \mathbf{0}\mathcal{A}\mathbf{0} \cap \mathcal{A}^2,$$

since $A^2 \subseteq eAe$ by (1.2).

We take an arbitrary element of the form

$$a = \sum_{i=1}^{n} \lambda_i e_i.$$

By equalities (1.1) and (1.3), we have

$$e \in^{\perp} \{a\} \cap \mathcal{A}^2 = \mathcal{A}^2 = e\mathcal{A}e \cap \mathcal{A}^2.$$

Now we take an element of the form $\lambda e + a$, $\lambda \neq 0$. Since $\mathcal{N}$ is an ideal, we have

$$(\lambda e + a)b^2(\lambda e + a) = \lambda e b^2 \lambda e + \lambda e b^2 a + a b^2 \lambda e + a b^2 a = \lambda e b^2 \lambda e = \lambda^2 b^2 \neq 0$$

for every $b \in \mathcal{N}$. We also have

$$(\lambda e + a)(\mu e + b)^2(\lambda e + a) = (\lambda e + a)(\mu^2 e + 2\mu e b + b^2)(\lambda e + a)$$
$$= \lambda e \mu^2 e \lambda e + c = \lambda^2 \mu^2 e + c,$$

where

$$c = 2[\lambda e \mu^2 e a + a \mu^2 e a + a(\mu^2 e + 2\mu e b + b^2)(\lambda e + a)]$$

and $c \in \mathcal{N}$ since $\mathcal{N}$ is an ideal. Therefore $(\lambda e + a)(\mu e + b)^2(\lambda e + a) \neq 0$. Hence,

$$^{\perp}\{\lambda e + a\} \cap \mathcal{A}^2 = \{0\} = \mathbf{0}\mathcal{A}\mathbf{0} \cap \mathcal{A}^2.$$

So, $\mathcal{A}$ is an inner Rickart algebra. □

Remark 2.1. *Note that the associative algebra, indicated in theorem 2.4 is a direct sum of an indecomposable non-nilpotent algebra and a nilpotent algebra. If, for example, the nilradical $\mathcal{N}$ of an inner Rickart algebra $\mathcal{A}$ is of codimension 2, then the quantity of non-nilpotent indecomposable algebras in the direct sum of $\mathcal{A}$ may be one or two. For example, the inner Rickart algebra*

$$\mathcal{A}_1 : e_1^2 = e_1, e_2^2 = e_2, e_1 n_1 = n_1, n_2 e_2 = n_2,$$

where $\{e_1, e_2, n_1, n_2\}$ is a basis of the algebra $\mathcal{A}_1$ and the omitted products vanish, is decomposable. The subalgebras

$$\mathcal{B}_1 : e_1^2 = e_1, e_1 n_1 = n_1, \mathcal{B}_2 : e_2^2 = e_2, e_2 n_2 = n_2$$

are inner Rickart algebras satisfying the conditions of theorem 2.4 and $\mathcal{A}_1 = \mathcal{B}_1 \oplus \mathcal{B}_2$. The associative algebra

$$\mathcal{A}_2 : e_1^2 = e_1, e_2^2 = e_2, n_1 e_1 = n_1, e_2 n_1 = n_1, n_2 e_1 = n_2,$$

where $\{e_1, e_2, n_1, n_2\}$ is a basis of the algebra $\mathcal{A}_2$ and the omitted products vanish, is indecomposable. The nilradical $\mathcal{N} =< n_1, n_2 >$ is an inner Rickart algebra by Theorem 2.3. Then $\mathcal{A}_2$ is also an inner Rickart algebra by Theorem 3.3. The subalgebras

$$\mathcal{C}_1 : e_1^2 = e_1, n_1 e_1 = n_1, n_2 e_1 = n_2, \mathcal{C}_2 : e_2^2 = e_2, e_2 n_1 = n_1$$

of the algebra $\mathcal{A}_2$ are inner Rickart algebras satisfying the conditions of Theorem 2.4.

Theorem 2.5. *Let $\mathcal{A}$ be an associative algebra with the degenerate radical $\mathcal{D}eg(\mathcal{A})$, which does not have nilpotent elements a such that $a^2 \neq 0$. Then, if the quotient $\mathcal{A}/\mathcal{D}eg(\mathcal{A})$ is an inner Rickart algebra, then $\mathcal{A}$ is an inner Rickart algebra.*

Proof. Let $\mathcal{B} = \mathcal{A}/\mathcal{D}eg(\mathcal{A})$ and a be an arbitrary nontrivial element in $\mathcal{A}$. Then there exists an idempotent e in $\mathcal{A}$ such that

$$^{\perp}\{a\} \cap \mathcal{B}^2 = e\mathcal{B}e \cap \mathcal{B}^2.$$

Hence $^{\perp}\{a\} \cap \mathcal{A}^2 = e\mathcal{A}e \cap \mathcal{A}^2$. Indeed, by the condition of the theorem, for any element $b \in \mathcal{D}eg(\mathcal{A})$, $b^2 = 0$. So

$$\mathcal{D}eg(\mathcal{A}) \cap \mathcal{A}^2 = \{0\} \text{ and } \mathcal{B}^2 \equiv \mathcal{A}^2.$$

Thus $^{\perp}\{a\} \cap \mathcal{A}^2 \equiv^{\perp} \{a\} \cap \mathcal{B}^2 = e\mathcal{B}e \cap \mathcal{B}^2 \equiv e\mathcal{A}e \cap \mathcal{B}^2 = e\mathcal{A}e \cap \mathcal{A}^2$. Hence $^{\perp}\{a\} \cap \mathcal{A}^2 = e\mathcal{A}e \cap \mathcal{A}^2$. Also, for every trivial element $a \in \mathcal{A}$ we have $^{\perp}\{a\} \cap \mathcal{A}^2 = \mathcal{A}^2 = \mathbf{1}\mathcal{A}\mathbf{1} \cap \mathcal{A}^2$, where $\mathbf{1}$ is an element of $\mathcal{A}$ such that $\mathbf{1}\mathcal{A}^2 = \mathcal{A}^2\mathbf{1} = \mathcal{A}^2$. So $\mathcal{A}$ is an inner Rickart algebra. The proof is complete. □

Corollary 2.6. *Let $\mathcal{A}$ be a finite-dimensional associative algebra over a perfect field $\mathbb{F}$, with a nilpotent radical $\mathcal{N}il(\mathcal{A})$, which does not have nonzero nilpotent elements with a square root in $\mathcal{A}$. If the quotient $\mathcal{A}/\mathcal{N}il(\mathcal{A})$ does not have nonzero nilpotent elements, then $\mathcal{A}$ is an inner Rickart algebra, and there exist pairwise orthogonal idempotents $e_1, e_2, \ldots, e_m$ in $\mathcal{A}$ such that*

$$\mathcal{A} = (e_1\mathbb{F} \oplus e_2\mathbb{F} \oplus \cdots \oplus e_m\mathbb{F}) \dot{+} \mathcal{N}il(\mathcal{A}).$$

Proof. By the Cartan-Frobenius-Molien theorem, we have $\mathcal{A} = \mathcal{S} \oplus \mathcal{N}il(A)$, where $\mathcal{S}$ is semi-simple. By the condition of the corollary, $\mathcal{S}$ does not have nonzero nilpotent elements. Hence, $\mathcal{S} = e_1\mathbb{F} \oplus e_2\mathbb{F} \oplus \cdots \oplus e_m\mathbb{F}$, where $e_1, e_2, \ldots, e_m$ are pairwise orthogonal idempotents. So, $\mathcal{S}$ is an inner Rickart algebra. Therefore, by Theorem 2.5, $\mathcal{A}$ is an inner Rickart algebra. The proof is complete. □

Theorem 2.7. *Let $\mathcal{A}$ be a finite-dimensional inner Rickart algebra over a perfect field $\mathbb{F}$, with a nilpotent radical $\mathcal{N}il(\mathcal{A})$. Then there exist pairwise orthogonal idempotents $e_1, e_2, \ldots, e_m$ in $\mathcal{A}$ such that*

$$\mathcal{A} = (e_1\mathbb{F} \oplus e_2\mathbb{F} \oplus \cdots \oplus e_m\mathbb{F}) \dot{+} \mathcal{N}il(\mathcal{A}).$$

Proof. By the Cartan-Frobenius-Molien theorem, every finite-dimensional associative algebra $\mathcal{A}$ over the real or complex numbers can be represented as $\mathcal{A} = \mathcal{S}\dot{+}\mathcal{N}$, where $\mathcal{N}$ is a nil-radical and $\mathcal{S}$ is a semi-simple algebra. In its turn $\mathcal{S}$ can be represented as

$$\mathcal{S} = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \ldots \mathcal{C}_n,$$

where $\mathcal{C}_i$, $i = 1, 2, \ldots n$, are simple algebras and $\mathcal{C}_i \cong M_{n_i}(\mathcal{D}_i)$, where $M_{n_i}(\mathcal{D}_i)$ is the algebra of $n_i \times n_i$ matrices with entries from a division algebra $\mathcal{D}_i$. Therefore, by item 2) of Remark 1.6 in [3], every finite-dimensional inner Rickart algebra $\mathcal{A}$ has the following form

$$\mathcal{A} = \mathcal{S}\dot{+}\mathcal{N}il(\mathcal{A}),$$

where $\mathcal{N}il(\mathcal{A})$ is the nilpotent radical of $\mathcal{A}$ and $\mathcal{S}$ is a semisimple subalgebra every simple subalgebra of which is isomorphic to only one of the following algebras: an abelian algebra, the algebra of 2×2-matrices over a division algebra, the algebra of 3×3-matrices over a division algebra.

Since the nilradical $\mathcal{N}il(\mathcal{A})$ does not contain nonzero idempotents the factor-algebra $\mathcal{A}/\mathcal{N}il(\mathcal{A})$ contains all nonzero idempotents of the algebra $\mathcal{A}$. So, the factor-algebra $\mathcal{A}/\mathcal{N}il(\mathcal{A})$ is an inner

Rickart algebra. Since $\mathcal{A}/\mathcal{N}il(\mathcal{A}) \cong \mathcal{S}$ we have $\mathcal{S}$ is also an inner Rickart algebra. At the same time, $\mathcal{S}$ is a direct sum of simple algebras, every of which is an inner Rickart algebra.

By the examples on page 35 of [2] and on page 8 of [3] the algebras $M_2(\mathbb{F})$, $M_3(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, are not inner Rickart algebras. Therefore, there exist pairwise orthogonal idempotents $e_1, e_2, \ldots, e_m$ in $\mathcal{A}$ such that

$$\mathcal{S} = e_1\mathbb{F} \oplus e_2\mathbb{F} \oplus \cdots \oplus e_m\mathbb{F}.$$

The proof is complete. □

Theorem 2.8. *Let $\mathcal{A}$ be an indecomposible inner Rickart associative algebra over a field $\mathbb{F}$, $\mathcal{N}$ be an n-dimensional commutative nilradical of the associative algebra $\mathcal{A}$ such that the codimension of $\mathcal{N}$ is 2 and $\mathcal{N}^2 = \{a^2 : a \in \mathcal{N}\} \neq \{0\}$. Then there exist two mutually orthogonal nonzero idempotents p_1, $p_2 \in \mathcal{A}$ such that $\mathcal{A} = (\mathbb{F}p_1 \oplus \mathbb{F}p_2)\dot{+}\mathcal{N}$ and for any basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathcal{N}$ one of the following two cases is valid*

the first case

$$e_ie_je_k = 0, \quad i,j,k = 1,2,\ldots,n, \tag{1.1}$$

$$p_1(e_ie_j) = (e_ie_j)p_1 = e_ie_j, \quad i,j = 1,2,\ldots,n, \tag{1.2}$$

$$e_ip_1e_j + e_jp_1e_i = 0, \quad i,j = 1,2,\ldots,n, \tag{1.3}$$

$$p_ke_i \in \mathcal{N}, e_ip_k \in \mathcal{N}, k = 1,2, \tag{1.4}$$

$$p_2(e_ie_j + e_je_i) = 0, i,j = 1,2,\ldots,n, \tag{1.5}$$

$$2p_1e_ip_1 = p_1e_i + e_ip_1, i = 1,2,\ldots,n, \tag{1.6}$$

the second case

$$e_ie_je_k = 0, \quad i,j,k = 1,2,\ldots,n, \tag{2.1}$$

$$e(e_ie_j) = (e_ie_j)e = e_ie_j, \quad i,j = 1,2,\ldots,n, \tag{2.2}$$

$$e_iee_j + e_jee_i = 0, \quad i,j = 1,2,\ldots,n, \tag{2.3}$$

$$ee_i \in \mathcal{N}, e_ie \in \mathcal{N}. \tag{2.4}$$

$$\forall b \in \mathcal{N} p_1b^2p_1 \neq 0, p_2b^2p_2 \neq 0 \text{ if } b^2 \neq 0. \tag{2.5}$$

Conversely, any associative algebra with the basis $\{e, e_1, e_2, \ldots, e_n\}$, where e is an idempotent element, satisfying the properties (1.1)–(1.6) or the properties (2.1)–(2.5), is an inner Rickart associative algebra.

Proof. By Theorem 2.7, there exist pairwise orthogonal idempotents p_1, p_2 in $\mathcal{A}$ such that

$$\mathcal{A} = (p_1\mathbb{F} \oplus p_2\mathbb{F})\dot{+}\mathcal{N}.$$

By the proof of Theorem 2.4, we have, for any elements a, b, c from $\mathcal{N}$,

$$a(bc) = (ab)c = 0. \tag{1.1}$$

Let a be an arbitrary nonzero element from the radical $\mathcal{N}$ such that $a^2 \neq 0$ and $\mathcal{N}^2 = \{x^2 : x \in \mathcal{N}\}$. Then there exists a nonzero idempotent $e \in \mathcal{A}$ such that

$$a^2 \in^{\perp} \{a\} \cap \mathcal{A}^2 = e\mathcal{A}e \cap \mathcal{A}^2, \ aea = 0.$$

Due to (1.1) we have $\mathcal{N}^2 \subseteq^{\perp} \{a\} \cap \mathcal{A}^2 = e\mathcal{A}e \cap \mathcal{A}^2$. Hence,

$$\mathcal{N}^2 \subseteq e\mathcal{A}e \cap \mathcal{A}^2.$$

As well as,

$$a^2 \in^{\perp} \{a^2\} \cap \mathcal{A}^2 =^{\perp} \{a\} \cap \mathcal{A}^2.$$

There are following three possible cases

$$\begin{cases} e = p_1, \\ e = p_2, \\ e = p_1 + p_2. \end{cases}$$

Now, suppose that $e = p_1$, i.e.,

$$^{\perp}\{a\} \cap \mathcal{A}^2 = p_1\mathcal{A}p_1 \cap \mathcal{A}^2.$$

The conditions (1.2)–(1.4) can be found similar to the proof of Theorem 2.4.

We consider the annihilator $^{\perp}\{p_2\}$. It is clear that $p_1 \in {}^{\perp}\{p_2\} \cap A^2$ since $p_1 \cdot p_2 = 0$ and $U_{p_2}p_1 = 0$. Also, from

$$U_{p_2}(U_{p_1}(A)) = U_{p_2p_1}(A) = 0$$

it follows that

$$\mathcal{N}^2 \subset U_{p_1}(A) \cap A^2 \subset {}^{\perp}\{p_2\} \cap A^2.$$

Hence, $p_2a^2p_2 = 0$ for all $a \in N$. From $p_2((a+b)^2 - a^2 - b^2)p_2 = 0$ for all a, $b \in \mathcal{N}$ it follows that $p_2(ab + ba)p_2 = 0$. Note that, since A is an inner Rickart algebra and $p_1 \in {}^{\perp}\{p_2\} \cap A^2$ we have

$$^{\perp}\{p_2\} \cap A^2 = U_{p_1}(A) \cap A^2.$$

Now, we take the annihilator $^{\perp}\{p_1\}$. It is clear that $p_2 \in {}^{\perp}\{p_1\} \cap A^2$ since $p_1 \cdot p_2 = 0$ and $U_{p_1}p_2 = 0$. Also, from

$$\mathcal{N}^2 \subset U_{p_1}(A))$$

it follows that

$$U_{p_2}(A) \cap A^2 = \{0, p_2\}.$$

Indeed, for any $a \in \mathcal{N}$,

$$p_2(\lambda p_1 + \mu p_2 + a)^2 p_2 \neq (\lambda p_1 + \mu p_2 + a)^2$$

if $\lambda \neq 0$ or $a \neq 0$. Hence, $U_{p_2}(A) \cap A^2 = \{0, p_2\}$. Thus, $p_2(N^2) = 0$, $p_2a_2 = 0$ for any a, $b \in \mathcal{N}$. From this it follows that $p_2(ab + ba) = 0$. In particular,

$$p_2(e_ie_j + e_je_i) = 0, i, j = 1, 2, \ldots, n. \tag{1.5}$$

Note that, since A is an inner Rickart algebra and $p_2 \in {}^{\perp}\{p_1\} \cap A^2$ we have

$$^{\perp}\{p_1\} \cap A^2 = U_{p_2}(A) \cap A^2.$$

Note that

$$p_2(\lambda p_1 + a)^2 p_2 = 0$$

for any $a \in \mathcal{N}$, $\lambda \in \mathbb{F}$. Hence, $(\lambda p_1 + a)^2 \in U_{p_1}(A) \cap A^2$, i.e.

$$p_1(p_1a + ap_1)p_1 = p_1ap_1 + p_1ap_1 = 2p_1ap_1 = p_1a + ap_1.$$

In particular,

$$2p_1e_ip_1 = p_1e_i + e_ip_1, i = 1, 2, \ldots, n. \tag{1.6}$$

Now let us prove that an associative algebra $\mathcal{A}$ with a basis $\{p_1, p_2, e_1, e_2, \ldots, e_n\}$, satisfying the properties (1.1)–(1.5), is an inner Rickart algebra.

We take the annihilator $^{\perp}\{p_1\}$. It is clear that $p_2 \in^{\perp} \{p_1\}$. We prove that

$$^{\perp}\{p_1\} \cap A^2 = U_{p_2}(\mathcal{A}) \cap \mathcal{A}^2.$$

Indeed, we have $U_{p_2}(\mathcal{A}) \subseteq^{\perp} \{p_1\}$. The equality

$$p_1(\lambda p_1 + \mu p_2 + a)^2 p_1 = 0,$$

where $a \in \mathcal{N}$, is valid if $\lambda = 0$. We have

$$p_1(\mu p_2 + a)^2 p_1 = p_1(\mu p_2^2 + \mu p_2a + a\mu p_2 + a^2)p_1 = a^2 \neq 0$$

by (1.2). Hence, $^{\perp}\{p_1\} \cap A^2 = \{0, p_2\}$. At the same time, by (1.5), $p_2a^2 = 0$ for any $a \in \mathcal{N}$, and

$$p_2(\lambda p_1 + \mu p_2 + a)^2 p_2 \neq (\lambda p_1 + \mu p_2 + a)^2$$

if $\lambda \neq 0$ or $a \neq 0$. Hence, $U_{p_2}(A) \cap A^2 = \{0, p_2\}$. So, $^{\perp}\{p_1\} \cap A^2 = U_{p_2}(\mathcal{A}) \cap \mathcal{A}^2$.

Now we take the annihilator $^{\perp}\{p_2\}$. It is clear that $p_1 \in^{\perp} \{p_2\}$. We prove that

$$^{\perp}\{p_2\} \cap A^2 = U_{p_1}(\mathcal{A}) \cap \mathcal{A}^2.$$

Indeed, we have $U_{p_1}(\mathcal{A}) \subseteq^{\perp} \{p_2\}$ and $\mathcal{N}^2 \subseteq U_{p_1}(\mathcal{A})$ by (1.2). Besides

$$p_1(\mu p_1 + a)^2 p_1 = p_1(\mu^2 p_1^2 + \mu p_1 a + a\mu p_1 + a^2)p_1 =$$
$$\mu^2 p_1 + \mu p_1(p_1 a + a p_1)p_1 + a^2 = \mu^2 p_1 + \mu 2 p_1 a p_1 + a^2 =$$
$$\mu^2 p_1 + \mu(p_1 a + a p_1) + a^2 = (\mu p_1 + a)^2$$

by condition (1.6). Thus,

$$U_{p_1}(\mathcal{A}) \cap \mathcal{A}^2 = \{(\mu p_1 + a)^2 : a \in \mathcal{N}^2, \lambda \in \mathbb{F}\}.$$

At the same time

$$p_2(\mu p_1 + a)^2 p_1 = 0.$$

Therefore, ${}^{\perp}\{p_2\} \cap A^2 = U_{p_1}(\mathcal{A}) \cap \mathcal{A}^2$.

Similar to the proof of theorem 2.4 we have, for any element of the form $\lambda e + a$, $\lambda \neq 0$,

$${}^{\perp}\{\lambda e + a\} \cap \mathcal{A}^2 = \{0\} = \mathbf{0}\mathcal{A}\mathbf{0} \cap \mathcal{A}^2.$$

Now we take an element of the form $\lambda p_1 + \mu p_2 + a$, $\lambda \neq 0$, $\mu \neq 0$ and consider the annihilator ${}^{\perp}\{\lambda p_1 + \mu p_2 + a\}$. Since $\mathcal{N}$ is an ideal, by the properties of the Peirce decomposition and (1.2), (1.5), we have

$$(\lambda p_1 + \mu p_2 + a)b^2(\lambda p_1 + \mu p_2 + a)$$
$$= (\lambda p_1 + \mu p_2)b^2(\lambda p_1 + \mu p_2) + (\lambda p_1 + \mu p_2)b^2 a + a b^2(\lambda p_1 + \mu p_2) + a b^2 a$$
$$= (\lambda p_1 + \mu p_2)b^2(\lambda p_1 + \mu p_2) = \lambda^2 p_1 b^2 p_1 = \lambda^2 b^2 \neq 0$$

if $b^2 \neq 0$, for every $b \in \mathcal{N}$. By (1.5), We also have

$$(\lambda p_1 + \mu p_2 + a)(\nu p_1 + \rho p_2 + b)^2(\lambda p_1 + \mu p_2 + a)$$
$$= (\lambda p_1 + \mu p_2 + a)(\nu^2 p_1 + \rho^2 p_2 + \nu b p_1 + \rho b p_2 + \nu p_1 b + \rho p_2 b + b^2)^2(\lambda p_1 + \mu p_2 + a)$$
$$= \lambda^2\nu^2 p_1 + \mu^2\rho^2 p_2 + c,$$

where $c \in \mathcal{N}$ since $\mathcal{N}$ is an ideal. Therefore

$$(\lambda p_1 + \mu p_2 + a)(\nu p_1 + \rho p_2 + b)^2(\lambda p_1 + \mu p_2 e + a) \neq 0$$

if $\nu \neq 0$ or $\rho \neq 0$. Hence,

$${}^{\perp}\{\lambda p_1 + \mu p_2 + a\} \cap \mathcal{A}^2 = \{0\} = \mathbf{0}\mathcal{A}\mathbf{0} \cap \mathcal{A}^2.$$

Now we take an element of the form $\mu p_2 + a$, $\mu \neq 0$ and consider the annihilator ${}^{\perp}\{\mu p_2 + a\}$. Since $\mathcal{N}$ is an ideal, by the properties of the Peirce decomposition and (1.2), (1.5), we have

$$(\mu p_2 + a)b^2(\mu p_2 + a) = \mu p_2 b^2 \mu p_2 + \mu p_2 b^2 a + a b^2 \mu p_2 + a b^2 a$$
$$= \mu p_2 b^2 \mu p_2 = 0$$

for every $b \in \mathcal{N}$.

By (1.3), (1.5), We also have

$$(\mu p_2 + a)(\nu p_1 + \rho p_2 + b)^2(\mu p_2 + a)$$
$$= (\mu p_2 + a)(\nu^2 p_1 + \rho^2 p_2 + \nu b p_1 + \rho b p_2 + \nu p_1 b + \rho p_2 b + b^2)^2(\mu p_2 + a)$$
$$= \mu^2\rho^2 p_2 + a\rho^2 p_2 a,$$

if $\rho \neq 0$. Hence,

$${}^{\perp}\{\mu p_2 + a\} \cap \mathcal{A}^2 = \{0, p_1, \mathcal{N}\}.$$

Since $p_1\mathcal{A}p_1 \cap \mathcal{A}^2 = \{0, p_1, \mathcal{N}^2\}$ by (1.2), we have

$${}^{\perp}\{\mu p_2 + a\} \cap \mathcal{A}^2 = p_1\mathcal{A}p_1 \cap \mathcal{A}^2.$$

So, $\mathcal{A}$ is an inner Rickart algebra.

Now, suppose that $e = p_1 + p_2$, i.e.,

$${}^{\perp}\{a\} \cap \mathcal{A}^2 = (p_1 + p_2)\mathcal{A}(p_1 + p_2) \cap \mathcal{A}^2.$$

Then, similar to the proof of Theorem 2.4 we have the conditions (2.1)–(2.4) are valid.

We have $\mathcal{N}^2 \subseteq (p_1+p_2)\mathcal{A}(p_1+p_2)$. Moreover, by (2.2) $\mathcal{A}^2 \subseteq (p_1+p_2)\mathcal{A}(p_1+p_2)$. If, for example, there exists $b \in \mathcal{N}$ such that

$$^{\perp}\{b\} \cap \mathcal{A}^2 = p_1\mathcal{A}p_1 \cap \mathcal{A}^2,$$

then we get the previous case. So we assume that, for every $b \in \mathcal{N}$,

$$^{\perp}\{b\} \cap \mathcal{A}^2 = (p_1+p_2)\mathcal{A}(p_1+p_2) \cap \mathcal{A}^2.$$

Hence, for every $b \in \mathcal{N}$ such that $b^2 \neq 0$, we have

$$p_1b^2p_1 \neq 0, p_2b^2p_2 \neq 0. \tag{2.5}$$

Now let us prove that an associative algebra $\mathcal{A}$ with a basis $\{p_1, p_2, e_1, e_2, \ldots, e_n\}$, satisfying the properties (2.1)–(2.5), is an inner Rickart algebra.

We take an arbitrary element of the form

$$a = \sum_{i=1}^{n} \lambda_i e_i.$$

By equalities (2.1) and (2.3), we have

$$e \in^{\perp} \{a\} \cap \mathcal{A}^2 = \mathcal{A}^2 = e\mathcal{A}e \cap \mathcal{A}^2.$$

We take the annihilator $^{\perp}\{p_1\}$. It is clear that $p_2 \in^{\perp} \{p_1\}$. We prove that

$$^{\perp}\{p_1\} \cap A^2 = U_{p_2}(\mathcal{A}) \cap \mathcal{A}^2.$$

Indeed, by the condition (2.5) $U_{p_2}(\mathcal{A}) \cap \mathcal{N}^2 = \oslash$. We have $U_{p_2}(\mathcal{A}) \subseteq^{\perp} \{p_1\}$. The equality

$$p_1(\lambda p_1 + \mu p_2 + a)^2 p_1 = 0,$$

where $a \in \mathcal{N}$, is valid if $\lambda = 0$. We have

$$p_1(\mu p_2 + a)^2 p_1 = p_1(\mu p_2^2 + \mu p_2 a + a\mu p_2 + a^2)p_1 = p_1 a^2 p_1 \neq 0$$

by (2.5). Hence, $^{\perp}\{p_1\} \cap A^2 = \{0, p_2\}$. At the same time, by (2.5), $p_2a^2p_2 \neq 0$ for any $a \in \mathcal{N}$ such that $a^2 \neq 0$, and

$$p_2(\lambda p_1 + \mu p_2 + a)^2 p_2 = (\lambda p_1 + \mu p_2 + a)^2$$

if $\lambda = 0$ and $a = 0$. Hence, $U_{p_2}(A) \cap A^2 = \{0, p_2\}$. So, $^{\perp}\{p_1\} \cap A^2 = U_{p_2}(\mathcal{A}) \cap \mathcal{A}^2$.

