MINISTRY OF SCIENCE AND
HIGHER EDUCATION OF
THE REPUBLIC OF KAZAKHSTAN

TWMS

AHMET YESEVİ
ÜNİVERSİTESİ

Halyk

Түркі әлемі математиктерінің VII Дүниежүзілік Конгресі (TWMS Congress-2023) МАТЕРИАЛДАРЫ

(II бөлім)

MATERIALS

of the VII World Congress of Turkic World Mathematicians (TWMS Congress-2023)

(Part II)

МАТЕРИАЛЫ

VII Всемирного Конгресса математиков тюркского мира (TWMS Congress-2023)

(Часть II)

20–23 қыркүйек, 2023, Түркістан, Қазақстан
September 20–23, 2023, Turkestan, Kazakhstan
20–23 сентября, 2023, Туркестан, Казахстан

Түркі әлемі математиктерінің
VII Дүниежүзілік Конгресі
(TWMS Congress-2023)
МАТЕРИАЛДАРЫ
(II бөлім)

MATERIALS
of the VII World Congress of Turkic
World Mathematicians
(TWMS Congress-2023)
(Part II)

МАТЕРИАЛЫ
VII Всемирного Конгресса математиков
тюркского мира
(TWMS Congress-2023)
(Часть II)

20–23 қыркүйек, 2023, Түркістан, Қазақстан
September 20–23, 2023, Turkestan, Kazakhstan
20–23 сентября, 2023, Туркестан, Казахстан

КОНГРЕСС ҰЙЫМДАСТЫРУШЫЛАРЫ

Түркі әлемінің математикалық қоғамы

Қазақстан Республикасының Ғылым және жоғары білім министрлігі

Түркістан облысының әкімдігі

Қазақстан Республикасының Ұлттық инженерлік академиясы

Қазақстан Республикасы Президентінің жанындағы Қазақстан Республикасының Ұлттық Ғылым Академиясы

Қожа Ахмет Ясауи атындағы Халықаралық қазақ-түрік университеті

Қазақстан инновациялық және телекоммуникациялық жүйелер университеті

Математика және математикалық модельдеу институты

Мұхтар Әуезов атындағы Оңтүстік Қазақстан университеті

Сәрсен Аманжолов атындағы Шығыс Қазақстан университеті

Оңтүстік Қазақстан педагогикалық университеті

М.Х.Дулати атындағы Тараз өңірлік университеті

Халықаралық инженерлік және технология университеті

«Халық» қайырымдылық қоры

Қорқыт-Ата атындағы Қызылорда университеті

Академик Ә. Қуатбеков атындағы Халықтар достығы университеті

БАҒДАРЛАМАЛЫҚ КОМИТЕТ

ТӨРАҒАЛАРЫ

Жұмағұлов Бақытжан Тұрсынұлы, ҚР ҰҒА академигі, TWMS Президенті, Қазақстан Республикасы Парламенті Сенатының депутаты, Қазақстан

Нұрбек Саясат, Қазақстан Республикасының Ғылым және жоғарғы білім министрі, Қазақстан

Сатыбалды Дархан Амангелдыулы, Түркістан облысының әкімі

ТЕҢ ТӨРАҒАЛАРЫ

Аюпов Шавкат Абдуллаұлы, Өзбекстан Ғылым академиясының академигі, Өзбекстан Ғылым академиясы В. И. Романовский атындағы математика институтының директоры, Өзбекстан

Борубаев Алтай Асылқанұлы, Қырғыз Республикасы ҰҒА академигі, Қырғыз Республикасы ҰҒА Математика институтының директоры, Қырғызстан

Әлиев Фикрет Ахмедали оглы, Әзірбайжан ҰҒА академигі, Баку мемлекеттік университетінің Қолданбалы математика институтының директоры, TWMS Journal of Pure and Applied Mathematics журналының редакторы, Әзірбайжан

Пенахов Этибар, профессор, Әзірбайжан

Темірбекова Жанар Амангелдіқызы, Х. А. Ясауи атындағы Халықаралық қазақ-түрік университетінің ректоры, Қазақстан

Хаджисалехоглы Хилми, профессор, Түркия

Шимшек Мухиттин, Х. А. Ясауи атындағы Халықаралық қазақ-түрік университетінің Өкілетті кеңесінің төрағасы

МҮШЕЛЕРІ:

Аббасов Әли, Әзірбайжан ҰҒА академигі, Әзірбайжан

Азамов Абдулла Адамович, Өзбекстан Ғылым академиясының академигі, Өзбекстан

Айдазаде Камил, Әзірбайжан ҰҒА корреспондент-мүшесі, Әзірбайжан

Айсағалиев Серікбай Әбдіғалиұлы, профессор, Қазақстан

Арипов Мерсаид, профессор, М. Ұлықбек атындағы Өзбекстан ұлттық университеті, Өзбекстан

Арсланов Марат Мирзаевич, Татарстан Республикасы Ғылым академиясының академигі, Ресей

Ахмед-Заки Дархан Жұмақанұлы, ҚР Ғылым және жоғары білім министрлігі Ғылым комитетінің төрағасы, Қазақстан

Ашыралыев Аллаберен, профессор, Бахчешехир университеті, Түркия

Багиров Адил, Австралия федералды университеті, Австралия

Бадаев Серікжан Ағыбайұлы, профессор, Қазақстан

Байжанов Бектұр Сембиұлы, ҚР ҰҒА корреспондент-мүшесі, Қазақстан

Башар Мұстафа Тамер, Иллинойс университетінің профессоры, Жетілдірілген зерттеулер орталығының директоры, АҚШ

Бектемесов Мақтағали Әбдімажитұлы, ҚР ҰИА корреспондент-мүшесі, Қазақ ұлттық аграрлық университеті, Қазақстан

Билялов Билял, Әзірбайжан ҰҒА корреспондент-мүшесі, Әзірбайжан

Блиев Назарбай Қадырұлы, ҚР ҰҒА академигі, Қазақстан

Боқаев Нұржан Әділханұлы, Гумилев атындағы Еуразия ұлттық университетінің профессоры, Қазақстан

Войцик Вальдемар, профессор, Люблин техникалық университеті, Польша

Гиви Пейман, профессор, Питтсбург университеті, АҚШ

Гогатишвили Амиран, Чехия ғылым академиясы Математика институтының бас ғылыми қызметкері

Гогташ Фахретдин, профессор, Фират университетінің ректоры, Түркия

Даирбеков Нұрлан Слямханұлы, профессор, Қазақстан

Дженалиев Мувашархан Танабайұлы, профессор, Қазақстан

Ебади Сеид, профессор, Иран

Егоров Иван Егорович, профессор, Саха Республикасы, Ресей

Ергут Махмут, профессор, Түркия

Жұмаділдаев Асқар Серқұлұлы, ҚР ҰҒА академигі, Қазақстан

Кабанихин Сергей Игоревич, РҒА корреспондент-мүшесі, РҒА Есептеу математикасы және математикалық геофизика институты, Ресей

Кальменов Тынысбек Шәріпұлы, ҚР ҰҒА академигі, Қазақстан

Кангужин Балтабек Есматұлы, профессор, Қазақстан

Канетов Бекболат Эменович, профессор, Қырғызстан

Караев Мубариз, профессор, Сауд Арабиясы

Карапинар Ердал, профессор, Тайвань

Кулиев Вагиф, Әзірбайжан ҰҒА корреспондент-мүшесі, Әзірбайжан

Қалимолдаев Мақсат Нұраділұлы, ҚР ҰҒА академигі, Ақпараттық және есептеу технологиялары институты, Қазақстан

Қалтаев Айдархан Жүсіпбекұлы, ҚР ҰИА академигі, Сатпаев атындағы Қазақ ұлттық техникалық университеті, Қазақстан

Маммадов Юсуф, Әзірбайжан ҰҒА академигі, Әзірбайжан

Марданов Мисир Джумаил оглы, Әзірбайжан ҰҒА корреспондент-мүшесі, Әзірбайжан ҰҒА математика және механика институтының директоры, Әзірбайжан

Матиева Гүлбадан, Қырғызстан, Қырғызстан ҰҒА корреспондент-мүшесі, Қырғызстан

Милошевич Хранислав, профессор, Приштина университеті, Сербия

Мусин Олег, профессор, АҚШ

Насыров Семен Рафаилович, Татарстан Республикасы Ғылым академиясының корреспондент-мүшесі, Ресей

Нигматулин Роберт Искандерович, Ресей Ғылым академиясының академигі, Ресей

Нұрсұлтанов Ерлан Даутбекұлы, Гумилев атындағы Еуразия ұлттық университетінің профессоры, Қазақстан

Өтелбаев Мұхтарбай Өтелбайұлы, ҚР ҰҒА академигі, Қазақстан

Панков Павел Сергеевич, Қырғызстан ҰҒА корреспондент-мүшесі, Қырғызстан

Ружанский Михаил Владимирович, профессор, Гент университеті, Бельгия

Саваш Екрем, профессор, Ушак университетінің ректоры, Түркия

Садыбеков Махмуд Әбдісаметұлы, ҚР ҰҒА корреспондент-мүшесі, ҚР ҒЖБМ математика институтының директоры, Қазақстан

Салимов Ариф, профессор, Әзірбайжан

Сұлтанаев Яудат, профессор, Башқұртстан Республикасы, Ресей

Тайманов Александр Асанович, РҒА академигі, РҒА СБ Соболев атындағы математика институты, Ресей

Тосун Мурат, Сакария университетінің профессоры, Түркия

Умирбаев Уалбай Утмахамбетович, ҚР ҰҒА академигі, профессор, АҚШ

Фазуллин Зиганур Юсупович, профессор, Башқұртстан Республикасы, Ресей

Хаджиева Назиле, Азербайджан

Харин Станислав Николаевич, ҚР ҰҒА академигі, Қазақстан

Хасиев Асеф, Әзірбайжан ҰҒА академигі, Әзірбайжан

Шокин Юрий Иванович, РҒА академигі, РҒА СБ Ақпараттық және есептеу технологияларының федералды зерттеу орталығы, Ресей

Шокри Әли, профессор, Иран
Эфендиев Месуд, профессор, Германия
Юксел Сердар, профессор, Канада

ҰЙЫМДАСТЫРУШЫ КОМИТЕТ

ТӨРАҒА
Темірбеков Нұрлан Мұқанұлы, ҚР ҰИА академигі, ҚР ҰҒА корреспондент-мүшесі, Қазақстан

МҮШЕЛЕРІ:
Адамов Әбілмәжін Әлирахымұлы, профессор, Л.Н.Гумилев атындағы Еуразия ұлттық университеті, Қазақстан
Акбаров Сурхай Джаббар оглы, профессор, Йылдыз техникалық университеті, Түркия
Акин Омер, профессор, Түркия
Байгереев Досан Рақымғалиұлы, қауымдастырылған профессор, ҚР ҰИА корреспондент-мүшесі, С. Аманжолов атындағы Шығыс Қазақстан университеті, Қазақстан
Байзақов Асан Байзақұлы, профессор, Б. Н. Ельцин атындағы Қырғыз-Ресей Славян университеті, Қырғызстан
Бақанов Галитдин Бақанұлы, Х.А.Ясауи атындағы Халықаралық қазақ-түрік университеті, Қазақстан
Бас Эрдал, профессор, Фират университеті, Түркия
Бахтизин Рамиль Назифович, профессор, Уфа мемлекеттік мұнай техникалық университеті, Башқұртстан
Бектемесов Жоламан Мақтағалиұлы, Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан
Бердышев Абдумаувлен Сүлейманұлы, профессор, Абай атындағы Қазақ ұлттық педагогикалық университеті, Қазақстан
Булут Хасан, профессор, Түркия
Гунгор Мехмет Али, профессор, Сакарья университеті, Турция
Гусейнзаде Сабир, профессор, Ресей
Ерсой Солей, профессор, Түркия
Ет Микаил, профессор, Түркия
Әбдібеков Уәлихан Сейділдайұлы, ҚР ҰИА корреспондент-мүшесі, әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан
Әлиев Араз Рафиг оглы, профессор, Баку мемлекеттік университеті, Әзірбайжан
Әлиев Зиятхан Сейфаддин оглы, профессор, Баку мемлекеттік университеті, Әзірбайжан
Әмірғалиев Еділхан Несіпханұлы, ҚР ҰИА академигі, Ақпараттық және есептеуіш технологиялар институты, Қазақстан
Жакебаев Дәурен Бақытбекұлы, ҚР ҰИА корреспондент-мүшесі, Астана IT университеті, Қазақстан
Зафер Агаджик, профессор, Таяу Шығыстағы Американдық университет, Кувейт
Ибрахимов Наби, профессор, Әзірбайжан
Иманғалиев Ернар Иманғалиұлы, ҚР ҰИА бас ғалым хатшысы, Қазақстан
Иманқұлов Тимур Сәкенұлы, Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан
Исахов Әлібек Әбдиашимұлы, профессор, ҚР ҰИА корреспондент-мүшесі, Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан

Исгандаров Низамеддин, профессор, Баку мемлекеттік университеті, Әзірбайжан
Калантаров Варга, профессор, Коч университеті, Түркия
Қасенов Сырым Еркінұлы, Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан
Лакестани Мехридад, профессор, Иран
Мадияров Мұратқан Набенұлы, ҚР ҰИА академигі, С. Аманжолов атындағы Шығыс Қазақстан университеті, Қазақстан
Маммадова Гамар, профессор, Баку мемлекеттік университеті, Әзірбайжан
Мукуддинов Мукаддес, профессор, Татарстан
Муталимов Муталим, профессор, Әзірбайжан
Мұхамбетжанов Салтанбек Талапеденұлы, әл-Фараби Қазақ ұлттық университеті, Қазақстан
Намазова Гүлиза Өмірбекқызы, доцент, Қырғызстан
Озбай Хитай, профессор, Түркия
Озбеклер Абдуллах, профессор, Атилим университеті, Түркия
Панахов Гейлани, Әзірбайжан ҰҒА корреспондент-мүшесі, Әзірбайжан
Сабитов Иджок, профессор, Ресей
Сафарова Наргиз, профессор, Әзірбайжан
Сәрсенби Әбдіжахан Манапұлы, ҚР ҰИА корреспондент-мүшесі, М. Әуезов атындағы Оңтүстік Қазақстан университетінің профессоры
Сұлтанов Мұрат Әбдіқадырұлы, профессор, Х. А. Ясауи Халықаралық қазақ-түрік университеті, Қазақстан
Тамабай Динара Оразбекқызы, әл-Фараби Қазақ ұлттық университеті, Қазақстан
Темірбеков Алмас Нұрланұлы, қауымдастырылған профессор, әл-Фараби Қазақ ұлттық университеті, Қазақстан
Тұрметов Батырхан Құдайбергенұлы, профессор, Х. А. Ясауи Халықаралық қазақ-түрік университеті, Қазақстан
Урмашев Байдаулет Амантайұлы, Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан
Фараджов Араз, профессор, Әзірбайжан
Ханмаммедов Агил, профессор, Әзірбайжан
Хасаноглу Эльман, профессор, Исик университеті, Түркия
Хомпыш Хонатбек, әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан
Шахин Байрам, профессор, Эге университеті, Түркия
Шәкенов Қанат Қожахметұлы, әл-Фараби Қазақ ұлттық университеті, Қазақстан
Юстюнель Сүлеймен, профессор, Түркия
Ягубов Али, профессор, Әзірбайжан

CONGRESS ORGANIZERS

Turkic World Mathematical Society

Ministry of Science and Higher Education of the Republic of Kazakhstan

Akimat of Turkestan region

National Engineering Academy of the Republic of Kazakhstan

National Academy of Sciences of the Republic of Kazakhstan under the President of the Republic of Kazakhstan

Khoja Akhmet Yassawi International Kazakh-Turkish University

Kazakhstan University of Innovative and Telecommunication Systems

Institute of Mathematics and Mathematical Modeling

Mukhtar Auezov South Kazakhstan University

Sarsen Amanzholov East Kazakhstan University

South Kazakhstan Pedagogical University

M.H. Dulati Taraz Regional University

International University of Engineering and Technology

PF Charitable Foundation «Halyk»

Korkyt Ata Kyzylorda University

Peoples' Friendship University named after Academician A.Kuatbekov

PROGRAM COMMITTEE

CHAIRMEN

Bakytzhan T. Zhumagulov, Academician of NAS RK, President of TWMS, Deputy of the Senate of the Parliament of the Republic of Kazakhstan, Kazakhstan

Sayasat Nurbek, Minister of Science and Higher Education of the Republic of Kazakhstan, Kazakhstan

Darkhan Satybaldy, Akim of the Turkestan region

CO–CHAIRS

Altai A. Borubayev, Academician of NAS of the Kyrgyz Republic, Director of the Institute of Mathematics of NAS of the Kyrgyz Republic, Kyrgyzstan

Fikret A. Aliev, Academician of NAS of Azerbaijan, Director of the Institute of Applied Mathematics of Baku State University, Editor of TWMS Journal of Pure and Applied Mathematics, Azerbaijan

Etibar Panahov, Professor, Azerbaijan

Hilmi Hacisalihoğlu, Professor, Turkey

Muhittin Şimşek, Chairman of the Board of Trustees of Khoja Akhmet Yassawi International Kazakh-Turkish University

Shavkat A. Ayupov, Academician of the Academy of Sciences of the Republic of Uzbekistan, Director of the V.I. Romanovsky Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan, Uzbekistan

Zhanar A. Temirbekova, Rector of the Yassawi International Kazakh-Turkish University, Kazakhstan

MEMBERS:

Abdulla A. Azamov, Academician of the Academy of Sciences of the Republic of Uzbekistan, Uzbekistan

Adil Bagirov, Federation University Australia, Australia

Aidarkhan Zh. Kaltayev, Academician of NEA RK, Satpayev Kazakh National Technical University, Kazakhstan

Ali Şökri, Professor, University of Maragheh

Allaberen Ashyralyev, Professor, Bahçeşehir University, Turkey

Amiran Gogatishvili, Chief Researcher at the Institute of Mathematics of the Czech Academy of Sciences

Arif Salimov, Professor, Azerbaijan

Asaf Hacıyev, Academician of NAS of Azerbaijan, Azerbaijan

Askar S. Dzhumadildayev, Academician of NAS of the Republic of Kazakhstan, Kazakhstan

Ali Abbasov, Academic of NAS of Azerbaijan, Azerbaijan

Baltabek E. Kanguzhin, Professor, Kazakhstan

Bekbolat E. Kanetov, Professor, Kyrgyzstan

Bektur S. Baizhanov, Corresponding Member of NAS of the Republic of Kazakhstan, Kazakhstan

Bilal Bilalov, Corresponding Member of NAS of Azerbaijan, Azerbaijan

Darkhan Zh. Ahmed-Zaki, Chairman of the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan, Kazakhstan

Ekrem Savaş, Professor, rector of Usak University, Turkey

Erdal Karapınar, Professor, Taiwan

Fahrettin Göktaş, Professor, rector of Fırat University, Turkey

Gulbadan Matieva, Corresponding Member of the National Academy of Sciences of the Kyrgyz Republic, Kyrgyzstan
Hranislav Milošević, Professor, University of Pristina, Serbia
Iskander A. Taimanov, Academician of RAS, Sobolev Institute of Mathematics RAS, Russia
Ivan E. Egorov, Professor, Sakha Republic, Russia
Kamil Aydazade, Corresponding member of NAS of Azerbaijan, Azerbaijan
Mahmud A. Sadybekov, Corresponding Member of NAS of the Republic of Kazakhstan, Director of the Institute of Mathematics of NAS of the Republic of Kazakhstan, Kazakhstan
Mahmut Ergüt, Professor, Turkey
Maksat N. Kalimoldayev, Academician of NAS RK, Institute of Information and Computational Technologies, Kazakhstan
Maktagali A. Bektemessov, Corresponding Member of NEA RK, Kazakh National Agrarian University, Kazakhstan
Marat M. Arslanov, Academician of the Academy of Sciences of the Republic of Tatarstan, Russia
Mersaid Aripov, Professor, M. Ulugbek National University of Uzbekistan, Uzbekistan
Messoud Efendiev, Professor, Germany
Michael V. Ruzhansky, Professor, Ghent University, Belgium
Misir Mardanov, Corresponding Member of NAS of Azerbaijan, Director of the Institute of Mathematics and Mechanics of NAS of Azerbaijan, Azerbaijan
Mubariz Karaev, Professor, Saudi Arabia
Mukhtarbai O. Otelbayev, Academician of NAS of Kazakhstan, Kazakhstan
Murat Tosun, Professor at Sakarya University, Turkey
Mustafa Tamer Başar, Professor at the University of Illinois, Director of the Center for Advanced Studies, USA
Muvasharkhan T. Dzhenaliyev, Professor, Kazakhstan
Nazarbay K. Bliyev, Academician of NAS of Kazakhstan, Kazakhstan
Nazile Hajiyeva, Azerbaijan
Nurlan S. Dairbekov, Professor, Kazakhstan
Nurzhan A. Bokayev, Professor of the Gumilev Eurasian National University, Kazakhstan
Oleg Musin, Professor, USA
Pavel S. Pankov, Corresponding Member of NAS of the Kyrgyz Republic, Kyrgyzstan
Peyman Givi, Professor, University of Pittsburgh, USA
Robert I. Nigmatulin, Academician of RAS, Russia
Seid Ebadi, Professor, Iran
Semen R. Nasyrov, Corresponding Member of the Academy of Sciences of the Republic of Tatarstan, Russia
Serdar Yüksel, Professor, Canada
Sergey I. Kabanikhin, Corresponding Member of RAS, Institute of Computational Mathematics and Mathematical Geophysics SB RAS, Russia
Serikbay A. Aisagaliyev, Professor, Kazakhstan
Serikzhan A. Badayev, Professor, Kazakhstan
Stanislav N. Kharin, Academician of NAN RK, Kazakhstan
Tynysbek Sh. Kalmenov, Academician of NAS RK, Kazakhstan
Ualbai U. Umirbayev, Academician of NAS RK, Professor, USA
Vaqif Quliyev, Corresponding Member of NAS of Azerbaijan, Azerbaijan
Waldemar Wójcik, Professor, Lublin Technical University, Poland
Yaudat Sultanaev, Professor, Republic of Bashkortostan, Russia
Yerlan D. Nursultanov, Professor of the Gumilev Eurasian National University, Kazakhstan
Yuri I. Shokin, Academician of RAS, Federal Research Center for Information and Computing Technologies SB RAS, Russia

Yusif Mammadov, Academician of NAS of Azerbaijan, Azerbaijan
Ziganur Yu. Fazullin, Professor, Republic of Bashkortostan, Russia

ORGANIZING COMMITTEE

CHAIRMAN
Nurlan M. Temirbekov, Academician of NEA RK, corresponding member of NAS RK, Kazakhstan

MEMBERS:
Abdizhakhan M. Sarsenbi, Corresponding member of NEA RK, Professor of M. Auezov South Kazakhstan University
Abdullah Özbekler, Professor, Atilim University, Turkey
Abdumauvlen S. Berdyshev, Professor, Abay Kazakh National Pedagogical University, Kazakhstan
Abilmazhin A. Adamov, Professor, L.N. Gumilyov Eurasian National University, Kazakhstan
Ağacık Zafer, Professor, American University of the Middle East, Kuwait
Ali Yaqubov, Professor, Azerbaijan
Alibek A. Issakhov, Professor, Corresponding Member of NEA RK, Al-Farabi Kazakh National University, Kazakhstan
Almas N. Temirbekov, Associate Professor, Al-Farabi Kazakh National University, Kazakhstan
Aqil Khanmammadov, Professor, Azerbaijan
Araz Aliyev, Professor, Baku State University, Azerbaijan
Araz Faradzhov, Professor, Azerbaijan
Asan B. Bayzakov, Professor, Kyrgyz-Russian Slavic University named after B.N. Yeltsin, Kyrgyzstan
Batyrkhan Kh. Turmetov, Professor, Yassawi International Kazakh-Turkish University, Kazakhstan
Baydaulet A. Urmashev, Al-Farabi Kazakh National University, Kazakhstan
Bayram Shahin, Professor, Ege University, Turkey
Dauren B. Zhakebayev, Corresponding Member of NEA RK, Astana IT University, Kazakhstan
Dinara O. Tamabay, Al-Farabi Kazakh National University, Kazakhstan
Dossan R. Baigereyev, Associate Professor, Corresponding Member of NEA RK, S. Amanzholov East Kazakhstan University, Kazakhstan
Elman Hasanoğlu, Professor, Isik University, Turkey
Erdal Bas, Professor, Firat University, Turkey
Galitdin B. Bakanov, K.A. Yassawi International Kazakh-Turkish University, Kazakhstan
Guliza O. Namazova, Associate Professor, Kyrgyzstan
Hasan Bulut, Professor, Turkey
Hitay Özbay, Professor, Turkey
İdjok Sabitov, Professor, Russia
Kanat K. Shakenov, Al-Farabi Kazakh National University, Kazakhstan
Khompysh Honatbek, Al-Farabi Kazakh National University, Kazakhstan
Mehmet Ali Gungor, Professor, Sakarya University, Turkey
Mehridad Lakestani, Professor, Iran
Mikail Et, Professor, Turkey
Mukaddes Mukuddinov, Professor, Tatarstan

Murat A. Sultanov, Professor, Yassawi International Kazakh-Turkish University, Kazakhstan
Muratkan N. Madiyarov, Academician of NEA RK, S. Amanzholov East Kazakhstan University, Kazakhstan
Mutallim Mutallimov, Professor, Azerbaijan
Nabi İbrahimov, Professor, Azerbaijan
Nargiz Safarova, Professor, Azerbaijan
Nizamaddin İsgandarov, Professor, Baku State University, Azerbaijan
Ömer Akın, Professor, Turkey
Panahov Geylani, Corresponding member of NAS of Azerbaijan, Azerbaijan
Qamar Mammadova, Professor, Baku State University, Azerbaijan
Ramil N. Bakhtizin, Professor, Ufa State Petroleum Technological University, Bashkortostan
Sabir Guseinzade, Professor, Russia
Saltanbek T. Mukhambetzhanov, Al-Farabi Kazakh National University, Kazakhstan
Soley Ersoy, Professor, Turkey
Suleyman Üstünel, Professor, Turkey
Surkhay Akbarov, Professor, Yildiz Technical University, Turkey
Syrym Ye. Kassenov, Al-Farabi Kazakh National University, Kazakhstan
Timur S. Imankulov, Al-Farabi Kazakh National University, Kazakhstan
Ualikhan S. Abdibekov, Corresponding member of NEA RK, Al-Farabi Kazakh National University, Kazakhstan
Varga Kalantarov, Professor, Koc University, Turkey
Yedilkhan N. Amirgaliyev, Academician of NEA RK, Institute of Information and Computational Technologies, Kazakhstan
Yernar I. Imangaliyev, Chief Scientific Secretary of NEA RK, Kazakhstan
Zholaman M. Bektemessov, Al-Farabi Kazakh National University, Kazakhstan
Ziyatxan Aliyev, Professor, Baku State University, Azerbaijan

ОРГАНИЗАТОРЫ КОНГРЕССА

Математическое общество тюркского мира

Министерство науки и высшего образования Республики Казахстан

Акимат Туркестанской области

Национальная инженерная академия Республики Казахстан

Национальная академия наук Республики Казахстан при Президенте Республики Казахстан

Международный казахско-турецкий университет имени Ходжи Ахмеда Ясави

Казахстанский университет инновационных и телекоммуникационных систем

Институт математики и математического моделирования

Южно-Казахстанский университет имени Ауэзова

Восточно-Казахстанский университет имени Сарсена Аманжолова

Южно-Казахстанский педагогический университет

Таразский региональный университет имени М.Х.Дулати

Международный инженерно-технологический университет

Благотворительный Фонд «Халык»

Кызылординский университет имени Коркыт-Ата

Университет дружбы народов имени академика А. Куатбекова

ПРОГРАММНЫЙ КОМИТЕТ

ПРЕДСЕДАТЕЛИ

Жумагулов Бакытжан Турсынович, академик НАН РК, Президент TWMS

Нурбек Саясат, Министр науки и высшего образования Республики Казахстан, Казахстан

Сатыбалды Дархан Амангелдыулы, Аким Туркестанской области

СОПРЕДСЕДАТЕЛИ

Алиев Фикрет Ахмедалиоглы, академик НАН Азербайджана, директор института прикладной математики Бакинского государственного университета, редактор журнала TWMS Journal of Pure and Applied Mathematics, Азербайджан

Аюпов Шавкат Абдуллаевич, академик АН Узбекистана, директор института математики им. В.И. Романовского АН Узбекистана, Узбекистан

Борубаев Алтай Асылканович, академик НАН Кыргызской Республики, директор Института математики НАН КР, Кыргызстан

Пенахов Этибар, профессор, Азербайджан

Темирбекова Жанар Амангельдиевна, ректор Международного казахско-турецкого университета им. Х.А. Ясави, Казахстан

Хаджисалехоглы Хилми, профессор, Турция

Шимшек Мухиттин, председатель Полномочного Совета Международного казахско-турецкого университета им. Ходжа Ахмета Яссави

ЧЛЕНЫ:

Аббасов Али, академик НАН Азербайджана, Азербайджан

Азамов Абдулла Азамович, академик АН Узбекистана, Узбекистан

Айдазаде Камил, член-корреспондент НАН Азербайджана, Азербайджан

Айсагалиев Серикбай Абдигалиевич, профессор, Казахстан

Арипов Мирсаид, профессор, Национальный университет Узбекистана им. М. Улугбека, Узбекистан

Арсланов Марат Мирзаевич, академик АН Республики Татарстан, Россия

Ахмед-Заки Дархан Жумаканович, Председатель Комитета науки Министерства науки и высшего образования РК, Казахстан

Ашыралыев Аллаберен, профессор, Бахчешехир университет, Турция

Багиров Адил, Федеративный университет Австралии, Австралия

Бадаев Серикжан Агыбаевич, профессор, Казахстан

Байжанов Бектур Сембиевич, член-корреспондент НАН РК, Казахстан

Башар Мустафа Тамер, профессор Университета Иллинойса, директор Центра перспективных исследований, США

Бектемесов Мактагали Абдимажитович, член-корреспондент НИА РК, Казахский национальный аграрный университет, Казахстан

Билялов Билял, член-корреспондент НАН Азербайджана, Азербайджан

Блиев Назарбай Кадырович, академик НАН РК, Казахстан

Бокаев Нуржан Адилханович, профессор Евразийского национального университета им. Гумилева, Казахстан

Войцик Вальдемар, профессор, Люблинский технический университет, Польша

Гиви Пейман, профессор, Университет Питтсбурга, США

Гогатишвили Амиран, главный научный сотрудник Института математики Чешской Академии наук

Гогташ Фахретдин, профессор, ректор Университета Фират, Турция

Даирбеков Нурлан Слямханович, профессор, Казахстан

Дженалиев Мувашархан Танабаевич, профессор, Казахстан
Джумадильдаев Аскар Серкулович, академик НАН РК, Казахстан
Ебади Сеид, профессор, Иран
Егоров Иван Егорович, профессор, Республика Саха, Россия
Ергут Махмут, профессор, Турция
Ефендиев Месуд, профессор, Германия
Кабанихин Сергей Игоревич, член-корреспондент РАН, Институт вычислительной математики и математической геофизики СО РАН, Россия
Калимолдаев Максат Нурадилович, академик НАН РК, Институт информационных и вычислительных технологий, Казахстан
Калтаев Айдархан Жусупбекович, академик НИА РК, Казахский национальный технический университет им. Сатпаева, Казахстан
Кальменов Тынысбек Шарипович, академик НАН РК, Казахстан
Кангужин Балтабек Есматович, профессор, Казахстан
Канетов Бекболат Эменович, профессор, Кыргызстан
Караев Мубариз, профессор, Саудовская Аравия
Карапинар Ердал, профессор, Тайвань
Кулиев Вагиф, член-корреспондент НАН Азербайджана, Азербайджан
Маммадов Юсуф, академик НАН Азербайджана, Азербайджан
Марданов Мисир Джумаилоглы, член-корреспондент НАН Азербайджана, директор Института математики и механики НАН Азербайджана, Азербайджан
Матиева Гульбадан, член-корреспондент НАН КР, Кыргызстан
Милошевич Хранислав, профессор, Университет Приштины, Сербия
Мусин Олег, профессор, США
Насыров Семен Рафаилович, член-корреспондент АН Республики Татарстан, Россия
Нигматулин Роберт Искандерович, академик РАН, Россия
Нурсултанов Ерлан Даутбекович, профессор Евразийского национального университета им. Гумилева, Казахстан
Отелбаев Мухтарбай Отелбаевич, академик НАН РК, Казахстан
Панков Павел Сергеевич, член-корреспондент НАН КР, Кыргызстан
Ружанский Михаил Владимирович, профессор, Университет Гент, Бельгия
Саваш Екрем, профессор, ректор Университета Ушак, Турция
Садыбеков Махмуд Абдысаметович, член-корреспондент НАН РК, директор института математики КН МНВО РК, Казахстан
Салимов Ариф, профессор, Азербайджан
Султанаев Яудат, профессор, Республика Башкортостан, Россия
Тайманов Искандер Асанович, академик РАН, Институт математики им. Соболева СО РАН, Россия
Тосун Мурат, профессор университета Сакария, Турция
Умирбаев Уалбай Утмахамбетович, академик НАН РК, профессор, США
Фазуллин Зиганур Юсупович, профессор, Республика Башкортостан, Россия
Хаджиева Назиле, Азербайджан
Харин Станислав Николаевич, академик НАН РК, Казахстан
Хасиев Асеф, академик НАН Азербайджана, Азербайджан
Шокин Юрий Иванович, академик РАН, Федеральный исследовательский центр информационных и вычислительных технологий СО РАН, Россия
Шокри Али, профессор, Иран
Юксел Сердар, профессор, Канада

ОРГАНИЗАЦИОННЫЙ КОМИТЕТ

ПРЕДСЕДАТЕЛЬ:

Темирбеков Нурлан Муханович, академик НИА РК, член-корреспондент НАН РК, Казахстан

ЧЛЕНЫ:

Абдибеков Уалихан Сейдильдаевич, член-корреспондент НИА РК, Казахский национальный университет им. аль-Фараби, Казахстан

Адамов Абилмажин Алирахымович, профессор, Евразийский национальный университет имени Л.Н. Гумилева, Казахстан

Акбаров Сурхай Джаббар оглы, профессор, Технический университет Йылдыз, Турция

Акин Омер, профессор, Турция

Алиев Араз Рафиг оглы, профессор, Бакинский государственный университет, Азербайджан

Алиев Зиятхан Сейфаддин оглы, профессор, Бакинский государственный университет, Азербайджан

Амиргалиев Едилхан Несипханович, академик НИА РК, Институт вычислительных и информационных технологий, Казахстан

Байгереев Досан Ракимгалиевич, ассоциированный профессор, член-корреспондент НИА РК, Восточно-Казахстанский университет им. С. Аманжолова, Казахстан

Байзаков Асан Байзакович, профессор, Кыргызско-Российский Славянский университет имени Б. Н. Ельцина, Кыргызстан

Баканов Галитдин Баканович, Международный казахско-турецкий университет им. Х.А. Ясави, Казахстан

Бас Эрдал, профессор, Университет Фират, Турция

Бахтизин Рамиль Назифович, профессор, Уфимский государственный нефтяной технический университет, Башкортостан

Бектемесов Жоламан Мактагалиевич, Казахский национальный университет им. аль-Фараби, Казахстан

Бердышев Абдумаувлен Сулейманович, профессор, Казахский национальный педагогический университет им. Абая, Казахстан

Булут Хасан, профессор, Турция

Гунгор Мехмет Али, профессор, Университет Сакарья, Турция

Гусейнзаде Сабир, профессор, Россия

Ерсой Солей, профессор, Турция

Ет Микаил, профессор, Турция

Жакебаев Даурен Бахытбекович, член-корреспондент НИА РК, Астана IT университет, Казахстан

Зафер Агаджик, профессор, Американский университет Ближнего Востока, Кувейт

Ибрахимов Наби, профессор, Азербайджан

Имангалиев Ернар Имангалиевич, Главный Ученый секретарь НИА РК, Казахстан

Иманкулов Тимур Сакенович, Казахский национальный университет им. аль-Фараби, Казахстан

Исахов Алибек Абдиашимович, профессор, член-корреспондент НИА РК Казахский национальный университет им. аль-Фараби, Казахстан

Исгандаров Низамеддин, профессор, Бакинский государственный университет, Азербайджан

Калантаров Варга, профессор, Коч университет, Турция

Касенов Сырым Еркинович, Казахский национальный университет им. аль-Фараби, Казахстан

Лакестани Мехридад, профессор, Иран

Мадияров Мураткан Набенович, академик НИА РК, Восточно-Казахстанский университет им. С. Аманжолова, Казахстан

Маммадова Гамар, профессор, Бакинский государственный университет, Азербайджан

Мукуддинов Мукаддес, профессор, Татарстан

Муталимов Муталим, профессор, Азербайджан

Мухамбетжанов Салтанбек Талапеденович, Казахский национальный университет им. аль-Фараби, Казахстан

Намазова Гулиза Омурбековна, доцент, Кыргызстан

Озбай Хитай, профессор, Турция

Озбеклер Абдуллах, профессор, Университет Атилим, Турция

Панахов Гейлани, член-корреспондент НАН Азербайджана, Азербайджан

Сабитов Иджок, профессор, Россия

Сарсенби Абжахан Манапулы, член-корреспондент НИА РК, профессор Южно-Казахстанского университета им. М.Ауезова

Сафарова Наргиз, профессор, Азербайджан

Султанов Мурат Абдукадырович, профессор, МКТУ им. Х.А. Ясави

Тамабай Динара Оразбекқызы, Казахский национальный университет им. аль-Фараби, Казахстан

Темирбеков Алмас Нурланович, ассоциированный профессор, Казахский национальный университет им. аль-Фараби, Казахстан

Турметов Батырхан Худайбергенович, профессор, МКТУ им. Х. А. Ясави, Казахстан

Урмашев Байдаулет Амантаевич, Казахский национальный университет им. аль-Фараби, Казахстан

Фараджов Араз, профессор, Азербайджан

Ханмаммадов Агил, профессор, Азербайджан

Хасаноглу Елман, профессор, Ишикский университет, Турция

Хомпыш Хонатбек, Казахский национальный университет им. аль-Фараби, Казахстан

Шакенов Канат Кожахметович, Казахский национальный университет им. аль-Фараби, Казахстан

Шахин Байрам, профессор, Эгейский университет, Турция

Юстюнел Сулеймен, профессор, Турция

Ягубов Али, профессор, Азербайджан

Ықтималдықтар теориясы және математикалық статистика

Probability theory and mathematical statistics

Теория вероятностей и математическая статистика

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON ASYMPTOTICALLY PROPERTIES OF ESTIMATES OF THE RELATIVE-RISK FUNCTION BY STRONG MIXING UNDER RANDOM CENSORING

NURLAN DUSHATOV[1], RUSTAMJON MURADOV[2]

[1]Almalyk branch of Tashkent State Technical university named after Islam Karimov, Almalyk, Uzbekistan
[2]Namangan Institute of Engineering and Technology, Namangan, Uzbekistan
e-mails: n_dushatov@rambler.ru, rustamjonmuradov@gmail.com

Abstract. In this article we consider the problems of estimation and studies of asymptotic properties of relative-risk ratio estimates in the case when the data is censored and strong mixing.

Keywords: Relative risk; censored data; strong mixing; cumulative hazard.

AMS Subject Classification: 62H10, 62H12

Let $X_1, X_2, ..., X_n$ be survival times with continuous marginal distribution function (d.f.) $F(t), F(0) = 0$, censored from the right by nonnegative times $Y_1, Y_2, ..., Y_n$ with continuous d.f. $G(t), G(0) = 0$, such that instead of X_i we observe the pairs (Z_i, δ_i), $i = 1, ..., n$, where $Z_i = \min(X_i, Y_i)$ and $\delta_i = I(X_i \le Y_i)$ with $I(A)$ standing for indicator of event A. We assume that survival times X_i are independent of censoring times Y_i and $\alpha-$mixing (or strong mixing) with mixing parameter

$$\alpha(n) = \sup_{k\ge 1}\left\{|P(A\cap B) - P(A)\cdot P(B)|,\ A\in A_1^k,\ B\in A_{k+n}^{\infty}\right\} \to 0,\ n\to\infty \tag{1}$$

where A_i^k is the $\sigma-$algebra of events generated by random variables $\{X_j,\ i\le j\le k\}$. Among several mixing condition used in the literature, $\alpha-$mixing has many practical applications (see, Doukhan [5]). Under $\alpha-$mixing assumption we consider some asymptotic properties for relative – risk function (see for example [1,2], for no mixing situation).

For introducing relative-risk function we need some notation. Let $H(t)$ be d.f. of observed random variables Z_i. Then $H(t) = 1 - (1 - F(t))(1 - G(t))$, $t \ge 0$. Let $[\tau_H, T_H]$ be a support of H, where $\tau_H = \sup\{t \ge 0 : H(t) = 0\}$ and $T_H = \inf\{t \ge 0 : H(t) = 1\}$. It is easy to see that $\tau_H = \max(\tau_F, \tau_G)$ and $T_H = \max(T_F, T_G)$. We define supdistribution functions $H^{(m)}(t) = P(Z_i \le t,\ \delta_m = m)$, $m = 0, 1$ with $H^{(0)}(t) + H^{(1)}(t) = H(t)$ for all t, where

$$H^{(0)}(t) = \int_0^t (1 - F(s))dG(s),\quad H^{(1)}(t) = \int_0^t (1 - G(s))dF(s).$$

From sample data $S^{(n)} = \{(Z_i, \delta_i),\ 1 \le i \le n\}$, we define empirical estimators of distributions $H(t)$ and $H^{(m)}(t)$, $m = 0, 1$, as

$$\begin{aligned} H_n(t) &= \tfrac{1}{n}\textstyle\sum_{j=1}^n I(Z_j \le t),\\ H_n^{(m)}(t) &= \tfrac{1}{n}\textstyle\sum_{j=1}^n I(Z_j \le t,\ \delta_j = m),\ m = 0, 1, \end{aligned} \tag{2}$$

This work was partially supported by grant no UZB-Ind-2021-97.

respectively. The cumulative hazard functions (c.h.f.-s) corresponding to the d.f.-s F, G and H can be expressed as follows

$$\begin{aligned}
&\Lambda^{(0)}(t) = \int_0^t \frac{dG(s)}{1-G(s)} = \int_0^t \frac{dH^{(0)}(s)}{1-H(s)},\\
&\Lambda^{(1)}(t) = \int_0^t \frac{dF(s)}{1-F(s)} = \int_0^t \frac{dH^{(1)}(s)}{1-H(s)},\\
&\Lambda(t) = \Lambda^{(0)}(t) + \Lambda^{(1)}(t) = \int_0^t \frac{dH(s)}{1-H(s)}.
\end{aligned} \tag{3}$$

We introduce the relative-risk function $R(t) = \Lambda^{(1)}(t)\Lambda^{-1}(t)$ and its estimator $R_n(t) = \Lambda_n^{(1)}(t)\Lambda_n^{-1}(t)$, $t \geq 0,\ \ n = 1, 2, ...,$ where corresponding empirical estimators of c.h.f-ns are

$$\begin{aligned}
&\Lambda_n^{(0)}(t) = \int_0^t \frac{dH_n^{(0)}(s)}{1-H_n(s)},\\
&\Lambda_n^{(1)}(t) = \int_0^t \frac{dH_n^{(1)}(s)}{1-H_n(s)},\\
&\Lambda_n(t) = \Lambda_n^{(0)}(t) + \Lambda_n^{(1)}(t) = \int_0^t \frac{dH_n(s)}{1-H_n(s)}.
\end{aligned} \tag{4}$$

Observe that $0 \leq R(t),\ R_n(t) \leq 1$ for all $t \geq 0$ and $n = 1, 2, ...$

2. Asymptotic results for estimator

We assume throughout that $\alpha-$ mixing condition (1) hold for survival data sequence. The first statement establishes the almost sure representation result for sums of random functions with rate.

Theorem 2.1. Let $\alpha(n) = O(n^{-\nu})$ for some $\nu > 3$. Then for $\tau_H < \tau \leq t \leq T < T_H$, and $n \to \infty$,

$$R_n(t) - R(t) = \frac{1}{n}\sum_{i=1}^{n} \Psi_t(Z_i, \delta_i) + \varepsilon_n(t), \tag{5}$$

where $\Psi_t(x, y) = \frac{1}{\Lambda(t)}\left[\widetilde{\Psi}_t(x) - \frac{R(t)}{\Lambda(t)}\widetilde{\widetilde{\Psi}}_t(x, y)\right]$,

$$\begin{aligned}
&\widetilde{\Psi}_t(x) = \frac{H(\min(x,t))}{1-H(\min(x,t))} - \frac{I(x\leq t)}{1-H(x)},\\
&\widetilde{\widetilde{\Psi}}_t(x, y) = d(\min(x, t)) - \frac{I(x\leq t,\, y=1)}{1-H(x)},\\
&d(t) = \int_0^t \frac{dH^{(1)}(s)}{(1-H(s))^2}, E\Psi_t(Z_i, \delta_i) = 0,
\end{aligned} \tag{6}$$

with

$$\sup_{\tau_H \leq t < T_H} |\varepsilon_n(t)| \overset{a.s.}{=} O(b_n),$$

and $b_n = n^{-1/2}(\log n)^{-\lambda},\ \lambda > 0$ depends on ν.

The asymptotic normality of estimator $R_n(t)$ is consents of the next theorem.

Theorem 2.2. Under conditions of theorem 2.1, as $n \to \infty$, we have

$$\sqrt{n}\,(R_n(t) - R(t)) \xrightarrow{d} N(0, \sigma^2(t)), \tag{7}$$

where

$$\sigma^2(t) = Var\left\{\Psi_t(Z_1, \delta_1)\right\} + 2\sum_{j=2}^{\infty} Cov\left(\Psi_t(Z_1, \delta_1),\ \Psi_t(Z_j, \delta_j)\right).$$

Theorem 2.3. Assume the conditions of theorem 2.1. Then on a rich enough probability space there is a two – parametrical mean zero Gaussian process $\{N(x, y),\ x, y \geq 0\}$ such that,

$$\sup_{\tau \leq t \leq T} \left|\sqrt{n}(R_n(t) - R(t)) - N(t; n)\right| \overset{a.s.}{=} O((\log n)^{-\lambda}). \tag{8}$$

Corollary 2.1. Under the assumption of theorem 2.3, we have

$$\sup_{\tau \leq t \leq T} \left|\Lambda_n^{(m)}(t) - \Lambda^{(m)}(t)\right| \overset{a.s.}{=} O(a_n),\ m = 0, 1,$$

where $a_n = n^{-1/2}(\log\log n)^{1/2}$.

3. Proofs of Theorems 2.1-2.3.

In order to prove our main theorems, we use the following results.

Lemma 3.1. (Lemma 1 in [4]). Let $\{X_n,\ n \geq 1\}$ be a sequence of $\alpha-$ mixing random variables with mixing coefficient $\alpha(n)$, independent of an independent and identically distributed sequence of random variables $\{Y_n,\ n \geq 1\}$, then $\{(X_n, Y_n),\ n \geq 1\}$ is a sequence of $\alpha-$mixing random variables with mixing coefficient $4\alpha(n)$. In particular, so is $\{\min(X_n, Y_n),\ n \geq 1\}$.

Corollary 3.1. Under the conditions of Lemma 3.1 it follows that $\{Z_n,\ n \geq 1\}$ and $\{(Z_n, \delta_n),\ n \geq 1\}$ are two stationary sequences of $\alpha-$mixing random variables.

Lemma 3.2. (Theorem 3.2 in [3]). Let $\{Z_n,\ n \geq 1\}$ be a stationary $\alpha-$mixing sequence of random variables with d.f. F and mixing coefficient $\alpha(n) = O(n^{-\nu})$ for some $\nu > 3$ and let $H_n(t)$ be the empirical d.f. (2) based on $Z_1, ..., Z_n$. Then

$$\limsup_{n\to\infty} \left[\left(\frac{n}{2\log\log n}\right)^{1/2} \cdot \sup_{t\geq 0} |H_n(t) - H(t)|\right] \stackrel{a.s.}{=} 1. \tag{9}$$

From (9) we also easy get following.

Corollary 3.2. Corollary 3.1 and Lemma 3.2, we have

$$\sup_{t\geq 0} \left|H_n^{(m)}(t) - H^{(m)}(t)\right| \stackrel{a.s.}{=} O(a_n), \tag{10}$$

where $a_n = \left(\frac{\log\log n}{n}\right)^{1/2}$.

Let's define $B(t;s) = \sum_{j\leq s} g_j(t)$, where $g_j(t) = I(Z_j \leq t) - H(t)$.

Lemma 3.3. (Theorem 3 in [6]). Let $\{Z_n,\ n \geq 1\}$ be a stationary $\alpha-$mixing sequence of random variables with arbitrary d.f. $H(t)$ and mixing coefficient $\alpha(n) = O(n^{-\nu})$ for some $\nu > 3$. Then we can redefine empirical process on a richer probability space on which there exist a Kiefer process $\{\ K(t;s),\ (t,s) \in R \times [0,\infty)\ \}$ with covariance function

$$E\left[K(t;s) \cdot K(t';s')\right] = \min(s;s')\Gamma(t;t')$$

and

$$\Gamma(t;t') = Cov\left(g_1(t),\ g_1(t')\right) + \sum_{j=2}^{\infty}\left[Cov\left(g_1(t),\ g_j(t')\right) + Cov\left(g_1(t'),\ g_j(t')\right)\right],$$

such that for some $\lambda > 0$ depending on ν and $L > 0$,

$$\sup_{0\leq s\leq L}\sup_{t\in R} |B(t;s) - K(t;s)| \stackrel{a.s.}{=} O\left(L^{1/2}(\log L)^{-\lambda}\right). \tag{11}$$

From (11) one can get a more suitable form of an approximation.

Corollary 3.3. Under the conditions of Lemma 3.3 and Corollary 3.1, we have

$$\sup_{t\geq 0} \left|\sqrt{n}\ (H_n(t) - H(t)) - n^{-1/2} \cdot K(t;n)\right| \stackrel{a.s.}{=} O\left((\log n)^{-\lambda}\right). \tag{12}$$

Proof of Theorem 2.1. By the simple algebra we get

$$\begin{aligned}\Lambda_n(t) = R_n(t) - R(t) = (R_n(t) - \tfrac{\Lambda^{(1)}(t)}{\Lambda_n(t)}) + (\tfrac{\Lambda^{(1)}(t)}{\Lambda_n(t)} - R(t)) = \tfrac{1}{\Lambda_n(t)} \cdot \left(\Lambda_n^{(1)}(t) - \Lambda^{(1)}(t)\right) + \\ +\Lambda^{(1)}(t) \cdot \left(\tfrac{1}{\Lambda_n(t)} - \tfrac{1}{\Lambda(t)}\right) = \tfrac{1}{\Lambda_n(t)} \cdot \left[\left(\Lambda_n^{(1)}(t) - \Lambda^{(1)}(t)\right) - \tfrac{R(t)}{\Lambda(t)} \cdot (\Lambda_n(t) - \Lambda(t))\right].\end{aligned} \tag{13}$$

For $\Lambda(t)$ estimator $\Lambda_n(t)$ is consistent for all $\tau \leq t \leq T$. Hence $\Lambda_n(t) = R_n(t) - R(t)$ is asymptotic equivalent to process $\widetilde{\Lambda}_n(t) = \frac{1}{\Lambda(t)} \cdot \left[\left(\Lambda_n^{(1)}(t) - \Lambda^{(1)}(t)\right) - \frac{R(t)}{\Lambda(t)} \cdot (\Lambda_n(t) - \Lambda(t))\right]$. It is not difficult to have following representations

$$\begin{aligned}\Lambda_n^{(1)}(t) - \Lambda^{(1)}(t) = \tfrac{\left(H_n^{(1)}(t) - H(t)\right)}{1-H(t)} + \int_0^t \tfrac{(H_n(u) - H(u))dH^{(1)}(u)}{(1-H(u))^2} - \\ - \int_0^t \tfrac{\left(H_n^{(1)}(u) - H^{(1)}(u)\right)dH(u)}{(1-H(u))^2} + r_{1n}(t) + r_{2n}(t),\end{aligned}$$

$$\Lambda_n(t) - \Lambda(t) = \frac{(H_n(t) - H(t))}{1 - H(t)} + r_{3n}(t) + r_{4n}(t),$$

where

$$r_{1n}(t) = \int_0^t \frac{(H_n(u) - H(u))^2 \, dH^{(1)}(u)}{(1 - H(u))^2 \, (1 - H_n(u))},$$

$$r_{2n}(t) = \int_0^t \left[\frac{1}{1 - H_n(u)} - \frac{1}{1 - H(u)}\right] d\left(H_n^{(1)}(u) - H^{(1)}(u)\right),$$

$$r_{3n}(t) = \int_0^t \frac{(H_n(u) - H(u))^2 \, dH(u)}{(1 - H(u))^2 \, (1 - H_n(u))},$$

$$r_{4n}(t) = \int_0^t \left[\frac{1}{1 - H_n(u)} - \frac{1}{1 - H(u)}\right] d\left(H_n(u) - H(u)\right).$$

Now, by Theorem 2 in [4], we get

$$\sup_{\tau \le t \le T} |r_{1n}(t) + r_{2n}(t)| \stackrel{a.s.}{=} O(b_n), \tag{14}$$

and

$$\sup_{\tau \le t \le T} |r_{3n}(t) + r_{4n}(t)| \stackrel{a.s.}{=} O(b_n). \tag{15}$$

Finally, using that Theorem 2 in [4], one have

$$\widetilde{\Lambda}_n(t) = -\frac{1}{n} \sum_{i=1}^{n} \Psi_t(Z_i, \delta_i) + \varepsilon_n(t), \tag{16}$$

with $\sup\limits_{\tau \le t \le T} |\varepsilon_n(t)| \stackrel{a.s.}{=} O(b_n)$.
Theorem 2.1 is proved.

Proof of Theorem 2.2. With probability 1

$$\sup_{\tau \le t \le T} \max_{1 \le t \le n} \Psi_t\left(Z_i, \delta_i\right) \le$$

$$\le \frac{1}{H(\tau)} \left[\frac{1}{\left(1 - G\left(\min\left(Z_{(n)}, T\right)\right)\right)} \int_0^{\min(Z_{(n)}, T)} d\left(\frac{1}{1 - F(s)}\right) + \frac{1}{\left(1 - H\left(\min\left(Z_{(n)}, T\right)\right)\right)} + \frac{2}{H(\tau)\left(1 - H\left(\min\left(Z_{(n)}, T\right)\right)\right)}\right] \le$$

$$\le C \cdot \frac{1}{H^2(\tau)(1 - H(T))},$$

where C is a some positive constant possible depending on d.f. H. Then by Corollary 3.1, $\{\Psi_t\left(Z_i, \delta_i\right),\ i \ge 1\}$ is a sequence of $\alpha-$mixing bounded random variables. Now in order to obtain the asymptotic normality (7), we just apply Theorem 18.5.4 in [8]. Thus Theorem 2.2 is proved.

Proof of Theorem 2.3. Almost sure representation (5) playes a key role on investigating of $\Lambda_n(t)$ through c.h.f-ns $\left\{\Lambda_n^{(m)}(t) - \Lambda^{(m)}(t),\ m = 0, 1,\ t \le T\right\}$. From Theorem 2.1 in [7] for each $n \ge 1$ we have

$$\sup_{\tau \le t \le T} \left|\sqrt{n}\left(\Lambda_n^{(m)}(t) - \Lambda^{(m)}(t)\right) - N(t; n)\right| \stackrel{a.s.}{=} O\left((\log n)^{-\lambda}\right). \tag{17}$$

Adding (13)-(16) and (17) we obtain (8). Theorem 2.3 is proved.

Proof of Corollary 2.1. From (3.9) in [6] for each $n \ge 1$ we have

$$\sup_{t \in R} |N(t; n)| \ \le \ C^* \cdot \sup_{t \in R} n^{-1/2} \cdot |K(t; n)| \stackrel{a.s.}{=} O\left(a_n\right), \tag{18}$$

where C^* is a positive constant and $\{K(t; s),\ (t, s) \in R \times [0, \infty)\}$ is the Kiefer process. Now Corollary 2.1 follows from (17) and (18).

References

[1] Abdushukurov A.A. Nonparametric estimation of distribution function based on relative risk function, *Commun.Statist.: Theory & Methods*, Vol.27, No.8, 1998, pp.1991-2012.

[2] Abdushukurov A.A. On nonparametric estimation of reliability indice by censored samples, *Theory Probab. Appl.*, Vol.49, No.1, 1999, pp.3-11.

[3] Cai Z., Roussas J.J. Uniform strong estimation under $\alpha-$mixing with rates. *Statistics & Probab. Letters.* 1992. Vol. 15. p.47-55.

[4] Cai Z. Asymptotic properties of Kaplan-Meier estimator for censored dependent data. *Statistics & Probab. Letters.* 1998. Vol. 37. p.381-389.

[5] Doukhan P. *Mixing: Properties and examples*, in: Lecture Notes in Statistics. Springer, Berlin. 1994.

[6] Dhompongsa S. A note on the almost sure approximation of the empirical process of weakly dependent random variables. *Yokohoma Math.* 1984. Vol.32. p.113-121.

[7] Fakoor V., Nanhai Rad N. Strong Gaussian Approximation of Product-Limit and Quantile Processes for Strong Mixing and Censored Data. *Commun. Statist.: Theory & Methods.* 2010. Vol. 39. p.2271-2279.

[8] Ibragimov I.A., Linnik Y.V. *Independent and Stationary Sequences of Random Variables.* Walters-Noordhoff, Groningen, the Netherlands. 1971.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON ESTIMATION OF SURVIVAL FUNCTION IN COX MODEL UNDER RANDOM RIGHT CENSORING AND ROBUST INFERENCE

RUSTAMJON MURADOV[1], ABDURAXIM ABDUSHUKUROV[2],
ABHIK GHOSH[3], AYANENDRANATH BASU[3]

[1]Namangan Institute of Engineering and Technology, Namangan, Uzbekistan
[2]Moscow State University Tashkent Branch, Tashkent, Uzbekistan
[3]Indian Statistical Institute, Kolkata, India

e-mails: rustamjonmuradov@gmail.com, aabdushukurov1710@gmail.com,abhik.ghosh@isical.ac.in, ayanbasu@isical.ac.in

Abstract. In this paper we present three types of parametric-nonparametric estimators for conditional survival function in Cox proportional hazards regression model when the lifetime of interest is subjected to random censorship from both sides. In the context of parametric proportional hazard model with randomly right censored responses, the idea of the minimum density power divergence estimator has been developed.

Keywords: Survival function; random censoring; covariates; robustness; Gaussian process.

AMS Subject Classification: 62H10, 62H12

1. Introduction and Notation

In the analysis of survival data one of widely applied regression models is the model of proportional hazards of Cox (Cox,1972;1975). Today, the Cox model is the most important model in survival analysis, reliability and quality of life research, clinical trials, epidemiology, and biomedical studies. There have also been widely applications of the Cox model in demography, econometrics, finance, pharmacology, biology, insurance, etc. According to this model, conditional cumulative hazard function(c.h.f.) of object tested for survival rate with life time Z satisfies to representation

$$\Lambda(x/\nu) = \Lambda_0(x)\exp((\beta^T, \nu)),\ x \geq 0, \tag{1}$$

where the baseline c.h.f. $\Lambda_0(x) = \Lambda(x/0)$ - is assumed to be continuous,

$$\Lambda(x/\nu) = \lim_{h\downarrow 0} P(Z \leq x + h/Z \geq x, V = \nu),$$

$V = (V_1..., V_p)-$ is the vector of covariates and $\beta = (\beta_1..., \beta_p)$ is the vector of regression parameters. In model of Cox the primary goal consists in estimation of baseline survival function

$$1 - H_0(x) = \exp(-\Lambda_0(x)),\ x \geq 0, \tag{2}$$

corresponding an initial condition of object at $V = O = (0, ..., 0)$, based on independent observations of vector $(Z; V)$. At present time there is an enormous literature on researches in this model. In particular, at research of abovementioned problem of estimating considered the cases, when a random variable (r.v.) Z either is observed completely, or is exposed only to censoring on the right (Bednarski,1993; Pons,2000; Tsiatis,1981). We consider more general scheme when Z is exposed to random censoring at the left and on the right by r.v.-s L and Y accordingly. In

This work was partially supported by grant no UZB-Ind-2021-97.

view of presence of censoring from the left the problem is reduced to estimation of conditional survival function $1 - H_{\tau 0}(x)$, where

$$H_{\tau 0}(x) = P(Z \le x / Z \ge \tau, V = O),\ x \ge \tau, \tag{3}$$

and number $\tau \in [0; \infty)$. Let us denote $a \wedge b = \min(a; b)$, $a \vee b = \max(a; b)$, $\bar{R}^+ = [0, \infty)$ and for right continuous function f jump at point x as $\triangle f(x) = f(x) - f(x-)$. Let $\pi(\nu) = P(V \le \nu) = P(V_1 \le \nu_1 ..., V_p \le \nu_p),\ \nu = (\nu_1 ..., \nu_p) \in R^p$,

$$H(x/\nu) = P(Z \le x / V = \nu),\ K(x/\nu) = P(L \le x / V = \nu),$$

$$G(x/\nu) = P(Y \le x / V = \nu),\ x \in \bar{R}^+,$$

- are joint distribution function(d.f.) of vector V and conditional d.f.-s of r.v.-s Z, L and Y correspondingly. It is easy to see that according to (1) and (2) the conditional d.f. of r.v. Z is:

$$\begin{aligned} H(x/\nu) = P(Z \le x / V = \nu) = 1 - (1 - H_0(x))^{\exp\left((\beta^T, \nu)\right)} = \\ = 1 - \exp\{-\exp\left((\beta^T, \nu)\right) \Lambda_0(x)\}. \end{aligned} \tag{4}$$

In considered model are assumed satisfied following a conditions:

(C1) Joint distribution of (Z, L, Y, V) is such, that components of random vector (Z, L, Y) are conditionally independent for given V.

(C2) Joint distribution of a vector (Z, L, Y, V) is such, that for numbers τ and T, $\tau < T$, $\Lambda_0(T) < \infty$ we have

$$\inf_{\tau \le x \le T} \int\limits_{R^p} P(L \le x \le Z \wedge Y / V = \nu) d\pi(\nu) > 0. \tag{5}$$

It is easy to see that according to (1) the condition (5) is equivalent to

$$\inf_{\tau \le x \le T} \int\limits_{R^p} K(x/\nu) \left(1 - G(x - /\nu)\right) \left(1 - H(x - /\nu)\right) d\pi(\nu) > 0.$$

(C3) The observation is available the vector $\lambda = (\zeta, L, \chi_1, \chi_2, \chi_3, V)$, where $\zeta = L \vee (Z \wedge Y)$, $\chi_1 = I(Z \wedge Y < L)$, $\chi_2 = I(L \le Y < Z)$ and $\chi_3 = I(L \le Z \le Y)$.