Similarly, we have

$$^{\perp}\{p_2\} \cap A^2 = U_{p_1}(\mathcal{A}) \cap \mathcal{A}^2.$$

Now we take an element of the form $\lambda p_1 + \mu p_2 + a$, $\lambda \neq 0$ or $\mu \neq 0$. Since $\mathcal{N}$ is an ideal, by the properties of the Peirce decomposition, we have

$$\begin{gathered}(\lambda p_1 + \mu p_2 + a)b^2(\lambda p_1 + \mu p_2 + a) \\ = (\lambda p_1 + \mu p_2)b^2(\lambda p_1 + \mu p_2) + (\lambda p_1 + \mu p_2)b^2 a + ab^2(\lambda p_1 + \mu p_2) + ab^2 a \\ = (\lambda p_1 + \mu p_2)b^2(\lambda p_1 + \mu p_2) \neq 0\end{gathered}$$

for every $b \in \mathcal{N}$. We also have

$$\begin{gathered}(\lambda p_1 + \mu p_2 + a)(\nu p_1 + \rho p_2 + b)^2(\lambda p_1 + \mu p_2 + a) = \\ (\lambda p_1 + \mu p_2 + a)(\nu^2 p_1 + \rho^2 p_2 + \nu b p_1 + \rho b p_2 + \nu p_1 b + \rho p_2 b + b^2)^2(\lambda p_1 + \mu p_2 + a) \\ = \lambda^2\nu^2 p_1 + \mu^2\rho^2 p_2 + c,\end{gathered}$$

where $c \in \mathcal{N}$ since $\mathcal{N}$ is an ideal. Therefore

$$(\lambda p_1 + \mu p_2 + a)(\nu p_1 + \rho p_2 + b)^2(\lambda p_1 + \mu p_2 + a) =\neq 0$$

if $\nu \neq 0$ or $\rho \neq 0$. Hence,

$$^{\perp}\{\lambda p_1 + \mu p_2 + a\} \cap \mathcal{A}^2 = \{0\} = \mathbf{0}\mathcal{A}\mathbf{0} \cap \mathcal{A}^2.$$

So, $\mathcal{A}$ is an inner Rickart algebra. □

Proposition 2.9. *The associative algebra $M_3(\mathbb{C})$ does not have nilpotent elements $a \in M_3(\mathbb{C})$ such that $a^3 \neq 0$.*

Proof. Let

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$$

be a matrix from $M_3(\mathbb{C})$. Then

$M^3 = (a+e+j) \cdot M^2 + (bd+cg+fh-aj-ae-je)M + (bfg+cdh-cge-bdj-afh+aje)E.$

Let $\alpha = a+e+j$, $\beta = bd+cg+fh-aj-ae-je$ and $\gamma = bfg+cdh-cge-bdj-afh+aje$. Then

$$M^3 = \alpha \cdot M^2 + \beta \cdot M + \gamma \cdot E,$$

where E is the unit matrix. Suppose that $M^k = 0$, $M^{k-1} \neq 0$ for an integer number $k > 2$. Multiplying the last equality by M^{k-1} on the right side yields

$$M^{k+2} = \alpha \cdot M^{k+1} + \beta \cdot M^k + \gamma \cdot M^{k-1}.$$

Hence, $\gamma \cdot M^{k-1} = 0$ and $\gamma = 0$ since $M^{k-1} \neq 0$. So,

$$M^3 = \alpha \cdot M^2 + \beta \cdot M.$$

Multiplying the last equality by M^{k-2} on the right side yields

$$M^{k+1} = \alpha \cdot M^k + \beta \cdot M^{k-1}.$$

From this, it follows that

$$0 = 0 + \beta \cdot M^{k-1}$$

since $M^k = 0$. At the same time $M^{k-1} \neq 0$. So, $\beta = 0$. Similarly, we have $\alpha = 0$. Hence, $M^3 = 0$. This ends the proof. □

Remark 2.2. *By Proposition 2.9 and the example on page 8 of* [3] *the condition of Lemma 2.1 is not sufficient for an associative algebra $\mathcal{A}$ to be an inner Rickart algebra.*

3. Description of inner Baer algebras with a nonzero nilradical

An associative algebra $\mathcal{A}$ is called an inner Baer algebra if it satisfies the following property: for each subset $S \subseteq \mathcal{A}$ there exists an idempotent $e \in \mathcal{A}$ such that ${}^{\perp_q}\mathcal{S} = e\mathcal{A}e$.

The following theorem is valid by Theorem 3.5 in [2].

Theorem 3.1. *The following conditions are equivalent:*

- *$\mathcal{A}$ is an inner Baer algebra;*
- *$\mathcal{A}$ is an inner Rickart algebra and the set of all idempotents of $\mathcal{A}$ is a complete lattice.*

By theorems 2.7 and 3.1, we have the following theorem.

Theorem 3.2. *Every finite-dimensional inner Rickart algebra over a perfect field is an inner Baer algebra.*

Theorem 3.3. *Let $\mathcal{A}$ be an associative algebra with the degenerate radical $\mathcal{D}eg(\mathcal{A})$, which does not have nilpotent elements a such that $a^2 \neq 0$. Then, if the quotient $\mathcal{A}/\mathcal{D}eg(\mathcal{A})$ is an inner Baer algebra, then $\mathcal{A}$ is an inner Baer algebra.*

Proof. Clearly $\mathcal{A}/\mathcal{D}eg(\mathcal{A})$ is an inner Rickart algebra. Hence, by Theorem 2.5, $\mathcal{A}$ is an inner Rickart algebra. By Theorem 3.1 the set of all idempotents in $\mathcal{A}/\mathcal{D}eg(\mathcal{A})$ forms a complete lattice. This lattice is also the lattice of all idempotents in $\mathcal{A}$. Hence by Theorem 3.1 $\mathcal{A}$ is an inner Baer algebra. The proof is complete. □

By Theorem 3.2 we have the following tables (cf. [5]).

Table 1
Two-dimensional associative algebras.

J	Multiplication table	Is this algebra an inner Baer algebra?	According to which theorem?
$\mathcal{A}_2^1$	$n_1n_1 = n_2$	-	Theorem 2.3
$\mathcal{A}_2^2$	$e_1e_1 = e_1$, $e_1n_1 = n_1$	+	Theorem 2.4, Theorem 3.3
$\mathcal{A}_2^3$	$e_1e_1 = e_1$, $n_1e_1 = n_1$	+	Theorem 2.4, Theorem 3.3
$\mathcal{A}_2^4$	$e_1e_1 = e_1$, $e_1n_1 = n_1$, $n_1e_1 = n_1$	+	Theorem 2.4, Theorem 3.3

Table 2
Three-dimensional associative algebras.

J	Multiplication table	Is this algebra an inner Baer algebra?	According to which theorem?
$\mathcal{A}_3^1$	$n_1n_3 = n_2$, $n_3n_1 = n_2$	-	Theorem 2.3
$\mathcal{A}_3^2(\alpha)$	$n_1n_3 = n_2$ $n_3n_1 = \alpha n_2, \alpha \in \mathbb{C} \setminus \{1, -1\}$	-	Theorem 2.3
$\mathcal{A}_3^2(\alpha)$	$n_1n_3 = n_2$ $n_3n_1 = \alpha n_2, \alpha = -1$	+	Theorem 2.2
$\mathcal{A}_3^3$	$n_1n_1 = n_2$, $n_1n_2 = n_3$ $n_2n_1 = n_3$	-	Theorem 2.3
$\mathcal{A}_3^4$	$n_1e_1 = n_2$, $n_2e_1 = n_2$ $e_1e_1 = e_1$	+	Theorem 2.4, Theorem 3.3
$\mathcal{A}_3^5$	$n_2e_1 = n_2$, $e_1n_1 = n_1$ $e_1e_1 = e_1$	+	Theorem 2.4, Theorem 3.3
$\mathcal{A}_3^6$	$e_1n_1 = n_2$, $e_1n_2 = n_2$ $e_1e_1 = e_1$	+	Theorem 2.4, Theorem 3.3
$\mathcal{A}_3^7$	$n_1e_1 = n_1$, $e_1e_1 = e_1$ $e_2n_1 = n_1$, $e_2e_2 = e_2$	+	Theorem 3.3
$\mathcal{A}_3^8$	$n_1e_1 = n_1$, $n_2e_1 = n_2$ $e_1n_1 = n_1$, $e_1e_1 = e_1$	+	Theorem 2.4, Theorem 3.3
$\mathcal{A}_3^9$	$n_2e_1 = n_2$, $e_1n_1 = n_1$ $e_1n_2 = n_2$, $e_1e_1 = e_1$	+	Theorem 2.4, Theorem 3.3
$\mathcal{A}_3^{10}$	$n_1e_1 = n_1$, $n_2e_1 = n_2$ $e_1n_1 = n_1$, $e_1n_2 = n_2$, $e_1e_1 = e_1$	+	Theorem 2.4, Theorem 3.3
$\mathcal{A}_3^{11}$	$n_1e_1 = n_2$, $n_2e_1 = n_2$ $e_1n_1 = n_2$, $e_1n_2 = n_2$, $e_1e_1 = e_1$	+	Theorem 2.4, Theorem 3.3
$\mathcal{A}_3^{12}$	$n_1n_1 = n_2$, $n_1e_1 = n_1$, $n_2e_1 = n_2$ $e_1n_1 = n_1$, $e_1n_2 = n_2$, $e_1e_1 = e_1$	-	Theorem 2.4

References

[1] Ayupov Sh. A., Arzikulov F. N., *Jordan counterparts of Rickart and Baer $*$-algebras*, Uzbek. Mat. Zh. no. 1, 13–3 (2016)

[2] Ayupov Sh. A., Arzikulov F. N., *Jordan counterparts of Rickart and Baer $*$-algebras, II*, S̃ao Paulo J. Math. Sci. **13**, 27-38 (2019)

[3] Arzikulov F. N., Khakimov U. I., *Description of finite-dimensional inner Rickart and Baer Jordan algebras*, Communications in Algebra (2023). https://doi.org/10.1080/00927872.2022.2164586

[4] Garces J., Li L., Peralta A., Tahlawi H., *A projection-less approach to Rickart Jordan structures*, Journal of Algebra **609**, 567–605 (2022)

[5] Rakhimov I. S., Rikhsiboev I. M., Basri W., *Complete lists of low dimensional complex associative algebras*, arXiv:0910.0932 [math.RA] (2009). https://doi.org/10.48550/arXiv.0910.0932

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON QUASI-DOUBLE LINES OF A PARTIAL MAPPING OF EUCLIDE SPACE

GULBADAN MATIEVA[1], CHOLPONAI ABDULLAYEVA[2], NURJAMAL KURBANBAEVA[1]

[1]Osh State University, Osh, Kyrgyzstan
[2]Kyrgyz-Uzbek International University named after B. Sydykov, Osh, Kyrgyzstan

Abstract. It is considered the problem related to partial mapping of 6- dimensional Euclidean space E_6 A family of smooth lines is given in the domain $\Omega \subset E_6$ so that through each point $X \in \Omega$ passes one line of a given family. A movable reper is chosen so that it was Frenet's reper for the line of the given family. The integral lines of the coordinate vectors fields of this reper form a Frenet's net. On a tangent to the line ω^3 of this net a point F_3^2 is defined in an invariant way. When the point X moves in the domain Ω the point F_3^2 describes its domain $\Omega_3^2 \subset E_6$. In this way we get a partial mapping $f_3^2 : \Omega \to \Omega_3^2$ such that $f_3^2(X) = F_3^2$. The subject of research is partial mapping of the six-dimensional Euclidean space E_6. The purpose of the study is to find the necessary and sufficient conditions for the existence of quasi-double lines of a partial mapping f_3^2. The study used: the method of external forms of Cartan and the method of moving reper. Necessary and sufficient conditions are found for lines belonging to 5dimensional distributions to be quasidouble lines of the partial mapping f_3^2.

Keywords: Euclidean space, Frenet frame, net of Frenet, partial mapping, distribution, quasi-double line.

In domain $\Omega \subset E_6$ it is considered a set of smooth lines such that through a point $X \in \Omega$ passed one line of given set. The moving reper $\mathfrak{R} = (X, \overrightarrow{e_i}) \quad (i, j, k = \overline{1,6})$ is reper of Frenet for the line ω^1 of the given set of smooth lines [1], [2]. Derivation formulas of the reper $\mathfrak{R}$ have a forme:

$$d\vec{X} = \omega^i \overrightarrow{e_i}, d\overrightarrow{e_i} = \omega_i^k \overrightarrow{e_k}.$$

The forms ω^i, ω_i^k satisfied structure equations of Euclidean space:

$$D\omega^i = \omega^k \wedge \omega_k^i, D\omega_i^k = \omega_i^j \wedge \omega_j^k, \omega_i^j + \omega_j^i = 0.$$

Integral lines of vector fields $\overrightarrow{e_i}$ are formed the net Σ_6 of Frenet [1] for the line ω^1 of the given set of lines. Since reper $\mathfrak{R}$ is constructed on tangent of lines of the net Σ_6, the forms ω_i^k are principal forms [1], in orther words:

$$\omega_i^k = \Lambda_{ij}^k \omega^j.$$

Taking into account (3) from (2) we have:

$$\Lambda_{ij}^k = -\Lambda_{kj}^i.$$

If we differentiate externally (3) then we have:

$$D\omega_i^k = d\Lambda_{ij}^k \wedge \omega^j + \Lambda_{ij}^k D\omega^j.$$

By using formula (2) from here we have got:

$$\omega_i^j \wedge \omega_j^k = d\Lambda_{ij}^k \wedge \omega^j + \Lambda_{ij}^k \wedge \omega^\ell \wedge \omega_\ell^j.$$

Taking in to account (3) from last formula is follows that:

$$\omega_i^j \wedge \Lambda_{j\ell}^k \omega^\ell = d\Lambda_{ij}^k \wedge \omega^j - \Lambda_{ij}^k \omega_\ell^j \wedge \omega^\ell$$

or

$$\Lambda_{j\ell}^k \omega_i^j \wedge \omega^\ell = d\Lambda_{ij}^k \wedge \omega^j - \Lambda_{ij}^k \wedge \omega_\ell^j \wedge \omega^\ell.$$

From here we found:

$$d\Lambda_{ij}^k \wedge \omega^j - \Lambda_{i\ell}^k \omega_j^\ell \wedge \omega^j - \Lambda_{j\ell}^k \omega_i^j \wedge \omega^\ell = 0$$

or

$$\left(d\Lambda_{ij}^k - \Lambda_{i\ell}^k \omega_j^\ell - \Lambda_{\ell j}^k \omega_i^\ell\right) \wedge \omega^j = 0.$$

By using lemma of Cartan [2] from here we have:

$$d\Lambda_{ij}^k - \Lambda_{i\ell}^k \omega_j^\ell - \Lambda_{\ell j}^k \omega_i^\ell = \Lambda_{ijm}^k \omega^m$$

or

$$d\Lambda_{ij}^k = \left(\Lambda_{ijm}^k + \Lambda_{il}^k \Lambda_{jm}^l + \Lambda_{lj}^k \Lambda_{im}^l\right) \omega^m.$$

were

$$B_{ikm}^j = \left(\Lambda_{ijm}^k + \Lambda_{il}^k \Lambda_{im}^l + \Lambda_{ij}^k \Lambda_{im}^l\right)$$

The system of variable $\left\{\Lambda_{ij}^k, \Lambda_{ijm}^k\right\}$ is formed geometrical object of second order. The formulas of Frenet [3] for the line ω^1 of given set have a form:

$$\begin{gathered}
d_1 \overrightarrow{e_1} = \Lambda_{11}^2 \overrightarrow{e_2}, \\
d_1 \overrightarrow{e_2} = \Lambda_{21}^1 \overrightarrow{e_1} + \Lambda_{21}^3 \overrightarrow{e_3}, \\
d_1 \overrightarrow{e_3} = \Lambda_{31}^2 \overrightarrow{e_2} + \Lambda_{31}^4 \overrightarrow{e_4}, \\
d_1 \overrightarrow{e_4} = \Lambda_{41}^3 \overrightarrow{e_3} + \Lambda_{41}^5 \overrightarrow{e_5}, \\
d_1 \overrightarrow{e_5} = \Lambda_{51}^4 \overrightarrow{e_4} + \Lambda_{51}^6 \overrightarrow{e_6}, \\
d_1 \overrightarrow{e_6} = \Lambda_{61}^5 \overrightarrow{e_5}.
\end{gathered}$$

and $\Lambda_{11}^3 = -\Lambda_{11}^3 = 0, \Lambda_{11}^4 = -\Lambda_{41}^1 = 0, \Lambda_{11}^5 = -\Lambda_{51}^1 = 0, \Lambda_{11}^6 = -\Lambda_{61}^1 = 0$
$\Lambda_{21}^5 = -\Lambda_{51}^2 = 0, \Lambda_{21}^4 = -\Lambda_{41}^2 = 0, \Lambda_{31}^5 = -\Lambda_{51}^3 = 0. \Lambda_{21}^6 = -\Lambda_{61}^2 = 0, \Lambda_{31}^6 = -\Lambda_{61}^3 = 0. \Lambda_{61}^4 = -\Lambda_{41}^6 = 0$

Here $k_1^1 = \Lambda_{11}^2, k_2^1 = \Lambda_{21}^3, k_3^1 = \Lambda_{31}^4, k_4^1 = \Lambda_{41}^5 = -\Lambda_{51}^4, k_5^1 = \Lambda_{51}^6 = -\Lambda_{61}^5 - \omega^1$ are the first, second, third, fourth and fifth curvatures of the line ω^1, respectively (where d_1 is the symbol of differentiation along the line ω^1).

The pseudo focus $F_i^j (i \neq j)$ of the tangent to ω^i line Σ_6 is determined by the following radius vector:

$$\overrightarrow{F_i^j} = \vec{X} - \frac{1}{\Lambda_{ij}^j} \overrightarrow{e_i} = \vec{X} + \frac{1}{\Lambda_{jj}^i} \vec{e}_i$$

There are five pseudofocuses on each tangent $(X, \vec{e}_i)$: on a tangent $(X, \vec{e}_1) - F_1^2, F_1^3, F_1^4, F_1^5, F_1^6$;
on a tangent $(X, \vec{e}_2) - F_2^1, F_2^3, F_2^4, F_2^5, F_2^6$;
on a tangent $(X, \vec{e}_3) - F_3^1, F_3^2, F_3^4, F_3^5, F_3^6$;

on a tangent $(X,\vec{e}_4)-F_4^1,F_4^2,F_4^3,F_4^5,F_4^6$;
on a tangent $(X,\vec{e}_5)-F_5^1,F_5^2,F_5^3,F_5^4,F_5^6$;
on a tangent $(X,\vec{e}_6)-F_6^1,F_6^2,F_6^3,F_6^4,F_6^5$.
A net Σ_6 in $\Omega\subset E_6$ is called a Frenet cyclic net, if the following repers
$\mathfrak{R}_1=(X,\vec{e}_1,\vec{e}_2,\vec{e}_3,\vec{e}_4,\vec{e}_5,\vec{e}_6)\,,\mathfrak{R}_2=(X,\vec{e}_2,\vec{e}_3,\vec{e}_4,\vec{e}_5,\vec{e}_6,\vec{e}_1)$,
$\mathfrak{R}_3=(X,\vec{e}_3,\vec{e}_4,\vec{e}_5,\vec{e}_6,\vec{e}_1,\vec{e}_2)\,,\mathfrak{R}_4=(X,\vec{e}_4,\vec{e}_5,\vec{e}_6,\vec{e}_1,\vec{e}_2,\vec{e}_3)$,
$\mathfrak{R}_5=(X,\vec{e}_5,\vec{e}_6,\vec{e}_1,\vec{e}_2,\vec{e}_3,\vec{e}_4)$ are, respectively, Frenet repers for lines $\omega^1,\omega^2,\omega^3,\,\omega^4,\omega^5,\omega^6$.
Let the net Σ_6 be a Frenet cyclic net. We denote it by $\tilde{\Sigma}_6$.

1. Materials of the study

Pseudo focus $F_3^2\in(X,\vec{e}_3)$ is determined by the radius vector:

$$\overrightarrow{F_3^2}=\vec{X}-\frac{1}{\Lambda_{32}^2}\overrightarrow{e_3}=\vec{X}+\frac{1}{\Lambda_{22}^3}\overrightarrow{e_3}$$

When a point X moves in a domain $\Omega\subset E_6$, pseudo focus F_3^2 describes its domain $\Omega_3^2\subset E_6$. A partial mapping $f_3^2:\Omega\to\Omega_3^2$ is defined such that $f_3^2(X)=F_3^2$.
We differentiate in the usual way (9) and using formulas (1), (2), (3) we get:

$$d\overrightarrow{F_3^2}=d\vec{X}-d\left(\frac{1}{\Lambda_{32}^2}\right)\overrightarrow{e_3}-\frac{1}{\Lambda_{32}^2}d\overrightarrow{e_3}=\omega i\overrightarrow{e_i}+\frac{d\Lambda_{32}^2}{\left(\Lambda_{32}^2\right)^2}\overrightarrow{e_3}-\frac{1}{\Lambda_{32}^2}\omega_3^i\overrightarrow{e_i}$$

or

$$d\overrightarrow{F_3^2}=\omega^m\overrightarrow{e_m}+\frac{C_{32m}^2\omega^m}{\left(\Lambda_{32}^2\right)^2}\overrightarrow{e_3}-\frac{\Lambda_{3m}^i\omega^m}{\Lambda_{32}^2}\overrightarrow{e_i},$$

here $d\Lambda_{32}^2=\left(\Lambda_{32m}^2+\Lambda_{3\ell}^2\Lambda_{2m}^\ell+\Lambda_{\ell 2}^2\Lambda_{3m}^\ell\right)\omega^m=C_{32m}^2\omega^m$.
From the last equality follows:

$$\begin{aligned}d\overrightarrow{F_3^2}=&\left[\overrightarrow{e_1}+\frac{C_{321}^2}{\left(\Lambda_{32}^2\right)^2}\overrightarrow{e_3}-\frac{\Lambda_{31}^i}{\Lambda_{32}^2}\overrightarrow{e_i}\right]\omega^1+\left[\overrightarrow{e_2}+\frac{C_{322}^2}{\left(\Lambda_{32}^2\right)^2}\overrightarrow{e_3}-\frac{\Lambda_{32}^i}{\Lambda_{32}^2}\overrightarrow{e_i}\right]\omega^2\\&+\left[\overrightarrow{e_3}+\frac{C_{323}^2}{\left(\Lambda_{32}^2\right)^2}\overrightarrow{e_3}-\frac{\Lambda_{33}^i}{\Lambda_{32}^2}\overrightarrow{e_i}\right]\omega^3+\left[\overrightarrow{e_4}+\frac{C_{324}^2}{\left(\Lambda_{32}^2\right)^2}\overrightarrow{e_3}-\frac{\Lambda_{34}^i}{\Lambda_{32}^2}\overrightarrow{e_i}\right]\omega^4+\\&+\left[\overrightarrow{e_5}+\frac{C_{325}^2}{\left(\Lambda_{32}^2\right)^2}\overrightarrow{e_3}-\frac{\Lambda_{35}^i}{\Lambda_{32}^2}\overrightarrow{e_i}\right]\omega^5+\left[\overrightarrow{e_6}+\frac{C_{326}^2}{\left(\Lambda_{32}^2\right)^2}\overrightarrow{e_3}-\frac{\Lambda_{36}^i}{\Lambda_{32}^2}\overrightarrow{e_i}\right]\omega^6\end{aligned}$$

We introduce the notations:

$$\overrightarrow{c_1} = \overrightarrow{e_1} + \frac{C^2_{321}}{\left(\Lambda^2_{32}\right)^2}\overrightarrow{e_3} - \frac{\Lambda^i_{31}}{\Lambda^2_{32}}\overrightarrow{e_i};$$
$$\overrightarrow{c_2} = \overrightarrow{e_2} + \frac{C^2_{322}}{\left(\Lambda^2_{32}\right)^2}\overrightarrow{e_3} - \frac{\Lambda^i_{32}}{\Lambda^2_{32}}\overrightarrow{e_i};$$
$$\overrightarrow{c_3} = \overrightarrow{e_3} + \frac{C^2_{323}}{\left(\Lambda^2_{32}\right)^2}\overrightarrow{e_3} - \frac{\Lambda^i_{33}}{\Lambda^2_{32}}\overrightarrow{e_i};$$
$$\overrightarrow{c_4} = \overrightarrow{e_4} + \frac{C^2_{324}}{\left(\Lambda^2_{32}\right)^2}\overrightarrow{e_3} - \frac{\Lambda^i_{34}}{\Lambda^2_{32}}\overrightarrow{e_i};$$
$$\overrightarrow{c_5} = \overrightarrow{e_5} + \frac{C^2_{325}}{\left(\Lambda^2_{32}\right)^2}\overrightarrow{e_3} - \frac{\Lambda^i_{35}}{\Lambda^2_{32}}\overrightarrow{e_i}$$
$$\overrightarrow{c_6} = \overrightarrow{e_6} + \frac{C^2_{326}}{\left(\Lambda^2_{32}\right)^2}\overrightarrow{e_3} - \frac{\Lambda^i_{36}}{\Lambda^2_{32}}\overrightarrow{e_i}$$

Since the given net $\tilde{\Sigma}_6$ is a Frenet cyclic net, the vectors $\vec{c}_i$ have the form:

$$\overrightarrow{c_1} = \overrightarrow{e_1} - \frac{\Lambda^2_{31}}{\Lambda^2_{32}}\overrightarrow{e_2} + \frac{C^2_{321}}{\left(\Lambda^2_{32}\right)^2}\overrightarrow{e_3} - \frac{\Lambda^4_{31}}{\Lambda^2_{32}}\overrightarrow{e_4};$$
$$\overrightarrow{c_2} = \overrightarrow{e_2} + \frac{C^2_{322}}{\left(\Lambda^2_{32}\right)^2}\overrightarrow{e_3} - \frac{\Lambda^4_{32}}{\Lambda^2_{32}}\overrightarrow{e_4};$$
$$\overrightarrow{c_3} = \left[1 + \frac{C^2_{323}}{\left(\Lambda^2_{32}\right)^2}\right]\overrightarrow{e_3} - \frac{\Lambda^4_{33}}{\Lambda^2_{32}}\overrightarrow{e_4};$$
$$\overrightarrow{c_4} = -\frac{\Lambda^2_{34}}{\Lambda^2_{32}}\overrightarrow{e_2} + \frac{C^2_{324}}{\left(\Lambda^2_{32}\right)^2}\overrightarrow{e_3} + \overrightarrow{e_4};$$
$$\overrightarrow{c_5} = -\frac{\Lambda^2_{35}}{\Lambda^2_{32}}\overrightarrow{e_2} + \frac{C^2_{325}}{\left(\Lambda^2_{32}\right)^2}\overrightarrow{e_3} - \frac{\Lambda^4_{35}}{\Lambda^2_{32}}\overrightarrow{e_4} + \overrightarrow{e_5}$$
$$\overrightarrow{c_6} = -\frac{\Lambda^2_{36}}{\Lambda^2_{32}}\overrightarrow{e_2} + \frac{C^2_{326}}{\left(\Lambda^2_{32}\right)^2}\overrightarrow{e_3} - \frac{\Lambda^4_{36}}{\Lambda^2_{32}}\overrightarrow{e_4} + \overrightarrow{e_6}$$

The $\mathfrak{R}' = \left(F^2_3, \vec{c}_i\right)$ moving reper is connected into the Ω^2_3 domain . In the general case vectors (10) are linearly independent.

Consider the line γ belonging to the five-dimensional distribution $\Delta_5 = (X, \overrightarrow{e_2}, \overrightarrow{e_2}, \overrightarrow{e_3}, \overrightarrow{e_4}, \overrightarrow{e_5})$. Its tangent vector has the form:

$\vec{\gamma} = \gamma^1\overrightarrow{e_1} + \gamma^2\overrightarrow{e_2} + \gamma^3\overrightarrow{e_3} + \gamma^4\overrightarrow{e_4} + \gamma^5\overrightarrow{e_5}$. Find the tangent vector of the line

$$\bar{\gamma} = f^2_3(\gamma) : \vec{\gamma} = \gamma^1\overrightarrow{c_1} + \gamma^2\overrightarrow{c_2} + \gamma^3\overrightarrow{c_3} + \gamma^4\overrightarrow{c_4} + \gamma^5\overrightarrow{c_5}.$$

Taking into account formulas (10) from here we have:

$$\vec{\gamma} = \gamma^1\overrightarrow{e_1} + \left(\gamma^1 c^2_1 + \gamma^2 + \gamma^4 c^2_4 + \gamma^5 c^2_5\right)\overrightarrow{e_2} + \left(\gamma^1 c^3_1 + \gamma^2 c^3_2 + \gamma^4 c^3_4 + \gamma^5 c^3_5\right)\overrightarrow{e_3} +$$
$$+ \left(\gamma^1 c^4_1 + \gamma^2 c^4_2 + \gamma^3 c^4_3 + \gamma^4 + \gamma^5 c^4_5\right)\overrightarrow{e_4} + \gamma^5\overrightarrow{e_5},$$

where C^j_i - coordinates of vector $\overrightarrow{C_i}$. It is this clear that the condition $\vec{\gamma}, \vec{\gamma}, \overrightarrow{XF^2_3} \in \Delta_5$ is always satisfied. Therefore, the line $\gamma \in \Delta_5$ is always a quasidouble line of the partial mapping f^2_3.

Similar to the above, it follows that the line β belonging to the distribution $\Delta_5' = (X, \overrightarrow{e_2}, \overrightarrow{e_3}, \overrightarrow{e_4}, \overrightarrow{e_5}, \overrightarrow{e_6})$ is always a quasidouble line of the partial mapping f_3^2.

Now consider the line δ belonging to the distribution $\Delta_5'' = (X, \overrightarrow{e_1}, \overrightarrow{e_3}, \overrightarrow{e_4}, \overrightarrow{e_5}, \overrightarrow{e_6})$. Its tangent vector $\vec{\delta}$ has the form: $\vec{\delta} = \delta^1\overrightarrow{e_1} + \delta^3\overrightarrow{e_3} + \delta^4\overrightarrow{e_4} + \delta^5\overrightarrow{e_5} + \delta^6\overrightarrow{e_6}$. The tangent vector $\overrightarrow{\vec{\delta}}$ of the line $\bar{\delta} = f_3^2(\delta)$ is defined as follows:

$$\overrightarrow{\vec{\delta}} = \delta^1\overrightarrow{c_1} + \delta^3\overrightarrow{c_3} + \delta^4\overrightarrow{c_4} + \delta^5\overrightarrow{c_5} + \delta^6\overrightarrow{c_6}$$

Using formulas (10), we get:

$$\overrightarrow{\vec{\delta}} = \delta^1\overrightarrow{e_1} + \left(\delta^1 c_1^2 + \delta^4 c_4^2 + \delta^5 c_5^2 + \delta^6 c_6^2\right)\overrightarrow{e_2} + \left(\delta^1 c_1^3 + \delta^3 c_3^3 + \delta^4 c_4^3 + \delta^5 c_5^3 + \delta^6 c_6^3\right)\overrightarrow{e_3} +$$
$$+ \left(\delta^1 c_1^4 + \delta^3 c_3^4 + \delta^4 + \delta^5 c_5^4 + \delta^6 c_6^4\right)\overrightarrow{e_4} + \delta^5\overrightarrow{e_5} + \delta^6\overrightarrow{e_6}.$$

From the condition $\vec{\delta}, \vec{\delta}, \overrightarrow{XF_3^2} \in \Delta_5''$ we have:

$$\delta^1 c_1^2 + \delta^4 c_4^2 + \delta^5 c_5^2 + \delta^6 c_6^2 = 0.$$

Using formulas (10) we get:

$$\Lambda_{31}^2\delta^1 + \Lambda_{34}^2\delta^4 + \Lambda_{35}^2\delta^5 + \Lambda_{36}^2\delta^6 = 0,$$

were, Λ_{31}^2-second curvature of the line ω^1;
Λ_{34}^2-fifth curvature of the line ω^4;
Λ_{35}^2-fourth curvature of the line ω^5;
Λ_{36}^2-third curvature of the line ω^6. On the contrary, if the condition (11) is fulfilled, then the line $\delta \in \Delta_5''$ is a quasidouble line of the partial mapping f_3^2.