Notice, that r.v. Z is observed only in case of $\chi_3 = 1$ and cosidered model includes a censoring by a random interval (Abdushukurov, 2009, 2015).

2. Construction of Estimators and Main Results

Let $S^{(n)} = \{(\zeta_j, L_j, \chi_{1j}, \chi_{2j}, \chi_{3j}, V_j), j = 1..., n\}$ is the independent repeated sample of observations of a vector λ. In order to constructing estimators of conditional d.f. (3) we propose some auxiliary d.f.-s. It is easy to establish equality for conditional d.f. of r.v. ζ that

$$\begin{aligned} E(x/\nu) = P(\zeta \le x / V = \nu) = K(x/\nu) \left(1 - (1 - G(x/\nu)) \left(1 - H(x/\nu)\right)\right) = \\ = T_1(x/\nu) + T_2(x/\nu) + T_3(x/\nu), \end{aligned} \tag{6}$$

where conditional sub-distribution functions

$$\begin{aligned} T_1(x/\nu) = \int\limits_{[0,x]} \left(1 - (1 - G(u - /\nu)) \left(1 - H(u/\nu)\right)\right) dK(u/\nu) = \\ = P(\zeta \le x, \chi_1 = 1 / V = \nu), \end{aligned}$$

$$T_2(x/\nu) = \int\limits_{[0,x]} K(u/\nu)\,(1-H(u/\nu))\,dG(u/\nu) = P(\zeta \le x, \chi_2 = 1/V = \nu),$$

$$T_3(x/\nu) = \int\limits_{[0,x]} K(u/\nu)\,(1-G(u/\nu))\,dH(u/\nu) = P(\zeta \le x, \chi_3 = 1/V = \nu).$$

Then according to (6) the d.f. of r.v. ζ is $E(x) = P(\zeta \le x) = \int\limits_{R^p} E(x/\nu)d\pi(\nu) = T_1(x)+T_2(x)+T_3(x)$, where

$$T_m(x) = \int\limits_{R^p} T_m(x/\nu)d\pi(\nu),\ m = 1,2,3. \tag{7}$$

Let us introduce corresponding empirical estimators for $E(x)$, $T_m(x)$, $m = 1,2,3$ and $\pi(t)$ by sample $S^{(n)}$:

$$E_n(x) = \frac{1}{n}\sum_{j=1}^{n} I(\zeta_j \le x) = \sum_{m=1,2,3} T_{mn}(x),\ x \in \bar{R}^+,$$

$$T_{mn}(x) = \frac{1}{n}\sum_{j=1}^{n} I(\zeta_j \le x, \chi_{mj} = 1),\ m = 1,2,3;\ x \in \bar{R}^+,$$

$$\pi_n(t) = \frac{1}{n}\sum_{j=1}^{n} I(V_{1j} \le t_1 ..., V_{pj} \le t_p),\ t \in R^p.$$

First we consider a case of one-dimensional covariates ($p = 1$). Let

$$\omega^{(s)}(u;\beta) = \int\limits_{R} \nu^s \exp(\beta\nu) P(L \le u \le Z \wedge Y/V = \nu)d\pi(\nu),\ s = 0,1,2.$$

It is easy to check up, that by definition of $T_3(x/\nu)$ and according to (1):

$$T_3(x/\nu) = \int\limits_{[0;x]} P(L \le u \le Z \wedge Y/V = \nu)d\Lambda(u/\nu) =$$

$$= \exp(\beta\nu) \int\limits_{[0;x]} P(L \le u \le Z \wedge Y/V = \nu)d\Lambda_0(u). \tag{8}$$

Integrating equality (8) on $\pi(\nu)$, and then having resolved the received integrated equation with respect to $\Lambda_0(x)$, we have following expression for $\Lambda_{\tau 0}(x) = \Lambda_0(x) - \Lambda_0(\tau-)$:

$$\Lambda_{\tau 0}(x) = \Lambda_{\tau 0}(x;\beta) = \int\limits_{[\tau;x]} \frac{dT_3(u)}{\omega^{(0)}(u;\beta)}. \tag{9}$$

Let us enter a following, intermediate estimator for c.h.f. (9), depending on β :

$$\Lambda^{(n)}_{\tau 0}(x) = \Lambda^{(n)}_{\tau 0}(x;\beta) = \int\limits_{[\tau;x]} \frac{dT_{3n}(u)}{\omega^{(0)}_n(u;\beta)}. \tag{10}$$

Here $\omega^{(0)}_n(u;\beta)$ an estimator for $\omega^{(0)}(u;\beta)$, defined at $s = 0$ from the formula

$$\omega^{(s)}_n(u;\beta) = \frac{1}{n}\sum_{j=1}^{n} \nu_j^s \exp(\beta\nu_j) I(L_j \le u \le Z_j \wedge Y_j),\ s = 0,1,2.$$

Let us enter the truncated version of likelihood function of sample $S^{(n)}$:

$$L_n(\beta)=\prod_{\substack{i=1\\ \zeta_i\le T}}^{n}\left\{\frac{\exp(\beta\nu_i)}{\sum\limits_{j=1}^{n}\exp(\beta\nu_j)I(L_j\le\zeta_i\le Z_j\wedge Y_j)}\right\}^{\chi_{3i}},$$

where $T=\sup\{x:E(x)<1\}$. Parameter β we shall estimate by solving the following log-likelihood equation:

$$U_n^T(\beta)=\frac{d\log L_n(\beta)}{d\beta}=\sum_{i=1}^{n}\int\limits_{[0;T]}\left(\nu_i-\frac{\omega_n^{(1)}(u;\beta)}{\omega_n^{(0)}(u;\beta)}\right)dN_i(u)=0, \tag{11}$$

where $N_i(u)=I(L_i\le Z_i\le Y_i\wedge u)-$ is the counting process. Let $\hat\beta_n$ is a solution of the equation (11). Then a corresponding estimation for $\Lambda_{\tau 0}(x)$ we shall receive by substitution: $\hat\Lambda_{\tau 0}^{(n)}(x)=\Lambda_{\tau 0}^{(n)}(x;\hat\beta_n)$, $x\ge\tau$. For $1-H_{\tau 0}(x)$ we propose the following three types of estimators:

$$1-\hat H_{\tau 0}^{(n)}(x;m)=\psi_{nm}(x;\hat\beta_n),\ m=1,2,3;\ x\ge\tau, \tag{12}$$

where

$$\begin{aligned}\psi_{n1}(x;\beta)&=\exp\left(-\Lambda_{\tau 0}^{(n)}(x;\beta)\right),\\ \psi_{n2}(x;\beta)&=\prod_{\tau\le u\le x}\left(1-\Delta\Lambda_{\tau 0}^{(n)}(x;\beta)\right),\\ \psi_{n3}(x;\beta)&=\left[\frac{\omega_n^{(0)}(x;\beta)}{\omega_n^{(0)}(\tau;\beta)}\right]^{R_n(x;\beta)},\end{aligned} \tag{13}$$

and $R_n(x;\beta)=\Lambda_{\tau 0}^{(n)}(x;\beta)\left(-\int\limits_{[\tau;x]}\frac{d\omega_n^{(0)}(u;\beta)}{\omega_n^{(0)}(u;\beta)}\right)^{-1}$ (assume that $\frac{0}{0}=0$ and $\frac{\infty}{\infty}=1$). Note that the estimators (12) are the analoques of exponential hazard of Breslow, product-limit of Kaplan-Meier and relative risk power estimators of Abdushukurov (Abdushukurov,1998;1999;2011;2015). Such three types of estimators of d.f.-s in several mixed competing risks and random censoring models are investigated in monography (Abdushukurov,2009). The classical model of Cox described by representation c.h.f. by formula (1) is parametrical-nonparametrical where parameter of regression β and baseline c.h.f. Λ_0 are unknown. Therefore estimation procedure is spent in two stages: at first we construct the intermediate estimators $\Lambda_{\tau 0}^{(n)}(x;\beta)$ by formula (10) for conditional c.h.f. $\Lambda_{\tau 0}(x)$ at fixed β, and then solving the equation (11) we find the maximum likelihood estimator(MLE) $\hat\beta_n$ and by substitution MLE $\hat\beta_n$ instead of β we obtain the final estimator $\Lambda_{\tau 0}^{(n)}(x;\hat\beta_n)$ for c.h.f.. Therefore, for research of estimators (12) for conditional baseline survival function $1-H_{\tau 0}(x)$, at first we investigate properties of estimators $\Lambda_{\tau 0}^{(n)}(x;\beta)$ and $\hat\beta_n$.

In Theorem 1 we declare the uniform consistency of $\Lambda_{\tau 0}^{(n)}(x;\beta)$. Let us denote $\mathfrak{L}_n(x)=\mathfrak{L}_n(x;\beta)=\Lambda_{\tau 0}^{(n)}(x;\beta)-\Lambda_{\tau 0}(x;\beta)$.

Theorem 1. In conditions (C1) - (C3) at $n\to\infty$

$$\sup_{\tau\le x\le T\wedge\zeta_{(n)}}|\mathfrak{L}_n(x)|\xrightarrow{p}0. \tag{14}$$

Property of a consistency of an estimator $\hat\beta_n$ contains in following theorem.

Theorem 2. Assume the conditions (C1) - (C3) are satisfied. Moreover let d.f. $\pi(\nu)$ is absolutely continuous and

$$\sup_{t\in B(\varepsilon_0;\beta)} E[V\exp(tV)]^2 \le K < \infty, \tag{15}$$

where $B(\varepsilon_0;\beta)-$ is the interval of lengths $2\varepsilon_0$ with the center at a point β (the true value). Then the equation (11) has a solution $\{\hat{\beta}_n, n \ge 1\}$ and at $n \to \infty$

$$\hat{\beta}_n \to \beta \text{ a.s.} \tag{16}$$

We define information function

$$\Im(\beta) = \int\limits_{[0;T]} \left[\frac{\omega^{(2)}(u;\beta)}{\omega^{(0)}(u;\beta)} - \left(\frac{\omega^{(1)}(u;\beta)}{\omega^{(0)}(u;\beta)}\right)^2\right] dT_3(u),$$

and its intermediate estimator

$$\Im_n(\beta) = \int\limits_{[0;T]} \left[\frac{\omega_n^{(2)}(u;\beta)}{\omega_n^{(0)}(u;\beta)} - \left(\frac{\omega_n^{(1)}(u;\beta)}{\omega_n^{(0)}(u;\beta)}\right)^2\right] dT_{3n}(u.)$$

Let us introduce following condition:

(C4) Functions $\{\omega^{(k)}(u;t), k = 0,1,2\}$ are uniform continuous functions on $u \in [0;T]$ and $t \in B(\varepsilon_0;\beta)$, and also are bounded on $[0;T] \times B(\varepsilon_0;\beta)$.

Theorem 3. Assume the conditions (C1) - (C4) and (15) are satisfied. Then at $n \to \infty$

$$n^{\frac{1}{2}}(\hat{\beta}_n - \beta) \xrightarrow{D} N\left(0;(\Im(\beta))^{-1}\right). \tag{17}$$

Remark 1. It is easy to see that in conditions of the Theorem 3 as consistent estimator of $\Im(\beta)$ statistics $\Im_n(\hat{\beta}_n)$ can also be used.

Remark 2. Note, that the solution $\hat{\beta}_n$ of equation (11) is consistent in Fisher sense, as to those is functional $U_n^T(\beta.)$ This property is easy established by repeating reasonings from (Bednarski, 1993).

Now we start investigate the basic estimator $\hat{\Lambda}_{\tau 0}^{(n)}(x) = \Lambda_{\tau 0}^{(n)}(x;\hat{\beta}_n)$, $x \ge \tau,$, for c.h.f. (9). At the same time with process $\mathfrak{L}_n(x)$ we consider also processes$\mathfrak{L}_n^*(x) = \hat{\Lambda}_{\tau 0}^{(n)}(x) - \Lambda_{\tau 0}(x;\beta)$ and $\mathcal{L}_n(x) = \hat{\Lambda}_{\tau 0}^{(n)}(x) - \Lambda_{\tau 0}^{(n)}(x;\beta)$, $x \ge \tau$.

Theorem 4. In conditions of Theorem 3:

$$\sup_{\tau \le x \le T\wedge\zeta_{(n)}} |\mathfrak{L}_n^*(x)| \xrightarrow[n\to\infty]{p} 0. \tag{18}$$

$$n^{\frac{1}{2}}\mathfrak{L}_n^*(x) \xrightarrow[n\to\infty]{D} \mathfrak{L}_0(x) \text{ in } D[\tau;T]. \tag{19}$$

Here Gaussian process $\{\mathfrak{L}_0(x), x \ge \tau\}$ is represented on distribution as $\mathfrak{L}_0(x) \stackrel{D}{=} \gamma_1(x) - V(x;\beta)\gamma_0$, where $\gamma_0 = N(0;(\Im(\beta))^{-1})$, and $\gamma_1(x)-$ is Gaussian process with independent increments with $E\gamma_1(x) = 0$, and for $x, y \in [\tau;T]$:

$$E\gamma_1(x)\gamma_1(y) = \int\limits_{[\tau;x\wedge y]} \frac{d\Lambda_0(u)}{\omega^{(0)}(u;\beta)} \text{ and } V(x;\beta) = \int\limits_{[\tau;x]} \frac{\omega^{(1)}(u;\beta)}{\omega^{(0)}(u;\beta)} d\Lambda_0(u).$$

Let us consider three classes of estimators defined by formulas (12) and (13) for the truncated survival functions $1 - H_{\tau 0}(x)$. As intermediate we shall consider estimators $1 - H_{\tau 0}^{(n)}(x;\beta;m) = \Psi_{nm}(x;\beta)$, $m = 1,2,3$ by using functionals (13). Let $1 - H_\tau(x/\nu_0) = P(Z \ge x/Z \ge \tau, V = \nu_0), x \ge \tau$, conditional truncated from the left survival function of object with lifetime Z and fixed covariate ν_0. In view of (4)

$$1 - H_\tau(x/\nu_0) = (1 - H_{\tau 0}(x))^{\exp(\beta\nu_0)}\,,\ x \geq \tau. \tag{20}$$

According to representation (20) we construct estimators for $1 - H_\tau(x/\nu_0)$ by following three types of statistics (see (12)):

$$1 - \hat{H}^{(n)}_{\tau m}(x/\nu_0) = \left(\Psi_{nm}(x;\hat{\beta}_n)\right)^{\exp(\hat{\beta}_n\nu_0)}\,,\ x \geq \tau,\ m = 1,2,3. \tag{21}$$

In the following theorem is established asymptotical equivalence of three estimators (21).

Theorem 5. Assume the conditions of the Theorem 3 are hold and in case of $m = 3$ the continuity of d.f.-s $K(x/\nu)$ and $G(x/\nu)$ on $x \in [\tau;T]$ is required. Then for each $m = 1,2,3$:

I) $\sup\limits_{\substack{\tau\leq x\leq T\\ x<\zeta_{(n)}}} \left|\hat{H}^{(n)}_{\tau m}(x/\nu_0) - H_\tau(x/\nu_0)\right| \xrightarrow[n\to\infty]{p} 0$;

II) $n^{\frac{1}{2}}\left(\hat{H}^{(n)}_{\tau m}(x/\nu_0) - H_\tau(x/\nu_0)\right) \xrightarrow[n\to\infty]{D} \chi_{\tau\nu_0}(x)$ in $D[\tau;T]$.

Here $\chi_{\tau\nu_0}(x)-$ is centered Gaussian process with covariance for $x, y \geq \tau$:

$$M\chi_{\tau\nu_0}(x)\chi_{\tau\nu_0}(y) = (1 - H_\tau(x/\nu_0))\,(1 - H_\tau(y/\nu_0))\,\Gamma_\tau(x, y/\nu_0),$$

where

$$\Gamma_\tau(x, y/\nu_0) = \exp(2\beta\nu_0)\left\{\int\limits_{[\tau;x\wedge y]} \frac{d\Lambda_0(u)}{\omega^{(0)}(u;\beta)} + \right.$$

$$\left. + (\Im(\beta))^{-1}\left[\int\limits_{[\tau;x]}\left(\frac{\omega^{(1)}(u;\beta)}{\omega^{(0)}(u;\beta)} - \nu_0\right)d\Lambda_0(u)\right]\left[\int\limits_{[\tau;y]}\left(\frac{\omega^{(1)}(u;\beta)}{\omega^{(0)}(u;\beta)} - \nu_0\right)d\Lambda_0(u)\right]\right\}.$$

Remark 3. Note, that limiting Gaussian process $\mathfrak{L}_0(x)$ from the Theorem 4 is same structures as corresponding process in the case of censoring from the right (see (Tsiatis,1981)). Then repeating calculations from the Appendix 2 in (Tsiatis,1981) it is possible to show, that r.v. γ_0 and process $\gamma_1(x)$, $x \in [\tau;T]$ are independent.

3. Robust inference under the Parametric proportional hazard models

One of the most widely used statistical tool for analyzing survival data is the famous Cox proportional hazard regression model, which helps us to examine the relationship of different important covariates with a randomly right censored response measuring some lifetime. The random right censored observations are extremely common in different practical application including those from medical sciences. Mathematically, we can represent such randomly right censored observations as $X_i = \min\{T_i, C_i\}$ for each $i = 1,\ldots,n$, where n is the sample size, $T_1,\ldots,T_n$ are independent and identically distributed (IID) true lifetimes for the i-th entry (e.g., patients, item on test, etc.) and $C_1,\ldots,C_n$ are n IID censoring time following a distribution independent of that of T_is. We also know which observations are censored with data $\delta_i = I(T_i \leq C_i)$ for each $i = 1,\ldots,n$. When we additionally have p covarites corresponding to each entry as given by $\nu_1,\ldots,\nu_n \in \mathbb{R}^p$. The semi-parametric Cox regression model is then given by specifying the hazard of the i-th entry/lifetime as

$$\lambda_i(t) = \lambda(t|\nu_i) = \lambda_0(t)e^{\boldsymbol{\beta}^T\nu_i},\ i = 1,\ldots,n, \tag{22}$$

where $\boldsymbol{\beta} \in \mathbb{R}^p$ are the target parameter (called regression coefficients) indicating the covariate effects on censored response, but baseline hazard remains unspecified as an unknown function λ_0 making the model semi-parametric.

However, whenever there are evidences from the data about a possible parametric model for the baseline hazard (e.g., exponential, Weibull, etc.), we can make more efficient inference by assuming a fully parametric version of the Cox regression model in (22) which specifies the the hazard of the i-th entry/lifetime as

$$\lambda_{i,\boldsymbol{\theta}}(t) = \lambda_{\boldsymbol{\theta}}(t|\nu_i) = \lambda(t,\boldsymbol{\gamma})e^{\boldsymbol{\beta}^T\nu_i},\ i = 1,\ldots,n,\ \boldsymbol{\theta} = (\boldsymbol{\gamma}^T,\boldsymbol{\beta}^T)^T \in \mathbb{R}^{q+p}, \tag{23}$$

where $\lambda(t,\boldsymbol{\gamma})$ denotes the assumed parametric baseline hazard function. In such cases, which is commonly referred to as the (fully) parametric proportional hazard (PPH) model, the inference about the unknown parameters are classically performed by the most efficient maximum likelihood estimator and associated likelihood based tests (e.g., likelihood ratio test or Wald test). However, modern complex datasets often contains some amount of data contamination (e.g., outliers) that is known to severely affects the likelihood based inference leading to incorrect inference. So, it is extremely important to develop and use a robust inference procedures to generate correct and stable insights even in the presence of data contamination and we will discuss one such robust yet highly efficient methodology in the present paper.

Our robust parameter estimator is derived based on the popular approach of minimum divergence method of robust inference (see, e.g., Basu et al., 2011, for details) utilizing the nice properties of a particular divergence measure known as the density power divergence (DPD) originally proposed by Basu et al. (1998). The DPD measure between two densities f and g are given by

$$d_\alpha(g,f) = \int f^{1+\alpha} - \frac{1+\alpha}{\alpha}\int f^\alpha g + \frac{1}{\alpha}\int g^{1+\alpha},$$

where $\alpha > 0$ is a tuning parameter controlling the trade-offs between robustness and efficiency of the resulting inference. In particular, as $\alpha \to 0$, the DPD measure coincides (in a limiting sense) with the Kullback-Leibler divergence and hence, the resulting inference exactly the same as the non-robust likelihood based inference. We achieve greater robustness by using the minimum DPD estimator (MDPDE) and associated DPD or MDPDE based tests with increasing values of $\alpha > 0$, but often a small positive value of α suffices to produce results with desired robustness/stability along with a little loss in (asymptotic) efficiency. Please refer to Basu et al. (1998, 2011) for more details about the minimum DPD inference.

In the context of PPH model (23) with randomly right censored responses, the idea of MDPDE has been developed and explored by Ghosh and Basu (2019). As described there, we can define the MDPDE of the parameters $\boldsymbol{\theta}$ under the PPH model (23), based on the sample data $\{X_i, \delta_i, \nu_i : i = 1,\ldots,n\}$, as the minimizer of the following objective function over the whole parameter space:

$$H_{n,\alpha}(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n}\left[\int f_{i,\boldsymbol{\theta}}^{1+\alpha} - \left(1+\frac{1}{\alpha}\right)\lambda_{i,\boldsymbol{\theta}}(X_i)^{\alpha\delta_i}S_{i,\boldsymbol{\theta}}(X_i)^\alpha\right],$$

where, for each $i = 1,\ldots,n$, we define the associated model density and survival functions respectively as given by

$$f_{i,\boldsymbol{\theta}}(x,\delta) = [\lambda_{i,\boldsymbol{\theta}}(x)]^\delta S_{i,\boldsymbol{\theta}}(x),\ S_{i,\boldsymbol{\theta}}(x) = \exp\left[-\Lambda_{\boldsymbol{\gamma}}(x)e^{\boldsymbol{\beta}^T\nu_i}\right],$$

with $\Lambda_{\boldsymbol{\gamma}}(t) = \int_0^t \lambda(s,\boldsymbol{\gamma})ds$ denoting the cumulative baseline hazard as per model (23). That the resulting MDPDE works well, robustly and efficiently as desired, under different practical situations is explored in detail in Ghosh and Basu (2019); they have also studied several asymptotic properties of the MDPDE under appropriate assumptions under the PPH model set-up. Their final results on the asymptotic consistency and normality of the MDPDE is replicated below from Ghosh and Basu (2019).

Theorem 6. Consider the set-up of randomly right censored data as mentioned above along with the PPH model assumption (23). Then, under suitable regularity conditions depending

on $\alpha \geq 0$, the MDPDE $\widehat{\boldsymbol{\theta}}_{n,\alpha}$ exists, is consistent for the "best fitting parameter" value $\boldsymbol{\theta}^g_\alpha$ and asymptotically normal with

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_{n,\alpha} - \boldsymbol{\theta}^g_\alpha\right) \xrightarrow{\mathcal{D}} N_{p+q}\left(\mathbf{0}_{q+p}, \boldsymbol{V}_\alpha(\boldsymbol{\theta}^g_\alpha)^{-1}\boldsymbol{K}_\alpha(\boldsymbol{\theta}^g_\alpha)\boldsymbol{V}_\alpha(\boldsymbol{\theta}^g_\alpha)^{-1}\right),$$

where $\boldsymbol{V}_\alpha$ and $\boldsymbol{K}_\alpha$ are as defined in the Supplementary Material of Ghosh and Basu (2019).

Once we know the asymptotic variance of the MDPDE from the above theorem, we can use to to perform further inference, like hypothesis testing and model selection, for the PPH model (23) using the robust MDPDE and a consistent estimates of its asymptotic variance matrix. These are explored in details by Nandy et al. (2020) who have developed robust Wald-type hypothesis testing procedures as well as a DPD based divergence information criterion for model selection under the PPH model (23) for randomly right censored lifetimes. These proposed MDPDE and associated inference procedures are seen to provide better accurate and stable insights in several real data examples, as well as different practically relevant simulation studies. So, we strongly recommend their usage in future applied researches involving randomly right censored data in survival analyses.

References

[1] Abhik Ghosh and Ayanendranath Basu, *Robust and efficient estimation in the parametric proportional hazards model under random censoring*, *Statistics in Medicine* Vol.38, No.27, 2019, pp.5283–5299.

[2] Abdushukurov A.A., Estimation of Survival Function in Cox Regression Model Under Random Censoring From Both Sides, *Commun. Stat.-Theory Methods*, Vol.44, No.3, 2015, pp.533-553.

[3] Abdushukurov A.A. Statistics of incomplete observations: Asymptotical theory for nonclassical models. Tashkent. National University of Uzbekistan. University Press. 2009.-267p. (In Russian).

[4] Abdushukurov A.A. Nonparametric estimation of the distribution function based on relative risk function. Commun. Statist.: Theory & Meth. 1998. v.27. N.8.p. 1991-2012.

[5] Abdushukurov A.A. On nonparametric estimation of reliability indices by censored samples. Theory Probab. Appl. 1999.v.43.N.1.p.3-11.

[6] Abdushukurov A.A. Nonparametric estimation based on incomplete observation. International Encyclopedia of Statistical Sciences.Springer 2011. Part 14. p. 962-964.

[7] Abdushukurov A.A. Estimation of survival function in Cox regression model under random censoring from both sides. Commun. in Statist.: Theory & Meth. 2015. v.44. p. 533-553.

[8] Amarnath Nandy, Abhik Ghosh, Ayanendranath Basu and Leandro Pardo, Robust Hypothesis Testing and Model Selection for Parametric Proportional Hazard Regression Models. arXiv preprint, 2020 arXiv:2009.12686.

[9] Ayanendranath Basu, Ian R. Harris, Nils L. Hjort and M. C. Jones., Robust and efficient estimation by minimising a density power divergence. Biometrika, 1989 N 85(3). p. 549–559.

[10] Ayanendranath Basu, Hiroyuki Shioya, and Chanseok Park,Statistical inference: the minimum distance approach. CRC press, 2011.

[11] Bednarski T. Robust estimation in Coxs regression model.- Scand. J. Statist. 1993. v.20. N3. p.213-225.

[12] Cox D.R. Regression models and life-tables. (with discussion) . J. Royal Statist. Soc. 1972. B.34. p.187-220.

[13] Cox D.R. Partial likelihood.-Biometrica. 1975. v.62. p.269-276.

[14] Gill R.D. Censoring and stochastic integrals. Math. Centre Tracts. N124: Math. Centrum Amsterdam. 1980.-178p.

[15] Hoeffding W. Probability inequalities for sums of bounded random variables. J.A.S.A. 1963. v.58. p.13-30.

[16] Pons O. Nonparametric estimation in a varying-coefficient Cox model.-Math. Meth. Statist. 2000. v.9. N4. p.376-398.

[17] Tsiatis A.A. A large sample study of a Coxs regression model.-Ann. Statist. 1981. v.9. N1. p.93-106.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON ESTIMATION OF MULTIVARIATE SURVIVAL FUNCTION WITH RANDOM CENSORING IN THE PRESENT OF PROGNOSTIC FACTORS

RUSTAMJON MURADOV[1], ABDURAXIM ABDUSHUKUROV[2], BISWABRATA PRADHAN [3]

[1]Namangan Institute of Engineering and Technology, Namangan, Uzbekistan
[2]Moscow State University Tashkent Branch, Tashkent, Uzbekistan
[3]Indian Statistical Institute, Kolkata, India

e-mails: rustamjonmuradov@gmail.com, aabdushukurov1710@gmail.com, bis@isical.ac.in

Abstract. The problem consist in estimation of conditional jointly survival function by right random censored observations in the presence of prognostic factors. We propose two estimators of conditional jointly survival function and study its large sample properties. We show an asymptotic equivalence of estimators. We derive an almost sure representation results for estimators. Also, present result of asymptotic normality with tending to the same limiting Gaussian process.

Keywords: Jointly survival function; random censoring; prognostic factors; copula function; Gaussian process.

AMS Subject Classification: 62H10, 62H12

1. Introduction

In such research areas as bio-medicine, engineering, insurance, social sciences researchers are interested in positive variables, which are expressed as a time until a certain event. For example, in medicine the survival time of individual, while in industrial trials, time until breakdown of a machine are non-negative random variables (r.v.-s) of interest. But in such practical situations, the observed data may be incomplete, that is censored. This is the case, for example, in medicine when the event of interest-death due to a given cause and the censoring event is death due to other cause. In industrial study, it may occur that some piece of equipment is taken away (that is censored) because it shows some sign of future failure. Moreover, the r.v.-s of interest (lifetimes, failure times) and censoring r.v.-s usually can be influenced by other variable, often called prognostic factor(covariate). In medicine, dose of a drug and in engineering some environmental conditions (temperature, pressure) are influenced to the observed variables. The basic problem consist in estimation of distribution of lifetime by such censored dependent data. The aim of paper is considering this problem in the case of right random censoring model in the presence of prognostic factor.

2. A short introduction to the concept of copulas

Without a doubt the dependence relations between random variables plays a very important role in many fields of mathematics and is one of the most widely studied subjects in probability and statistics. M. Fr´echet and G. Dall'Aglio (see Nelsen, 1999) did some interesting works about this matter in the fifties, studying the bivariate and trivariate distribution functions (d.f.-s) with given univariate margins. The answer to this problem for the univariate margins case

This work was partially supported by grant no UZB-Ind-2021-97.

was given by A. Sklar creating a new class of functions which he called copulas. The concept of copulas was introduced in 1959 to study the linkage between multivariate distribution functions and their univariate marginals. Since then, copulas have gained growing importance as a tool for modeling statistical dependence of random variables in many fields. We begin with a short introduction to the concept of copulas. For more details with an emphasis on the statistical and mathematical foundations of copulas see Nelsen (1999,2006).

Definition 1. A copula $C(u,v):[0,1]^2 \to [0,1]$ is a bivariate distribution function with uniform marginals.

A first example of copulas is the product copula $C(u,v) = uv$, which characterizes independent r.v.-s when the d.f.-s are continuous. The importance of copulas in statistics is described in Sklar's Theorem.

Theorem 1. (Sklar, 1959). Let H be a joint d.f. with margins F and G. Then there exists a copula C such that for all x, y in R,

$$H(x,y) = C(F(x), G(y)). \tag{1}$$

If F and G are continuous, then C is unique; otherwise, C is uniquely determined on $Ran(F) \times Ran(G)$. Conversely, if C is a copula and F and G are d.f.-s, then the function H defined by (1) is a joint d.f. with margins F and G. Thus copulas link joint d.f.-s to their one-dimensional margins. The representation (1) suggests that if the copula C were known, then substituting continuous marginal estimators for F and G would yield a plug-in estimate of their associated joint d.f. H. Moreover, in light of Sklar's result with arrive at the following functional definition of a copula.

Definition 2. Given a bivariate d.f. H with marginals F and G, the function defined as

$$C(u,v) = H\left(F^{-1}(u), G^{-1}(v)\right),$$

for $(u,v) \in [0,1]^2$, where $F^{-1}(u)$ and $G^{-1}(v)$ are the inverse functions of F and G respectively, is the copula corresponding to H.

In many applications, the r.v.-s of interest represent the lifetimes of individuals or objects in some population. The probability of an individual living or surviving beyond time x is given by the survival function $S(x) = P(X > x) = 1 - F(x)$, where, as before, F denotes the d.f. of X. Let C be the copula function of the bivariate distribution of (X,Y). We have

$$\bar{H}(x,y) = P(X > x, Y > y) = 1 - F(x) - G(y) + H(x,y) =$$

$$= S(x) + S(y) - 1 + C(1 - S(x), 1 - S(y)) = C^*(S(x), S(y)),$$

where $C^*(u,v) = u + v - 1 + C(1-u, 1-v)$-survival copula function.

Let φ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$such that $\varphi(1) = 0$.

Definition 3. The pseudo-inverse of φ is the function $\varphi^{[-1]}$ with $Dom\varphi^{[-1]} = [0,\infty]$ and given by

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & 0 < t < \varphi(0), \\ 0, & \varphi(0) \le t \le \infty. \end{cases}$$

Note that $\varphi^{[-1]}$ is continuous and no increasing on $[0,\infty]$, and strictly decreasing on $[0,\varphi(0)]$. Furthermore,

$$\varphi\left(\varphi^{[-1]}(t)\right) = \begin{cases} t, & 0 < t < \varphi(0), \\ 0, & \varphi(0) \le t \le \infty, \end{cases} = \min(t, \varphi(0)).$$

If $\varphi(0) = \infty$, then $\varphi^{[-1]} = \varphi^{-1}$.

Definition 4. Copulas of the form $C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$are called Archimedean copulas, where the function φ is called a generator of the copula $\varphi(1) = 0$.

3. Description of the model and asymptotic properties of estimators

The problem of estimation of multivariate distribution (or survival) function from incomplete data is considered with beginning of 1980's (Campbell (1981), Campbell & Földes (1982), Hanley & Parnes (1983), Horváth (1983), Tsay, Leurgang & Crowley (1986), Burke (1988), Dabrowska (1988, 1989), Gill (1992), Huang (2000), Pradhan (2010,2019), Abdushukurov (2004, 2011, 2022) etc.). In the special bivariate case, there are the numerous examples of paired data representing the times to death of individuals (twins or married couples), the failure times of components of system and others which subject to random censoring with possible dependence between the two censoring variables. At present time there are several approaches to estimation of survival functions of vectors of life times. However, some of these estimators either are inconsistent or not fully defined in range of joint survival functions and hence not applicable in practice. Almost all of these estimators have an exponential or product structures. In this work we present other type of estimator of power structure for bivariate survival function $F(t,s)$ of bivariate lifetime vector, which is censored from the right by censoring vector of random variables. We prove weak convergence and strong consistency results for estimators. The authors consider the problem of estimation of multivariate survival functions in dependent models of random censoring using copula functions and in the presence of prognostic factors(covariates).

Let's consider the case when the support of prognostic factors C is the interval $[0,1]$ and we consider our results on fixed design points $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1$. Thus, on a probability space $(\Omega,\, \mathcal{A},\, P)$let's consider two sequences $\mathbb{X} = \{(X_{1i},\, X_{2i}),\, i \geq 1\}$ and $\mathbb{Y} = \{(Y_{1i},\, Y_{2i}),\, i \geq 1\}$ of conditionally independent random vectors with conditional survival functions $F_x(t,s) = P(X_{11} > t,\, X_{21} > s/C = x)$ and $G_x(t,s) = P(Y_{11} > t,\, Y_{21} > s/C = x)$ at given prognostic factors $C = x$, where $x \in [0,1]$ and $(t,s) \in \bar{R}^{+2} = [o,\infty) \times [o,\infty)$. Here sequences $\mathbb{X}$ and $\mathbb{Y}$ can be dependent. Introduce a marginal conditional survival functions

$$S_{1x}^{X}(t) = P(X_{11} > t/C = x),\, S_{2x}^{X}(s) = P(X_{21} > s/C = x),$$

$$S_{1x}^{Y}(t) = P(Y_{11} > t/C = x),\, S_{2x}^{Y}(s) = P(Y_{21} > s/C = x), \tag{2}$$

which supposed to be continuous, $x \in [0,1]$, $t,s \geq 0$. Suppose that vector $\mathbb{X}$ is censored from the right by vector $\mathbb{Y}$ and observed data is consist on $\mathbb{V}^{(n)} = \{(Z_i,\, \delta_i, C_i),\ i = 1,\, \ldots, n\}$, where $Z_i = (Z_{1i}, Z_{2i})$, $Z_{ki} = \min(X_{ki}, Y_{ki})$, $\delta_i = (\delta_{1i}, \delta_{2i})$, $\delta_{ki} = I(Z_{ki} = X_{ki})$, $k = 1,2$; $i = \overline{1,\, n}$. Here $I(A)$ is stands for an indicator of event A. Introduce jointly conditional survival function of the vector $(X_{11}, X_{21,}Y_{11}, Y_{21})$ given $C = x$:

$$K_x(t,s,z,v) = P(X_{11} > t, X_{21} > s, Y_{11} > z, Y_{21} > v/C = x),\ (t,s,z,v) \in \overline{R}^{+4}.$$

Then we obtain representations at $x \in [0,1]$:

$$F_x(t,s) = K_x(t,s,0,0),\quad G_x(t,s) = K_x(0,0,t,s),\ (t,s) \in \overline{R}^{+2}, \tag{3}$$

$$H_x(t,s) = P(Z_{11} > t, Z_{21} > s/C = x) = P(X_{11} > t, Y_{11} > t, X_{21} > s, Y_{21} > s/C = x) =$$

$$= K_x(t,s,\, t,s),\ (t,s) \in \overline{R}^{+2}. \tag{4}$$

Then according to the Theorem of Sklar (see, Nelson (1999)), there exist on $[0,1]^4$ conditional copula survival function $C_x(\overline{u})$, $\overline{u} = (u_1,u_2,u_3,u_4) \in [0,1]^4$ such that we have following representation of survival function (3) for all $(t,s,z,v) \in \overline{R}^{+4}$ as

$$K_x(t,s,z,v) = C_x\left(S_{1x}^{X}(t), S_{2x}^{X}(s), S_{1x}^{Y}(z), S_{2x}^{Y}(v)\right). \tag{5}$$

Assume that at the fixed design value $x \in [0,1]$, C_x is Archimedean copula, i.e.

$$C_x(u_1,u_2,u_3,u_4) = \varphi^{[-1]}\left[\varphi_x(u_1) + \varphi_x(u_2) + \varphi_x(u_3) + \varphi_x(u_4)\right], \bar{u} = (u_1,u_2,u_3,u_4) \in [0,1]^4, \tag{6}$$

where, for each x, $\varphi_x : [0,1] \to [0,\infty]$ is a known continuous, convex, strictly decreasing function with $\varphi_x(1) = 0$, $\varphi_x^{[-1]}$ is a pseudo-inverse of φ_x (see, [2]) and given by

$$\varphi_x^{[-1]}(s) = \begin{cases} \varphi_x^{-1}(s), & 0 \le s \le \varphi_x(0), \\ 0, & \varphi_x(0) \le s \le \infty. \end{cases}$$

We assume that copula generator function φ_x is strict, i.e. $\varphi_x(0) = \infty$ and hence $\varphi_x^{[-1]} = \varphi_x^{-1}$. Thus from formulas (3)-(6) one can get for a fixed $x \in [0,1]$

$$F_x(t,s) = \varphi_x^{-1}\left[\varphi_x\left(S_{1x}^X(t)\right) + \varphi_x\left(S_{2x}^X(s)\right)\right],\ (t,s) \in \overline{R}^{+2},$$
$$G_x(t,s) = \varphi_x^{-1}\left[\varphi_x\left(S_{1x}^Y(t)\right) + \varphi_x\left(S_{2x}^Y(s)\right)\right],\ (t,s) \in \overline{R}^{+2}, \tag{7}$$

and

$$H_x(t,s) = \varphi_x^{-1}\left[\varphi_x\left(S_{1x}^X(t)\right) + \varphi_x\left(S_{2x}^X(s)\right) + \varphi_x\left(S_{1x}^Y(t)\right) + \varphi_x\left(S_{2x}^Y(s)\right)\right],\ (t,s) \in \overline{R}^{+2},$$

Since

$$S_{1x}^Z(t) = P(Z_{11} > t/C = x) = H_x(t;0) = P(X_{11} > t,\ Y_{11} > t/C = x),\ t \ge 0,$$

and

$$S_{2x}^Z(s) = P(Z_{21} > s/C = x) = H_x(0,s) = P(Y_{11} > s,\ Y_{21} > s/C = x),\ s \ge 0,$$

then from last formula in (7) we obtain

$$S_{1x}^Z(t) = \varphi_x^{-1}\left[\varphi_x\left(S_{1x}^X(t)\right) + \varphi_x\left(S_{1x}^Y(t)\right)\right],\ t \ge 0,$$
$$S_{2x}^Z(s) = \varphi_x^{-1}\left[\varphi_x\left(S_{2x}^X(s)\right) + \varphi_x\left(S_{2x}^Y(s)\right)\right],\ s \ge 0. \tag{8}$$

Now from (8) it follows that for fixed $x \in [0,1]$

$$\varphi_x(H_x(t,s)) = \varphi_x(F_x(t,s)) + \varphi_x(G_x(t,s)),\ (t,s) \in \overline{R}^{+2}.$$

Note that functionals (8) admits to estimation of one dimensional survival functions S_{1x}^X and S_{2x}^X correspondingly by subsamples $\mathbb{V}_1^{(n)} = \left\{(Z_{1i}, \delta_{1i}, C_i),\ i = \overline{1,n}\right\}$ and $\mathbb{V}_2^{(n)} = \left\{(Z_{2i}, \delta_{2i}, C_i),\ i = \overline{1,n}\right\}$ with $\mathbb{V}_1^{(n)} + \mathbb{V}_2^{(n)} = \mathbb{V}^{(n)}$ and then by $\mathbb{V}^{(n)}$ estimation of joint survival function $F_x(t,s)$ using first formula in (7). Let $H_{kx}^{(1)}(t) = P(Z_{ki} \le t, \delta_{ki} = 1/C_i = x)$ are subdistribution functions and $\Lambda_{kx}(t)$ is crude hazard functions of r.v. X_{ki}, $k = 1,2$ subjecting to censoring by Y_{ki} for given $C_i = x$. Then (see, Fleming and Harrington (1991)) we have

$$\Lambda_{kx}(dt) = \frac{P(X_{ki} \in dt,\ X_{ki} \le Y_{ki}/C_i = x)}{P(X_{ki} \ge t,\quad Y_{ki} \ge t/C_i = x)} = \frac{H_{kx}^{(1)}(dt)}{S_{kx}^Z(t-)},\ \ k = 1,2. \tag{9}$$

From (9) one can obtain following expressions of survival functions S_{kx}^X :

$$S_{kx}^X(t) = \varphi_x^{-1}\left\{-\int_0^t S_{kx}^Z(u-)\,\varphi_x'\left(S_{kx}^Z(u)\right)d\Lambda_{kx}(u)\right\} =$$

$$= \varphi_x^{-1}\left\{-\int_0^t \varphi_x'(S_{kx}^Z(u))dH_{kx}^{(1)}(u)\right\},\ t \ge 0,\ k = 1,2, \tag{10}$$

(see, for example, Breakers and Veraverbeke (2005)). In order to constructing the estimator of S_{kx}^X according to representation (10), we introduce some smoothed estimators of S_{kx}^Z and $H_{kx}^{(1)}$. Similarly to Breakers and Veraverbeke (2005), we will also use the Gasser-Müller weights

$$w_{ni}(x,h_n) = \frac{1}{q_n(x,h_n)}\int_{x_{i-1}}^{x_i}\frac{1}{h_n}\pi\left(\frac{x-u}{h_n}\right)du,\ i = 1,...,n,$$

with

$$q_n(x,h_n)=\int_0^{x_n}\frac{1}{h_n}\pi\left(\frac{x-u}{h_n}\right)du,$$

where $x_0=0$, π is a known probability density function (kernel) and $\{h_n,\ n\geq 1\}$ is a sequence of positive constants, tending to zero as $n\to\infty$, called bandwidth sequence. Let's introduce the weighted estimators of S_{kx}^Z and $H_{kx}^{(1)}$ respectively for $k=1,2$ as

$$S_{kx}^Z(t)=\sum_{i=1}^n w_{ni}(x,h_n)\,I(Z_{ki}>t)=1-H_{kxh}(t),$$

$$H_{kxh}(t)=\sum_{i=1}^n w_{ni}(x,h_n)\,I(Z_{ki}\leq t,\,\delta_{ki}=1). \tag{11}$$

$$S_{kx}^Z(t)=\sum_{i=1}^n w_{ni}(x,h_n)\,I(Z_{ki}>t)=1-H_{kxh}(t),$$

$$H_{kxh}(t)=\sum_{i=1}^n w_{ni}(x,h_n)\,I(Z_{ki}\leq t,\,\delta_{ki}=1).$$

$$S_{kxh}=\varphi_x^{-1}\left\{-\int_0^t\varphi_x'\left(S_{kxh}^Z(u)\right)dH_{kxh}^{(1)}(u)\right\},\ t\geq 0. \tag{12}$$

Remark that in case of no prognostic factors, estimator (11) reduces to estimator first obtained by Zeng and Klein (1995) for one sample case, which in case of the independent copula $\varphi(y)=-\log y$, reduces to a exponential hazard estimate. Now we propose second estimator from Abdushukurov (2020), the extended analogue of relative -risk power estimator introduced in Abdushukurov (1998, 1999) under independent censoring case for $k=1,2$:

$$\hat{S}_{kxh}^X(t)=\varphi_x^{-1}\left[\varphi_x\left(\hat{S}_{kxh}^Z(t)\right)\mu_{kxh}(t)\right],\ t\geq 0, \tag{13}$$

where

$$\mu_{kxh}(t)=\varphi_x\left(S_{kxh}^X(t)\right)\left[\varphi_x\left(\tilde{S}_{kxh}^Z(t)\right)\right]^{-1},$$

$$\varphi_x\left(S_{kxh}^X(t)\right)=-\int_0^t\varphi_x'\left(S_{kxh}^Z(u)\right)dH_{kxh}^{(1)}(u),$$

$$\varphi_x\left(\hat{S}_{kxh}^Z(t)\right)=-\int_0^t n\left[\varphi_x\left(S_{kxh}^Z(u)\right)-\varphi_x\left(S_{kxh}^Z(u)-\frac{1}{n}\right)\right]dH_{kxh}^{(1)}(u),$$

$$\varphi_x\left(\tilde{S}_{kxh}^Z(t)\right)=-\int_0^t\varphi_x'\left(S_{kxh}^Z(u)\right)dH_{kxh}(u).$$

Using estimators (12) and (13) we can propose two corresponding estimators of the first jointly survival function in (7):

$$F_{xh}(t,s)=\varphi_x^{-1}\left[\varphi_x\left(S_{1xh}^X(t)\right)+\varphi_x\left(S_{2xh}^X(s)\right)\right],\ (t,s)\in\overline{R}^{+2},$$

$$\hat{F}_{xh}(t,s)=\varphi_x^{-1}\left[\varphi_x\left(\hat{S}_{1xh}^X(t)\right)+\varphi_x\left(\hat{S}_{2xh}^X(s)\right)\right],\ (t,s)\in\overline{R}^{+2}. \tag{14}$$

In order to investigate estimators (14) we introduce some conditions. For the design point $x_1,...,x_n$ denote

$$\underline{\Delta}_n=\min_{1\leq i\leq n}(x_i-x_{i-1}),\ \bar{\Delta}_n=\max_{1\leq i\leq n}(x_i-x_{i-1}).$$

For the kernel π, let $\|\pi\|_2^2 = \int\limits_{-\infty}^{\infty} \pi^2(u)\,du$, $m_v(\pi) = \int\limits_{-\infty}^{\infty} u^v \pi(u)\,du$, $v = 1, 2$, $\|\pi\|_\infty = \sup\limits_{u\in\mathbb{R}} \pi(u)$. Moreover, we use next assumptions on design points and on the kernel function:

A1) As $n \to \infty$, $x_n \to 1$, $\underline{\Delta}_n = \mathrm{O}\left(\frac{1}{n}\right)$, $\bar{\Delta}_n - \underline{\Delta}_n = o\left(\frac{1}{n}\right)$.

A2) π is a probability density function with compact support $[-M, M]$ for some $M > 0$, with $m_1(\pi) = 0$ and $|\pi(u) - \pi(u')| \le C(\pi)|u - u'|$, where $C(\pi)$ is some constant. Let $T_k = \inf\left\{t \ge 0 : S_{kx}^Z(t) = 0\right\}$, $k = 1, 2$. For our result we need some smoothness conditions on functions $H_{kx}(t) = P(Z_{ki} \le t/C_i = x)$ and $H_{kx}^{(1)}(t)$, $k = 1, 2$. We formulate them for a general (sub) distribution function$N_x(t)$, $0 \le x \le 1$, $t \ge 0$ and for a fixed $T > 0$:

A3) $\frac{\partial}{\partial x} N_x(t) = \dot{N}_x(t)$ exist and continuous in $(x, t) \in [0, 1] \times [0, T]$.

A4) $\frac{\partial}{\partial t} N_x(t) = N'_x(t)$ exist and continuous in $(x, t) \in [0, 1] \times [0, T]$.

A5) $\frac{\partial^2}{\partial x^2} N_x(t) = \ddot{N}_x(t)$ exist and continuous in $(x, t) \in [0, 1] \times [0, T]$.

A6) $\frac{\partial^2}{\partial t^2} N_x(t) = N''_x(t)$ exist and continuous in $(x, t) \in [0, 1] \times [0, T]$.

A7) $\frac{\partial^2}{\partial x \partial t} N_x(t) = \dot{N}'_x(t)$ exist and continuous in $(x, t) \in [0, 1] \times [0, T]$.

A8) $\frac{\partial \varphi_x(u)}{\partial u} = \varphi'_x(u)$ and $\frac{\partial^2 \varphi_x(u)}{\partial u^2} = \varphi''_x(u)$ are Lipschitz in the $x-$ direction with a bounded Lipschitz constant and $\frac{\partial^3 \varphi_x(u)}{\partial u^3} = \varphi'''_x(u)$ exist and continuous in $(x, t) \in [0, 1] \times [0, T]$.

Firstly we establish the asymptotic equivalence of two estimators $\hat{F}_{xh}(t_1, t_2)$ and $F_{xh}(t_1, t_2)$ proposed by formulas (14). Let's denote $\nabla = [0, T_1] \times [0, T_2]$.

Theorem 2. Assume (A1), (A2), functions $H_{kxh}(t)$ and $H_{kxh}^{(1)}(t)$ satisfy (A5)-(A7) in $[0, T_k]$, $k = 1, 2$; φ_x satisfies (A8). Then as $n \to \infty$ and $x \in [0, 1]$:

$$\sup_{(t_1,t_2)\in\nabla} \left|\hat{F}_{xh}(t_1, t_2) - F_{xh}(t_1, t_2)\right| \overset{a.s.}{=} \mathrm{O}\left(\frac{1}{n}\right). \tag{15}$$

Proof of Theorem 2. For all $(t_1, t_2) \in \nabla$ we have

$$\left|\hat{F}_{xh}(t_1, t_2) - F_{xh}(t_1, t_2)\right| =$$

$$= \left|\varphi_x^{-1}\left[\varphi_x\left(\hat{S}_{1xh}^X(t_1)\right) + \varphi_x\left(\hat{S}_{2xh}^X(t_2)\right)\right] - \varphi_x^{-1}\left[\varphi_x\left(S_{1xh}^X(t_1)\right) + \varphi_x\left(S_{2xh}^X(t_2)\right)\right]\right| \le$$

$$\le \left|\varphi_x^{-1}\left[\varphi_x\left(\hat{S}_{1xh}^X(t_1)\right) + \varphi_x\left(\hat{S}_{2xh}^X(t_2)\right)\right] - \varphi_x^{-1}\left[\varphi_x\left(S_{1xh}^X(t_1)\right) + \varphi_x\left(\hat{S}_{2xh}^X(t_2)\right)\right]\right| +$$

$$+ \left|\varphi_x^{-1}\left[\varphi_x\left(S_{1xh}^X(t_1)\right) + \varphi_x\left(\hat{S}_{2xh}^X(t_2)\right)\right] - \varphi_x^{-1}\left[\varphi_x\left(S_{1xh}^X(t_1)\right) + \varphi_x\left(S_{2xh}^X(t_2)\right)\right]\right| \le$$

$$\le \frac{1}{\varphi'_x(\xi_{1xh}(t_1))}\left|\varphi_x\left(\hat{S}_{1xh}^X(t_1)\right) - \varphi_x\left(S_{1xh}^X(t_1)\right)\right| +$$

$$+ \frac{1}{\varphi'_x(\xi_{2xh}(t_2))}\left|\varphi_x\left(\hat{S}_{2xh}^X(t_2)\right) - \varphi_x\left(S_{2xh}^X(t_2)\right)\right| \le$$

$$\le \frac{1}{\varphi'_x(\xi_{1xh}(t_1))}\left|\varphi_x\left(\hat{S}_{1xh}^Z(t_1)\right)\mu_{1xh}(t_1) - \varphi_x\left(S_{1xh}^X(t_1)\right)\right| +$$

$$+ \frac{1}{\varphi'_x(\xi_{2xh}(t_2))}\left|\varphi_x\left(\hat{S}_{2xh}^Z(t_2)\right)\mu_{2xh}(t_2) - \varphi_x\left(S_{2xh}^X(t_2)\right)\right| \le$$

$$\le \frac{\mu_{1xh}(t_1)}{\varphi'_x(\xi_{1xh}(t_1))}\left|\varphi_x\left(\hat{S}_{1xh}^Z(t_1)\right) - \varphi_x\left(\tilde{S}_{1xh}^Z(t_1)\right)\right| +$$

$$+ \frac{\mu_{2xh}(t_2)}{\varphi'_x(\xi_{2xh}(t_2))}\left|\varphi_x\left(\hat{S}_{2xh}^Z(t_2)\right) - \varphi_x\left(\tilde{S}_{2xh}^Z(t_2)\right)\right|, \tag{16}$$

where $\xi_{kxh}(t) \in \left(\min\left\{\varphi_x\left(\hat{S}^Z_{kxh}(t)\right)\mu_{kxh}(t),\, S^X_{kxh}(t)\right\}, \max\left\{\varphi_x\left(\hat{S}^Z_{kxh}(t)\right)\mu_{kxh}(t),\, S^X_{kxh}(t)\right\}\right)$. But for all $(x,t) \in [0,1]\times[0,T_k]$ and $k=1,2$ in (15) $0 \le \mu_{kxh}(t) \le 1$ and it is not difficult to see, that

$$\sup_{0\le t\le T_k}\left|\varphi_x\left(\hat{S}^Z_{kxh}(t)\right)-\varphi_x\left(\tilde{S}^Z_{kxh}(t)\right)\right| \le$$

$$\le \sup_{0\le t\le T_k}\left|\int_0^{T_k}\left|n\left[\varphi_x\left(S^Z_{kxh}(u)\right)-\varphi_x\left(S^Z_{kxh}(u)-\frac{1}{n}\right)-\varphi_x'\left(S^Z_{kxh}(u)\right)\right]\right| dH_{kxh}(u)\right| \le$$

$$\le \frac{1}{2n}\sup_{0\le t\le T_k}\left|\varphi_x'\left(\theta_{kxh}(t)\right)\right| \overset{a.s.}{=} \mathrm{O}\left(\frac{1}{n}\right), \tag{17}$$

where $\theta_{kxh}(t) \in \left(\min\left\{S^Z_{kxh}(t),\, S^Z_{kxh}(t)-\frac{1}{n}\right\}, \max\left\{S^Z_{kxh}(t),\, S^Z_{kxh}(t)-\frac{1}{n}\right\}\right)$. Now from (16) and (17) follows (15). Theorem 2 is proved.

Thus for all $(t_1,t_2)\in\nabla$ and $x\in[0,1]$

$$\hat{F}_{xh}(t_1,t_2)-F_x(t_1,t_2)=F_{xh}(t_1,t_2)-F_x(t_1,t_2)+q_n(t_1,t_2),$$

where $q_n(t_1,t_2)=\hat{F}_{xh}(t_1,t_2)-F_{xh}(t_1,t_2)$ and by Theorem 2 $\sup\limits_{(t_1,t_2)\in\nabla}|q_n(t_1,t_2)| \overset{a.s.}{=} \mathrm{O}\left(\frac{1}{n}\right)$. Hence from two estimators in (13) it is enough to investigate only estimator $F_{xh}(t_1,t_2)$. The next result is on almost sure representation of estimator by sums of weighted functions with rate.

Theorem 3. Assume (A1), (A2), functions $H_{kxh}(t)$ and $H^{(1)}_{kxh}(t)$ satisfy (A5)-(A7) in $[0,T_k]$, $k=1,2$; φ_x satisfies (A8), $h_n\to 0$, $\frac{\log n}{nh_n}\to 0$ and $\frac{nh_n^5}{\log n}=\mathrm{O}(1)$. Then as $n\to\infty$,

$$F_{xh}(t_1,t_2)-F_x(t_1,t_2)=\sum_{k=1,2}\sum_{i=1}^{n} w_{ni}(x,h_n)\,\psi_{kxt_k}(Z_i,\delta_i)+r_n(t_1,t_2),$$

where for $k=1,2$

$$\psi_{kxt_k}(Z_i,\delta_i)=\frac{(-1)}{\varphi_x'(S^X_{kx}(t_k))}\left[\int_0^{t_k}\varphi_x''\left(S^Z_{kx}(u)\right)\left(I\left(Z_{ki}\le u\right)-H_{kx}(u)\right)dH^{(1)}_{kx}(u)-\right.$$

$$-\varphi_x'\left(S^Z_{kx}(t_k)\right)\left(I\left(Z_{ki}\le t,\,\delta_{ki}=1\right)-H^{(1)}_{kx}(t_k)\right)-$$

$$\left.-\int_0^{t_k}\varphi_x''\left(S^Z_{kx}(u)\right)\left(I\left(Z_{ki}\le u,\,\delta_{ki}=1\right)-H^{(1)}_{kx}(u)\right)dH_{kx}(u)\right],$$

$r_n(t_1,t_2)=r_{1n}(t_1)+r_{2n}(t_2)$ and

$$\sup_{0\le t\le T_k}|r_k(t_k)| \overset{a.s.}{=} \mathrm{O}\left(\left(\frac{\log n}{nh_n}\right)^{3/4}\right).$$

The weak convergence of the empirical process $(nh_n)^{1/2}\left(F_{xh}(t_1,t_2)-F_x(t_1,t_2)\right)$, $(t_1,t_2)\in\nabla$ in the space $\ell^\infty(\nabla)$ of uniformly bounded functions on ∇, endowed with the uniform topology is the contests of the next theorem.

Theorem 4. Assume (A1), (A2), functions $H_{kx}(t)$ and $H^{(1)}_{kxh}(t)$ satisfy (A5)-(A7) in $[0,T_k]$, $k=1,2$ and φ_x satisfies (A8).

I) If $nh_n^5\to 0$ and $\frac{(\log n)^3}{nh_n}\to 0$, then as $n\to\infty$,

$$(nh_n)^{1/2}\left(F_{xh}(t_1,t_2)-F_x(t_1,t_2)\right)\Rightarrow \mathrm{W}_x(t_1,t_2)\ \ in\ \ell^\infty(\nabla),$$

II) If $h_n = Cn^{-1/5}$ for some $C > 0$, then as $n \to \infty$,

$$(nh_n)^{1/2} \left(F_{xh}(t_1, t_2) - F_x(t_1, t_2) \right) \Rightarrow \mathrm{W}_x^*(t_1, t_2) \ \ in \ \ell^\infty(\nabla),$$

here W_x and W_x^* are Gaussian fields with means

$$E\mathrm{W}_x(t_1, t_2) = 0, \ \ E\,\mathrm{W}_x^*(t_1, t_2) = a_x(t_1, t_2),$$

and same covariation structure

$$Cov\left(\mathrm{W}_x(t_1, t_2), \mathrm{W}_x(s_1, s_2)\right) = Cov\left(\mathrm{W}_x^*(t_1, t_2), \mathrm{W}_x^*(s_1, s_2)\right) = \Gamma_x(t_1, t_2, s_1, s_2),$$

with

$$a_x(t_1, t_2) = \frac{-C^{5/2} m_2(\pi)}{2} \sum_{k=1,2} \left\{ \frac{1}{\varphi_x'\left(S_{kx}^X(t_k)\right)} \cdot \right.$$
$$\left. \cdot \int_0^{t_k} \left[\varphi_x''\left(S_{kx}^Z(u)\right) \ddot{H}_{kx}(u)\, dH_{kx}^{(1)}(u) - \varphi_x'\left(S_{kx}^Z(u)\right) dH_{kx}^{(1)}(u) \right] \right\},$$

and

$$\Gamma_x(t_1, t_2, s_1, s_2) = \sum_{k=1,2} \frac{\|\pi\|_2^2}{\varphi_x'\left(S_{kx}^X(t_k)\right) \varphi_x'\left(S_{kx}^X(s_k)\right)} \left\{ \int_0^{\min(t_k, s_k)} \left(\varphi_x'\left(S_{kx}^Z(z)\right) \right)^2 dH_{kx}^{(1)}(z) + \right.$$
$$+ \int_0^{\min(t_k, s_k)} \left[\varphi_x''\left(S_{kx}^Z(w)\right) S_{kx}^Z(w) + \varphi_x'\left(S_{kx}^Z(w)\right) \right] \cdot \int_0^w \varphi_x''\left(S_{kx}^Z(y)\right) dH_{kx}^{(1)}(y) dH_{kx}^{(1)}(w) +$$
$$+ \int_0^{\min(t_k, s_k)} \varphi_x''\left(S_{kx}^Z(w)\right) \int_w^{\max(t_k, s_k)} \left(\varphi_x''\left(S_{kx}^Z(y)\right) S_{kx}^Z(y) + \varphi_x'\left(S_{kx}^Z(y)\right) \right) dH_{kx}^{(1)}(y) dH_{kx}^{(1)}(w) -$$
$$- \int_0^{t_k} \left[\varphi_x''\left(S_{kx}^Z(y)\right) S_{kx}^Z(y) + \varphi_x'\left(S_{kx}^Z(y)\right) \right] dH_{kx}^{(1)}(y) \cdot$$
$$\left. \cdot \int_0^{s_k} \left[\varphi_x''\left(S_{kx}^Z(w)\right) S_{kx}^Z(w) + \varphi_x'\left(S_{kx}^Z(w)\right) \right] dH_{kx}^{(1)}(w) \right\}.$$

In order to proving the theorem 3 and 4 we need some auxiliary results for weighted empiricals $H_{kxh}(t)$ and $H_{kxh}^{(1)}(t)$. The next Lemma 1 (Lemma A4 from [29]) about the rates of strong uniform consistency of weighted empiricals is formulated only for $H_{kxh}(t)$, it still true also for $H_{kxh}^{(1)}(t)$ and proved exactly with the same way.

Lemma 1. [29] (I) Assume (A1), (A2), functions $H_{kx}(t)$ satisfy (A3), $h_n \to 0$, $nh_n \to \infty$ and $\frac{nh_n^3}{\log n} = \mathrm{O}(1)$. Then as $n \to \infty$,

$$\sup_{0 \le t \le T_k} |H_{kxh}(t) - H_{kx}(t)| \overset{a.s.}{=} \mathrm{O}\left(\left(\frac{\log n}{nh_n} \right)^{1/2} \right), \ \ k = 1, 2.$$

(II) Assume (A1), (A2), functions $H_{kx}(t)$ satisfy (A3),(A5), $h_n \to 0$, and $\frac{nh_n^5}{\log n} = \mathrm{O}(1)$. Then as $n \to \infty$,

$$\sup_{0 \le t \le T_k} |H_{kxh}(t) - H_{kx}(t)| \overset{a.s.}{=} \mathrm{O}\left(\left(\frac{\log n}{nh_n} \right)^{1/2} \right), \ \ k = 1, 2.$$

The next lemma 2 (Lemma 2 in [17]) provides the convergence rate of Theorem 3.