Now consider the line ρ belonging to the distribution $\Delta_5''' = (X, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_4}, \overrightarrow{e_5}, \overrightarrow{e_6})$. Its tangent vector:
$\vec{\rho} = \rho^1\overrightarrow{e_1} + \rho^2\overrightarrow{e_2} + \rho^4\overrightarrow{e_4} + \rho^5\overrightarrow{e_5} + \rho^6\overrightarrow{e_6}$.
The tangent vector of line $\bar{\rho} = f_3^2(\rho)$ has a form:

$$\vec{\rho} = \rho^1\overrightarrow{e_1} + \left(\rho^1 c_1^2 + \rho^2 + \rho^4 c_4^2 + \rho^5 c_5^2 + \rho^6 c_6^2\right)\overrightarrow{e_2} +$$
$$+ \left(\rho^1 c_1^3 + \rho^2 c_2^3 + \rho^4 c_4^3 + \rho^5 c_5^3 + \rho^6 c_6^3\right)\overrightarrow{e_3} +$$
$$+ \left(\rho^1 c_1^4 + \rho^2 c_2^4 + \rho^4 + \rho^5 c_5^4 + \rho^6 c_6^4\right)\overrightarrow{e_4} + \rho^5\overrightarrow{e_5} + \rho^6\overrightarrow{e_6}$$

From condition $\vec{\rho}, \vec{\rho}, \overrightarrow{XF_3^2} \in \Delta_5'''$ we get the following:
$\rho^1 c_1^3 + \rho^2 c_2^3 + \rho^4 c_4^3 + \rho^5 c_5^3 + \rho^6 c_6^3 = 0$.
If we take into account formulas (10), we get:

$$C_{321}^2\rho^1 + C_{332}^2\rho^2 + C_{324}^2\rho^4 + C_{325}^2\rho^5 + C_{326}^2\rho^6 = 0.$$

On the contrary, if the condition (12) is fulfilled, then the line $\rho \in \Delta_5'''$ is a quasidouble line of the partial mapping of f_3^2.

2. Conclusion

Based on the above, the following theorem was proved.

Theorem. a) Lines γ, β belonging to distributions $\Delta_5 = (X, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}, \overrightarrow{e_4}, \overrightarrow{e_5})$, $\Delta'_5 = (X, \overrightarrow{e_2}, \overrightarrow{e_3}, \overrightarrow{e_4}, \overrightarrow{e_5}, \overrightarrow{e_6})$ (respectively) are always quasidouble lines of partial mapping f_3^2; b) the line δ, belonging to distribution $\Delta_5'' = (X, \overrightarrow{e_1}, \overrightarrow{e_3}, \overrightarrow{e_4}, \overrightarrow{e_5}, \overrightarrow{e_6})$, is a quasidouble line of partial mapping f_3^2, if and only if the equality (11) holds;

c) the line ρ, belonging to distribution $\Delta_5''' = (X, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_4}, \overrightarrow{e_5}, \overrightarrow{e_6})$, is a quasidouble line of partial mapping f_3^2, if and only if the equality (12) holds.

References

[1] Rashevsky, P.K. Rimanova geometry and tensor analysis [Text]/ P.K Rashevskyi// M. Science. 1967.-S. 481-482.

[2] Schouten, I.A. Introduction to new methods of differential geometry [Text]/ I.A. Shouten, D.J. Stroik. // M. IL. 1948. T. II-348.

[3] Finikov, S.P. The method of external forms Kartana in differential geometry [Text]/ S.P. Finikov // M-L.: Gosttehizdat, 1948.- 432.

[4] Bazylev, V.T. On multidimensional net in Euclidean space [Text]/ V. T. Bazylev // Lithuanian mathematical collection, 1966.VI.№4.-S.475-491.

[5] Matieva, G. Geometry of partial images, net and distributions of Euclidean space [Text]/ G. Matieva // Monograph. Osh, 2003.-S. 212-219.

[6] Bazylev, V.T. On fundamental objects of planar multidimensional net [Text]/ V.T. Bazylev // / Izvestiya VUZov Mathematica, 1967. - S. 3-11.

[7] Abdullayeva, Ch.Kh. Necessary and sufficient conditions for the existence of a quasidouble line of the pair f in the Euclidean space E6 // Nauka. Education. Technique. - Osh: KUMU, 2021. - #2. -S. 13-20.

[8] Abdullaeva, Ch.Kh. On the existence of quasidouble lines of the pair f in the fourdimensional E4 Euclidean space // Nauka. Education. Technique. - Osh: KUMU, 2022. - No. 1. S. 52-58.

[9] Abdullayeva, Ch.Kh. Necessary and sufficient conditions for existence of quasidouble lines of partial mapping in Euclidean space // Nauka. Education. Technique. - Osh: KUMU, 2022. - No. 3(75). -S. 32-39.

[10] Abdullayeva, Ch.Kh. On the existence of quasidouble lines of partial mapping in the Euclidean space E5 // Nauka. Education. Technique. - Osh: KUMU, 2022. - No. 3(75). -S. 39-49.

[11] Matieva G. Existence of quasidouble lines of a pair $\left(f^5 \begin{matrix} 5 \\ 1 \end{matrix} \right)$ in Euclidean space E_5 // Journal of Physics. Conference Series, Volume 1988 , Simposium Kebangsaan Sains Matematik ke28 (SKSM28), 28-29 July 2021 g., Kuantan, Pahang.

[12] Matieva G. On the existence of quasidouble lines of partial mapping in the Euclidean space E_6 [Matieva G., Kurbanbaeva N.N., Abdullaeva Ch.Kh.] Bulletin of Osh State University, mathematics, physics, technology. 2023, No. 1. pp 14015.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

MODELLING A CONFLICT EVENT AS A RANDOM PROCESS BY A GENERAL LOGICAL-PROBABILISTIC METHOD

RANA AMIROVA[1], NATIQ ALIYEV[2]

[1]Azerbaijan Technical University, Baku, Azerbaijan
[2]Azerbaijan University of Oil and Industry, Baku, Azerbaijan
e-mail: [1]ramirovamath@gmail.com [2]nlaliyev@gmail.com

1. Introduction

Modeling a conflict event as a random process using a logical-probabilistic approach includes the analysis and representation of the conflict in the form of random variables and probability distributions. The logical-probabilistic method determines the parameters that describe various aspects of a conflict event in order to model a conflict event as a random process.

For example, it could be the number of parties involved in the conflict, the level of hostility between the parties, the resources available, strategies, and so on. For each variable, its probability distribution is determined. The choice of a particular distribution depends on the nature of the variable and the information available. Relationships between variables are determined and which variables are interconnected and how they affect each other. Using certain probability distributions and relationships between variables, it is possible to generate random events representing the course of the conflict. This can be done by choosing random values from the appropriate distributions and taking into account the probabilities. After generating random events, you can analyze the results. This may include assessing the likelihood of certain outcomes, exploring relationships between variables, or evaluating the effectiveness of different strategies. In the process of analyzing the results, it is possible to identify weaknesses or incorrect assumptions of the model. If necessary, the model can be adjusted to improve its accuracy and reality. It is important to note that the modeling of conflict events as a random process using the logical-probabilistic method is a simplified representation of the complex and dynamic processes of real conflicts. Such an approach can help in understanding and analyzing the main factors influencing the conflict and assessing the probabilities of various scenarios.

From a mathematical point of view, the assessment of reliability and safety can be carried out using identical mathematical tools.

Logical and probabilistic methods are designed to model and calculate indicators of various properties of reliability (non-failure operation), survivability and safety of functioning of structurally complex systems of various types, classes and purposes.

Danger is understood as a threat, opportunity, probability of damage, disaster, misfortune, catastrophe. Danger reflects the idea of the potential realizability of damage in certain conditions and situations. The concept of danger has a probabilistic nature, i.e., hazard characteristics must necessarily contain parameters that determine the probability or frequency of damage.

2. State Model

Create a state model where each state of the system is represented by Boolean variables. Boolean functions can be used to define state transitions based on the presence or absence of certain conflict events. Determination of input variables that affect the reliability of the system.

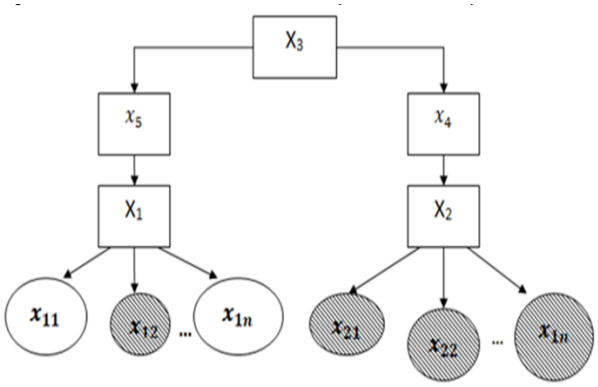


FIGURE 1. Logical-probabilistic methods and its use in the study of the safety of the functioning of complex systems

These variables can represent various factors such as system components, operating conditions, environment, and other factors that can affect system reliability. For each factor, create a Boolean variable to represent its presence or absence.

Logical links define dependencies between input variables and allow modeling their interaction.

We define probabilistic characteristics for each input variable, such as the probability of a component failure or the probability of a certain event occurring. These characteristics can be obtained on the basis of data from past experience, expert assessments [5].

In the theory of reliability in monotonous systems, there are no "harmful elements" for which the improvement of performance characteristics necessarily worsens the performance characteristics of the entire system under study as a whole.

However, all results are developed for monotonic functions describing the reliability (survivability, safety) of the system. The legitimacy of using these methods for nonmonotone functions of the algebra of logic has not been proven. In science, there are examples when the function of the algebra of logic that describes the scenarios for the development of danger will be nonmonotonic. Let us assume that the system under study consists of two systems X_1 and X_2. X_1 - is the current state of the system, that is, a conflict event. X_2- is the current state of the system generating the conflict event. Event X_3 -Audit represents the possible occurrence of a damaging factor (missile launch, fire, nuclear reactor accident, risks, wars). Events x_5 and x_4 represent destruction by the impact of the damaging factor of the first and second elements, respectively. This model considers inverse exits from vertices x_5 and x_4. They determine in the implementation of the non-destruction of the elements X_1 and X_2 in the event of a damaging effect. System X_3 creates a conflict zone in system X_1 through factor x_5 and system X_2. The result is a very complex threat to one element of the X_1 system. The X_3 system poses a threat to the X_2 system through the x_4 factor. As a result, an economic threat arises in all elements of the X_2 system. It is required to determine the stability of this system by the criterion of the performance of its functions by at least one element X_1 or X_2. When analyzing the expressed situations, the following results are obtained:

1. A complex and dangerous occupation zone has been created in one element of the X_1 system.

2. A complex and crisis economic threat has been created in all elements of the X_2 system. The logical function in this case looks like this [1-6]:

$$Y = X_1\bar{X}_3 \vee X_1\bar{x}_5 \vee X_1\bar{X}_2 \vee X_2\bar{X}_3 \vee X_2\bar{x}_4$$

This function is nonmonotonic.

$$Y_1 = X_1(x_{11} \vee \bar{x}_{12} \vee \ldots \vee x_{1n}),\ Y_2 = X_2(\bar{x}_{21} \vee \bar{x}_{22} \vee \ldots \vee \bar{x}_{2n})$$

Y_1- is a function that expresses the state of the system X_1 influenced by the audit system and a given factor x_5. Y_2- is a function that expresses the state of the system X_2 under the

influence of the audit system and a given factor x_4. The so-called event-logic approach is at the heart of the formulation and solution of all problems of modeling and calculating the reliability of systems using a general logical and probabilistic method. This approach provides for the sequential implementation of the following four main stages of the general logical probabilistic method [2].

1. The stage of structural-logical statement of the problem, which includes: - division of the entire system under consideration into a finite number of elements, each of which is represented in the reliability model by a simple (binary) event x_i with two possible states $\check{x}_l = \{x_i, \bar{x}_l\}$ for example, operability and failure, readiness and not ready, defeat and not defeat, given by probabilistic parameters, $p_i(t), q_i(t) = 1 - p_i(t)$;

2. Determination of the content and logical conditions for the implementation of y_i (and, or) non-realization $\bar{y}_l$ output (integrative) functions for each element in the system;

- logically strict verbal and graphic (analytical) description of the set X of individual elements of the system and the set of conditions Y for the implementation of their system functions by them, which together $G_F(X, Y)$ form a special scheme of the functional integrity of the system under consideration [4];

- logically rigorous description and setting with the help of individual or group output (integrative) functions of the logical criteria for the functioning of the system

$$Y_f = Y_f(\{\check{y}_l\}, i-1, \ldots, n), Y_{f_{x_1}} = X_1\bar{X}_3 \vee X_1\bar{x}_5 \vee X_1\bar{X}_2, \;\; Y_{(f_{(x_2)})} = X_2\bar{X}_3 \vee X_2\bar{x}_4$$

implementation of the main functions (and, or) the occurrence of dangerous states of the system

$Y_{f_{x_1}} = X_1\bar{X}_3 \vee X_1\bar{x}_5 \vee X_1\bar{X}_2$ -logical system health function X_1;

$Y_{(f_{(x_2)})} = X_2\bar{X}_3 \vee X_2\bar{x}_4$ -logical function of the health of the system X_2;

3. The stage of logical modelling, at which, with the help of special methods for transforming the circuit with functional integrity and logical criteria for functioning, the construction of a logical function of the system's operability is carried out. The logical system health function allows analytically strictly, but in a compact form, to determine all combinations of element states $Y_f = Y_f(\{\check{x}_l\}, i-1, \ldots, n)$, in which the system implements its output function $F = Y_{f_{x_1}} \vee Y_{f_{x_2}}$

Y_f - is a logical function of system health,

F -functioning of the system under study.

4. The stage of probabilistic modelling, at which, using special methods for transforming the system operability function, the construction of a polynomial of the calculated probabilistic function is carried out [1]:

$$P_F(\{p(t), q(t)\}, i = 1, \ldots, n, t), \implies P_F(t) = P_F(\{p(t), q(t)\}) = P_{FX_y}(t) \vee P_{FX_1}(t) \vee P_{FX_2}(t)$$

$$P_{FX_y}(t) = P_1(t)Q_3 \vee \; P_1(t)q_5(t) \vee \; P_1(t)Q_2(t) \vee \; P_2(t)q_4(t)$$

$$P_{FX_1}(t) = p_{x_{11}} + q_{x_{12}} + ... + p_{x_{1n}}, P_{FX_2}(t) = q_{x_{21}} + q_{x_{22}} + ... + p_{x_{2n}}$$

$P_{FX_1}(t)$- polynomial of the calculated probability function X_1;,

$P_{FX_2}(t)$ - is the polynomial of the calculated probability function X_2;.

As a calculated probabilistic model of system reliability, the polynomial of the probabilistic function $P_F(\{p(t), q(t)\}, i-1, \ldots, n, t)$, is used, which should accurately determine (within the accepted restrictions and assumptions) the law of distribution of the system's uptime, all health states of which are represented using the logical system health function

$$Y_f = Y_f(\{\check{x}_l\}, i-1, \ldots, n),.$$

The polynomial of the probabilistic function makes it possible to analytically and rigorously determine the law of distribution of the time of failure-free operation of the system for the implementation of the output function F, given by the logical criterion of operation.

The polynomial $P_{FX_1}(t)$- of the probability function is monotone. Although one of the elements of this X_1 system is a zone of serious and complex conflict, this system is monotonous. Despite the fact that one element of the system is included in the conflict zone, the system is

monotonous, which does not give the system security. The polynomial $P_{FX_2}(t)$- of the probability function of the system is non-monotonic, since all elements of this system X_2 are in the zone of economic crisis. All elements of the system are included in the zone of economic conflict, that is, the system is nonmonotonic and therefore the system X_2 is in a dangerous state.

The logical function of system health is a rigorous analytical form of describing a complex random event, the probability of which must be determined using the desired probability function. In this description, the conjunction operation represents the product of random events, the disjunction operation is their sum, and the inversion operation is the opposite outcome of the event. Therefore, to obtain a polynomial of a probabilistic function from the health function of the system, it is sufficient to perform symbolic transformations of the notation of logical variables and operations into the notation of probabilistic variables and arithmetic operations. These symbolic transformations must be carried out in strict accordance with the laws of probability theory for calculating the probabilities of products, sums and additions of random events.

3. Orthogonalization Method

Logical-Probabilistic Orthogonal Analysis methods are an approach to systems reliability analysis that combines logical modeling and probabilistic methods. They allow assessing and optimizing the reliability of the system, taking into account various factors and their relationships, orthogonalization of the analysis of relationships between input variables and assessing the reliability of the system. Orthogonalization allows you to separate the influence of various factors and determine their relative contribution to the reliability of the system. This can be done using mathematical methods such as orthogonal experiment methods or regression analysis methods. We consider modeling the reliability of a security system under the conditions of a conflict event using five Boolean variables[2]:.

The system under study consists of two elements X_1 and X_2. The X_3 event is somewhat of a shocking factor. The events x_4 and x_5 represent the destruction of elements X_1 and X_2 under the influence of X_3. Logical probabilistic model of system stability using a system of logical equations

$$y(X_1, X_2, ..., X_5) = X_1\bar{X}_3 \vee X_1\bar{x}_5 \vee X_1\bar{X}_2 \vee X_2\bar{X}_3 \vee X_2\bar{x}_4$$

which is nonmonotonic, iterative, and regular disjunctive normal form.

$$y(X_1, X_2, ..., X_5) = \begin{vmatrix} X_1\bar{X}_3 \\ X_1\bar{X}_5 \\ X_1\bar{X}_2 \\ X_2\bar{X}_3 \\ X_4\bar{X}_4 \end{vmatrix}$$

to do this, you need to make a transition from logical equality between events to algebraic equality between their probabilities. This transition is carried out by orthogonalization of the logical polynomial

$$A \vee\ B \vee\ C \vee\ D \vee ...$$

to the mind

$$A \vee \bar{A}B \vee \bar{A}\bar{B}C \vee \bar{A}\bar{B}\bar{C}D \vee ...$$

Both polynomials are logically equivalent, but differ in that the theorem on the probability of the sum of incompatible events is not applicable to the first of them, while the theorem on the probability of the sum of incompatible events is not applicable to the second.

$$P(A) + P(\bar{A}B) + P(\bar{A}\bar{B}C) + P(\bar{A}\bar{B}\bar{C}D)\ + ...$$

The question is how to calculate the probability $\{P\}$ of a complex statement $P\{y(X_1, X_2, ..., X_5) = 1\}$ (the probability of the truth of a function of the algebra of logic), if the probabilities of truth of the simplest statements are known

$$P\{x_i = 1\} = R_i,\ P\{x_i = 1\} = Q_i, \tag{1}$$

what is the logic of these probabilities?

Direct substitution of truth values of propositions x_i by their probabilities $P\{x_i = 1\}$ is possible only for simple structures (serial, parallel, tree-like).

Let's consider an example from the field of nonmonotonic structures designed to solve security problems.

In accordance with the logical-probabilistic calculus, the following formula follows, written in disjunctive-normal form,

$$y(X_1, X_2, \ldots, X_5) = \begin{vmatrix} X_1\bar{X}_3 \\ X_1\bar{x}_5 \\ X_1\bar{X}_2 \\ X_2\bar{X}_3 \\ X_2\bar{x}_4 \end{vmatrix} = \left| \begin{array}{c} X_1\bar{X}_3 \\ \overline{X_1\bar{X}_3}X_1\bar{x}_5 \\ \overline{X_1\bar{X}_3}\,\overline{X_1\bar{x}_5}X_1\bar{X}_2 \\ \overline{X_1\bar{X}_3}\,\overline{X_1\bar{x}_5}\,\overline{X_1\bar{X}_2}X_2\bar{X}_3 \\ \overline{X_1\bar{X}_3}\,\overline{X_1\bar{x}_5}\,\overline{X_1\bar{X}_2}\,\overline{X_2\bar{X}_3}X_2\bar{x}_4 \end{array} \right| =$$

$$= \left| \begin{array}{c} X_1\bar{X}_3 \\ \begin{vmatrix}\bar{X}_1 \\ X_3\end{vmatrix} X_1\bar{x}_5 \\ \begin{vmatrix}\bar{X}_1 \\ X_3\end{vmatrix}\begin{vmatrix}\bar{X}_1 \\ x_5\end{vmatrix} X_1\bar{X}_2 \\ \begin{vmatrix}\bar{X}_1 \\ X_3\end{vmatrix}\begin{vmatrix}\bar{X}_1 \\ x_5\end{vmatrix}\begin{vmatrix}\bar{X}_1 \\ X_2\end{vmatrix} X_2\bar{X}_3 \\ \begin{vmatrix}\bar{X}_1 \\ X_3\end{vmatrix}\begin{vmatrix}\bar{X}_1 \\ x_5\end{vmatrix}\begin{vmatrix}\bar{X}_1 \\ X_2\end{vmatrix}\begin{vmatrix}\bar{X}_2 \\ X_3\end{vmatrix} X_2\bar{x}_4 \end{array} \right| = \begin{vmatrix} X_1\bar{X}_3 \\ X_1\bar{x}_5X_3 \\ X_1\bar{X}_2x_5 \\ X_2\bar{X}_3\bar{X}_1 \\ \bar{X}_1X_3 \\ X_2\bar{x}_4X_1X_3x_5 \\ x_5X_2\bar{x}_4, X_3 \\ \bar{X}_1\bar{X}_2 \end{vmatrix}$$

Replacing Boolean Arguments with Their Truth Probabilities

$P\{X_1 = 1\} = p_1, P\{X_2 = 1\} = p_2, P\{X_3 = 1\} = p_3, P\{x_4 = 1\} = p_4, P\{x_5 = 1\} = p_5,$

We get:

$$P\{y(X_1, X_2, \ldots, X_5) = 1\} =$$

$$p_1q_3 + p_1q_5p_3 + p_1q_2p_5 + p_2q_3q_1 + q_1p_3 + p_2q_4p_1p_3p_5 + p_5p_2q_4p_3 + q_1q_2$$

$R_i = P(X_i)$ -Probability of the event. Assuming that all events are equally probable $R_1 = R_2 = R_3 = R_4 = R_5 = R = 0.5 = const$, we get the following polynomial that determines the probability of systems transition to a dangerous state: the operations of logical multiplication and addition are replaced by the operations of arithmetic multiplication and addition. Then the probability function of the transition of systems to a dangerous state will have the following form:

$$(P\{(X_1, X_2, \ldots, X_5) = 1\}) = 3R^2 + 3R^3 + R^4 + R^5$$

The weight of a logical function consisting of $n = 5$ elements is the relative proportion of sets of elements on which the function is equal to 1, among all 2^n sets of possible element values.

$R = 0.5$ corresponds to the function weight $P\{y(X_1, X_2, \ldots, X_5) = 1\}$

$$g_yP\{y(X_1, X_2, \ldots, X_5) = 1\} = \tfrac{27}{32}.$$

The number of dangerous states,

$$2^5g_yP\{y(X_1, X_2, \ldots, X_5) = 1\} = 27$$

among all 32 possible states of the system, written in perfectly disjunctive-normal form.

To eliminate variables from the Boolean function $P(x_1, x_2, ..., x_n)$ the best way is to find the derivative of the Boolean function. The derivative of a Boolean function is determined using the Boolean difference [3]:. The derivative of a Boolean function with respect to the argument x_i is the function

$$\tfrac{\partial \mathrm{P}}{\partial \mathrm{x_i}} = P(x_i = 1) \oplus (P(x_i = 0).$$

That is, derivatives with respect to the X_3, argument destroy the X_3 event in the system. Partial derivative of the function $P\{y(X_1, X_2, \ldots, X_5) = 1\}$ with respect to X_3:

$$\tfrac{\partial Py(X_1, X_2, ..., X_5)=1}{\partial(PX_3=1=p_3)} = -p_1 + p_1q_5 - p_2q_1 + q_1 + p_2q_4p_1p_5 + p_5p_2q_4$$

The significance of the argument xx_i in the monotonic logical function is numerically equal to the probability of the truth of the Boolean difference of the logic algebra function with respect to the argument x_i:

$$\xi_i = \frac{\partial Py(X_1,X_2,\ldots,X_5)=1}{\partial(PX_i=1=p_i)}$$

The significance of the argument X_3 in a nonmonotonic logical function of the type $\{y\,(X_1, X_2, \ldots, X_5) = 1\}$ is numerically equal to the probability of the truth of the Boolean difference of the logic algebra function with respect to the argument X_3 with a minus sign:

$$\xi_3 = -\frac{\partial Py(X_1,X_2,\ldots,X_5)=1}{\partial(PX_3=1=p_3)}$$

$$g_y\left(\frac{\partial P\{y(X_1,X_2,\ldots,X_5)=1\}}{\partial P(\{X_3=1\}=p_3)}\right) = R^4 + R^3 = \frac{3}{16}$$

Card $P\{y\,(X_1, X_2, \ldots, X_5) = 1\} = \frac{3}{16}32 = 6$

We will write the system health functions using the shortest paths of successful operation in matrix form as follows (Fig. 1).

$$Y_f = y(X_1, X_2, .., X_5) = \begin{vmatrix} X_1X_2 \\ X_1X_5X_3 \\ X_3X_4X_2 \\ X_3X_4X_2X_1 \end{vmatrix} \Longrightarrow$$

Let's use the orthogonalization algorithm. After transforming the functions into orthogonal disjunctive normal form, we will have:

$$\begin{vmatrix} X_1X_2 \\ \overline{X_1, X_2}X_1X_5X_3 \\ \overline{X_1, X_2}\,\overline{X_1, X_5, X_3}X_3X_4X_2 \\ \overline{X_1, X_2}\,\overline{X_1, X_5, X_3}\,\overline{X_3, X_4, X_2}X_3X_4X_2X_1 \end{vmatrix} \Longrightarrow$$

After applying the De Morgan theorem and replacing the negations of conjunctions by the sum of negations, we get:

$$\begin{vmatrix} X_1X_2 \\ \begin{vmatrix}\bar{X}_1\\ \bar{X}_2\end{vmatrix} X_1X_5X_3 \\ \begin{vmatrix}\bar{X}_1\\ \bar{X}_2\end{vmatrix}\begin{vmatrix}\bar{X}_1\\ \bar{X}_5\\ \bar{X}_3\end{vmatrix} X_3X_4X_2 \\ \begin{vmatrix}\bar{X}_1\\ \bar{X}_2\end{vmatrix}\begin{vmatrix}\bar{X}_1\\ \bar{X}_5\\ \bar{X}_3\end{vmatrix}\begin{vmatrix}\bar{X}_3\\ \bar{X}_4\\ \bar{X}_2\end{vmatrix} X_3X_4X_2X_1 \end{vmatrix} \Longrightarrow$$

The result of the inner cycle of matrix orthogonalization, that is, bringing the negations of conjunctions to an orthogonal form, will be a matrix of the form:

$$\begin{vmatrix} X_1X_2 \\ X_1\bar{X}_2X_5X_3 \\ \bar{X}_1X_3X_4X_2 \\ \bar{X}_5X_3X_4X_2X_1 \end{vmatrix} = X_1X_2 \vee X_1\bar{X}_2X_5X_3 \vee \bar{X}_1X_3X_4X_2 \vee \bar{X}_5X_3X_4X_2X_1$$

Using the probabilities (1) and knowing the logic of these probabilities, we calculate the probabilistic functions of the type (1):

$$Py\,(X_1, X_2, \ldots, X_5) = 1 = R_s = P_1P_2 + P_1Q_2P_5P_3 + Q_1P_3P_4P_2 + Q_5P_3P_4P_2P_1$$

Replacing $Q_i = 1 - R_i$ and assuming the equality $P_i = R$, we obtain the following one-parameter polynomials:

$$R_s = R^2 + 2R^3 - R^4 - R^5$$

Substituting $R = 0.5$ into them, we get the weights of the functions:

$$g_y = (\tfrac{1}{2})^2 + 2(\tfrac{1}{2})^3 - (\tfrac{1}{2})^4 - (\tfrac{1}{2})^5 = \tfrac{13}{32},$$

the number of sets on which the functions $y(X_1, X_2, \ldots, X_5)$ take the true value, i.e. powers of these sets and the number of dangerous states.

Card $y(X_1, X_2, \ldots, X_5) = \frac{13}{32}2^5 = 13$

among all 32 possible states of the system, written in perfectly disjunctive-normal form. These examples show:

– In the absolute correctness of the results obtained by the full set of all $2^5 = 32$ states

– Logical function of the healthy state $Y_f = y(X_1, X_2, \ldots, X_5)$- determines the reliability of systems X_1, X_2.

As a result, the main result of the work is the development of the logical probabilistic theory in terms of the analysis of the importance of individual arguments of the function of the algebra of logic. The results presented in the paper expand the possibilities of the logical-probabilistic calculus of estimating the indicators of importance to the area of consideration of nonmonotonic functions of the algebra of logic that describes the security of the system.

The use of logical-probabilistic methods of orthogonalization of systems reliability analysis allows more accurate modeling and analysis of the influence of various factors on system reliability, taking into account their interaction and probabilistic characteristics. This can help develop more reliable systems and make informed decisions to improve their reliability.

4. Analysis of the results

The simulation results were analyzed and interpreted in the context of the conflict. Various indicators can be studied, such as the intensity of the conflict or the likelihood of reaching certain situations. It should be noted that modeling a conflict as a random process is a simplified description of the real situation. Modeling results may be limited by available data and assumptions and therefore should be interpreted with caution. The logical-probabilistic approach can be a useful tool for analyzing and predicting the development of conflicts. The use of logical-probabilistic methods of orthogonalization of systems reliability analysis allows more accurate modeling and analysis of the influence of various factors on system reliability, taking into account their interaction and probabilistic characteristics. This can help develop more reliable systems and make informed decisions to improve their reliability.

Keywords: random process, conflict event, logical function, derivative of a Boolean function.

AMS Subject Classification: 03-04, 60A10, 60-08.

References

[1] Amirova, R.A., *Mathematical Logic*, PLAP Lambert Academic Publishing GmbH and Co. KG, 2019, Saarbrucken, Germany.

[2] Amirova R.A., Aliyev T.N., *Reliability Problem In Some Algebraic Structures*, PROCEEDINGS of the 7 th International Conference on CONTROL AND OPTIMIZATION WITH INDUSTRIAL APPLICATIONS (COİA 2020), Baku.

[3] Amirova, R.A., *Reliability Of The Dynamics Of Functioning Systems Of Objects With The Structure.*, ASIA LIFE SCIENCES The Asian International Journal of SUPPLEMENT 22(2) 28 AUGUST 2020. Printed in the Philippines. Scopus.

[4] Amirova R.A., Aliyev T.N., *Calculation of Reliability of Restored Power Supply Systems*, Proceedings of the 8 th International Conference on control and optimization with industrial applications, 24-26 August 2022, Baku.

[5] Amirova R.A., Aliyev N.L. , *Calculation of Reliability of Restored Power Supply Systems*, The Ministry of Education of the Republic of Kazakhstan, Altynsarin National Academy of Scientific and Pedagogical Journal BILIM №2 (105) 2023, Kazakhstan.

[6] Amirova R.A., *Reliability And Safety Of Structural Complex Systems*, Ed. St. Petersburg University, St. Petersburg, 2nd edition, 2007.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

SIMPLE REPRESENTATIONS OF $\mathfrak{o}_I^{(1)}(5)$ WITH RESTRICTED HIGHEST WEIGHTS IN CHARACTERISTIC TWO

SHERALI IBRAEV

Korkyt Ata Kyzylorda University, Kyzylorda, Kazakhstan
Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
e-mail: ibrayevsh@mail.ru

Abstract. In this paper we give a complete description of simple representations with restricted highest weights of the simple ten-dimensional Lie algebra $o_I^{(1)}(5)$ over an algebraically closed field k of characteristic $p = 2$. We also prove that all fundamental representations have non-trivial cohomology. As an example, a complete description of the second cohomology of the standard fundamental module is given.

Keywords: simple Lie algebra, representation, simple module, characteristic two.

AMS Subject Classification: 17B10, 17B20, 17B50.

1. Introduction

Let k be an algebraically closed field of characteristic $p = 2$. Over k, in [1], B. Weisfeiler and V. Kac constructed simple analogues of classical Lie algebras of type B_l as contragredient Lie algebras with the Cartan matrix

$$\Delta_l = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}.$$

In [2], they were described as derived algebras of Lie algebras that preserve bilinear forms with identity matrices of odd order and denoted by $\mathfrak{o}_I^{(1)}(2l+1)$. It is also known that the Lie algebra $\mathfrak{o}_I^{(1)}(2l+1)$ is isomorphic to the Kaplansky Lie algebra $Kap_3(2l+1)$ [3, Sec. 5]. For small $l \leq 6$, its outer derivations and central extensions are obtained in [4, Lemma 4.3(a)], and local deformations are calculated for $l \leq 4$, see, [5, Lemma 9.6, Conjecture 9.7]. The Lie algebra of the smallest dimension of this series is the three-dimensional Lie algebra $\mathfrak{o}_I^{(1)}(3)$. It has only two simple modules with nontrivial cohomology. There are one-dimensional trivial module and an adjoint module. The known results cover their cohomology, except for the third cohomology. It is easy to see that $H^3(\mathfrak{o}_I^{(1)}(3)) \cong k$ and $H^3(\mathfrak{o}_I^{(1)}(3), \mathfrak{o}_I^{(1)}(3)) = 0$. As noted above, the outer derivations, central extensions, and local derivations of the Lie algebra $\mathfrak{o}_I^{(1)}(5)$ are already known, and the global deformations have also been calculated [6]. All these described cohomologies correspond to the adjoint representation of $\mathfrak{o}_I^{(1)}(5)$. Other simple finite-dimensional representations of it, apart from the perhaps trivial one, have not been studied. In this paper, we

This work was partially supported by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (grant No 09259551).

describe all simple finite-dimensional representations with restricted weights of $\mathfrak{o}_I^{(1)}(5)$. The Lie algebra $\mathfrak{o}_I^{(1)}(5)$ has four simple modules of restricted highest weights. We will describe them as $\mathfrak{o}_I^{(1)}(5)$-submodules of the general Lie algebra $W(5,\mathbf{m})$ of Cartan type (see, Theorem 1 below). It is easy to show that all fundamental $\mathfrak{o}_I^{(1)}(5)$-modules have a non-trivial cohomology (see, Proposition 2 below).

Let L be the Lie algebra of type B_2 over the field of complex numbers. In the Lie algebra L, one can choose a basis such that all structure constants are integer and there is an isomorphism $L_{\mathbb{Z}} \bigotimes_{\mathbb{Z}_2} k \cong \mathfrak{o}_I^{(1)}(5)$ of Lie algebras, where $L_{\mathbb{Z}}$ is the integer linear span of this basis and k is an algebraically closed field of characteristic 2. Indeed, the generators $e_1, e_2, h_1, h_2, f_1, f_2$ with the defining relations

$$\begin{array}{l} Adh_1(e_1) = 2e_1, Adh_1(e_2) = -e_2, Adh_2(e_1) = -e_1, Adh_2(e_2) = e_2, \\ (Ade_1)^2(e_2) = 0, (Ade_2)^3(e_1) = 0, \\ (Ade_f)^2(f_2) = 0, (Adf_2)^3(f_1) = 0 \end{array} \tag{1}$$

continuous up to such a basis.

Considering $L_{\mathbb{Z}}$ as a Lie algebra over the ring $\mathbb{Z}$, we can introduce on the Cartan subalgebra

$$H_{\mathbb{Z}} = \langle h_1, h_2 \rangle_{\mathbb{Z}}$$

of $L_{\mathbb{Z}}$ an integer linear function $\lambda : H_{\mathbb{Z}} \to \mathbb{Z}$. Let us call them *weights* for $H_{\mathbb{Z}}$. We call a weight λ is *dominant* if $\lambda(h_1) \geq 0$ and $\lambda(h_2) \geq 0$.

For a finite-dimensional $\mathfrak{o}_I^{(1)}(5)$-module V, let

$$V_\lambda = \{v \in V \,:\, hv = \lambda(h)v \text{ for all } h \in H_{\mathbb{Z}}\}.$$

If $V_\lambda \neq 0$, then the subspace V_λ is called the *weight subspace*, and λ is called the *weight* for the subalgebra $H_{\mathbb{Z}}$ in V. Then we get the following decomposition of the space V into a direct sum of weight subspaces:

$$V = \bigoplus_{\lambda \in H_{\mathbb{Z}}^*} (V_\lambda \bigotimes_{\mathbb{Z}_2} k).$$

The highest vector with highest weight λ of a $\mathfrak{o}_I^{(1)}(5)$-module V is a non-zero vector $v_\lambda \in V_\lambda$ such that $xv_\lambda = 0$ for all $x \in \langle e_1, e_2, [e_1, e_2], [e_2, [e_2, e_1]] \rangle_{\mathbb{Z}}$. Let ω_1, ω_2 be the weights satisfying the equalities

$$\omega_1(h_1) = 1,\ \omega_1(h_2) = 0,\ \omega_2(h_1) = 0,\ \omega_2(h_2) = 1.$$

They are called the *fundamental weights*. Denote by $L(r,s)$ a simple $\mathfrak{o}_I^{(1)}(5)$-module with highest weight $\lambda = r\omega_1 + s\omega_2$. We call $\lambda = r\omega_1 + s\omega_2$ a *restricted weight* if $0 \leq r, s \leq 1$.

Example. The Lie algebra $\mathfrak{g} = \mathfrak{o}_I^{(1)}(5)$ as an adjoint module has the set of weights

$$\Lambda_{\mathfrak{g}} = \{0, \pm(2\omega_1 - \omega_2), \pm(-\omega_1 + \omega_2), \pm\omega_1, \pm\omega_2\}$$

and the following weight decomposition:

$$\mathfrak{g} = \bigoplus_{\lambda \in \Lambda_{\mathfrak{g}}} (\mathfrak{g}_\lambda \bigotimes_{\mathbb{Z}_2} k).$$

Indeed, since $[h_1, e_1] = 2e_1$ and $[h_2, e_1] = -e_1$, for $e_1 \in \mathfrak{g}$ we get $\lambda(h_1) = 2$ and $\lambda(h_2) = -1$. Hence, $\lambda = 2\omega_1 - \omega_2$. So, $e_1 \in \mathfrak{g}_{2\omega_1 - \omega_2}$. Similar calculations give us

$$f_1 \in \mathfrak{g}_{-2\omega_1+\omega_2}, e_2 \in \mathfrak{g}_{-\omega_1+\omega_2}, f_2 \in \mathfrak{g}_{\omega_1-\omega_2},$$

$$[e_1, e_2] \in \mathfrak{g}_{\omega_1}, [f_1, f_2] \in \mathfrak{g}_{-\omega_1}, [e_2, [e_2, e_1]] \in \mathfrak{g}_{\omega_2}$$

and

$$[f_2, [f_2, f_1]] \in \mathfrak{g}_{\omega_2}, h_1, h_2 \in \mathfrak{g}_0.$$

Let R be a ring and $\langle v_1, \ldots, v_m\rangle_R$ denote the R-span of the set $\{v_1, \ldots, v_m\}$ of elements of the R-module V. In order to explicit describe all simple $\mathfrak{o}_I^{(1)}(5)$-modules with highest weight, consider the Lie algebra $\mathfrak{o}_I^{(1)}(5)$ as a subalgebra of the Lie algebra of derivations for the divided power algebra $O(5, \overline{m})$. Let $\varepsilon_i = (\delta_{i1}, \cdots, \delta_{i5})$, where $\delta_{i,j}$ is Kronecker symbol, and $m_1, \cdots, m_5$ are positive integers. The divided power algebra $O(5, \mathbf{m})$ with height $\mathbf{m} = \sum_{i=1}^5 m_i\varepsilon_i$ is defined as follows:

$$O(5, \mathbf{m}) = \langle x^{(\alpha)} := \prod_{i=1}^5 x^{(\alpha_i)} \ : \ \alpha \in \Gamma_5(\mathbf{m}),\ x^{(\alpha)}x^{(\beta)} = \prod_{i=1}^5 \binom{\alpha_i + \beta_i}{\alpha_i} x^{(\alpha+\beta)}\rangle_k,$$

$$\Gamma_5(\mathbf{m}) = \{\sum_{i=1}^5 \alpha_i\varepsilon_i \ : \ \alpha_i \in \mathbb{Z}_0,\ 0 \leq \alpha_i < 2^{m_i},\ i = 1, \cdots, 5\},$$

where $\mathbb{Z}_0$ is the set of integers. For $\alpha = (\alpha_1, \cdots, \alpha_5) \in \Gamma_5(\mathbf{m})$, let $\mid \alpha \mid = \sum_{i=1}^5 \alpha_i$. Let us define the derivations ∂_i, $i = 1, \cdots, 5$, of the divided power algebra $O(5, \mathbf{m})$ by the formula

$$\partial_i x^{(\alpha)} = x^{(\alpha-\varepsilon_i)}.$$

Then the following derivations of $O(5, \mathbf{m})$ form a basis of $\mathfrak{o}_I^{(1)}(5)$:

$$h_1 = -x_2\partial_2 + x_3\partial_3 + x_4\partial_4 - x_5\partial_5, h_2 = x_2\partial_2 - x_4\partial_4,$$
$$e_1 = x_3\partial_2 - x_4\partial_5, e_2 = x_1\partial_4 - x_2\partial_1, e_3 = x_1\partial_5 - x_3\partial_1, e_4 = x_3\partial_4 - x_2\partial_5,$$
$$f_1 = x_2\partial_3 - x_5\partial_4, f_2 = x_1\partial_2 - x_4\partial_1, f_3 = x_1\partial_3 - x_5\partial_1, f_4 = x_4\partial_3 - x_5\partial_2.$$

The generators $e_1, e_2, h_1, h_2, f_1, f_2$ of this basis satisfy the relations (1).

2. Simple modules over $\mathfrak{o}_I^{(1)}(5)$

We can consider the algebra $O(5, \mathbf{m})$ as a $\mathfrak{o}_I^{(1)}(5)$-module with respect to the natural derivation action. Then simple finite-dimensional modules can be described as $\mathfrak{o}_I^{(1)}(5)$-submodules of $W(5, \mathbf{m})$. It is known that the Lie algebra $\mathfrak{o}_I^{(1)}(5)$ is not restricted [1, Table 1]. As in the case of the restricted classical Lie algebra of type B_2, it has four simple modules of the restricted highest weights with respect to the chosen Cartan subalgebra $H_{\mathbb{Z}}$. But their dimensions are slightly different. For example, $\dim L(1, 0) = 5$ and $\dim L(0, 1) = 10$, when the analogous dimensions for the Lie algebra of type B_2 are 5, 4, respectively. For simple $\mathfrak{o}_I^{(1)}(5)$-modules with restricted highest weights, we obtain the following description:

Theorem 1. *Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{o}_I^{(1)}(5)$ over an algebraically closed field k of characteristic 2. The following isomorphisms of $\mathfrak{g}$-modules hold:*

(a) $L(0, 0) \cong \langle 1\rangle_k \cong k$;
(b) $L(1, 0) \cong \langle x_1, \cdots, x_5\rangle_k$;
(c) $L(0, 1) \cong \bigwedge^2 L(1, 0) \cong \mathfrak{g}$;
(d) $L(1, 1) \cong L(0, 1) \bigotimes L(1, 0)/(L(1, 0) \bigoplus L(0, 1))$.

Proof. The statements (a) and (b) are obvious.

(c) According to the statement (b),

$$\bigwedge^2 L(1, 0) \cong \langle x_1, \cdots, x_5\rangle_k \bigwedge \langle x_1, \cdots, x_5\rangle_k \cong \langle x_i \wedge x_j \ : \ i, j = 1, \cdots, 5;\ i < j\rangle_k.$$

Note that the weight set of the $\mathfrak{g}$-module $\bigwedge^2 L(1, 0)$ coincides with the weight set $\Lambda_{\mathfrak{g}}$ of the adjoint module $\mathfrak{g}$, i.e. the following weight decomposition holds:

$$\bigwedge^2 L(1, 0) = \bigoplus_{\lambda \in \Lambda_{\mathfrak{g}}} (V_\lambda \bigotimes\nolimits_{\mathbb{Z}_2} k),$$

where

$$V_0 = \langle x_2 \wedge x_4, x_3 \wedge x_5, \rangle_{\mathbb{Z}}, V_{2\omega_1-\omega_2} = \langle x_3 \wedge x_4 \rangle_{\mathbb{Z}}, V_{-2\omega_1+\omega_2} = \langle x_2 \wedge x_5 \rangle_{\mathbb{Z}}, V_{-\omega_1+\omega_2} = \langle x_1 \wedge x_2 \rangle_{\mathbb{Z}},$$
$$V_{\omega_1-\omega_2} = \langle x_1 \wedge x_4 \rangle_{\mathbb{Z}}, V_{\omega_1} = \langle x_1 \wedge x_3 \rangle_{\mathbb{Z}}, V_{-\omega_1} = \langle x_1 \wedge x_5, \rangle_{\mathbb{Z}}, V_{\omega_2} = \langle x_2 \wedge x_3 \rangle_{\mathbb{Z}},$$
$$V_{-\omega_2} = \langle x_4 \wedge x_5, \rangle_{\mathbb{Z}}, V_{\omega_1} = \langle x_1 \wedge x_3, \rangle_{\mathbb{Z}}, V_{-\omega_1} = \langle x_1 \wedge x_5, \rangle_{\mathbb{Z}}.$$

It is easy to see that $e_i v \neq 0$ for all $i = 1, 2, 3, 4$ and for all $v \in (\bigwedge^2 L(1,0))_{\mathbb{Z}} \setminus V_{\omega_2}$. This shows that $\bigwedge^2 L(1,0)$ has no proper submodules. The isomorphism $\bigwedge^2 L(1,0) \cong \mathfrak{g}$ of $\mathfrak{g}$-modules is given by

$$x_2 \wedge x_4 \mapsto h_1 + h_2, x_3 \wedge x_5 \mapsto h_2, x_3 \wedge x_4 \mapsto e_1, x_2 \wedge x_5 \mapsto f_1, x_1 \wedge x_2 \mapsto e_2,$$
$$x_1 \wedge x_4 \mapsto f_2, x_1 \wedge x_3 \mapsto e_3, x_1 \wedge x_5 \mapsto f_3, x_2 \wedge x_3 \mapsto e_4, x_4 \wedge x_5 \mapsto f_4.$$

(*d*) Since $L(1,0) \cong \langle \partial_1, \cdots, \partial_5 \rangle_k$ and $L(0,1) \cong \bigwedge^2 L(1,0)$, we get the isomorphism

$$L(0,1) \bigotimes L(1,0) \cong (\bigwedge^2 L(1,0)) \bigotimes \langle \partial_1, \cdots, \partial_5 \rangle_k$$

of $\mathfrak{g}$-modules. Using the notation $x_i x_j \partial_k$ for the element $(x_i \wedge x_j) \otimes \partial_k$ of the space

$$(\bigwedge^2 L(1,0)) \bigotimes \langle \partial_1, \cdots, \partial_5 \rangle_k,$$

we obtain

$$L(0,1) \bigotimes L(1,0) \cong \langle x_i x_j \partial_k \, : \, i, j, k = 1, \cdots, 5; i < j \rangle_k.$$

Then the action of $\mathfrak{g}$ on $L(1,0) \bigotimes L(0,1)$ is given by $x(u\partial_k) = x(u)\partial_k - u\partial_k(x)$, where $x \in \mathfrak{g}$ and $u\partial_k \in L(1,0) \bigotimes L(0,1)$.

As we already know, the set of weights of the $\mathfrak{g}$-module is

$$\Lambda_{L(0,1)} = \Lambda_{\mathfrak{g}} = \{0, \pm(2\omega_1 - \omega_2), \pm(-\omega_1 + \omega_2), \pm\omega_1, \pm\omega_2\}.$$

Let us calculate the set of weights of the $\mathfrak{g}$-module $L(1,0) \cong \langle \partial_1, \cdots, \partial_5 \rangle_k$:

$$h_1\partial_1 = 0, h_1\partial_2 = \partial_2, h_1\partial_3 = -\partial_3, h_1\partial_4 = -\partial_4, h_1\partial_5 = \partial_5;$$
$$h_2\partial_1 = 0, h_2\partial_2 = -\partial_2, h_2\partial_3 = 0, h_2\partial_4 = \partial_4, h_2\partial_5 = 0.$$

Hence, $\Lambda_{L(1,0)} = \{0, \pm(\omega_1 - \omega_2), \pm\omega_1\}$. Then,

$$\Lambda_{L(0,1) \bigotimes L(1,0)} = \{a + b \, : \, a \in \Lambda_{L(0,1)}, \, b \in \Lambda_{L(1,0)}\} =$$
$$\Lambda_{L(0,1)} \cup \Lambda_{L(1,0)} \cup$$
$$\{\pm(3\omega_1 - 2\omega_2), \pm(3\omega_1 - \omega_2), \pm 2(\omega_1 - \omega_2), \pm 2\omega_1, \pm(\omega_1 + \omega_2), \pm(\omega_1 - 2\omega_2)\}.$$

Note that the following weights are dominant: $0, \omega_1, \omega_2, 2\omega_1, \omega_1 + \omega_2$.

The weight subspace $(L(0,1) \bigotimes L(1,0))_0$ is six-dimensional and is spanned by vectors

$$x_1x_2\partial_2, x_1x_3\partial_3, x_1x_4\partial_4, x_1x_5\partial_5, x_3x_5\partial_1, x_4x_5\partial_1.$$

Let

$$v_0 = a_1x_1x_2\partial_2 + a_2x_1x_3\partial_3 + a_3x_1x_4\partial_4 + a_4x_1x_5\partial_5 + a_5x_3x_5\partial_1 + a_6x_2x_4\partial_1$$

is a highest vector. Then it follows from the condition $e_1v_0 = 0$ that $a_1 = a_2$, $a_2 = a_6$, and $a_3 = a_4$. Similarly, from the condition $e_2v_0 = 0$, we get $a_2 = a_4 = a_5 = 0$, $a_1 = a_6$, and $a_3 = a_6$. Then $a_1 = a_3 = a_6 = 0$. Hence, the highest vector with zero highest weight is trivial. In a same way, it is easy to show that the highest vector with the highest weight $2\omega_1$ is also trivial.

The weight subspace $(L(0,1) \bigotimes L(1,0))_{\omega_1}$ is six-dimensional and is spanned by vectors

$$x_1x_3\partial_1, x_2x_3\partial_2, x_2x_4\partial_5, x_3x_4\partial_4, x_3x_5\partial_5, x_4x_5\partial_2.$$

Let

$$u_{\omega_1} = a_1x_1x_3\partial_1 + a_2x_2x_3\partial_2 + a_3x_2x_4\partial_5 + a_4x_3x_4\partial_5 + a_5x_3x_5\partial_5 + a_6x_4x_5\partial_2$$

is a highest vector. Then it follows from the conditions $e_1v_{\omega_1} = 0$ and $e_2v_{\omega_1} = 0$ that

$$a_1 = a_2 = a_4 = a_5, a_3 = a_6 = 0.$$

Therefore,

$$u_{\omega_1} = x_1x_3\partial_1 + x_2x_3\partial_2 + x_3x_4\partial_5 + x_3x_5\partial_5$$

is the highest vector with highest weight ω_1 and and all other highest vectors of the weight ω_1 are collinear to this vector. In a same way, it is easy to show that

$$v_{\omega_2} = x_1x_2\partial_5 + x_1x_3\partial_4 + x_2x_3\partial_1 + x_3x_5\partial_5$$

is the highest vector with the highest weight ω_2.

Thus, the $\mathfrak{g}$-module $L(0,1)\bigotimes L(1,0)$ has two simple submodules with highest weights ω_1 and ω_2. Acting by the reducing elements

$$f_1 = x_2\partial_3 - x_5\partial_4, f_2 = x_1\partial_2 - x_4\partial_1$$

on the highest vector u_{ω_1}, we obtain the following basis of the $\mathfrak{g}$-submodule $L(1,0)$ of the $\mathfrak{g}$-module $L(0,1)\bigotimes L(1,0)$:

$$u_{\omega_1} = x_1x_3\partial_1 + x_2x_3\partial_2 + x_3x_4\partial_5 + x_3x_5\partial_5,$$

$$u_{-\omega_1+\omega_2} = f_1u_{\omega_1} = x_1x_2\partial_1 + x_2x_3\partial_3 + x_2x_4\partial_4 + x_2x_5\partial_5,$$

$$u_0 = f_2f_1u_{\omega_1} = x_1x_2\partial_2 + x_1x_3\partial_3 + x_1x_4\partial_4 + x_1x_5\partial_5,$$

$$u_{\omega_1-\omega_2} = f_2^2f_1u_{\omega_1} = x_2x_4\partial_2 + x_3x_4\partial_3 + x_4x_5\partial_5 + x_1x_4\partial_1,$$

$$u_{-\omega_1} = f_1f_2^2f_1u_{\omega_1} = x_2x_5\partial_2 + x_3x_5\partial_3 + x_4x_5\partial_4 + x_1x_5\partial_1.$$

Similarly, we obtain the following basis for the $\mathfrak{g}$-submodule $L(0,1)$:

$$v_{\omega_2} = x_1x_2\partial_5 + x_1x_3\partial_4 + x_2x_3\partial_1 + x_3x_5\partial_5,$$

$$v_{\omega_1} = f_2v_{\omega_2} = x_2x_3\partial_2 + x_2x_4\partial_5 + x_3x_4\partial_4,$$

$$v_{-\omega_1+\omega_2} = f_1f_2v_{\omega_2} = x_2x_3\partial_3 + x_2x_5\partial_5 + x_3x_5\partial_4,$$

$$v_{2\omega_1-\omega_2} = f_2^2v_{\omega_2} = x_1x_3\partial_2 + x_1x_4\partial_5 + x_3x_4\partial_1,$$

$$v_0^1 = f_1f_2^2v_{\omega_2} = x_1x_2\partial_2 + x_1x_3\partial_3 + x_1x_4\partial_4 + x_1x_5\partial_5 + x_2x_4\partial_1 + x_3x_5\partial_1,$$

$$v_0^2 = f_2f_1f_2v_{\omega_2} = x_1x_3\partial_3 + x_1x_5\partial_5 + x_3x_5\partial_1,$$

$$v_{-2\omega_1+\omega_2} = f_1f_2f_1f_2v_{\omega_2} = x_1x_2\partial_3 + x_1x_5\partial_4 + x_2x_5\partial_1,$$

$$v_{\omega_1-\omega_2} = f_2f_1f_2^2v_{\omega_2} = x_3x_3\partial_3 + x_3x_5\partial_2 + x_4x_5\partial_5,$$

$$v_{-\omega_1} = f_2f_1f_2f_1f_2v_{\omega_2} = x_2x_4\partial_3 + x_2x_5\partial_2 + x_4x_5\partial_5,$$

$$v_{-\omega_2} = f_2^2f_1f_2f_1f_2v_{\omega_2} = x_1x_4\partial_3 + x_1x_5\partial_2 + x_4x_5\partial_1.$$

Basis vectors with the same weights of $\mathfrak{g}$-submodules L(1,0) and $L(0,1)$ are linearly independent. Therefore, the sum of these submodules is decomposable as $\mathfrak{g}$-submodules of $L(0,1)\bigotimes L(1,0)$. Since the weight subspaces corresponding to the highest vectors of $L(0,1)\bigoplus L(1,0)$ are one-dimensional, then we can describe the simple $\mathfrak{g}$-module $L(1,1)$ as a quotient of $L(0,1)\bigotimes L(1,0)$ by $L(0,1)\bigoplus L(1,0)$.

The proof of Theorem 1 is complete.

3. Simple peculiar modules of $\mathfrak{o}_I^{(1)}(5)$

In this section we prove that all fundamental modules over $\mathfrak{o}_I^{(1)}(5)$ are peculiar. Recall that a $\mathfrak{o}_I^{(1)}(5)$-module V is a *peculiar* if $H^n(\mathfrak{o}_I^{(1)}(5), V) \neq 0$ for some n.

Proposition 2. *All fundamental modules over $\mathfrak{o}_I^{(1)}(5)$ are peculiar.*

Proof. The peculiarity of the $\mathfrak{o}_I^{(1)}(5)$-module $L(0,1)$, as an adjoint module, follows from the result in [4, Lemma 4.3(a)]. Let us prove that simple modules $L(1,0)$ is also peculiar.

By the statement (b) of Theorem 1, $L(1,0) = \langle x_1, \cdots, x_5 \rangle$. Consider the following 2-cochain:

$$\psi(x,y) = \begin{cases} x_3 & \text{if } x = f_2 \text{ and } y = f_4, \\ x_2 & \text{if } x = f_3 \text{ and } y = f_4, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for all $x, y, z \in \mathfrak{o}_I^{(1)}(5)$,

$$d\psi(x,y,z) = -\psi([x,y],z) + \psi([x,z],y) - \psi([y,z],z) + x\psi(y,z) - y\psi(x,z) + z\psi(x,y) = 0.$$

Hence, ψ is a 2-cocycle. Now let us prove that ψ is not a coboundary. If ψ is a coboundary of some 1-cochain ω, then from the equality $d\omega(f_2, f_4) = \psi(f_2, f_4)$ we get that

$$f_2\omega(f_4) - f_4\omega(f_2) = x_3.$$

The last equality is impossible, since the images of the representation operators, corresponding to elements f_2 and f_4, do not contain x_3. Therefore, ψ is a coboundary. Thus,

$$H^2(\mathfrak{o}_I^{(1)}(5), L(1,0)) \neq 0.$$

The proof of Proposition 2 is complete.

Note that both fundamental modules have nontrivial second cohomology. The second cohomology of the $\mathfrak{g}$-module $L(0,1)$ was obtained in [5, Lemma 9.6]. For the second cohomology with coefficients in $\mathfrak{g}$-module $L(1,0)$, we obtain the following description:

Proposition 3. *Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{o}_I^{(1)}(5)$ over an algebraically closed field k of characteristic 2. The isomorphism $H^2(\mathfrak{g}, L(1,0)) \cong \langle [\psi_1^2], \cdots, [\psi_5^2] \rangle_k$, where*

$$\psi_1^2 = f_2^* \wedge f_4^* \otimes x_3 + f_3^* \wedge f_4^* \otimes x_2, \psi_2^2 = f_1^* \wedge f_3^* \otimes x_4 + e_2^* \wedge f_1^* \otimes x_3,$$

$$\psi_3^2 = e_3^* \wedge f_4 \otimes x_4 + e_4^* \wedge f_2^* \otimes x_3 + e_1^* \wedge e_2^* \otimes x_3 + e_3^* \wedge f_1^* \otimes x_2 + e_4^* \wedge f_4^* \otimes x_1 + e_3^* \wedge f_3^* \otimes x_1,$$

$$\psi_4^2 = e_2^* \wedge e_4^* \otimes x_5 + e_3^* \wedge e_4^* \otimes x_4, \psi_5^2 = e_1^* \wedge e_3^* \otimes x_2 + e_1^* \wedge f_2^* \otimes x_5$$

of vector spaces holds.