Lemma 2. [17] Under the conditions of theorem 3, as $n \to \infty$,

$$\sup_{0\leq t\leq T_k}\left|-\int_0^t\left[\varphi_x^{'}\left(S_{kxh}^Z(u)\right)-\varphi_x^{'}\left(S_{kx}^Z(u)\right)\right]d\left(H_{kx}^{(1)}(u)-H_{kx}^{(1)}(u)\right)\right|\overset{a.s.}{=}\mathrm{O}\left(\left(\frac{\log n}{nh_n}\right)^{3/4}\right),\ \ k=1,2.$$

Finally, in this paper we considered a nonparametric estimation of conditional jointly survival function under dependent right random censoring situation. We introduced two new estimators. Some basic asymptotic properties of estimators have been investigated: asymptotic equivalence of two estimators almost surely with rate tending to zero, approximation of estimators by weighted sum of random functions and weak convergence to Gaussian fields.

References

[1] Fleming T. R., Harrington D. R. , *Counting processes and survival analysis.*New York: Wiley, 1991.

[2] Nelsen R. B., *An introduction to copulas.* New York: Springer, 2006.

[3] Abdushukurov A. A., *Estimates of unknown distributions from incomplete observations and their properties.* Germany.: LAP, 2011.

[4] Abdushukurov A. A., Nonparametric estimation of the distribution function based on relative risk function, *Commun. Statist.: Theory & Methods.* **27**, 1998, pp. 1991–2012.

[5] Abdushukurov A. A., On nonparametric estimation of reliability indices by censored samples, *Theory. Probab. Appl.* **43**, 1999, pp. 3–11.

[6] Abdushukurov A.A., Nonparametrical estimators of survival function on the plane by censored observations, *Uzbek mathematical journal*, **2**, 2004, pp. 3–11.

[7] Abdushukurov A. A., Muradov R. S., Estimation of two-dimensional survival function by random right censored data, *Proceedings of XI th International Conference "Computer Data Analysis & Modeling"*, 2016, pp. 81–86.

[8] Abdushukurov A.A., Estimation of joint survival function from censored observations, *Zavod. Lab. Diagn. Mater.,* **82**, 2016, pp. 80–84.

[9] Abdushukurov A. A., Muradov R. S., On the estimates of the distribution function in a random censorship model, *Zavod. Lab. Diagn. Mater.*, **80**, 2014, pp. 62–67.

[10] Abdushukurov A. A., Muradov R. S., On Estimation of Conditional Distribution Function under Dependent Random Right Censored Data, *Journal of Siberian Federal University. Mathematics & Physics*, **7**, 2014, pp. 409–416.

[11] Abdushukurov A. A., Muradov R. S., Estimation of survival functions from dependent random right censored data, *Intern. J. Innovation in Science and Mathematics*, **2**, 2014, pp. 280–287.

[12] Abdushukurov A. A., Muradov R. S., Estimation of Conditional Jointly Survival Function under Dependent Right Random Censored Data, *Lobachevskii Journal of Mathematics,* **43(9)**, 2022, pp. 2360-2369.

[13] Abdushukurov A. A., Muradov R. S., Some algebraic properties of Archimedean copula functions and their applications in the statistical estimation of the survival function, *New Trends in Math. Sci.*, **4**, 2014, pp. 88–98.

[14] Abdushukurov A. A., Muradov R. S., Estimation of bivariate survival function from random censored data with Poisson sample size, *Calcutta Statistical Association Bulletin.*, **72(2)**, 2020, pp. 111-122.

[15] Abdushukurov A. A., "Estimation of conditional survival function under dependent right random censored data, "*Austrian J. of Statist.,* **49**, 1–8, (2020).

[16] Abdushukurov A. A., Muradov R. S., "On Estimation of Bivariate Survival Function from Random Censored Data, "*Journal of Siberian Federal University. Mathematics & Physics.,* **13(4)**, 422–430, (2020).

[17] Braekers R., Veraverbeke N., A copula graphic estimator for the conditional survival function under dependent censoring, *Canadian J. Statist.,* **33(3)**, 2005, pp. 429–447.

[18] Burke M. D., Estimation of a bivariate distribution function under random censorship, *Biometrika,* **75**, 1988, pp. 379–382.

[19] Campbell G., Nonparametric bivariate estimation with randomly censored data, *Biometrika,* **68**, 1981, pp. 417–422.

[20] Campbell G., Földes A., Lange sample properties of nonparametric bivariate estimators with censored data, *Colloquia Mathematica-Societatis János Bolyai,* **32**, 1982, pp. 103–122.

[21] Dabrowska D. M. , Kaplan Meier estimate on the plane, *Ann. Statist..,* **16**, 1988, pp.1475–1489.

[22] Dabrowska D. M., Kaplan Meier estimate on the plane: weak convergence, LIL and the bootstrap, *J. Multivar. Anal..,* **29**, 1989, pp. 308–325.

[23] Gill R. D., Multivariate Survival Analysis I., *Theory Probab. Appl..,* **37**, 1992, pp. 19–35.
[24] Gill R. D., Multivariate Survival Analysis II., *Theory Probab. Appl..,* **37**, 1992, pp. 307–328.
[25] Hanley J. A., Parnes M. N., Nonparametric estimation of a multivariate distribution in the presence of censoring, "*Biometrika,* **68**, 1983, pp. 129–139.
[26] Horváth L., The rate of strong uniform consistency for the multivariate product-limit estimator., "*Colloquia Mathematica-Societatis János Bolyai,* **13**, 1983, pp. 202–209.
[27] Huang Y., Two sample multistate accelerated sojourn times model, *J.A.S.A..*, **95**, 2000, pp. 619–627.
[28] Tsai W. Y., Leurgans S., Crowley J., Nonparametric estimation of a bivariate survival function in the presence of censoring, *Ann. Statist..*, **14**, 1986, pp. 1351–1365.
[29] Zeng M., Klein J. P., Estimates of marginal survival for dependent competing risks based on an assumed copula, *Biometrika,* **82**, 1995 pp. 122–128.
[30] Dauda, K. A., Pradhan, B., Uma Shankra, B. and Mitra, S. Decision tree for modeling survival data with competing risks, *Biocybernetics and Biomedical Engineering,* 39, 2019, pp. 697-708.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON CRAMER-RAO TYPE INEQUALITY IN COMPETING RISKS MODEL UNDER RANDOM CENSORING FROM UNOBSERVATION INTERVALS, EFFICIENCY AND SEQUENTIAL ESTIMATION

SABITBEK ERISBAEV[1]

[1]Karakalpak State University named after Berdakh, Nukus, Uzbekistan
e-mail: sabitbekn@mail.ru

Abstract. We consider competing risks when random censoring occurs by nonpreservation intervals. We present some useful representations of Fisher information matrix and at the establish efficiency measures for parameter estimator. We will also establish inequalities such as Cramer-Rao for a sequential estimation.

Keywords: random censoring, nonobservation intervals, Competing Risks Model, Fisher information, regularity conditions, Cramer-Rao inequality.

AMS Subject Classification: 62N01, 62N05.

1. Introduction

In Competing $/\!\!\rightarrow$ Risks Model (CRM) it is interesting to investigate the random variable (r.v.) X with values in measurable space $(\mathscr{X}, \mathscr{B})$ and events $(A^{(1)}, ..., A^{(k)})$ forming the complete group, where k is fixed. In practice r.v. X obvious means survival or a reliability time of some testing object (individual or physical system) exposing to k competing risks and getting out of work in case one of events $(A^{(1)}, ..., A^{(k)})$. In suck case pair $\{(X, A^{(i)}), i = 1, ..., k\}$ define the time and reason the object getting out of work (see more about CRM on [3, 4]). While repeat the experiment where the aggregate $(X, A^{(1)}, ..., A^{(k)})$ is observed under homogeneous conditions we took the sequence of copies $\{(X_m, A_m^{(1)}, ..., A_m^{(k)}), m \geq 1\}$. Let $\delta_m^{(i)} = I(A_m^{(i)})$ is indicator of event $A_m^{(i)}$. Every vector $\zeta_m = (X_m, A_m^{(1)}, ..., A_m^{(k)})$ induces statistical model with sample space $\mathscr{Y} = \mathscr{X} \times \{0,1\}^{(k)} = \mathscr{X} \times \{0,1\} \times ... \times \mathscr{X} \times \{0,1\}$ and σ-algebra $\mathscr{E}$ of the sets such as $B \times D_1 \times ... \times D_k$, where $B \in \mathscr{B}$ and $D_i \in \{0,1\}, i = \overline{1,k}$. We suppose that distribution of vector ζ_m on $(\mathscr{Y}, \mathscr{E})$ depends on unknown parameter $\theta = (\theta_1, ..., \theta_s) \in \Theta$:

$$Q_\theta^*(B \times D_1 \times ... \times D_k) = P_\theta(X_m \in B, \delta_m^{(1)} \in D_1, ..., \delta_m^{(k)} \in D_k), \tag{1}$$

where Θ is an open set in $R^{(s)}$. Let distribution (1) is absolutely continuous with respect to finite measure $\nu(x) = \mu(x) \times \varepsilon_1 \times ... \times \varepsilon_k$, where μ is Lebesque measure on $R^{(1)}$ and ε_i are counting measures, concentrated at points $y^{(i)} \in \{0,1\}, i = 1, ..., k$. Further we consider such statistical scheme, where aggregate $\{(X_m, A_m^{(1)}, ..., A_m^{(k)})$ is nonobservable inside of some interval $[Y_{1m}, Y_{2m}]$, where $D_m^{(0)} = \{\omega : Y_{1m}(\omega) \leq X_m(\omega) \leq Y_{2m}(\omega)\}$ and $D_m^{(i)} = A_m^{(i)} \bigcap (\{\omega : X_m(\omega) < Y_{1m}(\omega)\} \bigcup \{\omega : X_m(\omega) > Y_{2m}(\omega)\}), i = 0, 1, ..., k$. Let $\Delta_m^{(i)} = I(D_m^{(i)}), i = 0, 1, ..., k$ and $w_m = \varepsilon_{1m} + \varepsilon_{2m}$, where $\varepsilon_{1m} = I(X_m < Y_{1m})$ and $\varepsilon_{2m} = I(X_m > Y_{2m})$.It is obriously, that $\Delta_m^{(0)} = 1 - w_m$ and $\Delta_m^{(i)} = w_m \delta_m^{(i)}$. In CRM we interested in properties of pairs $\{(X_m, A_m^{(i)}), i = 1, ..., k\}, m \geq 1$ and

consequently consider the subdistributions

$$Q_{i\theta}(B) = Q^*_\theta(B \times \{0\} \times ... \times \{0\} \times \{1\} \times \{0\} \times ... \times \{0\}), i = 1, ..., k, \tag{2}$$

produced from (1) when $D_i = \{1\}$ and $D_l = \{0\}, i \neq l, l = 1, ..., k$. Let $Q_\theta(B) = \sum\limits_{i=1}^{k} Q_{i\theta}(B)$. By $h^{(i)}$ and h define the densities of subdistributions $Q_{i\theta}(B)$ and $Q_i(B)$:

$$\begin{aligned} Q_{i\theta}(B) &= \int\limits_B h^{(i)}(x;\theta)\mu(dx), i = 1, ..., k, \\ Q_\theta(B) &= \int\limits_B h(x;\theta)\mu(dx) = Q_{1\theta}(B) + ... + Q_{k\theta}(B), \end{aligned} \tag{3}$$

where $h = h^{(1)} + ... + h^{(k)}$. For $B = (-\infty, x]$ we put $Q_{i\theta}((-\infty, x]) = H^i(x;\theta), i = \overline{1,k}$, and $Q_\theta((-\infty, x]) = H(x;\theta)$. Later, we define the cumulative hazard functions (c.h.f.-s.) of the pairs $(X_m, A_m^{(i)})$:

$$\begin{aligned} \Lambda^{(i)}(x;\theta) &= \int\limits_{(-\infty,x]} lim_{\Delta\downarrow 0} P_\theta(t < X_m < t + \Delta, A^{(i)}/X_m > t)\mu(dt) = \\ &= \int\limits_{(-\infty,x]} \frac{dH^{(i)}(t;\theta)}{1 - H(t;\theta)}, i = 1, ..., k; x \in R^{(1)}. \end{aligned} \tag{4}$$

Then c.h.f corresponding the r.v. X_m is $\Lambda(x;\theta) = \sum\limits_{i=1}^{k} \Lambda^{(i)}(x;\theta)$. In CRM exponentional hazards functionals $F^{(i)}(x;\theta) = 1 - exp\{\Lambda^{(i)}(x;\theta)\}, i = 1, ..., k$, which describe the distribution of the pairs $(X_m, A_m^{(i)})$ of the i-th risk. In light of equality $\Lambda(x;\theta) = -log(1 - H(x;\theta))$, we have representation

$$1 - H(x;\theta) = P_\theta(X_m > t) = \prod_{i=1}^{k}(1 - F^{(i)}(x;\theta)). \tag{5}$$

Let define density $f^{(i)}(x;\theta) = \dfrac{\partial}{\partial\theta}F^{(i)}(x;\theta), i = \overline{1,k}$. Then the hazard rate density for i-risk is $f^{(i)}(1 - F^{(i)})$ and they by formulas (3)–(5) for every $(x;\theta) \in R^{(1)} \times \Theta$ and $i = 1, .., k$ admits the representation:

$$\frac{f^{(i)}(x;\theta)}{1 - F^{(i)}(x;\theta)} = \frac{h^{(i)}(x;\theta)}{1 - H^{(i)}(x;\theta)},$$

or equivalently

$$h^{(i)}(x;\theta) = f^{(i)}(x;\theta) \prod_{\substack{j=1 \\ j\neq i}} (1 - F^{(j)}(x;\theta)). \tag{6}$$

Let for n-th stage of experiments to observation is available the sample $\mathbb{Z}^{(n)} = (\mathbb{Z}_1, ..., \mathbb{Z}_n)$, where $\mathbb{Z}_m = W_m X_m + (1 - W_m)[Y_{1m}, Y_{2m}]$, that is every observation $\mathbb{Z}_m$ in sample $\mathbb{Z}^{(n)}$ is r.v. $X_{(m)}$ (when $W_m = 1$) or an interval $[Y_{1m}, Y_{2m}]$ (when $W_m = 0$). Let $P_n(\mathbb{Z}^{(n)};\theta) = \prod_{m=1}^{n} P_m(\mathbb{Z}^{(n)};\theta)$ is density function of data $\mathbb{Z}^{(n)}$, then we have following truncated likelihood function of sample

$\mathbb{Z}^{(n)}$, without including musance joint distribution $G(u,v)$. Then we have

$$P_n(\mathbb{Z}^{(n)};\theta)=\prod_{m=1}^{n}\left\{\left[\prod_{i=1}^{k}\left[f^{(i)}(X_m;\theta)\prod_{\substack{j=1\\ j\neq i}}(1-F^{(j)}(X_m;\theta))\right]^{\delta_m^{(i)}}\right]^{W_m}\times\right.$$
$$\left.\times\left[H(Y_{2m};\theta)-H(Y_{1m};\theta)\right]^{1-W_m}\right\}=$$
$$=\prod_{m=1}^{n}\left\{\left[\prod_{i=1}^{k}\left[h^{(i)}(X_m;\theta)\right]^{\delta_m^{(i)}}\right]^{W_m}\left[H(Y_{2m};\theta)-H(Y_{1m};\theta)\right]^{1-W_m}\right\}. \tag{7}$$

2. Main results

Let $N^{(i)}=\{x: h^{(i)}(x;\theta)>0\}$ and $N=\prod_{i=1}^{k}N^{(i)}$ are support sets of densities.

(C1) The support $\{N^{(i)}, i=\overline{1,k}\}$ are independent from θ and $N\neq\varnothing$;

(C2) There exists the derivatives $\dfrac{\partial^l h^{(i)}(x;\theta)}{\partial\theta_j^l}, l=1,2; i=1,...,k; j=1,...,s$, for all $\theta\in\Theta$;

(C3) $\int_{-\infty}^{+\infty}\left|\dfrac{\partial^l h^{(i)}(x;\theta)}{\partial\theta_j^l}\right|\mu(dx)<\infty, l=1,2; i=1,...,k; j=1,...,s$, for all $\theta\in\Theta$;

(C4) There are finite integrals $I_{lj}^{(i)}(\theta)=E_\theta\left[\dfrac{\partial}{\partial\theta_l}logh^{(i)}(X_m;\theta)\dfrac{\partial}{\partial\theta_j}logh^{(i)}(X_m;\theta)\right]$ for all $l,j=1,...,s, i=1,...,k$ and $\theta\in\Theta$;

(C5) The finite matrix $I_X(\theta)=\|I_j^X(\theta)\|_{l,j=\overline{1,s}}=\|\sum_{i=1}^{k}I_{lj}^{(i)}(\theta)\|_{l,j=\overline{1,s}}=\sum_{i=1}^{k}I^{(i)}(\theta)$ is positive define for all $\theta\in\Theta$.

It is clear that $I^{(i)}(\theta)$ is Fisher informational matrix according to pair $(X_m,\delta_m^{(i)})$ for a fixed m and $I_X(\theta)$ is same for r.v. X_m.

Let

$$S_n(\mathbb{Z}^{(n)};\theta)=\frac{\partial logP_n(\mathbb{Z}^{(n)};\theta)}{\partial\theta}=\sum_{m=1}^{n}l_\theta(X_m,Y_{1m},Y_{2m},W_m)$$

where $l_\theta(x,u,v,w)=w\sum_{i=1}^{k}\delta^{(i)}\dfrac{\partial logh^{(i)}(x;\theta)}{\partial\theta}+(1-w)\dfrac{\partial log(h(u;\theta)-h(v;\theta))}{\partial\theta}$, for $v\leq u$.

Theorem 1. *Suppose the conditions (C1) - (C4). Then information matrix is defined as*

$$J(\theta)=\int_{-\infty}^{+\infty}\left\{\int_{y_1}^{\infty}\left[\sum_{i=1}^{k}\left(\int_{-\infty}^{y_1}\left(\frac{\partial logh^{(i)}(x;\theta)}{\partial\theta}\right)\left(\frac{\partial logh^{(i)}(x;\theta)}{\partial\theta}\right)^T\right)P_\theta(x;\theta)dH(x;\theta)\right]+\right.$$
$$+\left[\int_{y_2}^{\infty}\left(\frac{\partial logh^{(i)}(x;\theta)}{\partial\theta}\right)\left(\frac{\partial logh^{(i)}(x;\theta)}{\partial\theta}\right)^T P_\theta(x;\theta)dH(x;\theta)\right]+$$
$$+\int_{y_1}^{\infty}\left(\frac{\partial log(H(u;\theta)-H(v;\theta))}{\partial\theta}\right)\left(\frac{\partial log(H(u;\theta)-H(v;\theta))}{\partial\theta}\right)^T\times$$
$$\left.\times(H(u;\theta)-H(v;\theta))\right\}dG(y_1,y_2).$$

Proof of Theorem 1. Under the condition of theorem for all $\theta \in \Theta$ by using of conditional expectation formulas, we obtain

$$
\begin{aligned}
&E_\theta l_\theta(X_m, Y_{1m}, Y_{2m}, W_m) = \\
&= E_\theta \left[W_m \sum_{i=1}^{k} \delta_m^{(i)} \frac{\partial log h^{(i)}(X_m;\theta)}{\partial \theta} \right] + \\
&+ E_\theta \left[(1 - W_m) \cdot \frac{\partial log(H(Y_{2m};\theta) - H(Y_{1m};\theta))}{\partial \theta} \right] = \\
&= E_\theta \left\{ \sum_{i=1}^{k} \left[E_\theta \left[\delta_m^{(i)} \cdot \frac{\partial log h^{(i)}(X_m;\theta)}{\partial \theta} \cdot I(X_m < Y_{1m}/Y_{1m}) \right] I(Y_{1m} < Y_{2m}) \right] + \right. \\
&+ E_\theta \left[\delta_m^{(i)} \cdot \frac{\partial log h^{(i)}(X_m;\theta)}{\partial \theta} \cdot I(X_m > Y_{2m}/Y_{2m}) \right] I(Y_{1m} \le Y_{2m}) \} + \\
&\left. + E_\theta \{ E_\theta \left[\frac{\partial log(H(Y_{2m};\theta) - H(Y_{1m};\theta))}{\partial \theta} \cdot I(Y_{1m} < X_m < Y_{2m})/(Y_{1m}, Y_{2m}) \right] I(Y_{1m} \le Y_{2m}) \right\} = \\
&= \int_{-\infty}^{+\infty} \int_{y_1}^{\infty} \sum_{i=1}^{k} \left(\int_{-\infty}^{y_1} \frac{\partial log h^{(i)}(x;\theta)}{\partial \theta} dH^{(i)}(x;\theta) + \int_{y_2}^{\infty} \frac{\partial log h^{(i)}(x;\theta)}{\partial \theta} dH^{(i)}(x;\theta) \right) dG(y_1, y_2) + \\
&+ E_\theta \left\{ \frac{\partial log(H(y_2;\theta) - H(y_1;\theta))}{\partial \theta} \cdot (H(y_2;\theta) - H(y_1;\theta))\, I(Y_{1m} \le Y_{2m}) \right\} = \\
&= \int_{-\infty}^{+\infty} \int_{y_1}^{\infty} \sum_{i=1}^{k} \left(\int_{-\infty}^{y_1} \frac{\partial h^{(i)}(x;\theta)}{\partial \theta} \mu(dx) + \int_{y_2}^{\infty} \frac{\partial h^{(i)}(x;\theta)}{\partial \theta} \mu(dx) \right) dG(y_1, y_2) + \\
&+ \int_{-\infty}^{+\infty} \int_{y_1}^{\infty} \frac{\partial}{\partial \theta} (H(y_2;\theta) - H(y_1;\theta)) dG(y_1, y_2) = \\
&= \int_{-\infty}^{+\infty} \int_{y_1}^{\infty} \left[-\frac{\partial}{\partial \theta} (H(y_2;\theta) - H(y_1;\theta)) + \frac{\partial}{\partial \theta} (H(y_2;\theta) - H(y_1;\theta)) \right] dG(y_1, y_2) = 0.
\end{aligned}
\tag{8}
$$

Hence for all $\theta \in \Theta$,

$$
E_\theta l_\theta(X_m, Y_{1m}, Y_{2m}, W_m) = 0, \tag{9}
$$

that is expectation of score function is zero.
Consequently for all $\theta \in \Theta$, $Var_\theta l_\theta(X_m, Y_{1m}, Y_{2m}, W_m) = E_\theta l_\theta^2(X_m, Y_{1m}, Y_{2m}, W_m) = \| E_\theta l_{\theta j} \cdot l_{\theta l} \|_{j,l=1}^{s}$, where

$$
\begin{aligned}
&E_\theta l_{\theta j}(X_m, Y_{1m}, Y_{2m}, W_m) l_{\theta l}(X_m, Y_{1m}, Y_{2m}, W_m) = \\
&= E_\theta \left[W_m \sum_{i=1}^{k} \frac{\partial log h^{(i)}(X_m;\theta)}{\partial \theta_j} \cdot \frac{\partial log h^{(i)}(X_m;\theta)}{\partial \theta_l} \right] + \\
&+ E_\theta \left[(1 - W_m) \frac{\partial (H(Y_{2m};\theta) - H(Y_{1m};\theta))}{\partial \theta_j} \cdot \frac{\partial (H(Y_{2m};\theta) - H(Y_{1m};\theta))}{\partial \theta_l} \right] = \\
&= J_{jl}^{(1)}(\theta) + J_{jl}^{(2)}(\theta),
\end{aligned}
$$

where for all $\theta \in \Theta$ and $j, l = 1, ..., s$

$$
\begin{aligned}
|J_{jl}^{(1)}(\theta)| \leq \\
&\leq 2\sum_{i=1}^{k} \int\limits_{-\infty}^{\infty} \left| \frac{\partial log h^{(i)}(x;\theta)}{\partial \theta_j} \right| \cdot \left| \frac{\partial log h^{(i)}(x;\theta)}{\partial \theta_l} \right| dH^{(i)}(x;\theta) \leq \\
&\leq 2\sum_{i=1}^{k} \left[\int\limits_{-\infty}^{\infty} \left(\frac{\partial log h^{(i)}(x;\theta)}{\partial \theta_j} \right)^2 dH^{(i)}(x;\theta) \right]^{\frac{1}{2}} \times \\
&\times \left[\int\limits_{-\infty}^{\infty} \left(\frac{\partial log h^{(i)}(x;\theta)}{\partial \theta_l} \right)^2 dH^{(i)}(x;\theta) \right]^{\frac{1}{2}} = \\
&= 2\sum_{i=1}^{k} \left[I_{jj}^{(i)}(\theta) I_{ll}^{(i)}(\theta) \right]^{\frac{1}{2}} < \infty,
\end{aligned}
\tag{10}
$$

where we used finiteness of elements of matrix from condition (C5) and inequality of Cauchy-Bunyakovsky. Similarly, we estimate $J_{jl}^{(2)}(\theta)$ for all $\theta \in \Theta$ and $j, l = 1, ..., s$:

$$
\begin{aligned}
|J_{jl}^{(2)}(\theta)| \leq \\
&\leq \int\limits_{-\infty}^{\infty} \int\limits_{y_1}^{\infty} \left| \int\limits_{y_1}^{y_2} \frac{\partial h(x;\theta)}{\partial \theta_j} \mu(dx) \right| \cdot \left| \int\limits_{y_1}^{y_2} \frac{\partial h(x;\theta)}{\partial \theta_l} \mu(dx) \right| \frac{dG(y_1;y_2)}{(H(y_2;\theta) - H(y_1;\theta))} = \\
&= \int\limits_{-\infty}^{\infty} \int\limits_{y_1}^{\infty} \left[\int\limits_{y_1}^{y_2} \left(\frac{\partial log h(x;\theta)}{\partial \theta_j} \sqrt{h(x;\theta)} \right) \sqrt{h(x;\theta)} \mu(dx) \right] \times \\
&\times \left[\int\limits_{y_1}^{y_2} \left(\frac{\partial log h(x;\theta)}{\partial \theta_l} \sqrt{h(x;\theta)} \right) \sqrt{h(x;\theta)} \mu(dx) \right] \frac{dG(y_1;y_2)}{(H(y_2;\theta) - H(y_1;\theta))} = \\
&= \int\limits_{-\infty}^{\infty} \int\limits_{y_1}^{\infty} \left[\int\limits_{y_1}^{y_2} \left(\frac{\partial log h(x;\theta)}{\partial \theta_j} \right)^2 dH(x;\theta) \right]^{\frac{1}{2}} \cdot \int\limits_{y_1}^{y_2} h(x;\theta) \mu(dx) \times \\
&\times \left[\int\limits_{y_1}^{y_2} \left(\frac{\partial log h(x;\theta)}{\partial \theta_l} \right)^2 dH(x;\theta) \right]^{\frac{1}{2}} \cdot \frac{dG(y_1;y_2)}{(H(y_2;\theta) - H(y_1;\theta))} \leq \\
&\leq \left[I_{jj}^{X}(\theta) \cdot I_{ll}^{X}(\theta) \right]^{\frac{1}{2}} < \infty.
\end{aligned}
$$

Now (9)–(11) imly the existence of $I_{jl}(\theta)$. Hence we by using of conditional expectation formulas as in (8), we have the chain of equalities

$$
\begin{aligned}
&E_\theta\left[\frac{\partial^2 l_\theta(X_m, Y_{1m}, Y_{2m}, W_m)}{\partial\theta_j\partial\theta_l}\right] = \\
&= \int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\sum_{i=1}^{k}\left(\int\limits_{-\infty}^{y_1}\frac{\partial^2 log h^{(i)}(x;\theta)}{\partial\theta_j\partial\theta_l}dH^{(i)}(x;\theta)+\right. \\
&\left.+\int\limits_{y_2}^{\infty}\frac{\partial^2 log h^{(i)}(x;\theta)}{\partial\theta_j\partial\theta_l}dH^{(i)}(x;\theta)\right)dG(y_1,y_2)+ \\
&+\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\frac{\partial^2 log(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_j\partial\theta_l}(H(y_2;\theta)-H(y_1;\theta))dG(y_1,y_2) = \\
&= \int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\sum_{i=1}^{k}\left\{\int\limits_{-\infty}^{y_1}\left[\frac{\partial^2 h^{(i)}(x;\theta)}{\partial\theta_j\partial\theta_l}h^{(i)}(x;\theta)+\frac{\partial h^{(i)}(x;\theta)}{\partial\theta_l}\frac{\partial h^{(i)}(x;\theta)}{\partial\theta_j}\right]\frac{\mu(dx)}{h^{(i)}(x;\theta)}+\right. \\
&\left.+\int\limits_{y_2}^{\infty}\left[\frac{\partial^2 h^{(i)}(x;\theta)}{\partial\theta_j\partial\theta_l}h^{(i)}(x;\theta)-\frac{\partial h^{(i)}(x;\theta)}{\partial\theta_l}\cdot\frac{\partial h^{(i)}(x;\theta)}{\partial\theta_j}\right]\frac{\mu(dx)}{h^{(i)}(x;\theta)}\right\}dG(y_1,y_2)+ \qquad (11)\\
&+\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left[\frac{\partial^2(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_j\partial\theta_l}(H(y_2;\theta)-H(y_1;\theta))-\right. \\
&\left.-\frac{\partial(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_j}\frac{\partial(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_l}\right]\cdot\frac{dG(y_1,y_2)}{(H(y_2;\theta)-H(y_1;\theta))} = \\
&= \int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\sum_{i=1}^{k}\left(\int\limits_{-\infty}^{y_1}\frac{\partial^2 h^{(i)}(x;\theta)}{\partial\theta_l\partial\theta_j}\mu(dx)+\int\limits_{y_2}^{\infty}\frac{\partial^2 h^{(i)}(x;\theta)}{\partial\theta_l\partial\theta_j}\mu(dx)\right)dG(y_1,y_2)+ \\
&+\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\frac{\partial^2(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_l\partial\theta_j}dG(y_1,y_2)-J_{lj}(\theta) = \\
&= \int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left[-\frac{\partial^2(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_l\partial\theta_j}+\frac{\partial^2(H(y_2;\theta)-H(y_1;\theta))}{\partial\theta_l\partial\theta_j}\right]dG(y_1,y_2)- \\
&-J_{lj}(\theta) = -J_{lj}(\theta).
\end{aligned}
$$

We denote regression of X_m on $\delta_m^{(i)} = 1$:

$$
\begin{aligned}
P(t;\theta) &= P_\theta(\delta_m^{(i)} = 1/X_m = t) = \\
&= E_\theta[\delta_m^{(i)}/X_m = t] = \\
&= \frac{P_\theta(\delta_m^{(i)} = 1/X_m = t)}{P_\theta(X_m = t)} = \\
&= \frac{dH^{(i)}(t;\theta)}{dH(t;\theta)}.
\end{aligned}
$$

Then $dH^{(i)}(t;\theta) = P(t;\theta)dH(t;\theta)$ and element of information matrix can be rewrite as

$$J_{lj}(\theta) = \int\limits_{-\infty}^{\infty} \left\{ \int\limits_{y_1}^{\infty} \left[\sum_{i=1}^{k} \left(\int\limits_{-\infty}^{y_1} \frac{\partial log h^{(i)}(x;\theta)}{\partial \theta_l} \cdot \frac{\partial log h^{(i)}(x;\theta)}{\partial \theta_j} + \right.\right.\right.$$
$$\left.\left.\left. + \int\limits_{y_2}^{\infty} \frac{\partial log h^{(i)}(x;\theta)}{\partial \theta_l} \cdot \frac{\partial log h^{(i)}(x;\theta)}{\partial \theta_j} \right) p(x;\theta) dH(x;\theta) \right] + \right. \tag{12}$$
$$+ \int\limits_{y_2}^{\infty} \frac{\partial log(H(y_2;\theta) - H(y_1;\theta))}{\partial \theta_l} \cdot \frac{\partial log(H(y_2;\theta) - H(y_1;\theta))}{\partial \theta_j} \times$$
$$\left. \times (H(y_2;\theta) - H(y_1;\theta)) \right\} dG(y_1, y_2)$$

which proves the theorem.