Proof. The set of weights of the space $\bigwedge^2 \mathfrak{g}^* \bigotimes L(1,0)$ is

$$\Lambda_{C^2(\mathfrak{g},M)} = \{\lambda + \mu : \lambda \in \Lambda_{L(1,0)}, \mu \in \Lambda_{\bigwedge^2 \mathfrak{g}^*}\} =$$

$$\Lambda_{L(1,0)} \cup \Lambda_{\mathfrak{g}} \cup \{\pm 2\omega_1, \pm 2\omega_2, \pm 2(\omega_1 - \omega_2), \pm 2(2\omega_1 - \omega_2)\} \cup$$

$$\{\pm(\omega_1 + \omega_2), \pm(\omega_1 - 2\omega_2), \pm(2\omega_1 + \omega_2), \pm(3\omega_1 - \omega_2), \pm(3\omega_1 - 2\omega_2)\} \cup$$

$$\{\pm 3\omega_1, \pm 3(\omega_1 - \omega_2), \pm(4\omega_1 - \omega_2), \pm(4\omega_1 - 3\omega_2), \}.$$

Then $2\Lambda_{C^2(\mathfrak{g},M)} \cap \Lambda_{C^2(\mathfrak{g},M)} = \{0, \pm 2\omega_1, \pm 2\omega_2, \pm 2(\omega_1 - \omega_2), \pm 2(2\omega_1 - \omega_2)\}$. Consequently, $H^2(\mathfrak{g}, M)$ as a Sp_4-module can be generated by classes of cocycles with weights $0, 2\omega_1$, and $2\omega_2$.

First consider the subspace $C^2(\mathfrak{g}, M)_{2\omega_1}$ of 2-cochains with the weight $2\omega_1$. Notice, that subspace $C^2(\mathfrak{g}, M)_{2\omega_1}$ is spanned by 2-cochains

$$h_1^* \wedge f_3^* \otimes x_3, h_2^* \wedge f_3^* \otimes x_3, f_1^* \wedge f_2^* \otimes x_3, f_1^* \wedge f_3^* \otimes x_2, f_1^* \wedge f_4^* \otimes x_1, f_3^* \wedge f_4^* \otimes x_4, e_2^* \wedge f_4^* \otimes x_3.$$

Then it follows from the cocyclicity condition that the $C^2(\mathfrak{g},M)_{2\omega_1}$ of 2-cocycles with weight $2\omega_1$ is one-dimensional and spanned by the cocycle

$$\psi^2 = f_1^* \wedge f_2^* \otimes x_3 + f_1^* \wedge f_3^* \otimes x_2 + f_3^* \wedge f_4^* \otimes x_4 + e_2^* \wedge f_4^* \otimes x_3.$$

This cocycle is the coboundary of the 1-cochain $f_3^* \otimes x_3$.

Similarly, it is easy to show that all cocycles with a weight in $\{-2\omega_1, \pm 2(\omega_1 - \omega_2)\}$ are coboundaries.

Now consider 2-cochains with weight $2\omega_2$, they spans the two-dimensional subspace

$$C^2(\mathfrak{g},M)_{2\omega_2} = \langle f_2^* \wedge f_4^* \otimes x_3, f_3^* \wedge f_4^* \otimes x_2 \rangle_k.$$

Then it follows from the cocyclicity condition that the subspace $C^2(\mathfrak{g})_{2\omega_2}$ of cocycles with weight 2ω is one-dimensional and is spanned by the cocycle $\psi_1^2 = f_2^* \wedge f_4^* \otimes x_3 + f_3^* \wedge f_4^* \otimes x_2$. Since $[f_2, f_4] = 0$, $f_2 u, f_4 u \notin L(1,0)_{\omega_1}$ for all $u \in L(1,0)$, this cocycle cannot be a coboundary. Hence, the cohomological class of the cocycle $\psi_1^2 = f_2^* \wedge f_4^* \otimes x_3 + f_3^* \wedge f_4^* \otimes x_2$ forms a basis of the subspace $H^2(\mathfrak{g},M)_{2\omega_2}$.

As in the previous case, it is easy to show that the cohomological classes of cocycles ψ_2^2, ψ_4^2 and ψ_5^2 are spanned the subspaces $H^2(\mathfrak{g},M)_{2(2\omega_1-\omega_2)}$, $H^2(\mathfrak{g},M)_{-2\omega_2}$, and $H^2(\mathfrak{g},M)_{-2(2\omega_1-\omega_2)}$, respectively.

Finally, we consider the subspace of zero-weight 2-cochains. It is 21-dimensional. Simple calculations show that the subspace of cocycles with zero weight is 6-dimensional and is spanned by cocycles

$$h_1^* \wedge f_3 \otimes x_5 + h_1^* \wedge f_2 \otimes x_4 + h_1^* \wedge e_3 \otimes x_3 + h_1^* \wedge e_2 \otimes x_2 + e_1^* \wedge f_1 \otimes x_1 + e_3^* \wedge f_3 \otimes x_1 + e_4^* \wedge f_4 \otimes x_1,$$

$$h_2^* \wedge f_3 \otimes x_5 + h_2^* \wedge f_2 \otimes x_4 + h_2^* \wedge e_3 \otimes x_3 + h_2^* \wedge e_2 \otimes x_2 + e_2^* \wedge f_2 \otimes x_1 + e_3^* \wedge f_3 \otimes x_1,$$

$$e_3^* \wedge f_4 \otimes x_4 + e_2^* \wedge f_4 \otimes x_3 + e_1^* \wedge e_2 \otimes x_3 + e_3^* \wedge f_1 \otimes x_2 + e_3^* \wedge f_3 \otimes x_1 + e_4^* \wedge f_4 \otimes x_1,$$

$$e_1^* \wedge f_3 \otimes x_4 + e_4^* \wedge f_2 \otimes x_3 + e_1^* \wedge e_2 \otimes x_3 + e_4^* \wedge f_3 \otimes x_2 + e_1^* \wedge f_1 \otimes x_1 + e_3^* \wedge f_3 \otimes x_1,$$

$$f_1^* \wedge f_2 \otimes x_5 + e_4^* \wedge f_2 \otimes x_3 + e_3^* \wedge f_1 \otimes x_2 + e_4^* \wedge f_3 \otimes x_2 + e_1^* \wedge f_1 \otimes x_1 + e_2^* \wedge f_2 \otimes x_1,$$

$$e_2^* \wedge f_4 \otimes x_5 + e_1^* \wedge e_2 \otimes x_3 + e_3^* \wedge f_1 \otimes x_2 + e_4^* \wedge f_3 \otimes x_2 + e_2^* \wedge f_2 \otimes x_1 + e_4^* \wedge f_4 \otimes x_1.$$

By the similar calculations we obtain

$$C^1(\mathfrak{g},M)_0 = \langle h_1^* \otimes x_1, h_1^* \otimes x_1, e_2^* \otimes x_2, e_3^* \otimes x_3, f_2^* \otimes x_4, f_3^* \otimes x_4 \rangle_k,$$

$$Z^1(\mathfrak{g},M)_0 = \langle e_2^* \otimes x_2 + e_3^* \otimes x_3 + f_2^* \otimes x_4 + f_3^* \otimes x_4 \rangle_k.$$

Then

$$\dim\, H^2(\mathfrak{g},M)_0 = \dim\, Z^2(\mathfrak{g},M)_0 + \dim\, Z^1(\mathfrak{g},M)_0 - \dim\, C^1(\mathfrak{g},M)_0 = 6 + 1 - 6 = 1.$$

As a basis of the subspace $H^2(\mathfrak{g},M)_0$, one can choose the cohomology class of the cocycle

$$\psi_3^2 = e_3^* \wedge f_4 \otimes x_4 + e_2^* \wedge f_4 \otimes x_3 + e_1^* \wedge e_2 \otimes x_3 + e_3^* \wedge f_1 \otimes x_2 + e_3^* \wedge f_3 \otimes x_1 + e_4^* \wedge f_4 \otimes x_1.$$

Since the Sp_4-module $L(2\omega_2) = L(\omega_2)^{(1)}$ is five-dimensional and has a weight decomposition

$$L(2\omega_2) = \bigoplus_{\lambda \in \{0, \pm 2\omega_2, \pm 2(2\omega_1 - \omega_2)\}} L(2\omega_2)_\lambda,$$

the cohomological classes of cocycles $\psi_1^2, \cdots, \psi_5^2$ form a basis for $H^2(\mathfrak{g},M)$ as an Sp_4-module.

The proof of Proposition 3 is complete.

References

[1] Weisfeiler B., Kac V., Exponentials in Lie algebras of characteristic p, *Mathematics of the USSR-Izvestiya*, Vol. 5, No.4, 1971, pp. 777-803.

[2] Lebedev A., Analog of the orthogonal, Hamiltonian, Poisson and kontact Lie superalgebras in characteristic 2, *J. Nonlinear Math. Phys.*, vol. 17, 2010, pp. 217-251.

[3] Bouarroudj S., Lebedev A., Leites D., Shchepochkina I., Lie Algebra deformations in characteristic 2, *arXiv:1301.2781[math.RT]*, 2015, https://doi.org/10.48550/arXiv.1301.2781.

[4] Bouarroudj S., Grozman P., Lebedev A., Leites D., Derivations and Central Extensions of Symmetric Modular Lie Algebras and Superalgebras (with an Appendix by Andrey Krutov), *Symmetry, Integrability and Geometry: Methods and Applications*, Vol. 19, 2023, 032, 73 pages.

[5] Bouarroudj S., Grozman P., Leites D., Deformations of Symmetric Simple Modular Lie (Super)Algebras, *Symmetry, Integrability and Geometry: Methods and Applications* Vol. 19, 2023, 031, 66 pages.

[6] Bouarroudj S., Lebedev A., Wagemann F., Deformations of the Lie algebra o(5) in characteristics 3 and 2, *Math. Notes*, Vol. 89, No. 6, 2011, pp. 777-791.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

SPINOR REPRESENTATIONS OF A CURVE WITH FRENET-TYPE FRAME IN MYLLER CONFIGURATION FOR 3-DIMENSIONAL LIE GROUPS

ZEHRA İŞBİLİR[1], BAHAR DOĞAN YAZICI[2], MURAT TOSUN[3]

[1]Düzce University, Faculty of Arts and Sciences, Department of Mathematics, Düzce, Türkiye
[2]Bilecik Şeyh Edebali University, Faculty of Science, Department of Mathematics, Bilecik, Türkiye
[3]Sakarya University, Faculty of Science, Department of Mathematics, Sakarya, Türkiye
e-mail: zehraisbilir@duzce.edu.tr, bahar.dogan@bilecik.edu.tr, tosun@sakarya.edu.tr

Abstract. In this paper, we introduce the representation of a curve with Frenet-type frame in Myller configuration for 3-dimensional Lie groups by using the spinors, which are a widely used notion from mathematics to physics. Also, we get quite a general spinor representation and have some special cases. Moreover, we obtain spinor equations of a curve with Frenet-type frame in Myller configuration for 3-dimensional Lie groups. Then, we construct some geometric properties, theories, and results with respect to this concept.

Keywords: Spinors, Versor field, Frenet-type frame, Myller configuration, Lie groups.

AMS Subject Classification: 15A66, 22E15, 53A04.

1. Introduction

One of the most popular and interesting concepts of differential geometry is curve theory for a great number of researchers in both Euclidean space and Minkowski space. From the establishing quite important and classical moving frame Frenet-Serret frame, which is constructed independently of each other by Frenet [27] and Serret [80], this topic has attracted the attention of researchers. A great number of studies have been done and also are ongoing in order to find and examine new moving frames such as; Darboux frame [15] and Bishop frame [3].

In Euclidean 3-space E_3, one determines some versor fields such as; tangent, principal normal, and binormal, along some plane fields such as; rectifying, normal, and osculating planes along a curve $\mathcal{C}$. A versor field (namely; a unit vector field) and a plane field are represented by means of the notations $(\mathcal{C},\zeta)$ and $(\mathcal{C},\pi)$, respectively. The couple $\{(\mathcal{C},\zeta),(\mathcal{C},\pi)\}$ where $\zeta\in\pi$ is called a Myller configuration and represented by using $\mathcal{M}(\mathcal{C},\overline{\zeta},\pi)$ in E_3 [52, 53, 57].

R. Miron completed some studies with respect to this pair $\left\{(C,\overline{\zeta}),(C,\pi)\right\}$ in 1960 [58], and we want to refer the very important and essential book with lots of information about Myller configuration [57]. If the plane π is tangent to the curve $\mathcal{C}$, a tangent Myller configuration which is represented by $\mathfrak{M}_t(\mathcal{C},\overline{\zeta},\pi)$ is obtained [53, 57]. A. Myller examined the parallelism of the versor field $(\mathcal{C},\overline{\zeta})$ in the plane field $(\mathcal{C},\pi)$ developing a generalization of Levi-Civita parallelism on the curved surfaces. Then, O. Mayer continued these concepts which included some new invariants for Myller configuration $\mathfrak{M}(C,\overline{\zeta},\pi)$. Also, Levi-Civita underlined the importance of these papers in his book [48]. Detailed terminology, definitions, theorems, and results of the geometry of Myller configurations $\mathcal{M}(\mathcal{C},\overline{\zeta},\pi)$ and tangent Myller configuration $\mathcal{M}_t(\mathcal{C},\overline{\zeta},\pi)$ have been written in [57].

Additionally, if $\mathcal{C}$ is a curve on the surface $\mathcal{S}$, the geometry of the field $(\mathcal{C},\zeta)$ on surface $\mathcal{S}$ is the geometry of the associated Myller configurations $\mathcal{M}_t(\mathcal{C},\overline{\zeta},\pi)$ [57]. Then, the Darboux frame

is defined for a Myller configuration $\mathcal{M}(\mathcal{C}, \overline{\zeta}, \pi)$ (see [4,55–57,64,69–72]. Note that the geometry of tangent Myller configuration $\mathcal{M}_t(\mathcal{C}, \overline{\zeta}, \pi)$ is a special case of general Myller configuration $\mathcal{M}(\mathcal{C}, \overline{\zeta}, \pi)$ [57].

Various studies have been completed related to the Myller configuration, and some of them can be listed such as; Myller configuration in nonholonomic manifolds E_3^2 in Euclidean space E_3 [28, 63, 65], Mark Krien's formula and Gauss-Bonnet formula for Myller configuration [59], Myller configuration in Riemannian geometry [60, 64, 66], Myller configuration in symplectic geometry [87], Myller configuration in infinite-dimensional Riemannian manifolds [10], Myller configuration in Finsler space [9], Myller configuration in Minkowski space [62, 68], and Myller configuration in Hamilton, Lagrange and Finsler [47], Myller configuration in hydromechanics in Euclidean space E_3 [29,31], respectively. Then, versor fields along a curve in a four-dimensional Lorentz space were examined [32]. In addition to these, we want to refer to the studies [57,61,67] about Myller configuration.

Also, special curves in a Myller configuration and properties of them [51], rectifying-type curves [53] and Bertrand curves [52] with Frenet-type frame in Myller configuration, and rectifying-type curves with rotation minimizing frame in n-dimensional space of $\mathbb{R}^n$ [41] were studied. Then, osculating-type curves in Myller configuration for Euclidean space E_3 [36] and E_4 [40], and rectifying-type curves in Myller configuration for Euclidean 4-space E_4 [39] were investigated. Moreover, Smarandache curves with Myller configuration [37] and Frenet-type frame in Myller configuration for 3-dimensional Lie groups and some special curves were determined [38].

In another respect, the concept of spinors was examined by É. Cartan in 1913, but P. Ehrenfest had earned the notion of spinor to the literature in the 1920s [88]. Spinors are one of the popular work frames for several researchers in mathematics and physics. From the point of view of physicists, the notion spinors are multilinear transformations and by this qualification, spinors are mathematical structures. Nevertheless, for mathematicians, this multilinear property does not matter and spinors are a vectorial concept [22, 24, 26].

One of the most used areas of spinors is Clifford algebra, which is called also geometric algebra by W. K. Clifford [8, 88]. According to physicists, the terminology spin states features of quantum particles that appeared in the course of various experiments, and if these features were considered, the spinor as a new mathematical concept is obtained [33]. Examining the spin of non-relativistic electron, the angular momentum of an electron and spin of the non-relavistic electron, other spin-1/2 particles, the state of relativistic many-particle systems in quantum field theory are expressed in some work frames by taking advantage of spinors [24, 44, 77]. W. Pauli examined the Pauli matrices, which can be said also the spin matrices, and described the spin in quantum theory [22, 79]. Also, the notion of spin is represented by Pauli matrices in the theory of quantum. By means of a vector with two complex components, which is spinor, the wave function of an electron was represented by Pauli, and then, Cartan studied the mathematical sense of spinors when working on the representation of groups [33]. According to Cartan, the concept of spinors develops a linear representation of the groups of rotations of a space of any dimension, and so relationships of spinors and physics can also be constructed with geometry [33]. Cartan's pioneer study [7] includes the geometric construction of spinor representations. The set of isotropic vectors establishes a two-dimensional surface in the space $\mathbb{C}^2$ and on the contrary, in $\mathbb{C}^2$, these vectors represent the same isotropic vectors. Also, these complex vectors are two-dimensional in the space $\mathbb{C}^2$ [7, 22]. In addition to these, spinors were examined by M. D. Vivarelli in [89], and the author established relations between the spinors and quaternions, and constructed the spinor representation of rotations in $\mathbb{E}^3$ by using the relations between the rotations in $\mathbb{E}^3$ and quaternions [22, 89].

Several studies completed and are ongoing related to spinors in various disciplines and working areas. One of the most attractive working areas for researchers related to spinors is geometry and Clifford algebra. In the geometric sense, the relationships among the differential geometry and spinor are established by del Castillo and Barrales [17], and this study is a pioneer study

for researchers and literature. Thanks to the study [17], lots of studies were completed with respect to the spinor representations of different moving frames and some special types of curves [20–22, 24, 25, 34, 35, 44, 82, 84–86]. Additionally, hyperbolic spinors and differential geometry were combined in several studies [2, 23, 42, 43]. Then, Fibonacci spinors [26] were determined by inspiring Vivarelli's study [89]. For terminology and detailed information about the spinors, we want to refer to some studies, as well [6, 16, 17, 33, 49, 50, 83, 88, 89].

On the other hand, the degenerate semi-Riemannian geometry of Lie groups was studied in [14] by Çöken and Çiftçi. Also, Çiftçi examined the generalization of Lancret's theorem and general helices [13]. Then, the rectifying, osculating and normal curves in three-dimensional compact Lie groups [5], and Smarandache curves in the three-dimensional Lie groups [76] were determined. Additionally, slant helices, general helices of AW(k)-type, Bertrand and Mannheim curve couples in the three-dimensional Lie group with a bi-invariant metric were obtained [30, 73, 74, 90]. Then, natural and conjugate mates of Frenet curves in 3-dimensional Lie group were introduced in [54]. In addition to these, inextensible flows of curves in three-dimensional Lie groups [75], generalized Bertrand and Mannheim curves in 3D Lie groups [78], spinor Frenet equations in the three-dimensional Lie groups [77], Bertrand curves of AW(k)-type in three-dimensional Lie groups and spherical indicatrices of them [12, 45] were scrutinized. Framed curves that might have singular points in the three-dimensional Lie groups and a Berry phase model of them were determined [19], and spinor representations of framed curves in the 3-dimensional Lie groups were defined [35]. Lie groups, which have several types of usage areas with respect to the theory and application in mechanics and physics, include some special cases such as; abelian Lie groups, $SO(3)$ and S^3. Then, incompressible inviscid fluid motion and rigid body motion correspond to the geodesic flow of left or right invariant metric have been determined on a Lie group [1, 11, 46].

In this study, we focus on the spinor representations of a curve with Frenet-type frame in Myller configuration for 3-dimensional Lie groups. In Section 1 and Section 2, we remind some basic notations and notions with respect to the spinors, Lie groups, Frenet-type frame in Myller configuration, and Frenet-type frame in Myller configuration for 3-dimensional Lie groups. In Section 3, we investigate the spinor equations of a curve with Frenet-type frame in Myller configuration for 3-dimensional Lie groups, and spinor components of them, as well. Then, we examine some special cases of them, and give some theories and results. Finally, we give conclusions in Section 4.

2. Preliminaries

In this section, we want to remind some basic and required notions and notations with respect to spinors, Lie groups, Frenet-type frame in Myller configuration, Frenet-type frame in Myller configuration for 3-dimensional Lie groups needed throughout the paper.

2.1. **Spinors.** A spinor is defined as follows:

$$\Omega = \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix}$$

with the help of the vectors $\lambda, \mu, \epsilon \in \mathbb{R}^3$ such that

$$\lambda + i\mu = \Omega^t \sigma \Omega, \tag{1}$$

$$\epsilon = -\widehat{\Omega}^t \sigma \Omega, \tag{2}$$

where "t" represents the transposition, $\widehat{\Omega}$ is the conjugation [7] (or mate [16]) of Ω, and also $\overline{\Omega}$ is the complex conjugation of Ω. Then, the followings are satisfied [17]:

$$\widehat{\Omega} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\Omega} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\Omega}_1 \\ \overline{\Omega}_2 \end{pmatrix} = \begin{pmatrix} -\overline{\Omega}_2 \\ \overline{\Omega}_1 \end{pmatrix}.$$

Additionally, Pauli matrices which are 2×2 cartesian components for the vector $\sigma=(\sigma_1,\sigma_2,\sigma_3)$ are written as follows [17]:

$$\sigma_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \sigma_2=\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix},\quad \sigma_3=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}. \tag{3}$$

Then, $\lambda+i\mu$ is an isotropic vector and ϵ is a real vector. Let $\eta=(\eta_1,\eta_2,\eta_3)\in\mathbb{C}^3$ be an isotropic vector if $\langle\eta,\eta\rangle=0$ where $\mathbb{C}^3$ is three-dimensional complex vector space. The isotropic vectors in $\mathbb{C}^3$ establish a two-dimensional surface in $\mathbb{C}^2$. If this surface is parameterized by means of the Ω_1 and Ω_2, then $\eta_1=\Omega_1^2-\Omega_2^2$, $\eta_2=i(\Omega_1^2+\Omega_2^2)$, $\eta_3=-2\Omega_1\Omega_2$ are expressed. Thus, $\Omega_1=\pm\sqrt{\frac{\eta_1-i\eta_2}{2}}$ and $\Omega_2=\pm\sqrt{\frac{-\eta_1-i\eta_2}{2}}$ can be written [7,17,20,22,34,35]. From $\lambda+i\mu=(\eta_1,\eta_2,\eta_3)$ the following equation is satisfied with the (1) and (3):

$$\eta_1=\Omega^t\sigma_1\Omega,\quad \eta_2=\Omega^t\sigma_2\Omega,\quad \eta_3=\Omega^t\sigma_3\Omega,$$

and

$$\begin{aligned}\lambda+i\mu&=\left(\Omega_1^2-\Omega_2^2,i(\Omega_1^2+\Omega_2^2),-2\Omega_1\Omega_2\right),\\ \epsilon&=\left(\Omega_1\overline{\Omega}_2+\overline{\Omega}_1\Omega_2,i(\Omega_1\overline{\Omega}_2-\overline{\Omega}_1\Omega_2),|\Omega_1|^2-|\Omega_2|^2\right).\end{aligned} \tag{4}$$

Because of the fact that $\lambda+i\mu\in\mathbb{C}^3$ is an isotropic vector, the vectors λ,μ,ϵ are mutually orthogonal vectors and $|\lambda|=|\mu|=|\epsilon|=\overline{\Omega}^t\Omega$. Then, $\langle\lambda\times\mu,\epsilon\rangle=\det(\lambda,\mu,\epsilon)>0$. Contraversely, if the vectors $\lambda,\mu,\epsilon\in\mathbb{R}^3$ are mutually orthogonal vectors of the same magnitude ($\det(\lambda,\mu,\epsilon)>0$), there is a spinor which is written in (1) [7,16,17,20,22,24,25,34,35,44,77,82,84]. Let Ω and Φ be any two spinors and the following properties are satisfied:

$$\Omega^t\sigma\Phi=\Phi^t\sigma\Omega, \tag{5}$$

$$\overline{\Omega^t\sigma\Phi}=-\widehat{\Omega}^t\sigma\widehat{\Phi}, \tag{6}$$

$$\widehat{(\alpha\Omega+\beta\Phi)}=\overline{\alpha}\widehat{\Omega}+\overline{\beta}\widehat{\Phi}, \tag{7}$$

where $\alpha,\beta\in\mathbb{C}$ [17]. Since the matrices $\sigma_1,\sigma_2,\sigma_3$ are symmetric, then (5) is satisfied. Because of the fact that the spinors Ω and $-\Omega$ correspond to the same ordered orthogonal basis $\{\lambda,\mu,\epsilon\}$ with $|\lambda|=|\mu|=|\epsilon|$ and $\det(\lambda,\mu,\epsilon)>0$, the correspondence between the spinors and orthogonal bases written in (1) is two-to-one. The ordered triads $\{\lambda,\mu,\epsilon\},\{\mu,\epsilon,\lambda\},\{\epsilon,\lambda,\mu\}$ correspond to different spinors. If $\Omega\neq 0$, the set $\{\Omega,\overline{\Omega}\}$ is linearly independent [7,16,17,20,22,24,25,34,35,44,77,82,84].

2.2. **Lie groups.** Let G be a Lie group with a bi-invariant metric $\langle,\rangle$ and D be the Levi-Civita connection of G. If the notation $\mathfrak{g}$ represents the Lie algebra of Lie group G, $\mathfrak{g}$ is isomorphic to T_eG where e is the neutral element of Lie group G. Also, if $\langle,\rangle$ is a bi-invariant metric on G, the followings is satisfied:

$$\langle\mathcal{X},[\mathcal{Y},\mathcal{Z}]\rangle=\langle[\mathcal{X},\mathcal{Y}],\mathcal{Z}\rangle$$

and

$$D_{\mathcal{X}}\mathcal{Y}=\frac{1}{2}[\mathcal{X},\mathcal{Y}]$$

for every $\mathcal{X},\mathcal{Y},\mathcal{Z}\in\mathfrak{g}$. Then, Lie bracket of any two vector fields $\mathcal{W}$ and $\mathcal{Z}$ is expressed as:

$$[\mathcal{W},\mathcal{Z}]=\sum_{i=1}^{n}w_iz_i\,[\mathcal{X}_i,\mathcal{X}_j]$$

where $\mathcal{W}=\sum_{i=1}^{n}w_i\mathcal{X}_i$ and $\mathcal{Z}=\sum_{i=1}^{n}z_i\mathcal{X}_i$ with the orthonormal basis $\{\mathcal{X}_1,\mathcal{X}_2,\ldots,\mathcal{X}_n\}$ of $\mathfrak{g}$, and $w_i=I\to\mathbb{R}$ and $z_i=I\to\mathbb{R}$. If $\delta:I\subset\mathbb{R}\to G$ be an arc-length regular curve, the covariant

derivative of the vector field $\mathcal{W}$ along the curve δ which is represented by the notation $D_{\delta'}\mathcal{W}$ is given as follows:

$$D_{\delta'}\mathcal{W} = D_{\mathcal{T}}\mathcal{W} = \dot{\mathcal{W}} + \frac{1}{2}[\mathcal{T}, \mathcal{W}]$$

where $\mathcal{T}$ is the unit tangent vector of the curve δ and $\dot{\mathcal{W}} = \sum_{i=1}^{n} \frac{dw}{dt}\mathcal{X}_i$. In addition to these, if the vector field $\mathcal{W}$ is the left-invariant vector field of the curve δ, $\dot{\mathcal{W}} = 0$ is obtained [11,19,35,77].

2.3. **Frenet-type frame in Myller configuration.** Let $(\mathcal{C}, \overline{\zeta})$ be a versor field and $\overline{r}(s)$ is a position vector of the curve $\mathcal{C}$ where s is the arc-length parameter on the curve $\mathcal{C}$. For Frenet-type frame of a versor field $\mathcal{R}_F = \{P, \overline{\zeta}_1, \overline{\zeta}_2, \overline{\zeta}_3\}$ of the versor field $(\mathcal{C}, \overline{\zeta})$, the followings are satisfied:

$$\frac{d\overline{r}}{ds} = \rho_1(s)\overline{\zeta}_1(s) + \rho_2(s)\overline{\zeta}_2(s) + \rho_3(s)\overline{\zeta}_3(s)$$

where $\rho_1^2(s) + \rho_2^2(s) + \rho_3^2(s) = 1$. Then, the following derivative formulas hold:

$$\begin{cases} \dfrac{d\overline{\zeta}_1(s)}{ds} = \mathcal{K}_1(s)\overline{\zeta}_2(s), \\ \dfrac{d\overline{\zeta}_2(s)}{ds} = -\mathcal{K}_1(s)\overline{\zeta}_1(s) + \mathcal{K}_2(s)\overline{\zeta}_3(s), \\ \dfrac{d\overline{\zeta}_3(s)}{ds} = -\mathcal{K}_2(s)\overline{\zeta}_2(s), \end{cases}$$

where $\mathcal{K}_1(s) > 0$, and $\mathcal{K}_1$-curvature and $\mathcal{K}_2$-torsion have the same geometrical interpretation as the curvature and torsion of a curve with Frenet frame in Euclidean 3-space E_3. Also, if the functions $\rho_1(s) = 1$, $\rho_2(s) = 0$ and $\rho_3(s) = 0$, we have Frenet equations of a curve in E_3 [27,57,80,81]. The fundamental theorem of invariants for the versor field $(\mathcal{C}, \overline{\zeta})$ is written as:

Theorem 2.1 ([57]). *If the invariants $\mathcal{K}_1(s), \mathcal{K}_2(s), \rho_1(s), \rho_2(s), \rho_3(s)$, with the condition $\rho_1^2(s)+\rho_2^2(s)+\rho_3^2(s) = 1$ are differentiable functions for the parameter $s \in [a, b]$, then there exists a curve $\mathcal{C} : [a, b] \to E_3$ parameterized by arc-length parameter s and a versor field $\overline{\zeta}(s)$, $s \in [a, b]$, whose the curvature, torsion and the functions $\rho_i(s)$ are $\mathcal{K}_1(s), \mathcal{K}_2(s)$ and $\rho_i(s), i = 1, 2, 3$. Any two such versor fields $(\mathcal{C}, \overline{\zeta})$ differ by a proper Euclidean motion.*

The quite informative book with respect to Myller configuration [57] by R. Miron can be examined in order to get more detailed information for terminology and notation about Myller configuration.

2.4. **Frenet-Type Frame in Myller Configuration for the 3-Dimensional Lie Groups.** Frenet-type frame in Myller configuration for 3-dimensional Lie groups were determined by İşbilir et al. in [38] by inspiring the studies [5, 13, 57, 77].

Definition 2.1 ([38]). *Let $(\mathcal{C}, \overline{\zeta})$ be a versor field and $\overline{r}(s) : I \mapsto G$ is a position vector of the curve $\mathcal{C}$ in Myller configuration for 3-dimensional Lie group G. The covariant derivative of $\mathcal{W}$ along the curve is determined as follows:*

$$\begin{aligned} D_{\hat{\zeta}(s)}\mathcal{W}(s) &= \dot{\mathcal{W}}(s) + \frac{1}{2}\left[\hat{\zeta}(s), \mathcal{W}(s)\right] \\ &= \dot{\mathcal{W}}(s) + \frac{1}{2}\rho_1(s)\left[\overline{\zeta}_1(s), \mathcal{W}(s)\right] + \frac{1}{2}\rho_2(s)\left[\overline{\zeta}_2(s), \mathcal{W}(s)\right] + \frac{1}{2}\rho_3(s)\left[\overline{\zeta}_3(s), \mathcal{W}(s)\right] \end{aligned}$$

and

$$\hat{\zeta}(s) = \frac{d\overline{r}}{ds} = \rho_1(s)\overline{\zeta}_1(s) + \rho_2(s)\overline{\zeta}_2(s) + \rho_3(s)\overline{\zeta}_3(s)$$

where $\dot{\mathcal{W}} = \sum_{i=1}^{n} \frac{dw}{ds}\mathcal{X}_i$ for the orthonormal basis $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}$ of $\mathfrak{g}$.

Theorem 2.2 ([38]). *Let $\overline{r}(s) : I \mapsto G$ be a curve with Frenet-type frame in Myller configuration for 3-dimensional Lie group G. Then, Frenet-type frame derivative formulas in Myller configuration for 3-dimensional Lie groups are given as follows:*

$$\begin{pmatrix} \dot{\overline{\zeta}}_1(s) \\ \dot{\overline{\zeta}}_2(s) \\ \dot{\overline{\zeta}}_3(s) \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{K}_1(s) - \rho_3(s)\flat_G & \rho_2(s)\flat_G \\ -\mathcal{K}_1(s) + \rho_3(s)\flat_G & 0 & \mathcal{K}_2(s) - \rho_1(s)\flat_G \\ -\rho_2(s)\flat_G & -\mathcal{K}_2(s) + \rho_1(s)\flat_G & 0 \end{pmatrix} \begin{pmatrix} \overline{\zeta}_1(s) \\ \overline{\zeta}_2(s) \\ \overline{\zeta}_3(s) \end{pmatrix} \tag{8}$$

where

$$\begin{cases} \mathcal{K}_1(s) - \rho_3(s)\flat_G = \left\langle \dot{\overline{\zeta}}_1(s), \overline{\zeta}_2(s) \right\rangle = -\left\langle \dot{\overline{\zeta}}_2(s), \overline{\zeta}_1(s) \right\rangle, \\ \mathcal{K}_2(s) - \rho_1(s)\flat_G = \left\langle \dot{\overline{\zeta}}_2(s), \overline{\zeta}_3(s) \right\rangle = -\left\langle \dot{\overline{\zeta}}_3(s), \overline{\zeta}_2(s) \right\rangle, \end{cases}$$

and

$$\flat_G = \frac{1}{2}\left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right], \overline{\zeta}_3(s) \right\rangle.$$

Special Cases 2.1 ([38]). *Let $\overline{r}(s) : I \mapsto G$ be a curve with the Frenet-type frame in Myller configuration for 3-dimensional Lie group G. Then, the followings hold:*

(i) *If $\flat_G = 0$, then the Frenet-type frame in Myller configuration for Euclidean 3-space is obtained [57].*
(ii) *If $\rho_1(s) = 1, \rho_2(s) = \rho_3(s) = 0$, then the Frenet-type frame in 3-dimensional Lie groups is obtained [5, 13, 77].*
(iii) *If $\rho_1(s) = 1, \rho_2(s) = \rho_3(s) = 0$ and $\flat_G = 0$, then the Frenet frame in Euclidean 3-space is obtained [27, 80, 81].*

Remark 2.1 ([38]). *Let $\overline{r}(s) : I \mapsto G$ be a curve with the Frenet-type frame in Myller configuration for 3-dimensional Lie group G. The following three special cases hold:*

(i) *Provided that G is abelian, then $\flat_G = 0$.*
(ii) *Provided that G is $SO(3)$, then $\flat_G = \frac{1}{2}$.*
(iii) *Provided that G is S^3, then $\flat_G = 1$*

where G is a Lie group with a bi-invariant metric $\langle , \rangle$ (see also [13, 18, 19, 35, 77]).

3. Spinor Representations of a Curve with Frenet-type Frame in Myller Configuration for 3-Dimensional Lie Groups

In this section, we introduce the spinor representations of a curve with Frenet-type frame in Myller configuration for 3-dimensional Lie groups G and also give some special cases of them. Then, we give the spinor equations of them in the three types special cases of G. Consequently, some geometric interpretations, theories, and results are presented with respect to them.

Definition 3.1. *Let $\overline{r}(s) : I \to G$ is a curve with the Frenet-type frame in Myller configuration for 3-dimensional Lie groups and the spinor φ corresponds to the triad $\{\overline{\zeta}_2, \overline{\zeta}_3, \overline{\zeta}_1\}$. Then, the spinor representations of the curve $\overline{r}$ with the Frenet-type frame in Myller configuration for 3-dimensional Lie groups are written as follows:*

$$\overline{\zeta}_2 + i\overline{\zeta}_3 = \varphi^t \sigma \varphi, \tag{9}$$

$$\overline{\zeta}_1 = -\widehat{\varphi}^t \sigma \varphi, \tag{10}$$

where $\overline{\varphi}^t \varphi = 1$.

Theorem 3.1. *Let $\overline{r}(s) : I \to G$ is a curve with the Frenet-type frame in Myller configuration for 3-dimensional Lie groups and the spinor φ corresponds to the triad $\{\overline{\zeta}_2, \overline{\zeta}_3, \overline{\zeta}_1\}$. Then, the following single spinor equation is obtained as follows:*

$$\frac{d\varphi}{ds} = \frac{1}{2}[-i\left(\mathcal{K}_2 - \rho_1 \flat_G\right)\varphi + \left(\mathcal{K}_1 - \rho_3 \flat_G + i\rho_2 \flat_G\right)\widehat{\varphi}]. \tag{11}$$

Proof. Taking the derivative of (9) according to the parameter s, we have:

$$\dot{\overline{\zeta}}_2 + i\dot{\overline{\zeta}}_3 = \left(\frac{d\varphi}{ds}\right)^t \sigma\varphi + \varphi^t \sigma \frac{d\varphi}{ds}. \tag{12}$$

The spinor $\dfrac{d\varphi}{ds}$ can be expressed as follows since $\{\varphi, \widehat{\varphi}\}$ is a basis for the spinors:

$$\frac{d\varphi}{ds} = a\varphi + b\widehat{\varphi} \tag{13}$$

where a and b are any complex-valued functions. By using (8), (12) and (13), we get:

$$\left(-\mathcal{K}_1 + \rho_3 \flat_G - i\rho_2 \flat_G\right)\overline{\zeta}_1 + \left(-i\mathcal{K}_2 + i\rho_1 \flat_G\right)\overline{\zeta}_2 + \left(\mathcal{K}_2 - \rho_1 \flat_G\right)\overline{\zeta}_3 = 2a\varphi^t\sigma\varphi + b\widehat{\varphi}^t\sigma\varphi + b\varphi^t\sigma\widehat{\varphi}.$$

With the help of (5), (9) and (10), we obtain:

$$\left(-\mathcal{K}_1 + \rho_3 \flat_G - i\rho_2 \flat_G\right)\overline{\zeta}_1 - i\left(\mathcal{K}_2 - \rho_1 \flat_G\right)\left(\overline{\zeta}_2 + i\overline{\zeta}_3\right) = 2a\left(\overline{\zeta}_2 + i\,\overline{\zeta}_3\right) - 2b\overline{\zeta}_1.$$

Also, we have:

$$a = -\frac{i\left(\mathcal{K}_2 - \rho_1 \flat_G\right)}{2} \quad \text{and} \quad b = \frac{\mathcal{K}_1 - \rho_3 \flat_G + i\rho_2 \flat_G}{2}. \tag{14}$$

If we substitute (14) in (13), we obtain:

$$\frac{d\varphi}{ds} = \frac{1}{2}[-i\left(\mathcal{K}_2 - \rho_1 \flat_G\right)\varphi + \left(\mathcal{K}_1 - \rho_3 \flat_G + i\rho_2 \flat_G\right)\widehat{\varphi}].$$

Hence, we complete the proof. □

From Remark 2.1, we obtain the Corollary 3.2-3.4.

Corollary 3.2. *Let $\overline{r}(s) : I \to G$ is a curve with the Frenet-type frame in Myller configuration for abelian Lie groups and the spinor φ corresponds to the triad $\{\overline{\zeta}_2, \overline{\zeta}_3, \overline{\zeta}_1\}$. Then, the following single spinor equation is given as follows:*

$$\frac{d\varphi}{ds} = \frac{1}{2}(-i\mathcal{K}_2\varphi + \mathcal{K}_1\widehat{\varphi}).$$

Corollary 3.3. *Let $\overline{r}(s) : I \to G$ is a curve with the Frenet-type frame in Myller configuration for Lie groups $SO(3)$ and the spinor φ corresponds to the triad $\{\overline{\zeta}_2, \overline{\zeta}_3, \overline{\zeta}_1\}$. Then, the following single spinor equation is given as follows:*

$$\frac{d\varphi}{ds} = \frac{1}{2}\left[-i\left(\mathcal{K}_2 - \frac{\rho_1}{2}\right)\varphi + \left(\mathcal{K}_1 - \frac{\rho_3}{2} + \frac{i\rho_2}{2}\right)\widehat{\varphi}\right].$$

Corollary 3.4. *Let $\overline{r}(s) : I \to G$ is a curve with the Frenet-type frame in Myller configuration for Lie groups S^3 and the spinor φ corresponds to the triad $\{\overline{\zeta}_2, \overline{\zeta}_3, \overline{\zeta}_1\}$. Then, the following single spinor equation is given as follows:*

$$\frac{d\varphi}{ds} = \frac{1}{2}[-i\left(\mathcal{K}_2 - \rho_1\right)\varphi + \left(\mathcal{K}_1 - \rho_3 + i\rho_2\right)\widehat{\varphi}].$$

Theorem 3.5. *Let $\overline{r}(s) : I \to G$ is a curve with the Frenet-type frame in Myller configuration for Lie groups and the spinor φ corresponds to the triad $\{\overline{\zeta}_2, \overline{\zeta}_3, \overline{\zeta}_1\}$. Then, the spinor equations of the versor fields $\overline{\zeta}_1, \overline{\zeta}_2, \overline{\zeta}_3$ in the 3-dimensional Lie groups are given as follows, respectively:*

$$\overline{\zeta}_1 = -\widehat{\varphi}^t \sigma \varphi, \tag{15}$$

$$\overline{\zeta}_2 = \frac{1}{2}\left(\varphi^t \sigma \varphi - \widehat{\varphi}^t \sigma \widehat{\varphi}\right), \tag{16}$$

$$\overline{\zeta}_3 = -\frac{i}{2}\left(\varphi^t \sigma \varphi + \widehat{\varphi}^t \sigma \widehat{\varphi}\right). \tag{17}$$

Proof. Assume that the spinor φ corresponds to the triad $\{\overline{\zeta}_2, \overline{\zeta}_3, \overline{\zeta}_1\}$ of the curve $\overline{r}$ in the 3-dimensional Lie group G. By (9), we get $\overline{\zeta}_2 = Re(\varphi^t \sigma \varphi)$ and $\overline{\zeta}_3 = Im(\varphi^t \sigma \varphi)$. With the help of the properties of complex numbers, we can write:

$$\overline{\zeta}_2 = \frac{1}{2}\left(\varphi^t \sigma \varphi + \overline{\varphi^t \sigma \varphi}\right),$$
$$\overline{\zeta}_3 = -\frac{i}{2}\left(\varphi^t \sigma \varphi - \overline{\varphi^t \sigma \varphi}\right).$$

By using (6), we have the equations (16) and (17). Also, we know $\overline{\zeta}_1 = -\widehat{\varphi}^t \sigma \varphi$ according to (10). □

Corollary 3.6. *Let $\overline{r}(s) : I \to G$ is a curve with the Frenet-type frame in Myller configuration for Lie groups and the spinor φ corresponds to the triad $\{\overline{\zeta}_2, \overline{\zeta}_3, \overline{\zeta}_1\}$. Then, the spinor components of the versor fields $\overline{\zeta}_1, \overline{\zeta}_2, \overline{\zeta}_3$ in the 3-dimensional Lie groups are written as follows:*

$$\overline{\zeta}_1 = \left(\varphi_1\overline{\varphi}_2 + \overline{\varphi}_1\varphi_2, i\left(\varphi_1\overline{\varphi}_2 - \overline{\varphi}_1\varphi_2\right), |\varphi_1|^2 - |\varphi_2|^2\right),$$
$$\overline{\zeta}_2 = \frac{1}{2}\left(\varphi_1^2 - \varphi_2^2 + \overline{\varphi}_1^2 - \overline{\varphi}_2^2, i\left(\varphi_1^2 + \varphi_2^2 - \overline{\varphi}_1^2 - \overline{\varphi}_2^2\right), -2\left(\varphi_1\varphi_2 + \overline{\varphi}_1\overline{\varphi}_2\right)\right),$$
$$\overline{\zeta}_3 = -\frac{i}{2}\left(\varphi_1^2 - \varphi_2^2 - \overline{\varphi}_1^2 + \overline{\varphi}_2^2, i\left(\varphi_1^2 + \varphi_2^2 + \overline{\varphi}_1^2 + \overline{\varphi}_2^2\right), 2\left(\overline{\varphi}_1\overline{\varphi}_2 - \varphi_1\varphi_2\right)\right).$$

Proof. For the spinor expression $\varphi^t \sigma \varphi$, we have:

$$\begin{cases} \varphi^t \sigma_1 \varphi = \begin{pmatrix}\varphi_1 & \varphi_2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \varphi_1^2 - \varphi_2^2, \\ \varphi^t \sigma_2 \varphi = \begin{pmatrix}\varphi_1 & \varphi_2\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = i(\varphi_1^2 + \varphi_2^2), \\ \varphi^t \sigma_3 \varphi = \begin{pmatrix}\varphi_1 & \varphi_2\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = -2\varphi_1\varphi_2. \end{cases} \tag{18}$$

Also, for the spinor expression $\widehat{\varphi}^t \sigma \widehat{\varphi}$, we have:

$$\begin{cases} \widehat{\varphi}^t \sigma_1 \widehat{\varphi} = \begin{pmatrix}-\overline{\varphi}_2 & \overline{\varphi}_1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-\overline{\varphi}_2\\ \overline{\varphi}_1\end{pmatrix} = \overline{\varphi}_2^2 - \overline{\varphi}_1^2, \\ \widehat{\varphi}^t \sigma_2 \widehat{\varphi} = \begin{pmatrix}-\overline{\varphi}_2 & \overline{\varphi}_1\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}-\overline{\varphi}_2\\ \overline{\varphi}_1\end{pmatrix} = i\left(\overline{\varphi}_2^2 + \overline{\varphi}_1^2\right), \\ \widehat{\varphi}^t \sigma_3 \widehat{\varphi} = \begin{pmatrix}-\overline{\varphi}_2 & \overline{\varphi}_1\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}-\overline{\varphi}_2\\ \overline{\varphi}_1\end{pmatrix} = 2\overline{\varphi}_1\overline{\varphi}_2. \end{cases} \tag{19}$$

Then, for the spinor expression $\widehat{\varphi}^t\sigma\varphi$, we have:

$$\begin{cases} \widehat{\varphi}^t\sigma_1\varphi = \begin{pmatrix} -\overline{\varphi}_2 & \overline{\varphi}_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = -\overline{\varphi}_2\varphi_1 - \overline{\varphi}_1\varphi_2, \\ \widehat{\varphi}^t\sigma_2\varphi = \begin{pmatrix} -\overline{\varphi}_2 & \overline{\varphi}_1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = i\left(-\overline{\varphi}_2\varphi_1 + \overline{\varphi}_1\varphi_2\right), \\ \widehat{\varphi}^t\sigma_3\varphi = \begin{pmatrix} -\overline{\varphi}_2 & \overline{\varphi}_1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = -|\overline{\varphi}_1|^2 + |\overline{\varphi}_2|^2. \end{cases} \tag{20}$$

If we substitute (18), (19) and (20) in (15), (16) and (17), we get what is wanted. □

Special Cases 3.1. *Let $\overline{r}(s) : I \to G$ is a curve with the Frenet-type frame in Myller configuration for Lie groups and the spinor φ corresponds to the triad $\{\overline{\zeta}_2, \overline{\zeta}_3, \overline{\zeta}_1\}$. Then, we have the followings:*

(i) *If $\rho_1 = 1, \rho_2 = \rho_3 = 0$ and $\flat_G = 0$, then we have spinor representation of a curve with Frenet frame in Euclidean 3-space [17].*
(ii) *If $\rho_1 = 1, \rho_2 = \rho_3 = 0$, then we get the spinor representation of a curve with Frenet frame in Lie groups [77].*
(iii) *If $\flat_G = 0$, then we obtain the spinor representation of a curve with Frenet-type frame in Myller configuration for Euclidean 3-space (cf. also Corollary 3.2).*

4. Conclusions

In this study, we investigated the representation of a curve with Frenet-type frame in Myller configuration for 3-dimensional Lie groups with the help of the spinors, which are quite popular and important concepts in basic disciplines such as mathematics and physics. Then, we gave some special cases and also, we obtain spinor equations of a curve with Frenet-type frame in Myller configuration for 3-dimensional Lie groups. In addition to these, we obtained some geometric properties, theories, and results related to the spinor representation of a curve with Frenet-type frame in Myller configuration for 3-dimensional Lie groups.

References

[1] Arnold V.I., Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, *Ann. Inst. Fourier (Grenoble)*, Vol.16, No.1, 1966, pp.319-361.
[2] Balcı Y., Erişir T., Güngör M.A., Hyperbolic spinor Darboux equations of spacelike curves in Minkowski 3-space, *J. Chungcheong Math. Soc.*, Vol.28, No.4, 2015, pp.525-535.
[3] Bishop R.L., There is more than one way to frame a curve, *Amer. Math. Monthly*, Vol.82, No. 3, 1975, pp.246-251.
[4] Bortolotti E., Su do una generalizzazione della teoria delle curve e sui sistemi conjugati di una V_2 in V_n, *Rend. R. Ist. Lombardo Sci. e Lett.*, Vol.LVIII, No.XI, 1925.
[5] Bozkurt Z., Gök İ., Okuyucu O.Z., Ekmekçi N., Characterizations of rectifying, normal and osculating curves in the three-dimensional compact Lie groups, *Life Science Journal*, Vol.10, No.3, 2013, 819-823.
[6] Brauer R., Weyl H., Spinors in n dimensions, *American Journal of Mathematics*, Vol.57, No.2, 1935, pp.425-449.
[7] Cartan E., *The Theory of Spinors*, Hermann, Paris, 1966 [Dover, New York (reprinted 1981)]).
[8] Clifford W.K., Applications of Grassmann's extensive algebra, *Am. J. Math.*, Vol.1, No.4, 1878, pp.350-358.
[9] Constantinescu O., Myller configurations in Finsler spaces, *Differential Geometry-Dynamical Systems*, Vol.8, 2006, pp.69-76.
[10] Craioveanu M., Topologia şi geometria varietăţilor Banach (in Romanian), Ph.D. Thesis, Iaşi, 1968.
[11] Crouch P., Silva L.F., The dynamic interpolation problem: on Riemannian manifolds, Lie groups, and symmetric spaces, *J. Dyn. Control Syst.*, Vol.1, No.2, 1995, 177-202.
[12] Çakmak A., Kızıltuğ S., Spherical indicatrices of a Bertrand curve in three-dimensional Lie groups, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, Vol.68, No.2, 2019, 1930-1938.
[13] Çiftçi Ü., A generalization of Lancert's theorem, *J. Geom. Phys.*, Vol.59, No.12, 2009, 1597-1603.
[14] Çöken A.C., Çiftçi Ü., A note on the geometry of Lie groups, *Nonlinear Anal.*, Vol.68, No.7, 2008, 2013-2016.

[15] Darboux G., *Leçons sur la thèorie gènèrale des surfaces I-II-III-IV*, Gauthier-Villars, Paris, 1896.
[16] del Castillo G.F.T., 3-*D Spinors, Spin-Weighted Functions and Their Applications*, Vol. 32, Springer Science & Business Media, Boston, 2003.
[17] del Castillo G.F.T., Barrales G.S., Spinor formulation of the differential geometry of curve, *Rev. Colomb. de Mat.*, Vol.38, No. 1, 2004, 27-34.
[18] do Espirito-Santo N., Fornari S., Frensel K., Ripoll J., Constant mean curvature hypersurfaces in a Lie group with a bi-invariant metric, *Manuscr. Math.*, Vol.111, 2003, 459-470.
[19] Doğan Yazıcı B., Okuyucu O.Z., Tosun M., Framed curves in three-dimensional Lie groups and a Berry phase model, *J. Geom. Phys.*, Vol.182, 2022, pp.104682.
[20] Doğan Yazıcı B., İşbilir Z., Tosun M., Spinor representation of framed Mannheim curves, *Turk. J. Math.*, Vol.46, No.7, 2022, pp.2690-2700.
[21] Eren K., Erişir T., Spinor representation of directional Q-frame, *Sigma J. Eng. Nat. Sci.* (in press).
[22] Erişir T., On spinor construction of Bertrand curves, *AIMS Mathematics*, Vol.6, No.4, 2021, pp.3583-3591.
[23] Erişir T., Güngör M.A., Tosun M., Geometry of the hyperbolic spinors corresponding to alternative frame, *Adv. Appl. Clifford Algebras*, Vol.25, 2015, pp.799-810.
[24] Erişir T., Kardağ N.C., Spinor representations of involute evolute curves in $\mathbb{E}^3$, *Fundam. J. Math. Appl.*, Vol.2, No.2, 2019, pp.148-155.
[25] Erişir T., Öztaş H.K., Spinor equations of successor curves, *Univers. J. Math. Appl.*, Vol.5, No.1, 2022, pp.32-41.
[26] Erişir T., Güngör M.A., On Fibonacci spinors, *Int. J. Geom. Methods Mod. Phys.*, Vol.17, No.04, 2020, pp.2050065.
[27] Frenet F., Sur les courbes a double courbure, *Journal de mathématiques pures et appliquées*, 1852, pp.437-447.
[28] Gheorghiev Gh., On dini nets (Romanian), Acad. Repub. Pop. Române, *Bal. 'Sti. Ser, Mat. Fiz. Chim.*, Vol. No.2, 1950, pp.17-20.
[29] Gheorghiev Gh., Cateva probleme geometrice legate de un camp de vectori unitari, *Bul. St. Acad. R.P.R.*, Vol.VI, 1955.
[30] Gök İ., Okuyucu O.Z., Ekmekçi N., Yaylı Y., On Mannheim partner curves in the three-dimensional Lie groups, *Miskolc Math. Notes*, Vol.15, No. 2, 2014, pp.467-479.
[31] Haimovici A., *Grupuri de transformări, Introducere elementară*, Ed. Didacticăşi Pedagogică, 1963.
[32] Heroiu B., Versor fields along a curve in a four dimensional Lorentz space, *Journal of Advanced Mathematical Studies*, Vol.4, No.1, 2011, pp.49-57.
[33] Hladik J., *Spinors in Physics*, Springer Science & Business Media, New York, 1999.
[34] İşbilir Z., Doğan Yazıcı B., Tosun M., The spinor representations of framed Bertrand curves, *Filomat*, Vol.37, No.9, 2023, 2831-2842.
[35] İşbilir Z., Doğan Yazıcı B., Tosun M., Spinor representations of framed curves in the three-dimensional Lie groups, Journal of Dynamical Systems and Geometric Theories, Vol.21, No.1, 2023, pp.61-83.
[36] İşbilir Z., Tosun M., On generalized osculating-type curves in Myller configuration, *An. St. Univ. Ovidius Constanta, Ser. Mat.*, 2023 (accepted).
[37] İşbilir Z., Tosun M., Generalized Smarandache curves with Frenet-type frame, *Honam Mathematical Journal*, 2023 (accepted).
[38] İşbilir Z., Doğan Yazıcı B., Tosun M., Frenet-type frame in Myller configuration for 3-dimensional Lie groups and some special curves (submitted) `https://www.researchgate.net/publication/370715980_Spinor_Representations_of_Frenet-Type_Frame_in_Myller_Configuration_for_3-Dimensional_Lie_Groups`.
[39] İşbilir Z., Tosun M., An extended framework for osculating-type curves in 4-dimensional Euclidean space, 2023 (submitted) `https://www.researchgate.net/publication/367284875_AN_EXTENDED_FRAMEWORK_FOR_OSCULATING-TYPE_CURVES_IN_4-DIMENSIONAL_SPACE`.
[40] İşbilir Z., Tosun M., A new insight on rectifying-type curves in 4-dimensional Euclidean space, 2023 (submitted) `https://www.researchgate.net/publication/367284868_A_NEW_INSIGHT_ON_RECTIFYING-TYPE_CURVES_IN_4-DIMENSIONAL_SPACE`.
[41] Keskin O., Yaylı Y., Rectifying-type curves and rotation minimizing frame $\mathbb{R}^n$, arXiv preprint, arXiv:1905.04540, 2019.
[42] Ketenci Z., Erişir T., Güngör M.A., A construction of hyperbolic spinors according to the Frenet frame in Minkowski space, *Journal of Dynamical Systems and Geometric Theories*, Vol.13, No.2, 2015, pp.179-193.
[43] Ketenci Z., Erişir T., Güngör M.A., Spinor equations of curves in Minkowski space, V. In: Congress of the Turkic World Mathematicians, Kyrgyzstan, June 05-07, 2014.
[44] Kişi İ., Tosun M., Spinor Darboux equations of curves in Euclidean 3-space, *Mathematica Moravica*, Vol.19, No.1, 2015, pp.87-93.
[45] Kızıltuğ S., Çakal S., Bertrand curves of AW(k)-type in three dimensional Lie groups, *J. Math. Comput. Sci.*, Vol.7, No.4, 2017, pp.806-816.
[46] Kolev B., Lie groups and mechanics: An introduction, *J. Nonlinear Math. Phys.*, Vol.11, No.4, 2004, 480-498.