Thus, considered model includes random censoring from one side (from left or/and right) and from both sides.

We can introduce an measure of efficiency of estimators by quadratical loss functions with respect to Cramer-Rao lower bound for unbiased estimators. For simplicity of corresponding formulas we consider later only scalar parameters. Let is $\widehat{\theta}_n$ is estimator of θ, not necessary unbiased. We took quadratical loss function

$$L(\widehat{\theta}_n, \theta) = \nu(\theta)(\widehat{\theta}_n - \theta)^2, 0 < \nu(\theta) < \infty.$$

Then the corresponding risks function is

$$R(\widehat{\theta}_n, \theta) = \nu(\theta) E_\theta\{(\widehat{\theta}_n - \theta)\} = \nu(\theta)[Var_\theta\{\theta_n\} + B_n^2(\theta)],$$

where $B_n(\theta) = E_\theta\{\widehat{\theta}_n\} - \theta$ is biased of estimator $\widehat{\theta}_n$ at point θ. In order to measure of efficiency of estimator $\widehat{\theta}_n$ it is necessary to obtain of lower bound for risk function $R(\widehat{\theta}_n, \theta)$. Then efficiency of estimator $\widehat{\theta}_n$ one can represent as a function of θ, that is as a relative efficiency

$$releff(\widehat{\theta}_n; \theta) = \frac{(g'(\theta))^2}{I_n(\theta) R(\widehat{\theta}_n; \theta)}, \theta \in \Theta,$$

where $g(\theta) = \theta + B_n(\theta)$ and $I_n(\theta) = nJ(\theta)$ is Fisher information function of sample and $J(\theta)$ corresponding function of one observation $\zeta_m = (X_m, \delta_m^{(1)}, ..., \delta_m^{(k)})$. If $B_n(\theta) \to 0$ as $n \to \infty$ and $\widehat{\theta}_n$ is quadratic consistent estimator of θ:

$$\lim_{n\to\infty} E_\theta\{(\widehat{\theta}_n - \theta)^2\} = 0,$$

then

$$releff(\widehat{\theta}_n; \theta) = \frac{1}{I_n(\theta) E_\theta\{(\widehat{\theta}_n - \theta)^2\}}.$$

For efficiency problems for simple right censoring and CRM we recommend our papers [1], [5].

Now we turn to the effectiveness of sequential estimation. Let the expression of the likelihood function $P_n(\mathbb{Z}^{(n)}; \theta)$ corresponding to the statistical sample introduced above $\mathbb{Z}^{(n)} = (Z_1, ..., Z_n)$ be determined by the following formula (7):

$$P_n(\mathbb{Z}^{(n)}; \theta) = \prod_{m=1}^{n} \left\{ \left[\prod_{i=1}^{k} \left[h^{(i)}(X_m; \theta) \right]^{\delta_m^{(i)}} \right]^{W_m} [H(Y_{2m}; \theta) - H(Y_{1m}; \theta)]^{1-W_m} \right\},$$

where, for simplicity, the parametric space is $\Theta \subseteq R^1$, i.e. $s = 1$. We denote the σ-algebra generated by random variables $Z_1, ..., Z_n$ by $\mathcal{F}_n$. Denote the stopping time relative to the $\{\mathcal{F}_n, n \geqslant 1\}$-irreducible flow by ν, i.e. $Z_1, ..., Z_n$ are observed until they fall into a certain set $\widetilde{\mathfrak{X}}_n$ and the experiment stops at some step $\nu = n$.

Where $\mathfrak{X}_n \in \mathcal{F}_n$ and

$$\widetilde{\mathfrak{X}}_n = \begin{cases} \mathfrak{X}_1, & n = 1, \\ \overline{\mathfrak{X}}_1 \cap \overline{\mathfrak{X}}_2 \cap ... \cap \overline{\mathfrak{X}}_n \cap \mathfrak{X}_n, & n = 2. \end{cases}$$

So, for the stopping time ν, relations $\nu \geqslant 1$, $P_\theta(\nu < \infty) = 1$ and $E_\theta v < \infty$ are fulfilled.

The following Wald assertion is valid for the stopping time v.

Lemma. (Wald's identity). Let $Y_1, Y_2, ...$be independent identically distributed random variables $E|Y_1| < \infty$ and v be the stopping time relative to flow $\{\mathcal{F}_n, n \geqslant 1\}$. Then

$$E\left(\sum_{m=1}^{v} Y_m\right) = EY_1 \cdot Ev. \tag{13}$$

If $EY_1^2 < \infty$, then

$$E\left[\sum_{m=1}^{v} Y_m - v \cdot EY_1\right]^2 = DY_1 \cdot Ev. \tag{14}$$

We use this Wald lemma to prove the assertion that the Cramer-Rao inequality holds for successive estimates. Denote the parameter function θ by $g(\theta)$ and its unbiased estimate by $\widehat{g}_n(\mathbb{Z}_n)$.

Theorem 2. *Let $\mathbb{Z}_1, \mathbb{Z}_2, ...$ be a sequence of independent identically distributed observations and $P_n(\mathbb{Z}^{(n)}; \theta)$ be the corresponding likelihood function. The regularity conditions (C1)-(C5) are met for these density functions. Let $\{\mathfrak{X}_n, n = 1, 2, ...\}$ be a sequence of stopping fields of sequential estimation. $g(\theta)$ is differentiable in Θ and its unbiased estimate $\widehat{g}_n(\mathbb{Z}_n)$ satisfies the following conditions:*

(C6)

$$\int \left|\widehat{g}_n(\mathbb{Z}^{(n)}) \frac{\partial}{\partial\theta} P_n\left(\mathbb{Z}^{(n)}; \theta\right)\right| \mu^{(n)}(d\mathbb{Z}^{(n)}) < \infty,$$

for all θ and $n = 1, 2, ...$;

(C7) $\sum\limits_{n=1}^{\infty} \dfrac{dg_n(\theta)}{d\theta}$ *is the series uniformly convergent to Θ, where*

$$g_n(\theta) = \int\limits_{\widetilde{\mathfrak{X}}_n} \widehat{g}_n(\mathbb{Z}^{(n)}) \frac{\partial}{\partial\theta} P_n\left(\mathbb{Z}^{(n)}; \theta\right) \mu^{(n)}(d\mathbb{Z}^{(n)}).$$

Then $E_\theta v < \infty$ and for all θ

$$D_\theta \widehat{g}_n(\mathbb{Z}_1, \mathbb{Z}_2, ...) \geqslant \frac{(g'(\theta))^2}{J(\theta) E_\theta v}. \tag{15}$$

Proof of Theorem 2. We denote the sample size corresponding to the sequential estimation process by v. According to the proof of Theorem 1 from conditions (C1)-(C4), we obtain:

$$E_\theta S(\mathbb{Z}^{(n)}; \theta) = 0$$

and

$$J(\theta) = E_\theta S^2(\mathbb{Z}^{(n)}; \theta) < \infty.$$

Hence, according to relations (13) and $E_\theta v < \infty$

$$E_\theta\left\{\sum_{m=1}^{v} l_\theta\left(X_m, Y_{1m}, Y_{2m}, W_m\right)\right\} = 0$$

for all $\theta \in \Theta$. Similarly, according to (14),

$$E_\theta\left\{\left[\sum_{m=1}^{v} l_\theta\left(X_m, Y_{1m}, Y_{2m}, W_m\right)\right]^2\right\} = J(\theta)\cdot E_\theta v \tag{16}$$

for all $\theta \in \Theta$.

Now, based on the Cauchy-Buniakovsky inequality, we write the following inequality for all $\theta \in \Theta$:

$$\begin{aligned}&\left|Cov_\theta\left\{\sum_{m=1}^{v} l_\theta\left(X_m, Y_{1m}, Y_{2m}, W_m\right), \widehat{g}_v\left(\mathbb{Z}_1, ..., \mathbb{Z}_v\right)\right\}\right| \leq \\ &\leq E_\theta\left\{\left[\sum_{m=1}^{v} l_\theta\left(X_m, Y_{1m}, Y_{2m}, W_m\right)\right]^2 \cdot E_\theta\left\{\left(\widehat{g}_v\left(\mathbb{Z}_1, ..., \mathbb{Z}_v\right) - g(\theta)\right)^2\right\}\right\}^{1/2}.\end{aligned} \tag{17}$$

(17) on the right-hand side of the inequality under the square root, based on (16) the variance of estimates $J(\theta)E_\theta v$ and $\widehat{g}_v$ with sequential estimation, and on the left-hand side of this inequality under conditions (C6) and (C7), as a result of differentiation under the integral sign, the following expression holds:

$$\begin{aligned}&E_\theta\left\{\sum_{m=1}^{v} l_\theta\left(X_m, Y_{1m}, Y_{2m}, W_m\right), \widehat{g}_v\left(\mathbb{Z}_1, ..., \mathbb{Z}_v\right)\right\} = \\ &= \sum_{n=1}^{\infty}\int\limits_{\widetilde{\mathfrak{X}}_n} \widehat{g}_n\left(\mathbb{Z}^{(n)}\right)\sum_{m=1}^{n} l_\theta\left(X_m, Y_{1m}, Y_{2m}, W_m\right) P_n\left(\mathbb{Z}^{(n)};\theta\right)\mu^{(n)}(d\mathbb{Z}^{(n)}) = \\ &= \sum_{n=1}^{\infty}\int\limits_{\widetilde{\mathfrak{X}}_n} \widehat{g}_n\left(\mathbb{Z}^{(n)}\right)\left(\frac{\partial}{\partial\theta}P_n\left(\mathbb{Z}^{(n)};\theta\right)\right)\mu^{(n)}(d\mathbb{Z}^{(n)}) = \\ &= \frac{\partial}{\partial\theta}\sum_{n=1}^{\infty}\int\limits_{\widetilde{\mathfrak{X}}_n} \widehat{g}_n\left(\mathbb{Z}^{(n)}\right) P_n\left(\mathbb{Z}^{(n)};\theta\right)\mu^{(n)}(d\mathbb{Z}^{(n)}) = g'(\theta).\end{aligned} \tag{18}$$

If we square both sides of inequality (18) and substitute (17) into the left-hand side of inequality, we obtain inequality (15). This completes the proof of the theorem.

In sequential evaluation, the following criteria are generally used:

(a) In condition $E_\theta v \leqslant m$ (m is any given number), the variance $E_\theta\left\{[\widehat{g}_v - g(\theta)]^2\right\}$ is minimized over all θ of unbiased estimate $\widehat{g}_v$;

(b) The average volume $E_\theta v$ for all θ is minimized under condition $E_\theta\left\{[\widehat{g}_v - g(\theta)]^2\right\} \leqslant C < \infty$ ($C$is the fixed number);

(c) Sum $CE_\theta v + E_\theta\left\{[\widehat{g}_v - g(\theta)]^2\right\}$ is minimized over θ.

References

[1] Abdushukurov A.A., Kim L.V. Lower Cramer-Rao and Bhattacharya bounds for randomly censored data. J. Soviet. Math., 1987. v.38, N.5. p. 2171 - 2185.

[2] Prakasa Rao, B.L.S. Remarks on Cramer-Rao type integlas inequalities for randomly censored data. In Analysis of Censored Data. Lecture Notes-Monograph Series. 1995. N. 27. p. 163-175.

[3] Prakasa Rao, B.L.S. Improved Cramer-Rao inequality revisited for randomly censored data. JIRSS. 2018. v. 17. N. 02. p. 1-12.

[4] Prakasa Rao, B.L.S. Cramer-Rao inequality revisited for randomly censored data. Proceedings of scientific–applied conf. STATISTICS and its application -V. 17-19 oktober, 2019. Tashkent. p. 19-28.

[5] Abdushukurov A. A., Erisbaev S. A. Fisher information and Cramer–Rao type inequalities for competting risks model. Bulletin of Instituts of Mathematics of Uzb., 2021. v. 4. N. 5. p. 50-58.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

THE METHOD OF ACCOMPANYING DISTRIBUTIONS IN ESTIMATING THE PROXIMITY OF SEQUENCES OF PROBABILITY MEASURES

SHAKIR FORMANOV[1], NURSADYK AKANBAY[2], LOLA SHARIPOVA[3]

[1]National University of Uzbekistan, Tashkent, Uzbekistan
[2]Al-Farabi Kazakh National University, Almaty, Republic of Kazakhstan
[3]Tashkent State Transport University, Tashkent, Uzbekistan
e-mail: lolaxon@gmail.com

Abstract. Let two sequences of probability distributions be given. These sequences are considered asymptotically close if the sequence composed of the difference of the corresponding distributions in a certain sense tends to zero. The study of the asymptotic proximity of two probability distributions is a more general problem compared to problems of weak convergence of sequences of probability distributions. Firstly, the limit of sequences of differences in probability distributions can also exist in the case when both given sequences of probability measures do not have weak limits. Secondly, one can easily make sure that the limit relation of weak convergence of sequences to the limit distribution is a special case of asymptotic proximity of sequences to the limit distribution, when one of the sequences is stationary, consisting of one limit distribution. Taking advantage of this, a non-classical version of the Lindeberg theorem is proved in the triangular array scheme of random variables.

Keywords: distribution function, characteristic function, the proper or weak convergence, the continuity (convergence) theorem, accompanying distributions, the Lindeberg theorem.

AMS Subject Classification: 60F05

1. Introduction. The tightness property of sequences of distributions

The basic concepts of the summation theory of independent random variables (r.v.'s) are given in the monographs [1, Chapter 1, §2, §3, pp.11–16] and [2, Chapter IX, §1, §2, §3, pp. 367–387]. In [3], the problems of Gaussian approximation of the distribution of sums of independent identically distributed r.v.'s are considered. In [4, Chapter 8, §2, pp. 157–164], [5, Chapter III, §4, pp. 457–463], and [6, Part II, Chaters 5, 18, pp. 301–318], the Lindeberg and Rotar numerical characteristics are given, which are used in the proofs of various versions of the Central Limit Theorem. In [7], general representations for these characteristics are proved. In [8], Ch. Stein proposed a universal method for proving the Central Limit Theorem based on one property that characterizes the cumulative distribution function (CDF) of the normal distribution. The paper [9] is devoted to the problems of weak convergence of sequences of distribution functions convolutions. In the monograph [10] and papers [11], [12], V. Senatov studied the properties of the Chebyshev – Hermite moment characteristics and their applications to asymptotic expansions in the Central Limit Theorem. Limit theorems for sums of indicator random variables have found application in technical problems considered in [13]. In [14], the validity of the Berry – Esseen-type estimates for the remainder term in the Central Limit Theorem is proved, and in [15], the validity of the Donsker – Prokhorov invariance principle is proved for sequences of r.v.'s connected in a homogeneous Markov chain with an arbitrary set of states. Papers [16], [17] are devoted to estimates of the convergence rate in the multidimensional Central Limit Theorem and in the absence of the condition for the existence of third-order moments for r.v.'s connected

in a homogeneous Markov chain in transition probabilities. The results of [18]–[20] are obtained using the Stein method modified in terms of characteristic functions. In [21], a nonclassical version of the Lindeberg – Feller theorem was proved, and in [22], this theorem was applied in the study of transient phenomena of branching random processes. The works [23]–[25] by I. Shevtsova are devoted to the study of a local version of the Central Limit Theorem and moment estimates of characteristic functions with their application to von Mises-type inequalities. In [26], an analytical proof of the well-known theorem of A. Sakhanenko on a strengthened version of the law of large numbers is given.

Let $\{F_n\}$ be an arbitrary sequence of distribution functions (d.f.'s). Recall that we say that the sequence $\{F_n\}$ converges properly or weakly to a d.f. F as $n \to \infty$ if the equalities

$$\lim_{n\to\infty} F_n(x) = F(x) \tag{1}$$

hold for each point of continuity x of $F(x)$ (i.e. the equalities hold at each point x for which the equalities $F(x+0) - F(x-0) = F(x+0) - F(x) = 0$ are correct). Relation (1) is written as $F_n \to F$. Let

$$f_n(t) = \int_{-\infty}^{\infty} e^{itx} dF_n(x), \quad n = 1, 2, \dots$$

be the corresponding sequence of characteristic functions (ch.f.'s). The following theorem is called the continuity (convergence) theorem.

Theorem A ([4, Chapter 7, § 3, p. 137]). *A necessary and sufficient condition for the convergence $F_n \to F$ is that $f_n(t) \to f(t)$ for any t, $f(t)$ being the ch.f. corresponding to $F(x)$.*

The above theorem A has only one drawback – one needs to know in advance that the function $f(t)$, to which ch.f.'s $f_n(t)$ converge, is a characteristic function itself. Meanwhile, such information may not be available (see [4, Chapter 8, § 8.9, p. 188]. In this regard, there arises a question under what conditions the limiting function will be characteristic. The answer to this question is contained in the following Theorem B. Let us first give definitions that play an important role in the summation theory of independent random variables (r.v.'s).

Definition 1. Let $\{P_n\}$ be a sequence of probability measures corresponding to the sequence of d.f.'s $\{F_n\}$, i.e.

$$P_n(A) = \int_{\mathbb{R}} I_A(x) F_n(dx), \ n = 1, 2, \dots$$

where $F_n(\cdot)$ is the probability measure generated by the d.f. F_n, $I_A(\cdot)$ is the indicator of an event A (the Borel set $A \subseteq \mathbb{R}$).

The sequence of probability measures $\{P_n\}$ (or d.f.'s $\{P_n\}$) is said to be *tight* if, for any $\varepsilon > 0$, there exists an $N = N(\varepsilon)$ such that

$$\inf_n P_n([-N, N]) > 1 - \varepsilon \tag{2}$$

(see [4, Chapter 6, § 3, p. 127]).

Definition 2. Let $\{X_n\}$ be a sequence of r.v.'s corresponding to the sequence of d.f.'s $\{F_n\}$, i.e.

$$F_n(x) = P(X_n < x), \ n = 1, 2, \dots, \ x \in \mathbb{R}.$$

A sequence of r.v.'s $\{X_n\}$ is said to be *stochastically bounded* if

$$P(|X_n| > L) \to 0 \ \text{ as } \ L \to \infty$$

uniformly in n (see [6, Chapter 2, § 8.3, p. 169]).

By virtue of (2), the equality

$$P\left(|X_n| > L\right) = F_n\left(\{x,\ |x| > L\}\right) = F_n(-L) + 1 - F(L) \to 0,\ L \to \infty \tag{3}$$

uniformly in n.

Relations (2) and (3) imply that the tightness condition of a sequence of distributions $\{P_n\}$ is equivalent to the condition of stochastic boundedness of the corresponding sequence of r.v.'s. Moreover, it is evident, condition (3) can be verified easily.

The contents of Theorem A is made more precise in the following

Theorem B. *Let*

$$f_n(t) = \int\limits_{-infty}^{\infty} e^{itx} dF_n(x), \quad n = 1, 2, \ldots$$

be a sequence of ch.f.'s, and $f_n(t) \to f(t)$ as $n \to \infty$ and at each t. Then the following conditions are equivalent:

a) $f(t)$ is a ch.f.,

b) $f(t)$ is continuous at the point $t = 0$,

c) the sequence $\{F_n\}$ is tight.

Here we give another proof of the equivalence of conditions b) and c).

Lemma 1. *Let a sequence of distributions $\{F_n\}$ be such that the corresponding sequence of ch.f.'s $\{f_n(t)\}$ converges pointwise to a function $f(t)$ continuous at zero. Then $\{F_n\}$ is tight.*

Proof. If $f(t)$ is the ch.f. of a r.v. X, then, for any $u > 0$,

$$P\left(|X| > \frac{1}{u}\right) \le \frac{7}{u} \int\limits_0^u \left(1 - Ref(t)\right) dt \tag{4}$$

(see [4, Chapter 7, §3, p. 137]).

By virtue of (4), for any $u > 0$,

$$\int\limits_{|x| > \frac{1}{u}} F_n(dx) \le \frac{7}{u} \int\limits_0^u \left(1 - Ref(t)\right) dt + \int\limits_0^u \left(f_n(t) - f(t)\right) dt. \tag{5}$$

The first term in the right-hand side of inequality (5) tends to zero at $u \to 0$ since $f(t)$ is continuous at zero and $f(0) = \lim\limits_{n\to\infty} f_n(0) = 1$. According to the well-known Lebesgue theorem on passage to the limit under the integral sign,

$$\lim_{n\to\infty} \frac{1}{u} \int\limits_0^u \left(f_n(t) - f(t)\right) dt = 0 \tag{6}$$

at any $u > 0$. By the tightness definition of a sequence of distributions, the lemma will be proved if we prove that the left-hand side of inequality (5)

$$\int\limits_{|x| > \frac{1}{u}} F_n(dx) \to 0 \ \text{ as } \ u \to 0 \tag{7}$$

at any $n \ge 1$.

Indeed, let $\varepsilon > 0$. There is $u(\varepsilon)$, for which the first term in (5)

$$\frac{1}{u(\varepsilon)} \int_0^{u(\varepsilon)} (1 - f(t))\, dt < \frac{\varepsilon}{2} \tag{8}$$

since $f(t)$ is continuous at zero. Further, for already chosen $u(\varepsilon)$, there exists $n_0(\varepsilon)$ such that, for all $n \geq n_0(\varepsilon)$, the second term

$$\frac{1}{u(\varepsilon)} \int_0^{u(\varepsilon)} |f_n(t) - f(t)|\, dt < \frac{\varepsilon}{2}. \tag{9}$$

So, by virtue of (8) and (9), for each $\varepsilon > 0$, there is $u(\varepsilon)$ such that, for all $n \geq n_0(\varepsilon)$,

$$\int_{|x|>\frac{1}{u}} F_n(dx) \leq \varepsilon. \tag{10}$$

There will always be $\overline{u}(\varepsilon)$ such that, for $n \leq n_0(\varepsilon)$,

$$F_n\left\{x;\ |x| \geq \frac{1}{\overline{u}(\varepsilon)}\right\} < \varepsilon. \tag{11}$$

Taking into account the latter and (10), we obtain that this will hold for all $\overline{u}(\varepsilon) < u(\varepsilon)$. So, for any $\varepsilon > 0$, there is $u(\varepsilon)$ such that inequality (11) holds for all $\overline{u}(\varepsilon) < u(\varepsilon)$.

Now, taking into account relations (10) and (11), we can conclude that, for any $\varepsilon > 0$, there is $u(\varepsilon)$ such that

$$\int_{|x|>\frac{1}{u}} F_n(dx) < \varepsilon,\ n = 1, 2, \ldots \tag{12}$$

for all $u > \overline{u}(\varepsilon)$. The last inequality (12) is equivalent to the statement about the tightness of the sequence of d.f.'s $\{F_n\}$. Lemma 1 is proved. □

2. The method of accompanying distributions. The proximity of sequences of probability distributions

According to Theorem 1 given in [6, Part II, Chapter 2, § 8.4], the following three assertions are equivalent:

a′) for any bounded continuity interval $[a, x)$ of F,

$$F_n([a, x)) \to F([a, x)) \quad \text{as} \quad n \to \infty;$$

b′) for any bounded continuity set B of the distribution F,

$$F_n(B) \to F(B) \quad \text{as} \quad n \to \infty;$$

c′) for any continuous and bounded function $u(x)$ satisfying the condition $\lim\limits_{|x|\to\infty} u(x) = 0$,

$$\int_{-\infty}^{\infty} u(x) d(F_n - F) \to 0 \quad \text{as} \quad n \to \infty.$$

Here $\{F_n\}$ is a sequence of distributions, $F(\cdot)$ is a finite measure.

Definition 3. It is said that a sequence of distributions $\{F_n\}$ *converge improperly* to the distribution F ($F_n \to F$) if one of the conditions a′), b′), c′) (and, consequently, all conditions) holds [6, Part II, Chapter 2, § 8.4, p. 170].

Lemma 2. *Let F_n improperly converge to some finite measure F. Then:*

1^o) *If F is a probability measure, then the convergence $(F_n \to F)$ is proper and the sequence $\{F_n,\ n \geq 1\}$ is tight.*

2^o) *If $\{F_n\}$ is tight, then F is a probability measure, and the convergence $(F_n \to F)$ is proper.*

The proof of Lemma 2 is given in [6, Subsection 8.8, p. 173]. In this case, the equivalence of statements a′), b′), c′) is used in combination with Definition 3. This proof is simple, but far from short due to the numerous implications to be proved.

Assume that we need to establish the convergence of any sequence of probability distributions $\{F_n\}$ to the distribution F. It turns out that it is often easier to solve such problems not directly, but by choosing such distributions G_n, for which the convergence $G_n \to F$ is almost obvious, and then proving that the difference of distributions $F_n - G_n \to 0$ in a certain sense. The matched distributions G_n are called *accompanying distributions.*

The convergence problem $F_n - G_n \to 0$ is more general than the convergence problem of distributions $(F_n \to F)$, since F_n and G_n can be close even in the case when the sequences $\{F_n\}$ and $\{G_n\}$ have no limit.

In addition to what has been said, it should be noted that it is no longer so easy to determine the convergence $F_n - G_n \to 0$ by conditions similar to the conditions of Theorems A and B above, since it is not entirely clear with respect to which distribution the sets B should be declared to be continuity sets.

It follows from the above that when applying the method of accompanying distributions, the convergence of the difference of distributions $F_n - G_n \to 0$ should be specified. This is done in Definition 4.

Definition 4. A difference of distributions $F_n - G_n \to 0$ *properly* or *weakly* if

$$\int_{-\infty}^{\infty} u(x)\,(F_n\,(dx) - G_n\,(dx)) \to 0 \tag{13}$$

for any continuous and bounded function $u(x)$ ([6, Chapter 2, § 8.5, p. 171].

It should be noted that Definition 4 of the convergence $F_n - G_n \to 0$ by formula (13) turns out to be the most convenient for passing to probability measures in spaces of a more general nature than $\mathbb{R}^1$ or $\mathbb{R}^k$.

Theorem 1. *Let $F_n - G_n \to 0$ properly. Then, if any subsequence $F_{n_k} \to F$ properly, then $G_{n_k} \to F$ properly. In particular, the convergence $F_n \to F$ implies the convergence $G_n \to F$.*

This theorem is given in [6, Chapter 2, Section 8, § 8.5, Lemma 5, p. 171], where it is written that the proof of Lemma 5 is left to the reader.

Proof of Theorem 1. Indeed, the conditions of Theorem 1 imply that

$$|F_{n_k} - G_{n_k}| \to 0 \ \text{ as } \ n_k \to \infty.$$

Now the proof of Theorem 1 is obtained by passing to the limit in the inequalities

$$|G_n - F| \leq |F_n - G_n| + |F_n - F|\,,$$

$$|G_{n_k} - F| \leq |F_{n_k} - G_{n_k}| + |F_{n_k} - F|$$

as $\min(n, n_k) \to \infty$. Theorem 1 is proved. □

3. Theorems on the convergence $F_n - G_n \to 0$

Let $\{F_n\}$ and $\{G_n\}$ be two sequences of probability distributions defined on the same probability space $(\Omega, \mathcal{F}, P)$. In this section, we give some theorems on the convergence $F_n - G_n \to 0$

properly or weakly in the sense of Definition 4.

Theorem 2. *Let $\{F_n\}$ and $\{G_n\}$ be two sequences of probability d.f.'s, and*

$$f_n(t) = \int\limits_{-\infty}^{\infty} e^{itx} dF_n(x), \;\; g_n(t) = \int\limits_{-\infty}^{\infty} e^{itx} dG_n(x)$$

be the corresponding sequences of characteristic functions (ch.f.'s). Then if

$$\delta_n = \left| \int\limits_{-\infty}^{\infty} u(x) \left(F_n(dx) - G_n(dx)\right) \right| = \left| \int\limits_{-\infty}^{\infty} u(x) d\left(F_n - G_n\right) \right| \to 0 \quad as \quad n \to \infty$$

i.e. condition (13) *is satisfied for any continuous and bounded function $u(x)$, then*

$$f_n(t) - g_n(t) \to 0 \quad as \quad n \to \infty \tag{14}$$

for any $t \in \mathbb{R}$.

Proof. According to Definition 4, there is an equivalence of implications

$$\{\delta_n \to 0\} \;\; \rightleftarrows \;\; \{F_n - G_n \to 0\}. \tag{15}$$

Note that Theorem 2 will be proved if we prove the implication

$$(13) \;\; \to \;\; (14). \tag{16}$$

Further, we have

$$I = |f_n(t) - g_n(t)| = \left| \int\limits_{-\infty}^{\infty} e^{itx} d(F_n - G_n) \right| = \left| \int\limits_{-\infty}^{\infty} \left(e^{itx} - 1\right) d(F_n - G_n) \right| \le I_1 + I_2 + I_3, \tag{17}$$

where

$$I_1 = \left| \int\limits_{|x| \le \varepsilon} \left(e^{itx} - 1\right) d(F_n - G_n) \right|, \tag{18}$$

$$I_2 = \left| \int\limits_{\varepsilon < |x| \le N} \left(e^{itx} - 1\right) d(F_n - G_n) \right|, \tag{19}$$

$$I_3 = \left| \int\limits_{|x| \ge N} \left(e^{itx} - 1\right) d(F_n - G_n) \right|, \;\; 0 < \varepsilon \le N. \tag{20}$$

Obviously, the inequality

$$\left|e^{itx} - 1\right| \le \min\left(|tx|, 2\right)$$

holds at any $t, x \in \mathbb{R}$. By virtue of the last inequality, it is easy to verify the validity of the following estimates

$$|I_1| \le |t| \left| \int\limits_{|x| \le \varepsilon} |x| \left(dF_n(x) + dG_n(x)\right) \right| \le 2|t| \cdot \varepsilon \le 2T \cdot \varepsilon, \;\; T > 0, \; |t| \le T. \tag{21}$$

Next, the integral

$$|I_2| \le 2 \left| \int\limits_{\varepsilon < |x| \le N} d\left(F_n - G_n\right) \right| = 2 \left| \int\limits_{-\infty}^{\infty} u_1(x) d\left(F_n - G_n\right) \right| \to 0 \;\; \text{as} \;\; n \to \infty, \tag{22}$$

where the continuous and bounded function

$$u_1(x) = \begin{cases} 0 & \text{at } \varepsilon < |x| \le N, \\ 1 & \text{in other cases.} \end{cases}$$

Now, repeating the reasoning given when obtaining (22) and setting

$$u_2(x) = \begin{cases} 0 & \text{at } |x| \le N, \\ 1 & \text{at } |x| > N, \end{cases}$$

we have the estimate

$$|I_3| \le 2\left|\int_{-\infty}^{\infty} u_2(x) d\left(F_n - G_n\right)\right| \le 2\left|\int_{|x|\ge N} u_2(x)\left(dF_n(x) + dG_n(x)\right)\right|$$

$$\le 4\max\left(\left(1 - F(N) + F(-N)\right), \left(1 - G(N) + G(-N)\right)\right).$$

For any $\varepsilon > 0$ and sufficiently large $N \ge N_0(\varepsilon)$, we have

$$1 - F(N) + F(-N) < \frac{\varepsilon}{2}, \quad 1 - G(N) + G(-N) < \frac{\varepsilon}{2}$$

for any $n = 1, 2, \ldots$ Therefore, for $N \ge N_0(\varepsilon)$, we have the estimate

$$|I_3| \le 4\varepsilon. \tag{23}$$

Taking into account all relations (17) – (23), we obtain that

$$\limsup_{n\to\infty} |f_n(t) - g_n(t)| \le (2T \cdot \varepsilon + 4\varepsilon) = (T + 2\varepsilon) \cdot 2\varepsilon. \tag{24}$$

Since $\varepsilon > 0$ is arbitrary, we obtain the validity of implication (16) from the last limit relation (24). Theorem 2 is proved. □

Theorem 3. *Let sequences of distributions $\{F_n\}$ and $\{G_n\}$ satisfy the following conditions:*
1) One of the sequences is tight.
2) The difference of the characteristic functions

$$f_n(t) - g_n(t) \to 0 \ \text{ as } \ n \to \infty. \tag{25}$$

Then the difference of distributions $F_n - G_n \to 0$ properly and both sequences are tight.