[47] Levi-Civita T., Rendiconti del Circolo di Palermo, Vol.XLII, 1917, pp.173.
[48] Levi-Civita T., *Lezioni di calcolo differentiate assoluto*, Zanichelli, 1925.
[49] Lounesto P., *Clifford Algebras and Spinors*, In: Clifford Algebras and Their Applications in Mathematical Physics, Vol.183, Springer, Dordrecht, 1986.
[50] Lounesto P., *Clifford Algebras and Spinors*, Cambridge University Press, 2001.
[51] Macsim G.F., Mihai A., Olteanu A., Curves in a Myller configuration, *International Conference on Applied and Pure Mathematics (ICAPM 2017)*, Iaşi, November 2-5, 2017.
[52] Macsim G.F., Mihai A., Olteanu A., Special curves in a Myller configuration, *Proceedings of the 16th Workshop on Mathematics, Computer Science and Technical Education, Department of Mathematics and Computer Science*, Vol.2, 2019.
[53] Macsim G., Mihai A., Olteanu A., On rectifying-type curves in a Myller configuration, *Bulletin of the Korean Mathematical Society*, Vol.56, No.2, 2019, pp.383-390.
[54] Mak M., Natural and conjugate mates of Frenet curves in three-dimensional Lie group, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.,* Vol.70, No.1, 2021, pp.522-540.
[55] Mayer O., Étude sur les réseaux de M. Myller, *Ann. St. de l'Univ. de Jassy*, Vol.XIV, 1926.
[56] Mayer O., Une interpretation geometrique de la seconde forme quadratique d'une surface, *C.R. Paris*, Vol.178, 1924, pp.1954.
[57] Miron R., *The geometry of Myller configurations. Applications to theory of surfaces and nonholonomic manifolds*, Romanian Academy, 2010.
[58] Miron R., Geometria unor configuraţii Myller, Analele Sţ. Univ. Iaşi, Vol.VI, No.3, 1960.
[59] Miron R., Despre torsiunea totală a unei suprafete, *Gazeta Matematică S. A.*, Vol.8, 1955.
[60] Miron R., Les configurations de Myller $\mathfrak{M}(C, \xi_i^1, T^{n-1})$ dans les espaces de Riemann V_n (I), *Tensor*, Vol.12, No.3, 1962 (Japonia).
[61] Miron R., Branzei D., Backgrounds of arithmetic and geometry, *S. Pure Math., World Scientific Publishing*, Vol.23, 1995.
[62] R. Miron, Myller configurations and Vranceanu nonholonomic manifolds, *Scientific Studies and Research*, Vol.21, No.1, 2011.
[63] Miron R., Asupra sferei neolonome şi planului neolonom, *An. Sţ. Univ. "Al. I. Cuza" din Iaşi, Seria Nouă, Sect. 1*, Vol.I, No.1-2, 1955, pp.43-53.
[64] Miron R., Configuraţii Myller $\mathfrak{M}(C, \xi_1^i, T^{n-1})$ in spaţii Riemann V_n. Aplicatii la studiul hipersuprafetelor din V_n, *Studii şi Cerc. Şt. Mat. Iaşi*, Vol.XII, No.1, 1962.
[65] Miron R., Despre reducerea la forma canonică a grupului intrinsec al unui spaţiu neolonom, *Buletin Ştiinţific. Secţia de St. Matem. şi Fiz.*, Vol.VIII, No.3, 1956, pp.631-645.
[66] Miron R., Configuraţii Myller $\mathfrak{M}(C, \xi_i^1, T^m)$ in spaţii Riemann V_n, *Aplicaţii la studiul varietăţilor V_m din V_n, Studii şi Cerc. St. Mat. Iaşi*, Vol.XIII, No.1, 1962.
[67] Miron R., Pop I., *Topologie Algebrică: Omologie, Omotopie, Spaţii de Acoperire*, Ed. Academiei, 1974.
[68] Miron R., *Geometria Configurat¸iilor Myller*, Editura Tehnică, Bucureşti, 1966.
[69] Myller A.I., Le parallelisme au sens de Levi-Civita dans un systeme de plans, *Ann. Sci. Univ. Jassy*, Vol.XIII, 1924, pp.1-2.
[70] Myller A.I., Quelques proriétés des surfaces reglées en liaison avec la théorie du parallélisme de Levi-Civita, *C.R. Paris*, Vol.174m, 1922, 997.
[71] Myller A., Surfaces réglées remarquables passant par une courbe donnée, *C.R. Paris*, Vol.175, 1922, pp.937.
[72] Myller A.I., Les systèmes de corubes sur une surface et le parallélisme de Levi-Civita, *C.R. Paris*, Vol.176, 1923, pp.433.
[73] Okuyucu O.Z., Gök İ., Yaylı Y., Ekmekçi N., Bertrand curves in the three-dimensional Lie groups, *Miskolc Math. Notes*, Vol.17, No.2, 2017, pp.999-1010.
[74] Okuyucu O.Z., Gök İ., Yaylı Y., Ekmekçi N., Slant helices in the three-dimensional Lie groups, *Appl. Math. Comput.*, Vol.221, 2013, pp.672-683.
[75] Yıldız Ö.G., Okuyucu O.Z., Inextensible flows of curves in Lie groups, *Casp. J. Math. Sci.*, Vol.2, No.1, 2013, pp.23-32.
[76] Okuyucu O.Z., Değirmen C., Yıldız Ö.G., Smarandache curves in the three-dimensional Lie groups, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, Vol.68, No.1, 2019, pp.1175-1185.
[77] Okuyucu O.Z., Yıldız Ö.G., Tosun M., Spinor Frenet equations in the three-dimensional Lie groups, *Adv. Appl. Clifford Algebras*, Vol.26, 2016, pp.1341-1348.
[78] Okuyucu O. Z., Doğan Yazıcı B. Generalized Bertrand and Mannheim curves in 3D Lie groups, *Fundam. J. Math. Appl.*, Vol.5, No.3, 2022, pp.201-209.
[79] Pauli W., Zur Quantenmechanik des magnetischen elektrons, *Zeitschrift fr Physik*, Vol.43, 1927, pp.601-632.
[80] J. A. Serret, Sur quelques formules relatives à la théorie des courbes à double courbure, *Journal de Mathématiques Pures et Appliquées*, 1851, pp. 193-207.

[81] Shifrin T., *Differential Geometry: A First Course in Curves and Surfaces*, University of Georgia, Preliminary Version, 2008.
[82] Şenyurt S., Çalışkan A., Spinor formulation of Sabban frame of curve on S^2, *Pure Math. Sci.*, Vol.4, No.1, 2015 pp.37-42.
[83] Tomonaga S.I., *The story of spin*, University of Chicago Press, 1997.
[84] Ünal D., Kişi İ., Tosun M., Spinor Bishop equations of curves in Euclidean 3-space, *Adv. Appl. Clifford Algebras*, Vol.23, 2013, pp.757-765.
[85] Ünal D., Spinor Q-equations in Lorentzian 3-space E_1^3, *Bitlis Eren Üniversitesi Fen Bilimleri Dergisi*, Vol.11, 2022, pp.294-300.
[86] Ünal D., Ünlütürk, Y. Spinor q-equations in Euclidean 3-space $\mathbb{E}^3$, *Soft Comput.*, Vol.27, 2023, pp.7023-7031.
[87] Vaisman I., *Simplectic Geometry and Secondary Characteristic Classes*, Progress in Mathematics, 72, Birkhauser Verlag, Basel, 1994.
[88] Vaz J., da Rocha R., *An Introduction to Clifford Algebras and Spinors*, Oxford University Press, 2016.
[89] Vivarelli M.D., Development of spinors descriptions of rotational mechanics from Euler's rigid body displacement theorem, *Celestial Mechanics*, Vol.32, 1984, 193-207.
[90] Yoon D.W., General helices of AW(k)-type in the Lie group, *J. Appl. Math.*, Vol.10, 2012, pp.535123.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

О РАСПРЕДЕЛЕНИЯХ, ОПРЕДЕЛЯЕМЫХ ЗАДАННОЙ СЕТЬЮ ФРЕНЕ

ГҮЛМИРА А. ШАМШИЕВА[1], НУРКЫЗ А. САРЫГУЛОВА[2]

[1] [2] ОшГУ, Ош, Кыргызстан
e-mail: [1]gshamshieva@mail.ru , [2]n.sarygulova@mail.ru

Аннотация. В области Ω евклидова пространства E_5 задано семейство гладких линий так, что через каждую точку $X \in \Omega$ проходит одна линия заданного семейства. Подвижной ортонормированный репер $\Re = (X, \vec{e_i})(i, j, k = \overline{1,5})$ в области Ω выбран так, чтобы он был репером Френе [1], для линии ω^1 заданного семейства. Интегральные линии векторных полей $\vec{e_i}$ образуют сеть Френе Σ_5. В области Ω имеем p-мерные $(p, q, s = 2, 3, 4)$ распределения, определяемые векторными полями $\vec{e_i}$. Найдены связи между векторами средних кривизн p-мерных распределений и кривизнами линий сети Френе.

Ключевые слова: евклидово пространство, распределение, циклическая сеть Френе, вектор средней кривизны.

1. Введение

В области Ω евклидова пространства E_5 задано семейство гладких линий так, что через каждую точку $X \in \Omega$ проходит одна линия заданного семейства. Подвижной ортонормированный репер $\Re = (X, \vec{e_i})(i, j, k = 1, 2, 3, 4, 5)$ в области Ω выбран так, чтобы он был репером Френе для линии ω^1 заданного семейства.Деривационные формулы репера $\Re$ имеют вид:

$$d\vec{X} = \omega^i \vec{e_i}, d\vec{e_i} = \omega_i^k \vec{e_i}. \tag{1}$$

Формы ω^i, ω_i^k удовлетворяют структурным уравнениям евклидова пространства:

$$D\omega^i = \omega^k \wedge \omega_k^i, D\omega_i^k = \omega_i^j \wedge \omega_j^k, \omega_i^j + \omega_j^i = 0. \tag{2}$$

Интегральные линии векторных полей $\vec{e_i}$ образуют сеть Френе Σ_5 для линии заданного семейства. Поскольку репер $\Re$ построен на касательных к линиям сети Σ_5, формы ω_i^k становятся главными, т.е.

$$\omega_i^k = \Lambda_{ij}^k \omega^j. \tag{3}$$

В силу последнего равенства из формулы (2) имеем:

$$\Lambda_{ij}^k = -\Lambda_{kj}^i. \tag{4}$$

Дифференцируя внешним образом равенства (3) получим:

$$D\omega_i^k = d\Lambda_{ij}^k \wedge \omega^j + \Lambda_{ij}^k D\omega^j.$$

Применяя формул (2) отсюда имеем:

$$\omega_i^j \wedge \omega_j^k = d\Lambda_{ij}^k \wedge \omega^j + \Lambda_{ij}^k \wedge \omega^l \wedge \omega_l^j.$$

В силу равенства (3) последнее равенство имеет вид:

$$\omega_i^j \wedge \Lambda_{il}^k \omega^l = d\Lambda_{ij}^k \wedge \omega^j - \Lambda_{ij}^k \omega_l^j \wedge \omega^l$$

или

$$\Lambda^k_{jl}\omega^j_i \wedge \omega^l = d\Lambda^k_{ij} \wedge \omega^j - \Lambda^k_{ij} \wedge \omega^j_l \wedge \omega^l.$$

Отсюда найдем:

$$d\Lambda^k_{ij} \wedge \omega^j - \Lambda^k_{il}\omega^l_j \wedge \omega^j - \Lambda^k_{jl}\omega^j_i \wedge \omega^l = 0$$

или

$$(d\Lambda^k_{ij} - \Lambda^k_{il}\omega^l_j - \Lambda^k_{lj}\omega^l_i) \wedge \omega^j = 0.$$

Применяя лемму Картана [2] отсюда имеем:

$$d\Lambda^k_{ij} - \Lambda^k_{il}\omega^l_j - \Lambda^k_{lj}\omega^l_i = \Lambda^k_{ilm}\omega^m$$

или

$$d\Lambda^k_{ij} = (\Lambda^k_{ijm} + \Lambda^k_{il}\Lambda^l_{jm} + \Lambda^k_{ij}\Lambda^l_{im})\omega^m. \tag{5}$$

Система величин $\{\Lambda^k_{ij}, \Lambda^k_{ijm}\}$ образуют геометрический объект второго порядка.

Формулы Френе для линии ω^1 заданного семейства имеют вид:

$$d_1\vec{e_1} = \Lambda^2_{11}\vec{e_2},$$
$$d_1\vec{e_2} = \Lambda^1_{21}\vec{e_1} + \Lambda^3_{21}\vec{e_3},$$
$$d_1\vec{e_3} = \Lambda^2_{31}\vec{e_2} + \Lambda^4_{31}\vec{e_4},$$
$$d_1\vec{e_4} = \Lambda^3_{41}\vec{e_3} + \Lambda^5_{41}\vec{e_5},$$
$$d_1\vec{e_5} = \Lambda^4_{51}\vec{e_4},$$

и

$$\Lambda^3_{11} = -\Lambda^3_{11} = 0, \Lambda^4_{11} = -\Lambda^1_{41} = 0, \Lambda^5_{11} = -\Lambda^1_{51} = 0, \tag{6}$$
$$\Lambda^5_{21} = -\Lambda^2_{51} = 0, \Lambda^4_{21} = -\Lambda^2_{41} = 0, \Lambda^5_{31} = -\Lambda^3_{51} = 0. \tag{7}$$

Здесь $k^1_1 = \Lambda^2_{11}, k^1_2 = \Lambda^3_{21}, k^1_3 = \Lambda^4_{31}, k^1_4 = \Lambda^5_{41} = \Lambda^4_{51}$ - первая, вторая, третья и четвертая кривизны линии ω^1 соответственно (где d_1 - символ дифференцирования вдоль линии ω^1).

Псевдофокус [3] $F^j_i (i \neq j)$ касательной к линии ω^i сети Σ_5 определяется следующим радиус-вектором:

$$\overrightarrow{F^j_i} = \vec{X} - \frac{1}{\Lambda^j_{ij}}\vec{e_i} = \vec{X} + \frac{1}{\Lambda^i_{jj}}\vec{e_i}. \tag{8}$$

На каждой касательной $(X, \overrightarrow{e_i})$ существуют по четыре псевдофокуса. На прямой $(X, \overrightarrow{e_1})$ существуют псевдофокусы $F^2_1, F^3_1, F^4_1, F^5_1$, на прямой $(X, \vec{e_2})$ - $F^1_2, F^3_2, F^4_2, F^5_2$, на прямой $(X, \vec{e_3})$ - $F^1_3, F^2_3, F^4_3, F^5_3$, на прямой $(X, \vec{e_4})$ - $F^1_4, F^2_4, F^3_4, F^5_4$, на прямой $(X, \vec{e_5})$ - $F^1_3, F^2_3, F^4_3, F^5_3$,

Сеть Σ_5 в $\Omega \subset E_5$ называется циклической сетью Френе [4], если реперы $\Re_1 = (X, \vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4}, \vec{e_5}), \Re_2 = (X, \vec{e_2}, \vec{e_3}, \vec{e_4}, \vec{e_5}, \vec{e_1}), \Re_3 = (X, \vec{e_3}, \vec{e_4}, \vec{e_5}, \vec{e_1}, \vec{e_2}), \Re_4 = (X, \vec{e_4}, \vec{e_5}, \vec{e_1}, \vec{e_2}, \vec{e_3}), \Re_5 = (X, \vec{e_5}, \vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4})$ являются соответственно реперами Френе для линий $\omega^1, \omega^2, \omega^3, \omega^4, \omega^5$ сети Σ_5 одновременно.

Пусть сеть Σ_5 является циклической сетью Френе. Ее обозначим через $\widetilde{\Sigma}_5$.

2. Материалы исследования

Касательные векторы к линиям сети Френе Σ_5 определяют следующие трехмерные распределения: $\Delta^{(ijk)}_3 = (X, \vec{e_i}, \vec{e_j}, \vec{e_k})$ $(i, j, k -$ различные$)$. В случае, когда сеть Френе Σ_5 является циклической сетью Френе, найдем векторы средних кривизн [5] этих распределений.

Через $\vec{M}^{(ijk)}_3$ обозначим вектор средней кривизны распределения $\Delta^{(ijk)}_3$. Тогда

$$\vec{M}^{(123)}_3 = \frac{1}{3}\Lambda^4_{33}\vec{e_4}, \tag{9}$$

где Λ^4_{33} - первая кривизна линии ω^3 сети Σ_5;

$$\vec{M}^{(134)}_3 = \frac{1}{3}(\Lambda^2_{11}\vec{e_2} + \Lambda^5_{44}\vec{e_5}), \tag{10}$$

где Λ_{11}^2 - первая кривизна линии ω^1 сети Σ_5, Λ_{44}^5 - первая кривизна линии ω^4 сети Σ_5.

$$\vec{M}_3^{(135)} = \frac{1}{3}(\Lambda_{11}^2\vec{e_2} + \Lambda_{33}^4\vec{e_5}), \tag{11}$$

$$\vec{M}_3^{(234)} = \frac{1}{3}\Lambda_{44}^5\vec{e_5}; \tag{12}$$

$$\vec{M}_3^{(235)} = \frac{1}{3}\Lambda_{11}^2\vec{e_2}; \tag{13}$$

$$\vec{M}_3^{(345)} = \vec{0}.$$

$$\vec{M}_3^{(145)} = \frac{1}{3}\Lambda_{33}^4\vec{e_4}; \tag{14}$$

$$\vec{M}_3^{(245)} = \frac{1}{3}\Lambda_{22}^3\vec{e_3}; \tag{15}$$

где Λ_{22}^3 - первая кривизна линии ω^2 сети $\widetilde{\Sigma}_5$;

$$\vec{M}_3^{(125)} = \frac{1}{3}\Lambda_{22}^3\vec{e_3}; \tag{16}$$

$$\vec{M}_3^{(124)} = \frac{1}{3}(\Lambda_{22}^3\vec{e_3} + \Lambda_{44}^3\vec{e_5}). \tag{17}$$

Аналогично найдем вектор средней кривизны 4-мерных распределений

$\Delta_4^{(1234)} = (X, \vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4}); \Delta_4^{(1235)} = (X, \vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_5}); \Delta_4^{(2345)} = (X, \vec{e_2}, \vec{e_3}, \vec{e_4}, \vec{e_5})$:

$$\vec{M}_4^{(1234)} = \frac{1}{4}\Lambda_{44}^5\vec{e_5}; \tag{18}$$

$$\vec{M}_4^{(1235)} = \frac{1}{4}\Lambda_{33}^4\vec{e_4}; \tag{19}$$

$$\vec{M}_4^{(2345)} = \vec{0}.$$

Векторы средних кривизн двумерных распределений $\Delta_2^{(ij)}(i \neq j)$ имеют вид:

$$\vec{M}_2^{(12)} = \frac{1}{2}\Lambda_{22}^3\vec{e_3}; \tag{20}$$

$$\vec{M}_2^{(13)} = \frac{1}{2}(\Lambda_{11}^2\vec{e_2} + \Lambda_{33}^4\vec{e_4}); \tag{21}$$

$$\vec{M}_2^{(14)} = \frac{1}{2}(\Lambda_{11}^2\vec{e_2} + \Lambda_{44}^5\vec{e_5}); \tag{22}$$

$$\vec{M}_2^{(15)} = \frac{1}{2}\Lambda_{11}^2\vec{e_2}; \tag{23}$$

$$\vec{M}_2^{(23)} = \frac{1}{2}\Lambda_{33}^4\vec{e_4}; \tag{24}$$

$$\vec{M}_2^{(24)} = \frac{1}{2}(\Lambda_{22}^3\vec{e_3} + \Lambda_{44}^5\vec{e_5}); \tag{25}$$

$$\vec{M}_2^{(25)} = \frac{1}{2}\Lambda_{22}^3\vec{e_3}; \tag{26}$$

$$\vec{M}_2^{(34)} = \frac{1}{2}\Lambda_{44}^5\vec{e_5}; \tag{27}$$

$$\vec{M}_2^{(35)} = \frac{1}{2}\Lambda_{55}^1\vec{e_1}; \tag{28}$$

$$\vec{M}_2^{(45)} = \frac{1}{2}\Lambda_{55}^1\vec{e_1}. \tag{29}$$

Теорема 1.
Если сеть Френе Σ_5 является циклической сетью Френе, то

а) распределения $\Delta_3^{(345)}, \Delta_4^{(2345)}-$ являются минимальными;

б) векторы средних кривизн распределений $\Delta_3^{(123)}, \Delta_3^{(235)}, \Delta_2^{(23)}, \Delta_4^{(1235)}-$ коллинеарны (см. (9), (13), (24), (19)), в том числе векторы средних кривизн распределений $\Delta_3^{(123)}, \Delta_3^{(235)}-$ равны;

в) векторы средних кривизн распределений $\Delta_3^{(234)}, \Delta_3^{(1234)}, \Delta_2^{(34)}-$коллинеарны (см. (12), (18), (27));

г) векторы средних кривизн распределений $\Delta_3^{(135)}, \Delta_2^{(13)}-$коллинеарны (см. (21), (11));

д) векторы средних кривизн распределений $\Delta_3^{(134)}, \Delta_2^{(14)}-$коллинеарны (см. (22), (10)).

Псевдофокус $F_3^2 \in (X, \vec{e}_3)$ определяется радиус-вектором:

$$\vec{F}_3^2 = \vec{X} - \frac{1}{\Lambda_{32}^3}\vec{e_3}. \tag{30}$$

Когда точка $X \in \Omega$ смещается в области Ω, псевдофокус F_3^2 описывает свою область $\Omega_3^2 \subset E_5$. В результате получим частичное отображение $f_3^2 : \Omega \to \Omega_3^2$ такое, что $f(X) = F_3^2$. Продифференцируя равенство (30) и применяя формулы (1), (2), (3) имеем:

$$d\vec{F}_3^2 = \omega^1\vec{c_1} + \omega^2\vec{c_2} + \omega^3\vec{c_3} + \omega^4\vec{c_4} + \omega^5\vec{c_5},$$

где

$$\begin{gathered}
\vec{c_1} = \vec{e_1} - \frac{\Lambda_{31}^2}{\Lambda_{32}^2}\vec{e_2} + \frac{C_{321}^2}{(\Lambda_{32}^2)^2}\vec{e_3} - \frac{\Lambda_{31}^4}{\Lambda_{32}^2}\vec{e_4};\\
\vec{c_2} = \vec{e_2} + \frac{C_{322}^2}{(\Lambda_{32}^2)^2}\vec{e_3} - \frac{\Lambda_{32}^4}{\Lambda_{32}^2}\vec{e_4};\\
\vec{c_3} = [1 + \frac{C_{323}^2}{(\Lambda_{32}^2)^2}]\vec{e_3} - \frac{\Lambda_{32}^4}{\Lambda_{32}^2}\vec{e_4};\\
\vec{c_4} = -\frac{\Lambda_{34}^2}{\Lambda_{32}^2}\vec{e_2} + \frac{C_{324}^2}{(\Lambda_{32}^2)^2}\vec{e_3} + \vec{e_4};\\
\vec{c_5} = -\frac{\Lambda_{35}^2}{\Lambda_{32}^2}\vec{e_2} + \frac{C_{325}^2}{(\Lambda_{32}^2)^2}\vec{e_3} - \frac{\Lambda_{35}^4}{\Lambda_{32}^2}\vec{e_4} + \vec{e_5}.
\end{gathered} \tag{31}$$

Формулы найдены в работе [6]. Присоединим к области Ω_3^2 подвижной репер $\Re' = (F_2^3, \vec{c_i})$.

Линии $\omega^i, g(\omega^i) = \overline{\omega}^i$ называются двойными линиями отображения g, если касательные прямые к ним, взятые в соответствующих точках X и $g(X)$ пересекаются, либо параллельны [7].

Линии $\omega^i, g(\omega^i)$ в E_5 называются квазидвойными линиями частичного отображения g, если касательные прямые к ним, взятые в соответствующих точках X и $g(X)$ принадлежат одному и тому же подпространству пространства E_5.

Рассмотрим линию γ, принадлежащую распределению $\Delta_3^{(145)}$.Её касательный вектор имеет вид:$\vec{\gamma} = \gamma^1\vec{e_1} + \gamma^4\vec{e_4} + \gamma^5\vec{e_5}$. Найдем касательный вектор $\vec{\bar{\gamma}}$ линии $f_2^3(\gamma) = \bar{\gamma}$ в виде: $\vec{\bar{\gamma}} = \gamma^1\vec{c_1} + \gamma^4\vec{c_4} + \gamma^5\vec{c_5}$. Учитывая формул (31) отсюда получим: $\vec{\bar{\gamma}} = \gamma^1\vec{e_1} + (\gamma^1 c_1^2 + \gamma^4 c_4^2 + \gamma^5 c_5^2)\vec{e_2} + (\gamma^1 c_1^3 + \gamma^4 C_4^3 + \gamma^5 C_5^3)\vec{e_3} + (\gamma^1 C_1^4 + \gamma^4 C_4^4 + \gamma^5 C_5^4)\vec{e_4} + \gamma^5\vec{e_5}$, где c_i^j- $j-$тая координата вектора $\vec{c_i}$.

Из условия $\vec{\gamma}, \vec{\bar{\gamma}}, X\vec{F}_2^3 \in \Delta_3^{(145)}$ имеем:

$$\gamma^1 c_1^2 + \gamma^4 c_4^2 + \gamma^5 c_5^2 = 0; \tag{32}$$

$$\gamma^1 c_1^3 + \gamma^4 c_4^3 + \gamma^5 c_5^3 = 0. \tag{33}$$

Учитывая формул (31) отсюда получим:

$$\Lambda_{31}^2\gamma^1 + \Lambda_{34}^2\gamma^4 + \Lambda_{35}^2\gamma^5 = 0; \tag{34}$$

$$C_{321}^2\gamma^1 + C_{324}^2\gamma^4 + C_{325}^2\gamma^5 = 0; \tag{35}$$

Обратно, если имеют места условия (34), (35), то линии γ и $\bar{\gamma}$ являются квазидвойными линиями частичного отображения f_2^3.

Аналогично, рассмотрим линию β, принадлежащую распределению $\Delta_3^{(245)}$. Её касательный вектор: $\vec{\beta} = \beta^1\vec{e_1} + \beta^4\vec{e_4} + \beta^5\vec{e_5}$. Касательный вектор $\vec{\bar{\beta}}$ линии $\bar{\beta} = f_2^3(\beta)$ имеет вид:

$$\vec{\bar{\beta}} = (\beta^2 + \beta^4 c_4^2 + \beta^5 c_5^2)\vec{e_2} + (\beta^2 c_2^3 + \beta^4 c_4^3 + \beta^5 c_5^3)\vec{e_3} + (\beta^2 c_2^4 + \beta^4 + \beta^5 c_5^4)\vec{e_4} + \beta^5\vec{e_5}.$$

Из условия $\vec{\beta}, \vec{\bar{\beta}}, X\vec{F}_3^2 \in \Delta_3^{(245)}$ получим:

$$\beta^2 c_2^3 + \beta^4 c_4^3 + \beta^5 c_5^3 = 0.$$

Учитывая формул (31), отсюда имеем:

$$C_{322}^2\beta^2 + C_{324}^2\beta^4 + C_{325}^2\beta^5 = 0. \tag{36}$$

Верно и обратное, т.е. если имеет место равенство (36), то линии β и $\bar{\beta}$ являются квазидвойными линиями частичного отображения f_2^3.

3. Вывод

Таким образом, доказана следующая теорема.

Теорема 2.

а) линия γ, принадлежащая распределению $\Delta_3^{(145)}$, является квазидвойной линией частичного отображения f_2^3 тогда и только тогда, когда выполнены условия (34), (35);

б) линия β, принадлежащая распределению $\Delta_3^{(245)}$, является квазидвойной линией частичного отображения f_2^3 тогда и только тогда, когда выполнены условия (36).

Список литературы

[1] Рашевский П.К. Риманова геометрия и тензорный анализ [текст]/П.К.Рашевский //Москва, Наука, 1967. – с.481-482

[2] Фиников С.П. Метод внешних форм Картана в дифференциальной геометрии. -М-Л.: Гостехиздат, 1948. 432 с.

[3] Базылев В.Т. О многомерных сетях в евклидовом пространстве [текст]/В.Т.Базылев//Литовский математический сборник, 1966. VI.№4. -с.475-491.

[4] Матиева Г. Геометрия частичных отображений, сетей и распределений евклидова пространства [текст]/Г.Матиева//Монография. – Ош, 2003. – с.212-219

[5] Кузьмин М.К. Сети, определяемые распределениями в евклидовом пространстве E_n//Проблемы Геометрии - Москва: ВИНИТИ,1975 - т.7.с. 215-229.

[6] Матиева Г. О существовании квазидвойных линий частичного отображения в евклидовом пространстве E_6/ Г.Матиева, Н.Н.Курбанбаева, Ч.Х.Абдуллаева // Вестник ОшГУ, математика, физика, технология.2023- №1.с.145-15.

[7] Базылев В.Т. О фундаментальных объектах плоских многомерных сетей [текст] / В.Т.Базылев // Известия вузов. Математика, 1967. Т.9. – с.3-11.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ГАУССОВА КРИВИЗНА ПОВЕРХНОСТЕЙ ПОРОЖДЕННЫХ ОРБИТАМИ ВЕКТОРНЫХ ПОЛЕЙ

ЭЛЬДОР РАДЖАБОВ[1], АБДИГАППАР НАРМАНОВ[1]

[1]Национальный университет Узбекистана имени Мирзо Улугбека, г. Ташкент, Узбекистан
e-mail: rajabov_2019@bk.ru

Аннотация. В работе изучается геометрия орбит векторных полей в трехмерном и четырехмерном евклидовых пространствах. Гауссова кривизна является объектом внутренней геометрии поверхностей, то есть она не изменяется при изометрических изгибаниях. Геометрия двумерных поверхностей четырехмерного евклидова пространства R^4 является очень важной частью дифференциальной геометрии и изучается многими авторами. По этому мы посвящали эту работу изучению внутренней геометрии поверхностей то есть изучению Гауссова кривизну поверхностей порожденную орбитами векторных полей. Мы доказали семейства орбит векторных полей на многообразии порождают двумерное слоение, слои которого являются поверхностями нулевой кривизны. Описана геометрия сингулярного слоения, порожденного орбитами систем двух векторных полей. Доказано, что регулярные слои слоения являются двумерными поверхностями отрицательной гауссовой кривизны и нулевой нормальной кривизны. Напомним, что если нормальная кручение поверхности во всех точках по всему направлению равно нулю тогда поверхность лежит в гиперплоскости.

Ключевые слова: Гауссова кривизна, геометрия орбит векторных полей, многообразие, слоение порожденное орбитами, нормальное кручение.

1. Введение

Пусть $M-$ гладкое многообразие размерности n, $A-$ максимальный атлас, определяющий на M структуру гладкого многообразия класса C^r, где $r \geq 0$. Многообразие M является также многообразием класса C^s, если $0 \leq s \leq r$. Систему локальных криволинейных координат на C^s многообразия M обозначим через A^s. Пусть теперь целое k удовлетворяет неравенствам $0 < k < n$.

[2] Семейство $F = \{L_\alpha; \alpha \in B\}$ линейно связных подмножеств M^n называется $k-$ мерным C^s слоением, если оно удовлетворяет следующим трем условиям:

(F_I) $\bigcup\limits_{\alpha \in B} L_\alpha = M^n$.

(F_{II}) для всех $\alpha, \beta \in B$ если $\alpha \neq \beta$ обязательно $L_\alpha \cap L_\beta = \varnothing$.

(F_{III}) для всякой точки $p \in M^n$ можно выбрать локальные координаты

$(U_\lambda, \varphi_\lambda) \in A^{(s)}, p \in U_\lambda$ так, что если $U_\lambda \cap L_\alpha \neq \varnothing$ для некоторого $\alpha \in B$, то компоненты линейной связности множества $\varphi_\lambda(U_\lambda \cap L_\lambda)$ имеют вид $\{(x_1, x_2, ..., x_n) \in \varphi_\lambda(U_\lambda) : x_{k+1} = c_{k+1}, x_{k+2} = c_{k+2}, ..., x_n = c_n\}$, где числа $c_{k+1}, c_{k+2}, ..., c_n$ постоянны на компонентах линейной связности.

Множество L_α называется слоем слоения F.

В описанной ситуации $k-$ мерное C^s- слоение называется также C^s- слоением коразмерности $q = n - k$. Наличие слоения F в многообразии M выражается символом (M, F). Условия (F_I), (F_{II}) означают, что M состоит из взаимно непересекающихся слоев. Условие (F_{III}) означает, что локально слои устроены как параллельные плоскости.

Если выполняется условие (F_{III}) координатные окрестности $(U_\lambda, \varphi_\lambda) \in A^{(s)}$ называются расслоенными, а совокупность всех расслоенных координатных окрестностей обозначается через A^s_F и называется системой расслоенных координатных окрестностей.

[7]. Подмножество L многообразия M называется $k-$ мерным слоем, если существует дифференциальная структура σ на L такая, что:

1. (L, σ) есть связное $k-$ мерное погруженное подмногообразие в M;

2. Если $N-$ произвольное локально связное топологическое пространство,

$$f : N \to M \text{ - непрерывное отображение}$$

такое, что $f(N) \subset L$, то $f : N \to (L, \sigma)$ непрерывно.

Из определения погружений следует, что если $f : N \to M$ дифференцируемое отображение и $f(N) \subset L$, то $f : N \to (L, \sigma)$ также дифференцируемо. Поэтому, в частности, σ является единственной дифференциальной структурой, по отношению к которой L является $k-$ мерным подмногообразием.

Теорема 1.1. [1]. *Пусть G связная группа преобразований, действующая на многообразии M. Гладкая вещественнозначная функция $f : M \to R$ является инвариантной функцией группы G если и только если*

$$X(f) = 0 \quad для \quad всех \quad x \in M$$

и каждой инфинитезимальной образующей X группы G.

[7]. Разбиение F многообразия M на слои называется гладким (из класса C^r) сингулярным слоением (т.е. слоением с особенностями), если выполнены следующие условия:

1. для каждой точки $x \in M$ существует C^r- карта (ψ, U), содержащая точку x такая, что $\psi(U) = V_1 \times V_2$, где V_1- окрестность начала в R^k, V_2- окрестность начала в R^{n-k}, $k-$ есть размерность слоя, проходящего через точку x;

2. $\psi(x) = (0, 0)$;

3. для каждого слоя L такого, что $L \cap U \neq \varnothing$, имеет место равенство $L \cap U = \psi^{-1}(V_l \times l)$, где $l = \{v \in V : \psi^{-1}(0, v) \in L\}$

Если размерности слоев слоения с особенностями одинаковы, то оно является регулярным слоением в смысле определения, данного в [2].

[5]. Направление $\overrightarrow{t} \in T_P F$ и нормальная плоскость N поверхности в точке $P \in F$ определяют гиперплоскость $R^3(P, t, N)$ в R^4 которая пересекает поверхность F по некоторой кривой γ. Кривую γ называют нормальным сечением поверхности F в точке P по направлению t. По построению кривая γ является трехмерной кривой. Кривизну $k_N(P, t)$ и кручение $\sigma_N(P, t)$ кривой γ в точке P называют соответственно нормальной кривизной и нормальным кручением поверхности в точке P по направлению t.

2. Геометрия орбит векторных полей

Рассмотрим семейство $D = \{X_1, X_2\}$ гладких векторных полей в R^3,

$X_1 = \{x, y, z\}, X_2 = \{-y + \alpha x, x + \alpha y, \alpha z\}$,

где $\alpha-$ действительное число.

Векторное поле $X_1 = \{x, y, z\}$ порождает следующую однопараметрическую группу конформных преобразований

$X^t : (x, y, z) \to \{xe^t, ye^t, ze^t\}, t \in R$,

поток векторного поля $X_2 = \{-y + \alpha x, x + \alpha y, \alpha z\}$ состоит из следующих конформных преобразований пространства.

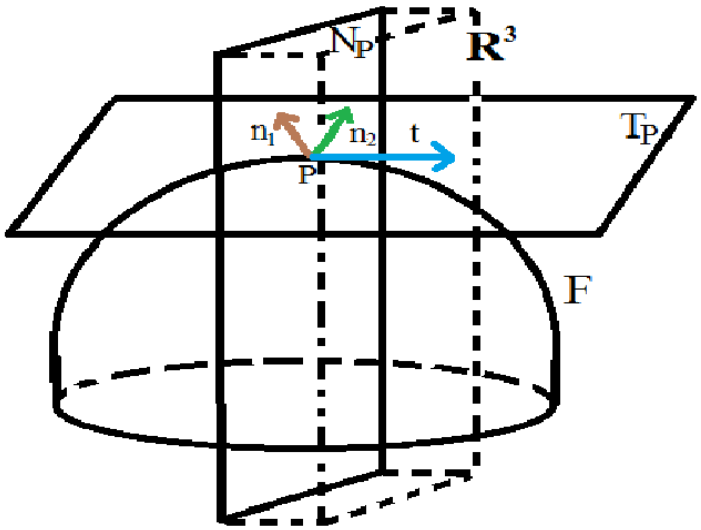


Рисунок 1. Гиперплоскость $R^3(P,t,N)$ в R^4

$X^s:(x,y,z)\to\{e^{\alpha s}(x\cos s-y\sin s),e^{\alpha s}(x\sin s+y\cos s),ze^{\alpha s}\}$, $s\in R$.
Переходим к изучению геометрии орбит этих векторных полей.

Теорема 2.1. *Орбиты семейства D векторных полей порождают сингулярное слоение, регулярные слои которого являются поверхностями нулевой Гауссого кривизны.*

Доказателство. Начало координат является неподвижной точкой отно-сительно конформных преобразований, порожденных векторными полями X_1,X_2. Поэтому начало координат является одной из орбит. Берем точку $(0,0,z_0)$ из множества $L_2=\{(0,0,z):z>0\}$ (или $L_3=\{(0,0,z):z<0\}$), тогда точки $X_1^t(0,0,z)=\left(0,0,ze^t\right),X_2^s(0,0,z)=(0,0,ze^{\alpha s})$ также принадлежат множеству L_2 (или L_3). Кроме того, из точки $(0,0,z_0)$ можем попасть в любую точку $(0,0,z)$ множества $L_2=\{(0,0,z):z>0\}$. Следовательно, множества L_2, L_3 являются орбитами. Теперь рассмотрим множество $L_4=\left\{(x,y,0):x^2+y^2>0\right\}$. Берем точку $(x,y,0)$ из множества $L_4=\left\{(x,y,0):x^2+y^2>0\right\}$, тогда

$$X_1^t(x,y,0)=\left(xe^t,ye^t,0\right), \tag{1}$$

$$X_2^s(x,y,0)=\left(e^{\alpha s}\left(x\cos s-y\sin s\right),e^{\alpha s}\left(x\sin s+y\cos s\right),0\right) \tag{2}$$

тоже принадлежит множеству $L_4=\left\{(x,y,0):x^2+y^2>0\right\}$. Берем две произвольные точки из множества L_4, т.е. $A(x_1,y_1,0),B(x_2,y_2,0)\in L_4$. Покажем, что найдутся такие параметры t,s что выполняется следующее равенство $X^s\left(X^t(x_1,y_1,0)\right)=(x_2,y_2,0)$. Это соотношение перепишем в координатном виде:

$$\begin{cases}x_1e^te^{\alpha s}\cos s-y_1e^te^{\alpha s}\sin s=x_2\\ x_1e^te^{\alpha s}\sin s+y_1e^te^{\alpha s}\cos s=y_2\end{cases}. \tag{3}$$

Так как ${x_i}^2+{y_i}^2>0,i=1,2$, то можем разделить первое уравнение на второе уравнение системы

$$\frac{x_1\cos s-y_1\sin s}{x_1\sin s+y_1\cos s}=\frac{x_2}{y_2}$$

или

$$\frac{x_1\sin s+y_1\cos s}{x_1\cos s-y_1\sin s}=\frac{y_2}{x_2}.$$

Отсюда находим:

$$\frac{y_2x_1-x_2y_1}{x_2x_1+y_1y_2}=\operatorname{tg}s.$$

Из первого уравнения находим значение параметра t:

$$t = \ln \frac{x_2}{x_1 \cos s - y_1 \sin s} - \alpha s.$$

Значит, существует значения параметров t, s такие, что $B = X_2^s(X_1^t(A))$, т.е. точка $A(x_1, y_1, 0) \in L_4$ может быть переведена в точку $B(x_2, y_2, 0) \in L_4$ с помощью потоков векторных полей

$$X_1 = \{x, y, z\}$$

,

$$X_2 = \{-y + \alpha x, x + \alpha y, \alpha z\}.$$

Таким образом, множество

$$L_4 = \left\{(x, y, 0) : x^2 + y^2 > 0\right\}$$

является орбитой семейства векторных полей. Функция

$$F = \frac{x^2 + y^2}{z^2}$$

является инвариантной функцией, по теореме 1, так как $X_i\,(F) = 0$ для i=1,2. Поэтому поверхности уровня этой функции являются инвариантными множествами для преобразований, порожденных векторными полями X_1, X_2. Поверхность уровня

$$L_5^+ = \{(x, y, z) : \frac{x^2 + y^2}{z^2} = c, z > 0, c \in R^+\}$$

является верхней частью конуса с выколотой вершиной. Берем две произвольные точки из множества L_5^+, т.е.

$$A(x_1, y_1, z_1) \in L_5^+, B(x_2, y_2, z_2) \in L_5^+$$

.

Покажем, что найдутся такие параметры t, s, что выполняется следующее равенство

$$X^s\left(X^t\,(x_1, y_1, z_1)\right) = (x_2, y_2, z_2)\,.$$

Это соотношение перепишем в координатном виде:

$$\begin{cases} x_1 e^t e^{\alpha s} \cos s - y_1 e^t e^{\alpha s} \sin s = x_2 \\ x_1 e^t e^{\alpha s} \sin s + y_1 e^t e^{\alpha s} \cos s = y_2 \\ z_1 e^t e^{\alpha s} = z_2 \end{cases}. \tag{4}$$

Из третьего уравнения этой системы получим следующее равенство: $t = \ln \frac{z_2}{z_1} - \alpha s$. Подставляя это выражение в первое и второе уравнения системы, получим:

$$\begin{cases} \frac{z_2}{z_1} e^{-\alpha s} e^{\alpha s}(x_1 \cos s - y_1 \sin s) = x_2 \\ \frac{z_2}{z_1} e^{-\alpha s} e^{\alpha s}(x_1 \sin s + y_1 \cos s) = y_2 \end{cases}. \tag{5}$$

Так как ${x_i}^2 + {y_i}^2 > 0, i = 1, 2$, можем разделить первое уравнение на второе уравнение системы

$$\frac{x_1 \cos s - y_1 \sin s}{x_1 \sin s + y_1 \cos s} = \frac{x_2}{y_2}$$

или

$$\frac{x_1 \sin s + y_1 \cos s}{x_1 \cos s - y_1 \sin s} = \frac{y_2}{x_2}.$$

Отсюда находим:

$$\frac{y_2 x_1 - x_2 y_1}{x_2 x_1 + y_1 y_2} = \operatorname{tg} s.$$

Значит, существуют значения параметров t, s, т.е. от любой точки $A(x_1, y_1, z_1) \in L_5^+$ с помощью потоков векторных полей

$$X_1 = \{x, y, z\},$$

$$X_2 = \{-y + \alpha x, x + \alpha y, \alpha z\}$$

можем переместиться в любую другую точку $B(x_2, y_2, z_2) \in L_5^+$. Таким образом, множество L_5^+ является орбитой семейства D.

Аналогично доказывается множество

$$L_5^- = \{(x, y, z) : \frac{x^2 + y^2}{z^2} = c, z < 0, c \in R^+\},$$

которое также является орбитой семейства D. Множество L_1 является точкой, множества L_2, L_3 являются лучами, множество L_4 является плоскостью с дыркой, множества L_5^+, L_5^- являются конусами без вершин. Поэтому орбиты семейства D векторных полей порождают сингулярное слоение, слои которого являются поверхностями нулевой кривизны. Теорема доказана.

Если эти векторные поля рассмотрим на многообразии

$$M = R^3 \backslash \{(x, y, z) : x = 0, y = 0\},$$

то получим следующую теорему.

Теорема 2.2. . *Орбиты семейства D векторных полей на многообразии M порождают двумерное слоение, слои которого являются поверхностями нулевой кривизны.*

Доказателство. Из теоремы 2 известно, что орбита семейства D порождает множества $L_i, i = 1, ..., 6$. Так как мы рассматриваем многообразие M, то орбита состоит из множеств $L_i, i = 4, 5, 6$. Размерность этих множеств равна двум. Поэтому орбиты семейства D векторных полей на многообразии M порождают двумерное слоение. Теорема доказана.

Рассмотрим семейство векторных полей $D = \{X_1, X_2\}$, которые являются инфинитезимальными образующими группы симметрии двумерного уравнения теплопроводности на четырехмерном евклидовом пространстве R^4 с декартовыми координатами x_1, x_2, t, u, где

$$X_1 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, X_2 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - 2u\frac{\partial}{\partial u}. \tag{6}$$

Теорема 2.3. . *Орбиты семейства векторных полей (6) порождают сингулярное слоение, регулярные слои которого являются поверхностями с отрицательной гауссовой кривизной и нулевым нормальным кручением.*

Доказателство. Сначала вычислим скобку Ли и найдом $[X_1, X_2] = 0$. Из теоремы Хермана следует, что семейство Dявляется вполне интегрируемым. Векторное поле

$$X_1 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$$

порождает следующую однопараметрическую группу преобразований

$$(x_1, x_2, t, u) \to (x_1, x_2, te^s, ue^s), s \in R, \tag{7}$$

векторное поле

$$X_2 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - 2u\frac{\partial}{\partial u}.$$

порождает следующую однопараметрическую группу преобразований

$$(x_1, x_2, t, u) \to \left(x_1e^s, x_2e^s, t, ue^{-2s}\right), s \in R \tag{8}$$

Найдем инвариантные функции этих преобразований. Известно, что [1] гладкая функция $f : M \to R$ является инвариантной функцией группы преобразований G, действующей на

M тогда и только тогда, когда $X(f) = 0$ для каждой инфинитезимальной образующей X группы G. Используя этот критерий, находим, что функции

$$I_1(x_1, x_2, t, u) = \frac{t}{x_1^2 u},$$
$$I_2(x_1, x_2, t, u) = \frac{x_2}{x_1}, \tag{9}$$

являются инвариантными функциями группы преобразований (7), (8), что следует из следующих равенств

$$X_1(I_1) = 0, \quad X_1(I_2) = 0, \quad X_2(I_1) = 0, \quad X_2(I_2) = 0. \tag{10}$$

Эти инвариантные функции дают нам семейство двумерных поверхностей

$$\frac{t}{x_1^2 u} = C_1, \qquad \frac{x_2}{x_1} = C_2, \qquad \text{где} \quad C_1, C_2 - \text{константы}.$$

Для заданных C_1, C_2 обозначим через F^C поверхность, которая определяется системой уравнений (10).

Для определенности будем считать, что $C_1 > 0$.

Если

$$p^0\left(x_1^0, x_2^0, t^0, u^0\right) \in F^C,$$

то из равенства (9) орбита $L(p^0)$ семейства D векторных полей через точку p^0 содержится на поверхности F^C.

Если точка p^0 является началом системы координат, то она является неподвижной точкой для векторных полей, и в этом случае орбита $L(p^0)$ является самой точкой.

Если $t^0 = 0$, то $u^0 = 0$. В этом случае векторное поле X_1 обращается в нуль в точке p^0, а орбита $L(p^0)$ является интегральной линией векторного поля X_2.

Пусть $t^0 > 0$, для определенности. Отсюда следует, что $u^0 > 0$.

Если $x_1^0 = 0$, имеем $x_2^0 = 0$. В этом случае орбита $L(p^0)$ является областью в плоскости Otu.

Предположим, что $x_1^0 \neq 0$, и покажем, что орбита $L(p^0)$ является двумерной поверхностью в F^C.

Пусть $p(x_1, x_2, t, u) \in F^C$ - любая точка, отличная от точки p^0 с $t > 0$.

На интегральной кривой векторного поля через точку p^0 лежит точка

$$p^1\left(x_1^0 e^{s_1}, x_2^0 e^{s_1}, t^0, u^0 e^{-2s_1}\right), \quad \text{где} \quad s_1 = \ln\sqrt{\frac{u^0 t}{u t^0}}.$$

Из точки p^1 по интегральной прямой векторного поля X_1 за "время"s_2 можно получить точку p, где $s_2 = \ln\frac{t}{t^0}$.

Действительно,

$$x_i^0 e^{s_1} = x_i^0\sqrt{\frac{u^0 t}{u t^0}} = x_i^0\sqrt{\frac{x_i^2}{\left(x_i^0\right)^2}} = x_i,$$
$$t^0 e^{s_2} = t, u^0 e^{s_2 - 2s_1} = u.$$

Таким образом, мы можем утверждать, что орбита $L(p^0)$ есть двумерное многообразие, состоящее из точек

$p(x_1, x_2, t, u) \in F^C$, с $t > 0, u > 0$ и x_1 имеет тот же знак, что и x_1^0.

Описание гиперплоских двумерных поверхностей в множестве S дано в [6]. Далее находим первую и вторую квадратичные формы орбиты $L(p^0)$. Параметризуем поверхность F^C следующими уравнениями

$$\begin{cases} x_1 = s \\ x_2 = C_2 s \\ t = q \\ u = C_1 \frac{q}{s^2} \end{cases}$$

и найдем

$$\frac{\partial r}{\partial s} = r_1 = \left\{1; -C_2; 0; -2C_1\frac{q}{s^3}\right\}$$

$$\frac{\partial r}{\partial q} = r_2 = \left\{0; 0; 1; \frac{C_1}{s^2}\right\}$$

и коэффициенты первой квадратичной формы

$$g_{11} = \langle r_1, r_1\rangle = 1 + C_2^2 + 4C_1^2\frac{q^2}{s^6}$$

$$g_{12} = g_{21} = \langle r_1, r_2\rangle = -\frac{2C_1^2 q}{s^5}$$

$$g_{22} = \langle r_2, r_2\rangle = 1 + \frac{C_1^2}{s^4}.$$

Нам нужны два вектора нормалей, чтобы найти коэффициенты вторых квадратичных форм

$$n_1 = \{C_2; -1; 0; 0\}$$

и

$$n_2 = \left\{1; C_2; -\frac{s\left(1-C_2^2\right)}{2q}; \frac{s^3\left(1-C_2^2\right)}{2C_1 q}\right\}.$$

Теперь мы можем найти две вторые квадратичные формы орбиты $L(p^0)$. Коэффициенты первой из них рассчитываются по формуле

$$b_{ij} = -\frac{1}{|n_1|}\langle \partial_i r, \partial_j n_1\rangle.$$

Поскольку вектор нормали n_1 является постоянным вектором, мы имеем

$$b_{11} = b_{22} = b_{12} = 0.$$

Коэффициенты второй из них рассчитываются по формуле

$$c_{ij} = -\frac{1}{|n_2|}\langle \partial_i r, \partial_j n_2\rangle.$$

Отсюда следует

$$c_{11} = -\frac{1}{|n_2|}\frac{3\left(1-C_2^2\right)}{s}, c_{12} = c_{21} = \frac{1}{|n_2|}\frac{\left(1-C_2^2\right)}{q}, c_{22} = 0.$$

Известно, что гауссова кривизна двумерной поверхности в R^4 вычисляется по формуле [3, 4]:

$$K = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2} + \frac{c_{11}c_{22} - c_{12}^2}{g_{11}g_{22} - g_{12}^2}.$$

Из этой формулы имеем

$$K = -\frac{1}{t^2|n_2|^2}\frac{\left(1-C_2^2\right)^2}{g_{11}g_{22} - g_{12}^2}.$$

Отсюда следует, что кривизна Гаусса является отрицательной.

Теперь рассмотрим нормальное кручение орбиты.

Пусть ξ - вектор касательной плоскости $T_pL(p^0)$ в точке $p\,(x_1, x_2, t, u)$ .s

Вектор ξ и нормальная плоскость N поверхности в точке p определяют гиперплоскость $R^3\,(p, \xi N)$ в R^4, пересекающую поверхность $L(p^0)$ вдоль некоторой кривой γ. Кривая γ также лежит в полугиперплоскости $x_2 = C_2x_1$. Отсюда следует, что кривая γ является

плоской кривой и не имеет кручения. Отсюда следует, что орбита является поверхностью $L(p^0)$ с нулевым нормальным кручением.

Теорема 2 доказана.

3. Благодарности

Авторы благодарны анонимному рецензенту, замечания которого помогли улучшить текст статьи.

Список литературы

[1] Olver P. Application of Lie Groups to Differential Equations // Second edition, Springer, 1993.
[2] Tamura I. Topology of foliations: an introduction // Translations of mathematical monographs, American Mathematical Soc., 2006.
[3] Narmanov A.Ya., Ergashova Sh.R. On the geometry of Liouville foliations // – E3S Web of Conferences, TransSiberia 2023. – 2023. – 402. 03032(2023), Pp. 1-8.
[4] Fomenko V. Некоторые свойства двумерных поверхностей с нулевым нормальным кручением в E4 // Матем. сб. – 1978. – 148(4). – С. 589–603
[5] Fomenko V. Classification of Two-Dimensional Surfaces with Zero Normal Torsion in Four-Dimensional Spaces of Constant Curvature // Math. Notes. – 2004. – Vol. 75. – Pp 690-701.
[6] Кадомцев С. Б. Исследование некоторых свойств нормального кручения двумерной поверхности в четырехмерном пространстве // Итоги науки и техн. Сер. Пробл. геом.,. – 1975. – том 7. – Pp. 267-278.
[7] Stefan P. Accessible sets, orbits, and foliations with singularities // Proc. London Mathematical Society. – 1974. – 29. – Pp. 694-713.

МАЗМҰНЫ
CONTENTS
СОДЕРЖАНИЕ

Дифференциалдық теңдеулер және математикалық физика теңдеулері
Differential equations and equations of mathematical physics
Дифференциальные уравнения и уравнения математической физики

Investigation of solutions of partial integro-differential equation of fourth order by means of a new way
Aizharkyn Ashirbaeva, Chebire Zholdoshova 19

Solving of system of partial differential equations of second order by means of the method of additional argument
Aizharkyn Ashirbaeva, Malika Bekieva 23

Integration of the modified Korteweg-de Vries-Liouville-Sine-Gordon equation (MKDF-L-SG) in the class of periodic infinite-gap functions
Aknazar Khasanov, Khodjimurod Normurodov, Ulugbek Xudayorov 27

The criterion of invertibility of the mixed type minimal operator
Alexander Rogovoy, Tynysbek Kalmenov 40

One problem for integro-differential equation
Asal Yuldasheva 49

A new functional relation for the Riemann zeta functions
Asset Durmagambetov 56

About system of moment equations and moment boundary conditions depending on the speed and surface temperature of the aircraft
Auzhan Sakabekov, Saltanat Madalieva, Ryskul Yergazina 61

Controlled equations to present transforming verbs with functional relations
Elaman Kenenbaev 74

Mixed problem for a parabolic integro-differential equation with involution
Farhod Rakhmanov 81

Investigation of system of nonlinear partial integro-differential equations of the first order by means of the method of additional argument
Gulkhan K. Sadykova 89

Optimization of Adams-type difference formulas in the Hilbert space $W_2^{(2,1)}(0,1)$
Kholmat Shadimetov, Rozik Karimov 93

On the continuum of exact solutions to the system of Maxwell's equations
Maksut Abenov, Turar Abzhaliuly 108

Solving an initial-boundary value problem for a loaded fractional-order diffusion equation
Minzilya Kosmakova, Danna Akhmanova, Kamila Izhanova 113

An explicit positivity-preserving nonstandard finite difference scheme

Mohammad Mehdizadeh Khalsaraei, Ali Shokri, Reza Shokri Jahandizi 117

On a spectral problem for a biharmonic operator in a rectangular domain
Muvasharkhan Jenaliyev, Madi Yergaliyev 126

The Cauchy problem for the modified Korteweg-de Vries equation with finite density in the class of periodic functions
Raykhonbek Eshbekov, Gayrat Mannonov 135

A criterion for correct solvability of a first order quasilinear the Verhulst logistic growth difference equation
Seyit Kerimkhulle, Ulzhan Makhazhanova, Ainur Saliyeva, Aislu Kassekeyeva, Alibek Adalbek, Roman Taberkhan 146

Solution of systems of higher order linear ordinary differential equations
Syrgak Kydyraliev, Anarkul Urdaletova, Elena Burova 155

Nonlinear n – Velocity Singularly Perturbed Inverse Problem of Kac-Type
Taalaibek Omurov, Zhyldyz Sarkelova 163

Analytical solutions of integral and integro-differential equations
Venera Muratalieva 171

On the Discrete Spectrum of the Schrödinger operator of the system of three-particles with masses $m_1 = m_2 = \infty$ and $m_3 < \infty$ on the three dimensional lattice
Zahriddin Muminov 177

Construction of hypergeometric type inhomogeneous second-order system solutions
Zhanar Ubayeva, Meiramgul Talipova, Ryskul Zhakhina, Altynzer Baiganova 192

Multidimensional normal-regular solutions of degenerate systems associated with Bessel functions of many variables
Zhaxylyk Tasmambetov 197

Об одной задаче для уравнения третьего порядка с кратными характеристиками в трехмерном пространстве в полуограниченной области
Азизбек Хамитов 212

Применение систем дифференциальных уравнений в эпидемиологии
Балжан Жумартова,Роза Ысмагул 219

Реализация метода малого параметра для исследования многопериодических решений систем с оператором дифференцирования по диагонали
Жайшылык Сартабанов 226

Мультистабильность в негладких системах
Жаныбай Жусубалиев, Айтибек Абдирасулов, Уланбек Сопуев 229

Симметрия дифференциальных уравнений с разрывной правой частью вриводит к бистабильности
Жаныбай Жусубалиев, Уланбек Сопуев, Дилмурат Турсунов, Нурсултан Бекназаров 235

Оценка числа предельных циклов уравнения $\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$, где P и Q полиномы
Маммет Г. Худай-Веренов 240

Дифференциалдық теңдеулерді шешуде графтар әдісін қолдану
Нургуль Шаждекеева, Айнагуль Адиева 249

Третья краевая задача для уравнения третьего порядка с переменными коэффициентами
Рахматилла Умаров 256

О разрешимости третьей краевой задачи для неоднородного уравнения четвертого порядка с кратными характеристиками
Юсупжон Апаков, Санжарбек Мамажонов 266

Функциялар теориясы және функционалдық талдау
Theory of functions and functional analysis
Теория функций и функциональный анализ

The infinite summation formulas of the multiple Lauricella's hypergeometric functions
Anvar Hasanov, Tuhtasin Ergashev 275

Power series representation and applications in the space of four-dimensional numbers
Anvarjon Akhmedov, Cholpon Abdullayeva, Topchubay Isakov 285

Описание оператора взятия поляры
Али Ягубов 296

Алгебра, математикалық логика және геометрия
Algebra, mathematical logic and geometry
Алгебра, математическая логика и геометрия

On the geometry of Hamiltonian systems
Abdigappar Narmanov, Shohida Ergashova 300

On the geometry of vector fields
Abdigappar Y. Narmanov, Bekzod R. Diayarov 308

On the minimization of k-valued logic functions in the class of disjunctive normal forms
Anvar Kabulov, Abdussattar Baizhumanov, Mansur Berdimurodov 314

The study of geometric laws in the patterns of the Khoja Ahmed Yasawi complex
Auyez Baidabekov, Elmira Kemelbekova, Zhanna Sadykova 321

On generalized quasi-type rectifying curves
Bahar Doğan Yazıcı, Murat Tosun 331

On inner and local inner derivations of four-dimensional Jordan algebras
Farhodjon Arzikulov, Furkat Urinboyev 337

On some counterparts of Rickart and Baer $*$-algebras
Farhodjon Arzikulov, Utkirbek Khakimov 349

On quasi-double lines of a partial mapping of Euclide space
Gulbadan Matieva, Cholponai Abdullayeva, Nurjamal Kurbanbaeva 363

Modelling a conflict event as a random process by a general logical-probabilistic method
Rana Amirova, Natiq Aliyev 369

Simple representations of $\mathfrak{o}_I^{(1)}(5)$ with restricted highest weights in characteristic two
Sherali Ibraev 376

Spinor representations of a curve with Frenet-type frame in Myller configuration for 3-dimensional Lie groups
Zehra İşbİlİr, Bahar Doğan Yazıcı, Murat Tosun 384

О распределениях, определяемых заданной сетью Френе
Гүлмира А. Шамшиева, Нуркыз А. Сарыгулова 396

Гауссова кривизна поверхностей порожденных орбитами векторных полей
Эльдор Раджабов, Абдигаппар Нарманов 401