Proof. Let $u(x)$ be continuous and bounded function. We prove that when conditions 1) and 2) are satisfied, we have

$$\delta_n = \left|\int_{-\infty}^{\infty} u(x) dF_n(x) - \int_{-\infty}^{\infty} u(x) dG_n(x)\right| \to 0 \ \text{ as } \ n \to \infty. \tag{26}$$

Suppose the opposite, i.e. there exists $\varepsilon_0 > 0$ and a sequence of integer numbers $n_k \to \infty$ such that

$$\delta_{n_k} \ge \varepsilon_0 \ \text{ for all } \ k. \tag{27}$$

Let the sequence $\{G_n\}$ be tight. According to Theorem 3 from § 8 of [6, Part II, Chapter 2, § 8.3, p. 151], one can choose from the subsequence $\{G_{n_k}\}$ a subsequence $\{G_{m_k}\}$ such that $\{m_1, m_2, \ldots\} \subset \{n_1, n_2, \ldots\}$ and $\{G_{m_k}\} \to G$ properly. Here G is some distribution. Let $g(t) = \int_{-\infty}^{\infty} e^{itx} dG(x)$ be the ch.f. of this distribution. Then, by the continuity (convergence) theorem,

$$g_{m_k}(t) \to f(t)$$

at any $t \in \mathbb{R}$. According to the same theorem, it follows from (26) that

$$F_{m_k} \to G$$

properly. Therefore,

$$F_{m_k} - G_{m_k} \to 0 \text{ properly as } m_k \to \infty.$$

Thus,

$$\delta_{m_k} \to 0 \text{ as } m_k \to \infty,$$

what contradicts to (27). Limit relation (26) is proved. From what has been proved, we obtain that when condition 1) is fulfilled, limit relations (25) and (26) are equivalent.

We now turn to the last part of the assertion of Theorem 3. Since the sequence $\{G_n\}$ is tight, any convergent subsequence $\{G_{n_k}\}$ converges properly. But by the fact proved in Theorem 2, the corresponding subsequence $\{F_{n_k}\}$ also converges properly. By virtue of the second assertion of Theorem $3'$ ([6, § 8.4, p. 152]), all such subsequences converge properly if and only if the sequence $\{F_n\}$ is tight. Theorem 3 is proved. □

4. The method of accompanying distributions and the Lindeberg theorem

Let, for $n \geq 1$, two sequences of random variables

$$X_{n1},\ X_{n2},\ \ldots,\ X_{nn}, \tag{28}$$

$$Y_{n1},\ Y_{n2},\ \ldots,\ Y_{nn} \tag{29}$$

be defined on some probability space $(\Omega, \mathcal{F}, P)$ and

$$F_{nj}(x) = P\left(X_{nj} < x\right),\ \ G_{nj}(x) = P\left(Y_{nj} < x\right),\ j = 1, 2, \ldots, n$$

be the corresponding distribution functions.

In most cases it is said that sequences of independent r.v.'s (28) and (29) forming "triangular arrays" are given (see [4, Chapter 8, § 3, p.157]).

Recall that the convolution of d.f.'s $F(x)$ and $G(x)$ is the distribution function

$$\left(F * G\right)(x) = \int\limits_{-\infty}^{\infty} F(x-u)dG(u) = F * G = \int\limits_{-\infty}^{\infty} G(x-u)dF(u) = \left(G * F\right)(x) = G * F.$$

Then, by virtue of the last equalities,

$$\overline{F}_n(x) = P\left(X_{n1} + \cdots + X_{nn} < x\right) = \prod_{j=1}^{n} (*)\, F_{nj}(x) = \overline{F}_n,$$

$$\overline{G}_n(x) = P\left(Y_{n1} + \cdots + Y_{nn} < x\right) = \prod_{j=1}^{n} (*)\, G_{nj}(x) = \overline{G}_n.$$

Next, we put

$$S_n^X = X_{n1} + \cdots + X_{nn},\ \ S_n^Y = Y_{n1} + \cdots + Y_{nn}$$

and suppose that r.v.s included in arrays (28) and (29) satisfy the following conditions for $j = 1, \ldots, n$:

$$\mathbf{E}X_{nj} = \mathbf{E}Y_{nj} = 0, \quad \sigma_{nj}^2 = \mathbf{E}X_{nj}^2 = \int\limits_{-\infty}^{\infty} x^2 dF_{nj}(x) = \mathbf{E}Y_{nj}^2 = \int\limits_{-\infty}^{\infty} x^2 dG_{nj}(x) < \infty. \tag{30}$$

Under conditions (30), without any loss of generality, we can assume that

$$\mathrm{Var}\left(S_n^X\right) = \mathrm{Var}\left(S_n^Y\right) = \sum_{j=1}^{n} \sigma_{nj}^2 = 1.$$

Beforehand we introduce the necessary notations to the characteristic functions of random variables included in "triangular arrays" (28) and (29). Let

$$f_{nj}(t) = \int\limits_{-\infty}^{\infty} e^{itx} dF_{nj}(x), \;\; g_{nj}(t) = \int\limits_{-\infty}^{\infty} e^{itx} dG_{nj}(x),$$

where the index $j = 1, 2, \ldots, n$. Then the characteristic functions of the sums S_n^X and S_n^Y of sequences of random variables (28) and (29)

$$\overline{f}_n(t) = \mathbf{E}e^{itS_n^X} = \int\limits_{-\infty}^{\infty} e^{itx} d\overline{F}_n(x) = \prod_{j=1}^{n} f_{nj}(t),$$

$$\overline{g}_n(t) = \mathbf{E}e^{itS_n^Y} = \int\limits_{-\infty}^{\infty} e^{itx} d\overline{G}_n(x) = \prod_{j=1}^{n} g_{nj}(t).$$

The following theorem takes place.

Theorem 4. *Let one of the sequences* $\left\{\overline{F}_n, n \geq 1\right\}$ *and* $\left\{\overline{G}_n, n \geq 1\right\}$ *be tight. Suppose that, for each* t,

$$\overline{f}_n(t) - \overline{g}_n(t) \to 0 \;\; as \;\; n \to \infty. \tag{31}$$

Then $\overline{F}_n - \overline{G}_n \to 0$ *properly and both sequences are tight. If the sequence* $\left\{\overline{G}_n, n \geq 1\right\}$ *consists of a single distribution* F $(G = \{F, \ldots, F\})$*, then the convergence* $\overline{F}_n - \overline{G}_n \to 0$ *properly turns into a limit theorem*

$$F_n(x) \to F(x) \;\; \left(\overline{F}_n \to F\right) \;\; as \;\; n \to \infty$$

at each point of continuity of the function $F(x)$.

From the last theorem at $G = \{\Phi, \ldots, \Phi\}$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-\frac{u^2}{2}} du,$$

one can obtain the classical version of the Lindeberg theorem.

References

[1] Ibragimov I. A., Linnik Yu. V., *Independent and Stationary Sequences of Random Variables*, Wolters–Noordhoff, Groningen, 1971.

[2] Linnik Yu. V., Ostrovskiy I. V., *The Decomposition of Variables and Vectors*, AMS, Providence, 1977.

[3] Ibragimov I. A., On the accuracy of Gaussian approximation to the distribution functions of sums of independent random variables, *Theory of Probability and its Applications*, **11** (4), 1966, pp.559–579.

[4] Borovkov A. A., *Probabilty Theory*, Springer, London, 2013.

[5] Shiryaev A. N., *Probabilty-1* (in Russian), Moscow Center for Continuous Mathematical Education, Independent Moscow Univ. of Moscow, Moscow, 2011.

[6] Rotar V. I., *Probability Theory*, World Scientific, Singapore, New Jersey, London, Hong Kong, 1997.

[7] Ibragimov I. A., Formanov Sh. K.,Presman E. L., On modifications of the Lindeberg and Rotar' conditions in the central limit theorem, *Theory of Probability and its Applications*, **65** (4), 2021, pp.648–651.

[8] Stein Ch., A bound for the normal approximation of a sum of dependent random variables, *In: Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, Univ. California Press, Berkley, pp.583–602, 1972.

[9] Rotar'V. I., Sholomitskii A., A condition for Convergence of Convolutions, *Theory of Probabability and its Applications*, **37** (2), 1993, pp.398–401.

[10] Senatov V. V., *Central Limit Theorem* (in Russian), Knizhniy Dom "LIBROKOM", Moscow, 2009.

[11] Senatov V. V., The Chebyshev – Hermite Moments and Their Applications to Asymptotic Expansions, *Journal of Mathematical Sciences*, **106** (2), 2001, pp.2873–2889.

[12] Senatov V. V., On Estimation of the Approximation Error in Asymptotic Expansions in the Central Limit Theorem, *Journal of Mathematical Sciences*, **112** (2), 2002, pp.4174–4197.

[13] Formanov Sh. K., M. M. Mamajonov, On one criterion for choosing a Dodge plan with a finite number of checks, *Nadezhnost i kontrol kachestva*, **8**, 1983, pp.60–66.
[14] Formanov Sh. K., On rate of convergence in the multidimensional central limit theorem for homogeneous Markov chains, *Soviet Math. Dokl.*, **204** (1), 1972, pp.49–51.
[15] Formanov Sh. K., On invariance principles for homogeneous Markov chains," Soviet Math. Dokl., **221** (1), 42–44 (1975).
[16] Formanov Sh. K., Some estimates of the rate of convergence in multidimensional limit theorems for homogeneous Markov processes, *In: Maruyama, G., Prokhorov, J.V. (eds), Proceedings of the Third Japan – USSR Symposium on Probability Theory. Lecture Notes in Mathematics*, Vol. 550, 311–329, Springer, Berlin, Heidelberg, 1976.
[17] Sirazhdinov S. H., Formanov Sh. K., On the estimates of the rate of convergence in the central limit theorem for homogeneous Markov chains, *Theory Probab. Appl.*, **28** (2), 1984, pp.229–239.
[18] Formanov Sh. K., The Stein–Tikhomirov Method and a Nonclassical Central Limit Theorem, *Math. Notes*, **71** (4), 2002, pp.550–555.
[19] Formanov Sh. K., Sharipova L. D., A modified variant of the Stein-Chen method and the nonclassical limit theorem on the convergence to a Poisson distribution, *Doklady Akademii Nauk*, **412** (1), 2007, pp.29-30.
[20] Formanov Sh. K.,Formanova T. A., The Stein – Tikhomirov Method and Berry – Esseen Inequality for Sampling Sums from a Finite Population of Independent Random Variables, *In: Shiryaev, A., Varadhan, S., Presman, E. (eds) Prokhorov and Contemporary Probability Theory. Springer Proceedings in Mathematics & Statistics*, vol 33. Part I, Springer, Berlin, Heidelberg, pp.261–275, 2013.
[21] Presman E. L., Formanov Sh. K., On Lindeberg-Feller Limit Theorem, *Doklady Mathematics*, **99** (2), 2019, pp.204–207.
[22] Formanov Sh. K., Juraev Sh. Yu., On transient phenomena in branching random processes with discrete time, *Lobachevskiy Journal of Mathematics*, **42** (12), 2021, pp.2777–2784.
[23] Shevtsova I. G., On Convergence Rate in the Local Limit Theorem for Densities, *Journal of Mathematical Sciences*, 205 (1), 2015, pp.105–112.
[24] Shevtsova I. G., Moment-Type Estimates for Characteristic Functions with Application to Von Mises Inequality, *Journal of Mathematical Sciences*, **214** (1), 2016, pp.119–131.
[25] Shevtsova I. G., On Convergence Rate in the Local Limit Theorem for Densities Under Various Moment Conditions, *Journal of Mathematical Sciences*, **221** (4), 2017, pp.588–608.
[26] Formanov Sh. K.,Khusainova B. B., Analytical proof of one strong version of the law of large numbers, *Global and Stochastic Analysis*, **8** (3), 2021, pp.137–142.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON ASYMPTOTIC EXPANSION FOR A CERTAIN CLASS OF STATISTICS IN GENERALIZED URN MODELS

SHERZOD MIRAKHMEDOV

Institute of Mathematics, Academy of Sciences. Tashkent. Uzbekistan
e-mail: shmirakhmedov@yahoo.com

Abstract. The generalized urn model is defined by the random vector of frequencies which is viewed as conditionally independent random variables, given their sum. Such a representation is exploited here to derive asymptotic approximation for the distribution function of a statistics that has a form of a series of functions of frequencies. Among others, the random allocation of particles of several types into infinitely many cells as well as random allocation of particles in sets are considered as a variants of this generalized urn model.

Keywords: Asymptotic expansion, urn models, random allocation, Poisson distribution, binomial distribution.

AMS Subject Classification: 62G20, 60F05.

1. Introduction

Urn models (also known as a random allocation schemes) are a useful tool which allows to formulate and better understand many combinatorial problems in probability and statistics, see for instance, [1] and [2]. Most general definition of urn models has been introduced by Mirakhmedov et al [3], where the asymptotic theory and higher-order expansions for the statistic of the form of a sum of functions of frequencies are presented. Although their work covers many specific urn models it does not cover, such probabilistic models as, for instance, random allocation of particles into infinitely many cells, random allocation of particles in sets, and the statistics based on several samples from a population(s), intensively studied in the literature, see [4],[5],[6], [7], [8],[9]. The generalized random allocation scheme we are interested here is as follows.

Let $\xi_l(n_l) = (\xi_{l,1}(n_l), \xi_{l,2}(n_l), ...)$, $l = 1, ..., s$, be a collection of sequences (random vectors) of independent non-negative integer random variables (r.v.s) , distribution of $\xi_{l,m}(n_l)$ depend on parameter $n_l \in \aleph = \{1, 2, ...\}$ and such that for each n_l the series $\zeta_l(n_l) = \xi_{l,1}(n_l) + \xi_{l,2}(n_l) + ...$ is a.s. converges, moreover this parameter n_l is such that $Pr\{\zeta_l(n_l) = n_l\} > 0$,$l = 1, ..., s$. Further, let $\eta_l(n_l) = (\eta_{l,1}(n_l), \eta_{l,2}(n_l), ...)$ be a r.vec., distribution of which can be represented as the joint conditional distribution of $\xi_l(n_l)$ given $\zeta_l(n_l) = n_l$, viz.,

$$\Im(\eta_l(n_l)) = \Im(\xi_l(n_l) \mid \zeta_l(n_l) = n_l), \quad l = 1, ..., s, \tag{1}$$

where $\Im(X)$ stands for the distribution of the r.vec. X. This equality implies that integer $\eta_{l,m}(n_l) \geq 0$, $Pr\{\eta_{l,1}(n_l) + \eta_{l,2}(n_l) + ... = n_l\} = 1$, and hence $\eta_l(n_l)$ should be viewed as r.vec. of frequencies in the random allocation of n_l particles into infinitely many cells labeled by $\{1, 2, ...\}$. The r.v. $\eta_{l,m}(n_l)$ then is the number of particles falling into the m-th cell after allocation of all n_l particles. The independent r.vec.s $\eta_1(n_1(n_1), ..., \eta_s(n_s)$ all together can be viewed, for example, as a random allocation of particles of s types, where n_l is the number of

particles of the l- th type, as well as random allocation of particles in sets, where now n_l is the number of particles of l-th set. Our goal here is to present a unified approach to derive asymptotic (as $min(n_1, ..., n_s) \to \infty$) approximation for distribution function of general classes of statistics of the form

$$R(n) = \sum_{m=1}^{\infty} f_m(\eta_{1,m}(n_1), ..., \eta_{s,m}(n_s)), \tag{2}$$

where $f_m(x_1, ..., x_s)$, $m = 1, 2, ...,$ is a sequence of functions (may be random) defined for $x_1 \geq 0, ..., x_s \geq 0$, such that the series (2) is a.s. converges for every $n = (n_1, ..., n_s)$. We emphasize that the allocation scheme is determined by the distribution of r.v,s $\xi_{l,m}(n_l)$. The generalized urn model and statistics considered in [3] follows if $s = 1$ and $Pr\{\xi_{1,m}(n_1) = 0\} = 1$ for $m > N$. some $N = N(n) \to \infty$.

Before proceeding to general results, we list the following special cases of probabilistic model (1) and statistics (2), which are most often encountered in applications.
(i) Independent random allocation scheme. Into an infinite number of cells,labelled by $\{1, 2, ...\}$, we throw particles of s types, where n_k particles of k-th type, $k = 1, ..., s$. The particles are thrown randomly one by one and independently of each other. The probability of a particle of k-th type falling into m-th cell is $p_{k,m} \geq 0$ and $p_{k,1} + p_{k,2} + ... = 1$. The r.vec. $\eta_l(n_l)$ has the multinomial distribution with infinitely many outcomes, viz.,

$$Pr\{\eta_{l,1}(n_l) = k_1, \eta_{l,2}(n_l) = k_2, ...\} = n_l! \prod_{m=1}^{\infty} \frac{(p_{l,m})^{k_m}}{k_m!},$$

for arbitrary non-negative integers $k_1, k_2, ...$ such that $k_1 + k_2 + ... = n_l$. This model determines by $\Im(\xi_{l,m}(n_l)) = Poi(n_l p_{l,m})$, the Poisson distribution with parameter $n_l p_{l,m}$. For instance, the number of occupied cells and the number of cells containing exactly a given number of particles of each types are examples of $R(n)$.For more examples see [4],,[6], [8]. The special case when $p_{l,m} > 0$, $m = 1, ..., N$ and $p_{l,m} = 0$ for all $m > N$, $l = 1, ..., s$, some $N = N(n_1, ..., n_s) \to \infty$ as $min(n_1, ..., n_s) \to \infty$,is well studied in the literature multinomial random allocation model, alternatively known as a sample scheme with replacement from a finite population of size N. The classical chi-square, likelihood-ratio statistic, and the empty-cells statistic are examples of $R_N(n)$. Here and below $R_N(n)$ stands for the statistics $R(n)$, where $Pr\{\xi_{l,m}(n_l) = 0\} = 1$ for all $m > N$,$l = 1, ..., s$.
(ii) Sample scheme without replacement from a finite population . Assume from a stratified population of size $\Omega_N = \omega_1 + ... + \omega_N$, where $\omega_m > 0$ is the size of m-th stratum, s independent samples of sizes $n_1, ..., n_s$, respectively, are drawn. Each sample is carried out according to the sample scheme without replacement. All $C_{\Omega_N}^{n_k}$ possible variants of the kth sample has the same probability equal to $\left(C_{\Omega_N}^{n_k}\right)^{-1}$. This urn model corresponds to the case where the r.vec.of frequencies $\eta_l(n_l) = (\eta_{l,1}(n_l), ..., \eta_{l,N}(n_l))$ has the multivariate hyper-geometric distribution:

$$Pr\{\eta_{l,1}(n_l) = k_1, ..., \eta_{l,N}(n_l) = k_N\} = \frac{C_{\omega_1}^{k_1} ... C_{\omega_N}^{k_N}}{C_{\Omega_N}^{n_l}},$$

for arbitrary non-negative integers $k_1, ..., k_N$ such that $k_1 + ... + k_N = n_l$. This urn model determines by $\Im(\xi_{l.m}(n_l)) = Bi(\omega_m, n_l/\Omega_N)$, the binomial distribution with indicated parameters .For instance the number of untouched strata and the sum of the elements of the population that appeared in the samples are important variants of $R_N(n)$. Note that the random allocation of particles in sets also is a variant of this model .
(iii) Multicolor Pólya- Egenberger urn model which corresponds to the case where $\eta_l(n_l) =$

$(\eta_{l,1}(n_l),...,\eta_{l,N}(n_l))$ has the generalized Pólya- Egenberger distribution:

$$Pr\{\eta_{l,1}(n_l)=k_1,...,\eta_{l,N}(n_l)=k_N\}=\left(C^{n_l}_{D_N+n_l-1}\right)^{-1}\prod_{m=1}^{\infty}C^{k_m}_{d_m+k_m-1}.$$

for arbitrary non-negative integers $k_1,...,k_N$ such that $k_1+...+k_N=n_l$, where $d_m>0$, $m=1,...,N$, $D_N=d_1+...d_N$. For this model $\Im(\xi_{l.m}(n_l))=NBi(d_m,\vartheta)$, the negative binomial distribution with parameters d_m and $\vartheta_l=n_l/(D_N+n_l)$. For example, the number of colors that appear in the sample exactly r times and the number of pairs having the same color, are statistics of the type $R_N(n)$. We note that sum of functions of "spacings-frequencies" under the hypothesis of homogeneity of two samples can be formulated as a variant of $R_N(n)$ in this model; see for instance, Holst and Rao [13].

We emphasis that we are dealing with triangular arrays where all the parameters of urn models (or what is the same random allocation schemes) vary (including the distribution of the r.v.s $\xi_{l,m}(n_l)$) when $min(n_1,...,n_s)\to\infty$ and $N\to\infty$ in case when we are dealing with finite number of cell (finite population), formally through a non-decreasing sequence of positive integers $n_{l,v}$ and N_V , as $v\to\infty$, hence it is important to express the remainder terms in our asymptotic results which show their explicit dependence on the $n_1,...,n_s,N$, the distribution of r.v.s $\xi_{l,m}(n_l)$ and the kernel functions f_m .

The paper is organized as follows. In Section 2 we present a systematic procedure for obtaining an asymptotic expansion for the characteristic function of $R(n)$ to terms of any order. The main results are presented in Section 3. Section 4 is devoted to demonstrate application of the general results in the random allocation of particles of several type into infinitely many cells. In what follows c , C are universal positive constants which may depend on the argument and may be different at different places; all asymptotic relations are considered as $min(n_1,...,n_s)\to\infty$.

2. Asymptotic expansion of the characteristic function of $R(n)$

In what follows, we will assume that the considered moments of the r.v.s exist, the series of r.v.s are a.s. convergent, while the numerical series converge absolutely.

Set $n=(n_1,...,n_s)$, $\xi^s_m(n)=(\xi_{1,m}(n_1),...,\xi_{s,m}(n_s))$, $\eta^s_m(n)=(\eta_{1,m}(n_1),...,\eta_{s,m}(n_s))$, $x^s(n)=(x(n_1),...,x(n_s))$, where

$$x(n_k)=\left(n_k-\lambda(n_k)\right)/B(n_k),\quad \lambda(n_k)=\sum_{m=1}^{\infty}E\xi_{k,m}(n_k),\quad B^2(n_k)=\sum_{m=1}^{\infty}Var\xi_{k,m}(n_k).$$

We adopt also the following notation:

$$\Lambda(n)=\sum_{m=1}^{\infty}Ef_m(\xi^s_m(n)),\gamma_k(n)=\frac{1}{B^2(n_k)}\sum_{m=1}^{\infty}cov\left(f_m(\xi^s_m(n)),\xi_{k,m}(n_k)\right),$$

$$g_m(y_1,...,y_s)=f_m(y_1,...,y_s)-Ef_m(\xi^s_m(n))-\sum_{k=1}^{s}\gamma_k(n)(y_k-E\xi_{k,m}(n_k)),$$

$$G^{\eta}(n)=\sum_{m=1}^{\infty}g_m(\eta^s_m(n)),\quad G^{\xi}(n)=\sum_{m=1}^{\infty}g_m(\xi^s_m(n)),$$

$$\sigma^2(n)=VarG^{\xi}(n)=\sum_{m=1}^{\infty}Varf_m(\xi^s_m(n))-\sum_{k=1}^{s}B^2(n_k)\gamma^2_k(n). \tag{3}$$

We have

$$G^{\eta}(n)=R(n)-\Lambda(n)-\sum_{k=1}^{s}x(n_k)B(n_k)\gamma_k(n),$$

$$Eg_m(\xi_m^s(n)) = 0, m \in \aleph, \ \sum_{m=1}^{\infty} cov\left(g_m(\xi_m^s(n), \xi_{k,m}(n_k)\right) = 0, \ k = 1, ..., s. \tag{4}$$

Under some mild conditions, one can show that as $min(n_1, ..., n_s) \to \infty$

$$ER(n) = \Lambda(n)$$

$$+\sum_{k=1}^{s}\left[x(n_k)B(n_k)\gamma_k(n) - \frac{1-x^2(n_k)}{2B^2(n_k)}\sum_{m=1}^{\infty} E\left(g_m(\xi_m^s(n))(\xi_{k,m}(n_k) - E\xi_{k,m}(n_k))^2\right)(1+o(1))\right],$$

and

$$VarR(n) = \sigma^2(n)(1+o(1)).$$

In addition to above notation we put: $\zeta^s(n) = (\zeta_1(n_1), ..., \zeta_s(n_s))$, $\lambda^s(n) = (\lambda(n_1), ..., \lambda(n_s))$, $\tau^s = (\tau_1, ..., \tau_s)$, $\parallel a \parallel$ and (a, b) stand for the Euclid norm and scalar product,respectively, of vectors $a, b \in R^s$. Equality (subtraction) of two vectors is defined as component-wise.
Due to (1) and the fact that $\eta_1(n_1), ..., \eta_s(n_s)$ is a collection of independent sequences, we have

$$E\exp\left(itG^{\eta}(n)\right) = E\left[\exp\left(itG^{\xi}(n)\right) \mid \zeta_1(n_1) = n_1, ..., \zeta_s(n_s) = n_s\right]. \tag{5}$$

On the other hand by the total expectation formula

$$E\exp\left(itG^{\xi}(n) + i(\tau^s, (\zeta^s(n) - n))\right)$$

$$= \sum_{k_1=0}^{\infty} ... \sum_{k_s=0}^{\infty} E\left[\exp\left(itG^{\xi}(n)\right) \mid \zeta_1(n_1) = n_1, ..., \zeta_s(n_s) = n_s\right] \prod_{l=1}^{s} e^{i\tau_l(k_l - n_l)} Pr\{\zeta_l(n_l) = k_l\}.$$

This together with (5) implies, by Fourier inversion

$$E\exp\left(itG^{\eta}(n)\right) = \frac{1}{(2\pi)^s \prod_{l=1}^{s} Pr\{\zeta_l(n_l) = n_l\}} \int_{-\pi}^{\pi} ... \int_{-\pi}^{\pi} E\exp\left(itG^{\xi}(n) + i(\tau^s, (\zeta^s(n) - n))\right) d\tau_1 ... d\tau_s \tag{6}$$

Denote

$$\tilde{\xi}_{l,m}(n_l) = B^{-1}(n_l)(\xi_{l,m}(n_l) - E\xi_{l,m}(n_l)), \quad \tilde{\xi}_m^s(n) = (\tilde{\xi}_{1,m}(n_1), ..., \tilde{\xi}_{s,m}(n_s)),$$

$$\tilde{g}_m(\xi_m^s(n)) = \sigma^{-1}(n) g_m(\xi_m^s(n)),$$

$$\Psi(n, t, \tau^s) = \prod_{m=1}^{\infty} E\exp\left(it\tilde{g}_m(\xi_m^s(n)) + i(\tau^s, \tilde{\xi}_m^s(n))\right),$$

$$\Theta(n, x_s(n), t) = \frac{1}{(2\pi)^{s/2}} \int_{-\pi B(n_1)}^{\pi B(n_1)} ... \int_{-\pi B(n_s)}^{\pi B(n_s)} e^{-i(\tau^s, x^s(n))} \Psi(n, t, \tau^s) d\tau_1 ... d\tau_s \tag{7}$$

Then equation (6) together with the inversion formula for the local probabilities $Pr\{\zeta_l(n_l) = n_l\}$ gives us the following
Lemma 1. It holds

$$\varphi(n, t) =: E\exp\left(it\sigma^{-1}(n)G^{\eta}(n)\right) = \frac{\Theta(n, x_s(n), t)}{\Theta(n, x_s(n), 0)}. \tag{8}$$

This lemma provides the crucial formula of interest. Special variants of (8) show up in literature; see e.g. [3],[6],[8],[11].
A formal construction of the asymptotic expansion for $\varphi(n, t)$ defined in (8) proceeds as follows: The $\Psi(n, t, \tau^s)$ is the characteristic function (ch.f.) of the sum (the series) of independent $(s+1)$ -dimensional r.vec.s $(\tilde{g}_m(\xi_m^s(n)), \tilde{\xi}_m^s(n))$, . Because of (3) and (4) this sum has zero expectation and uncorrelated components, and it has a unit covariance matrix. Keeping in the mind these important facts, we now use the well-known method of asymptotic expansion of the ch.f. of finite sum of independent r.vec.s, see Bhattacharya and Rao [14, Chapter 2] to derive the asymptotic

expansion of $\Psi(n,t,\tau^s)$. In our case under suitable conditions the terms of this expansion has form of polynomials, say $P_k(n,t,\tau^s)$,$k=1,2,...$, wrt $t,\tau_1,...,\tau_s$ containing the common factor $\exp\left(-(t^2+\tau_1^2+...+\tau_s^2)/2\right)$. The degree of $P_k(n,t,\tau^s)$ is $3k$ and the minimal degree is $k+2$. Hence these terms multiplied by $\exp\left(-i(\tau^s,x^s(n))\right)$ can be integrated wrt each of $\tau_1,...,\tau_s$ over the interval $(-\infty,\infty)$. A result of the integrations, say $H(n,x^s(n),t)$, can be considered as a formally asymptotic expansion of $\Theta(n,x^s(n),t)$. Finally, we get the asymptotic expansion of $\varphi(n,t)$ by dividing $H(n,x^s(n),t)$ by $H(n,x^s(n),0)$.
Define the polynomials $G_k(n,x^s(n),t)$ wrt t for $k=1,2,...$ as

$$G_k(n,x^s(n),t)=\frac{1}{(2\pi)^{s/2}}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty}P_k(n,t,\tau_1,...,\tau_s)\exp\left(-\sum_{l=1}^{s}(\tau_l+ix_l(n_l))^2/2\right)d\tau_1...d\tau_s \tag{9}$$

Then

$$H(n,x^s(n),t)=\exp\left(-\frac{t^2+\parallel x^s(n)\parallel^2}{2}\right)\left(1+G_1(n,x^s(n),t)+G_2(n,x^s(n),t)+...\right).$$

We emphasis that the coefficients of $P_k(n,t,\tau^s)$, and hence of $G_k(n,x^s(n),t)$, only involve the quantities $\prod_{l=1}^{k}\left(\alpha_{j,j_1,...,j_l}\right)^{\nu_l}$, where $j+j_1+...+j_k=l+2$ for the non-negative integers $j,j_1,...,j_r$ and $\nu_1+2\nu_2+...+k\nu_k=k$, $\alpha_{j,j_1,...,j_l}=\Sigma_{m=1}^{\infty}E\left(\tilde{g}_m^j(\xi_m^s(n))\tilde{\xi}_{1,m}^{j_1}(n_1)\cdot...\cdot\tilde{\xi}_{s,m}^{j_s}(n_s)\right)$.
For the details see [14, Chapter 2, Section 7]. For instance,

$$P_1(n,t,\tau^s)=\frac{i^3}{6}\sum_{m=1}^{\infty}E\left(t\tilde{g}_m(\xi_m^s(n))+(\tau^s,\tilde{\xi}_m^s(n))\right)^3,$$

$$P_2(n,t,\tau^s)=\frac{i^4}{24}\sum_{m=1}^{\infty}\left[E\left(t\tilde{g}_m(\xi_m^s(n))+(\tau^s,\tilde{\xi}_m^s(n))\right)^4-3\left(E\left(t\tilde{g}_m(\xi_m^s(n))+(\tau^s,\tilde{\xi}_m^s(n))\right)^2\right)^2\right]+$$

$$+\frac{1}{2}P_1^2(n,t,\tau^s).$$

Set

$$a_{i,j}^{(l)}(n)=\sum_{m=1}^{\infty}E\left(g_m^i(\xi_m^s(n))\xi_{l,m}^j(n_l)\right),a_k(n)=\sum_{m=1}^{\infty}Eg_m^k(\xi_m^s(n)).$$

Then, taking into account the fact that $\tilde{\xi}_{l,m}(n_l)$ are independent r.v.s and $E\tilde{\xi}_{l,m}(n_l)=0$, we obtain

$$P_1(n,t,\tau^s)=\frac{(it)^3}{6}\frac{a_3(n)}{\sigma^3(n)}+\frac{(it)^2}{2\sigma^2(n)}\sum_{l=1}^{s}\frac{(i\tau_l)a_{2,1}^{(l)}(n)}{B(n_l)}+\frac{it}{2\sigma(n)}\sum_{l=1}^{s}\frac{(i\tau_l)^2a_{1,2}^{(l)}(n)}{B^2(n_l)}+\sum_{l=1}^{s}\frac{(i\tau_l)^3a_{0,3}^{(l)}(n)}{6B^3(n_l)},$$

and hence by (9)

$$G_1(n,x^s(n),t)=\frac{(it)^3}{6}\frac{a_3(n)}{\sigma^3(n)}+\frac{(it)^2}{2\sigma^2(n)}\sum_{l=1}^{s}\frac{x(n_l)a_{2,1}^{(l)}(n)}{B(n_l)}-\frac{it}{2\sigma(n)}\sum_{l=1}^{s}\frac{(1-x^2(n_l))a_{1,2}^{(l)}(n)}{B^2(n_l)}$$

$$-\sum_{l=1}^{s}\frac{((x^3(n_l)-3x(n_l))a_{0,3}^{(l)}(n)}{6B^3(n_l)}.$$

Obviously, from the above formula for $P_2(n,t,\tau^s)$ one can derive an explicit formula for $G_1(n,x^s(n),t)$, but at the expense of a long formula, so we omit it.
Assume $E|g_m(\xi_m^s(n))|^d<\infty$ and $E|\xi_{l,m}(n_l)|^d<\infty$, $m\in\aleph$, $l=1,...,s$, some $d\geq 3$, and define $Q_k(n,x^s(n))$ from the equation

$$\sum_{k=1}^{\infty}(-1)^k\left(\sum_{\nu=0}^{d-3}G_\nu(n,x^s(n),0)\right)^k=\sum_{m=0}^{\infty}Q_m(n,x^s(n)).$$

Then

$$Q_m(n,x^s(n))=m!\sum\prod_{l=1}^{d-3}\frac{1}{m_l!}G_l^{m_l}(n,x^s(n),0) \tag{10}$$

where the summation is over all non-negative integer $m_1,...,m_{d-3}$ such that $m_1+2m_2+...+(d-3)m_{d-3}=m$. Let

$$W^{(d)}(n,x^s(n),t)=\sum_{m=0}^{d-3}\sum_{\nu=0}^{m}G_\nu(n,x^s(n),t)Q_{m-\nu}(n,x^s(n)). \tag{11}$$

Note that $G_0(n,x^s(n),t)=Q_0(n,x^s(n))=1$, so that $W^{(3)}(n,x^s(n),t)=1$. For example we have

$$W^{(4)}(n,x^s(n),t)=1+\left(G_1(n,x^s(n),t)-G_1(n,x^s(n),0)\right), \tag{12}$$

$$W^{(5)}(n,x^s(n),t)=W^{(4)}(n,x^s(n),t)$$
$$+\left(G_2(n,x^s(n),t)-G_2(n,x^s(n),0)-G_1(n,x^s(n),0)(G_1(n,x^s(n),t)-G_1(n,x^s(n),0))\right). \tag{13}$$

Finally we can conclude that

$$\varphi(n,t)=H(n,x^s(n),t)H^{-1}(n,x^s(n),0)=\exp\left(-t^2/2\right)W^{(d)}(n,x^s(n),t)+remainder.$$

Let us to present a result on asymptotic approximation of the ch.f. $\varphi(n,t)$. To this end we need the following additional notation.

$$\beta_k(n)=\frac{1}{\sigma^k(n)}\sum_{m=1}^{\infty}E|g_m(\xi_m^s(n))|^k,\kappa_k(n_l)=\frac{1}{B^k(n_l)}\sum_{m=1}^{\infty}E|\xi_{l,m}(n_l)-E\xi_{l,m}(n_l)|^k,$$

$$L_k(n)=\beta_k(n)+\kappa_1(n_l)+...+\kappa_s(n_l),$$

$$M_l(n,T)=\inf_{T\le u\le\pi}\sum_{m=1}^{\infty}\left(1-|E\exp\left(iu\xi_{l,m}(n_l)\right)|^2\right),\ \text{if } T\le\pi,\ \text{else } M_l(n,T)=\infty \tag{14}$$

$$\Upsilon_d=L_d(n)+\prod_{l=1}^{s}B(n_l)\exp\left(-\frac{1}{8s}M_l(n_l,0.4(B(n_l)\kappa_{3,l}(n_l))^{-1})\right), \tag{15}$$

$$\nabla(n)=\left(\min_{1\le l\le s}M_l(n_l,0.4(B(n_l)\kappa_{3,l}(n_l))^{-1})\right)^{-1/2}+\max_{1\le l\le s}\frac{B(n_l)}{M_l(n_l,0.4(B(n_l)\kappa_{3,l}(n_l))^{-1})} \tag{16}$$

$$T(n)=min\left(\beta_3^{-1}(n),\nabla^{-1}(n)\right).$$

Proposition 1. Let $|x(n_l)|\le C$, $E|g_m(\xi_m^s)|^d<\infty$, $E|\xi_{l,m}(n_l)|^d<\infty$, some $d\ge3$, $l=1,...,s$, $m\in\aleph$, and $\Upsilon_d(n)\le0.01$. Then there exist constants C and $C(s,d)$ such tha for $\mid t\mid\le cT(n)$ one has

$$\mid\varphi(n,t)-\exp\left(-t^2/2\right)W^{(d)}(n,x^s(n),t)\mid\le C(s,d)\exp\left(-t^2/8s\right)\Upsilon_d(n).$$

3. Approximation of the distribution of $R(n)$

We use the notations of Sec.2. We assume that $|x(n_l)| \le C$, $E|g_m(\xi_m^s)|^d < \infty$, $E|\xi_{l,m}(n_l)|^d < \infty$, some $d \ge 3$, $l = 1, ..., s$, $m \in \aleph$. Define $\mathbb{W}^{(d)}(n, x^s(n), u)$ so that

$$\int_{-\infty}^{\infty} e^{itu} d\mathbb{W}^{(d)}(n, x^s(n), u) = W^{(d)}(n, x^s(n), t)e^{-t^2/2} \tag{17}$$

The function $\mathbb{W}^{(d)}(n, x^s(n), u)$ can be obtained by formally substituting

$$(-1)^v \frac{d^v}{du^v}\Phi(u) = -e^{-t^2/2}H_{v-1}(u)/\sqrt{2\pi}, \quad where \quad \Phi(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-t^2/2}dt,$$

instead of $(it)^v$ for each v in the expression $W^{(d)}(n, x^s(n), t)$, where $H_v(u)$ is he v-th order Hermite -Chebuishev polynomial(see [14], p. 136-137).Note that $\mathbb{W}^{(3)}(n, x^s(n), u) = \Phi(u)$ and by(12)

$$\mathbb{W}^{(4)}(n, x^s(n), u) = \Phi(u)$$

$$+\frac{1}{\sqrt{2\pi}}e^{-u^2/2}\left[\frac{(1-u^2)a_3(n)}{6\sigma^3(n)} - \frac{u}{2\sigma^2(n)}\sum_{l=1}^{s}\frac{x(n_l)a_{2,1}^{(l)}(n)}{B(n_l)} + \frac{1}{3\sigma(n)}\sum_{l=1}^{s}\frac{(1-x^2(n_l))a_{1,2}^{(l)}(n)}{B^2(n_l)}\right].$$

We emphasis that in all above listed examples of the generalized urn models, the distributions of the r.v.'s $\xi_{l,m}(n_l)$ are such that $\lambda(n_l) = n_l$, and hence $x(n_l) = 0$,$l = 1, ..., s$. so the terms of asymptotic expansion, i.e. the function $\mathbb{W}^{(4)}(n, 0, u)$ is considerably simplified. For example

$$\mathbb{W}^{(4)}(n, 0, u) = \Phi(u) + \frac{1}{\sqrt{2\pi}}e^{-u^2/2}\left[\frac{(1-u^2)a_3(n)}{6\sigma^3(n)} + \frac{1}{3\sigma(n)}\sum_{l=1}^{s}\frac{a_{1,2}^{(l)}(n)}{B^2(n_l)}\right]. \tag{18}$$

Set

$$\Delta^{(d)}(n) = \sup_{-\infty<u<\infty}|Pr\{R(n) < u\sigma(n) + \Lambda(n) + \sum_{l-1}^{s} x(n_l)B(n_l)\gamma_l(n)\} - \mathbb{W}^{(d)}(n, x^s(n), u)|$$

$$\chi(n, a, b) = I\{a < b\}\int_a^b \frac{|\varphi(n, t)|}{|t|}dt \quad , \quad \bar{\chi}(n, a, b) = I\{a < b\}\int_a^b |\varphi(n, t)|dt$$

Theorem 1.There exists a constant $C(s)$ such that

$$\Delta^{(3)}(n) \le C(s)\left(L_3(n) + \nabla(n)\right).$$

Theorem 2. There exist constants c and $C(d, s)$ such that

$$\Delta^{(d)}(n) \le C(d, s)\Upsilon_d(n) + \chi(n, cT(n), \beta_d^{-1}(n)).$$

Theorem 3 . Let the statistic $R(n)$ be a lattice r.v. with span h and a set of possible values $\Re$. There exist constants c and $C(d, s)$ such that .

$$\sup_{z\in\Re}|\frac{\sigma(n)}{h}Pr\{R(n) = z\} - \frac{d}{du}\mathbb{W}^{(d)}(n, x^s(n), u)| \le C(d, s)\Upsilon_d(n) + \bar{\chi}(cT(n), \pi\sigma(n)/h)$$

The proof of the assertions of section 2 and 3 goes on the line of the method of Mirakhmedov et al [14]) along with modifications and complex algebra. So we omit it.

4. Applications

Assuming $Pr\{\xi_{l,m}(n_l)=0\}=1$, for all $m>N$ and $l=1,...,s$, where $N=N(n_1,...,n_s)\to\infty$ as $min(n_{1,...,n_s})\to\infty$ one can obtain from Theorems 1-3 results corresponding to the models of random allocation into finite number N of cells. Of these, in particular, follow the results of [3], [9], corresponding to special variants of statistics $R_N(n)$. To keep the length of the paper reasonable here we consider just three examples of statistic (2).

Example 1. Into an infinitely many cells labelled by $\aleph=1,2,...$ we throw particles of s types, where n_k particles are of the kth type. The particles are thrown randomly one at a time and independently of each other. The probability that a particle of kth type falling into mth cell is $p_{k,m}=p_{k,m}(n_l)\geq 0$, $p_{k,1}+p_{k,2}+...=1$. Then $\eta_{k,m}(n_l)$ is the nmber of particles of kth type falling into mth cell after allocating all of particles.We consider the r.v. $\mu_r(n)$, where $r=(r_1,...,r_s)$, integers $r_j\geq 0$, $r_1+...+r_s\geq 1$, which is the number of cells containing exactly r_k particles of kth type,$k=1,...,s$, after allocating all particles of all types, viz.,

$$\mu_r(n)=\sum_{m=1}^{\infty} I\{\eta_{1,m}(n_1)=r_1,...,\eta_{s,m}(n_s)=r_s\}$$

where $I\{A\}$ denotes the indicator of the event A. For this model $\Im(\xi_{k,m}(n_k))=Poi(\theta_{k,m})$, where $\theta_{k,m}=n_kp_{k,m}$, hence $\lambda(n_k)=n_k$, $x(n_k)=0$, $B(n_k)=n_k$. Set $\pi_k(a)=e^{-a}a^k/k!$, $a>0$,

$$\Pi_m(n,r)=\prod_{k=1}^{s}\pi_{r_r}(\theta_{k,m}),\quad P_{i,j}=\sum_{m=1}^{\infty}p_{i,m}^j.$$

The notation (14), (15)becomes as follows

$$\Lambda_r(n)=\sum_{m=1}^{\infty}\Pi_m(n,r)\ ,\ \gamma_l(n)=\frac{1}{n_l}\sum_{m=1}^{\infty}(r_l-\theta_{l,m})\Pi_m(n,r)$$

$$g_{r,m}(y_1,...,y_s)=I\{y_1=r_1,...,y_s=r_s\}-\Pi_m(n,r)-\sum_{l=1}^{s}\gamma_l(n)(y_l-\theta_{l,m}),$$

$$\sigma_r^2(n)=\sum_{m=1}^{\infty}\Pi_m(n,r)(1-\Pi_m(n,r))-\sum_{l=1}^{s}n_l\gamma_l^2(n),$$

$$\beta_{r,k}(n)=\sigma_r^{-k}(n)\sum_{m=1}^{\infty}E|g_{r,m}(\xi_m^s(n))|^k,\quad \kappa_{3,l}(n_l)\leq\sqrt{\kappa_{4,l}(n_l)}=\sqrt{3P_{l,2}(n)+n_l^{-1}}$$

Alike inequality (12) of Mirakhmedov [9] we have

$$M_l(n_l,0.4(B(n_l)\kappa_{3,l}(n_l))^{-1})\geq 0.1(1-e^{-1})n_l/(3n_lP_{l,2}+1).$$

Hence (see (16))

$$\nabla(n)=\max_{1\leq l\leq s}(\sqrt{n_l}P_{l,2}+n_l^{-1}),\quad T(n)=\min_{1\leq l\leq s}(\sqrt{n_l}/(1+n_lP_{l,2}),$$

and

$$\Upsilon_{r,4}(n)=\beta_{r,4}(n)+\sum_{l=1}^{s}\left(P_{l,2}+\frac{1}{n_l}\right)+\prod_{l=1}^{s}\exp\left(-\frac{n_l}{80s(3n_lP_{l,2}+1}\right).$$

Denote $\delta(n)=|\{m:\theta_{1,m}+...+\theta_{s,m}\geq 1\}|$, the number of elements of the indicated set.

Corollary 1 There exists a constant C(s,r) such that

$$\sup_{-\infty<u<\infty}|\ Pr\{\mu_r(n)<u\sigma_r(n)+\Lambda_r(n)\}-\Phi(u)\ |\leq C(s,r)\left(\beta_3(n)+\sum_{l=1}^{s}\left(\sqrt{n_l}P_{l,2}+\frac{1}{\sqrt{n_l)}}\right)\right).$$

Let $\mathbb{W}_r^{(4)}(n,0,u)$ defined as in (18)where instead of function g_m the above function $g_{r,m}$ is putted.
Corollary 2. There exist a positive constant $C(s,r)$ and $\rho \in (0,1)$ such that

$$\sup_{-\infty<u<\infty} \mid Pr\{\mu_r(n) < u\sigma_r(n) + \Lambda_r(n)\} - \mathbb{W}^{(4)}(n,0,u) \mid \leq C(s,r)\Upsilon_{r,4}(n) + \rho^{\delta(n)} \log(\beta_{r,4}^{-1}(n)).$$

Corollary 3. There exist a positive constant $C(s,r)$ and $\rho \in (0,1)$ such that

$$\sup_{z\in\mathbb{Z}^+} |Pr\{\mu_r(n) = z\} - \frac{d}{du_z}\mathbb{W}^{(4)}(n,0,u_z)| \leq C(s,r)\Upsilon_{r,4}(n) + \rho^{\delta(n)}\sigma_r(n),$$

where $u_z = (z - -\Lambda_r(n))/\sigma_r(n)$, $\mathbb{Z}^+$ is the set of non-negativ integers.
Example 2. Consider the allocation scheme of Example 1, where now $p_{l,m} > 0$ for $m = 1, ..., N$ and $p_{l,m} = 0$ for all $m > N$, $l = 1, ..., s$, $n = N(n_1, ..., n_s) \to \infty$ as $min(n_1, ..., n_s) \to \infty$. That is each n_l particles of lth type are allocated into a finite number N of cels by a multinomial allocation scheme,see [7]. We are interested here in r.v. $\mu_0(n)$, the number of empty cells after allocating of particles of all types, viz.,

$$\mu_0(n) = \sum_{m=1}^{N} I\{\eta_{1,m}(n_1) = 0, ..., \eta_{s,m}(n_s) = 0\}$$

We assume that

$$Np_{l,m} \leq C. \tag{19}$$

We will use the above notation, keeping in the mind the conditions of this example and adding suffixes 0 and N. whenever necessary. Recall notation $\theta_{k,m} = n_k p_{k,m}$ and set $\theta_m = \theta_{i,m} + ... + \theta_{s,m}$

$$\Lambda_0(n) = \sum_{m=1}^{N} \exp(-\theta_m), \quad \gamma_l = \frac{1}{n_l}\sum_{m=1}^{N} \theta_{l,m}\exp(-\theta_m),$$

$$\sigma_0^2(n) = \sum_{m=1}^{N}\left[\exp(-\theta_m)(1-\exp(-\theta_m)) - N^{-1}\sum_{j=1}^{s} n_j\gamma_j^2\right]$$

Corollary 4. There exist a constant $C(s)$ such that

$$\sup_{-\infty<u<\infty} |Pr\{\mu_0(n) < u\sigma_0(n) + \Lambda_0(n)\} - \Phi(u)| \leq C(s)\left(\frac{1}{\sigma_0(n)} + \sum_{j=1}^{s}\frac{1}{\sqrt{n_j}}\right).$$

Define $\mathbb{W}_0^{(4)}(n,0,u)$ as in (18), where the function g_m replaced by $g_{0,m} = I\{\xi_{1,m}(n_1) + ... + \xi_{s,m}(n_s) = 0\} - \exp(-\theta_m) + \sum_{j=1}^{s}\gamma_j(\xi_{j,m}(n_j) - \theta_{j,m})$.

Corollary 5. There exist a constant $C(s)$ and $\rho \in (0,1)$ such that

$$\sup_{-\infty<u<\infty} |Pr\{\mu_0(n) < u\sigma_0(n) + \Lambda_0(n)\} - \mathbb{W}^{(4)}(n,0,u_u))| \leq C(s)\left(\frac{1}{\sigma_0^2(n)} + \sum_{j=1}^{s}\frac{1}{n_j} + \rho^N\right).$$

Example 3. Consider a random allocation of particles in sets, which is a variant of the urn model presented in example (ii)of introduction, where it is assumed that $\omega_1 = ... = \omega_N = 1$. Namely, let the cells be indexed by $1, 2, ..., N$ and the sets of particles be indexed by $1, 2, ..., s$.The set of particles with index j contains n_j particles. We assume

• Particles are allocated one set at a time. • Sets of particles are allocated independently of each other. • No two particles from a given set can be allocated to a common cell. • All $C_N^{n_j}$ possible variants of the allocation of particles from the jth set have the same probability equal

to $\left(C_N^{n_j}\right)^{-1}$, $j = 1, ..., s$. Again, we consider the number of empty cells left after allocatingall s sets of particles, viz.,

$$\nu_0(n) = \sum_{m=1}^{N} I\{\eta_{1,m}(n_1) + ... + \eta_{s,m}(n_s) = 0\}.$$

We recall that this allocation scheme corresponds to an urn model when we deal with s independent samples of sizes $n_1, ..., n_s$, respectively, where each sample is drawn by simple sample scheme without replacement from a population of size N; here the r.v. $\nu_0(n)$ is a number of untouched elements of the population. Denote $p_l = n_l/N$, $q_l = 1 - p_l$, $Q_s = q_1 \cdot ... \cdot q_s$, $\sigma^2 = Q_s\left(1 - Q_s\left(1 + \sum_{l=1}^{s} p_l q_l^{-1}\right)\right)$. Set $\Im(\xi_l) = Bi(1, p_l)$ and $g = g(\xi_1, ..., \xi_s) = I\{\xi_1 + ... + \xi_s = 0\} - Q_s + Q_s \sum_{l=1}^{s} q_l^{-1}(\xi_l - p_l)$.

Corollary 6. There exists a constant $C(s)$ such that

$$\sup_{-\infty<u<\infty} |Pr\{\nu_0(n) < u\sigma\sqrt{N} + NQ_s\} - \Phi(u)| \le C(s)\left(\frac{E|g|^3}{\sigma^3\sqrt{N}} + \sum_{l=1}^{s}\frac{1}{\sqrt{n_l p_l q_l}}\right)$$

We emphasis that for this example $a_{1,2}^{(l)}(n) = 0$, see above notation.

Corollary 7. Assume $\sigma^2 E|g|/E|g|^3 \le c \min_{1\le l\le s}(p_l q_l)$. Then there exists a constant $C(s)$ such that

$$\sup_{-\infty<u<\infty} |Pr\{\nu_0(n) < u\sigma\sqrt{N} + NQ_s\} - \Phi(u) + \frac{(1-u^2)e^{-u^2/2}Eg^3}{6\sqrt{2\pi N}\sigma^3}|$$
$$\le C(s)\left(\frac{E|g|^4}{N\sigma^4} + \sum_{l=1}^{s}\frac{1}{n_l p_l q_l} + N^{(s+1)/2}e^{-cN\sigma}\right).$$

References

[1] Kotz S. and Balakrishnan N., (1997). Advances in urn models during the past two decades. In Advances in Combinatorial Methods and Appl. to Probab. and Statist., 203-257. Birkhauser, Boston. MA.

[2] Gnedin A. Hansen B. and Jim Pitman (2007). Notes on the occupancy problem with infinitely many boxes: general asymptotics and power laws. Probability Survey, v. 4, pp. 146-171. http://dx.doi.org/10.1214/07-PS092

[3] Mirakhmedov S.M. • Jammalamadaka S.R.• Ibrahim B. M (2014), On Edgeworth Expansions in Generalized Urn Models.. J Theor Probab (2014) 27:725–753 DOI 10.1007/s10959-012-0454-z

[4] Mikhaylov V. (1993). Polynomial and polynomial like allocation: Resent developments. In: Probab. Methods in Discrete Math., Proc., TVP/VSP Utrecht/Moscow, V.F. Kolchin et al (eds), 40-59.

[5] Vatutin, V.A., Mikhailov, V.G.(1982), Limit theorems for the number of empty cells in an equiprobable scheme for group allocation of particles. Theory Probab. Appl. 27, 734–743

[6] Karlin, S., (1967), Central limit theorems for certain infinite urn schemes, J. Math. Mech., 17: pp. 373-401

[7] Kolchin V.F., Sevast'yanov, B.A., and Chistyakov, V.P. (1978). Random Allocations, V.H. Winston and Sons, Washington, DC.

[8] Mirakhmedov Sh. A.(1992).Randomized decomposable statistics in the scheme of independent allocating particles into boxes. Discrete Math. Appl. 2 , 91–108.

[9] Mirakhmedov Sh. A.(1996). Limit theorems on decomposable statistics in a generalized allocation schemes. Discrete Math. Appl., 6, 379-404.

[10] Mirakhmedov Sh. M.(2007). Asymptotic normality associated with generalized occupancy problem. Statist. Probab. Letters, 77, 1549-1558.

[11] Mirakhmedov S.S and Miraakhmedov S.M. (2010) On Asymptotic Expansion in the Random Allocation of Particles by Sets. J Theor Probab., v.23: pp. 904–919 DOI 10.1007/s10959-009-0225-7

[12] Holst L., (1979). A unified approach to limit theorems for urn models. J. Appl., Probab., 16,Issue 1, 154-162.

[13] Holst L. and Rao J. S. (1980). Asymptotic theory for families of two-sample nonparametric statistics, Sankhya, 42, Ser. A, 19-52.

[14] Bhattacharya R.N. and R. Ranga Rao (1976). Normal approximation and asymptotic Expansions. John Wiley and Sons, New-York.
[15] Petrov, V.V.: Limit Theorems of Probability Theory. Oxford University Press, London (1995)

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ЛОКАЛЬНАЯ АСИМПТОТИЧЕСКАЯ НОРМАЛЬНОСТЬ ЭКСПЕРИМЕНТОВ ПРИ НЕОДНОРОДНОМ СЛУЧАЙНОМ ЦЕНЗУРИРОВАНИИ ИНТЕРВАЛОМ НЕНАБЛЮДЕНИЯ

АБДУРАХИМ АБДУШУКУРОВ[1], НАРГИЗА НУРМУХАМЕДОВА[2]

[1] Филиал МГУ имени М.В. Ломоносова в городе Ташкенте, Ташкент, Узбекистан
[2] Национальный университет Узбекистана, Ташкент, Узбекистан
e-mail: rasulova_nargiza@mail.ru

Аннотация. Статистика отношения правдоподобия (СОП) играет ключевую роль в теории принятия решений. Так, например при проверке простой гипотезы H_0 против сложной альтернативы H_1 о неизвестном законе распределения критерии, построенные на основе СОП согласно фундаментальной лемме Неймана-Пирсона является равномерно – наиболее мощными при любом объёме n наблюдений (см. [1,2]). Довольно интересные задачи возникают, когда альтернативы H_1 зависят от n и являются «близкими» к H_0, т.е. $H_1 = H_{1n} \to H_0$ при $n \to \infty$. В таких ситуациях проявляются асимптотические свойства СОП, полезные в теории оценивания неизвестных параметров и проверки гипотез. К таким свойствам относятся локальная асимптотическая нормальность СОП. Имеется достаточное количество литературы, посвященной исследованию свойства локальной асимптотической нормальности СОП и его использованию в статистике. Следует особо отметить работы [3-7], в которых показано, что локальная асимптотическая нормальность даёт возможность развития асимптотической теории оценок максимального правдоподобия и байесовских оценок, а также контигуальности семейств вероятностных распределений. В работах авторов [8-10] свойства локальная асимптотическая нормальность для СОП установлены в некоторых моделях неполных наблюдений, получаемых случайным цензурированием наблюдений в модели конкурирующих рисков (МКР). В данной работе также исследованы свойства локальной асимптотической нормальности СОП в МКР при неоднородном случайном цензурировании интервалом ненаблюдения.

Ключевые слова: конкурирующие риски, случайное цензурирование, статистика отношения правдоподобия, локальная асимптотическая нормальность.

Предметная классификация AMS: 62F05

Рассмотрим неоднородную МКР при случайном цензурировании случайных величин (с.в.) интервалом ненаблюдений. Пусть $\{X_m,\, m \geq 1\}$ - последовательность независимых с.в. с соответствующей функцией распределения (ф.р.) $\{H_m(x;\theta),\, m \geq 1\}$, определенную на вероятностном пространстве $(\Omega,\, A,\, P)$ и со значениями в измеримом пространстве $(X_m,\, B_m)$, $m \geq 1$ где X_m множество всевозможных значений с.в. X_m и $B_m = \sigma(X_m)$. Здесь θ неизвестный s-мерный параметр и $\Theta \subseteq R^s$-параметрическое пространство, открытое множество в R^s. Наряду с с.в. X_m наблюдается также и полная группа событий

$\left\{A_m^{(1)}, ..., A_m^{(k)}\right\}$, $m \geq 1$. Пусть $H_m^{(i)}(x;\theta) = P_\theta\left(X_m < x,\, \delta_m^{(i)} = 1\right), i = 1, ...k$, $m \geq 1$, субраспределения такие, что $H_m^{(1)}(x;\theta) + ... + H_m^{(n)}(x;\theta) = H(x;\theta)$ для всех $(x;\theta) \in R^1 \times \Theta$. Через $h_m^{(i)}(x;\theta) = \frac{\partial H_m^{(i)}(x;\theta)}{\partial x}$, $i = 1, ...k$, $m \geq 1$, плотности субраспределений такие, что $h_m^{(1)}(x;\theta) + ... + h_m^{(k)}(x;\theta) = \frac{\partial H_m(x;\theta)}{\partial x} = h_m(x;\theta)$ для всех $(x;\theta) \in R^1 \times \Theta$. Рассмотрим случай, согласно которой совокупность $\left\{\left(X_m;\ A_m^{(1)}, ..., A_m^{(k)}\right),\, m \geq 1\right\}$ наблюдается вне интервала $\{[Y_{1m}, Y_{2m}],\, m \geq 1\}$, где (Y_{1m}, Y_{2m}) последовательность независимых случайных векторов с совместной ф.р. $G_m(u,v) = P(Y_{1m} \leq u, Y_{2m} \leq v)$, где $(u;v) \in R^2 = R^1 \times R^1$. Пусть пары (Y_{1m}, Y_{2m}) независят от совокупности $\left(X_m;\ A_m^{(1)}, ..., A_m^{(k)}\right)$, $m = 1, ..., n$. Предположим, что $P(Y_{1m} \leq Y_{2m}) = 1$, $m = 1, ..., n$. Реально это может соответствовать случаю, когда наблюдения над m - объектом с временем жизни X_m, подвергается k конкурирующим рискам, могут быть прикрашены в случайный момент Y_{1m} и восстановлены в случайный момент Y_{2m}. Очевидно такая схема наблюдений содержит в себе модели случайного цензурирования с одной стороны (слева и/или справа) и с двух сторон. Тогда вместо совокупности $\left(X_m;\ A_m^{(1)}, ..., A_m^{(k)}\right)$ наблюдаются вектора

$$\left\{\left(Z_m;\ D_m^{(0)}, D_m^{(1)}, ..., D_m^{(k)}\right),\, m = 1, ..., n\right\},$$

где $Z_m = w_m X_m + (1 - w_m)[Y_{1m}, Y_{2m}]$, $w_m = \varepsilon_{1m} + \varepsilon_{2m}$, $\varepsilon_{1m} = I(X_m < Y_{1m})$ $\varepsilon_{2m} = I(X_m > Y_{2m})$, $D_m^{(0)} = \{\omega : Y_{1m}(\omega) \leq X_m(\omega) \leq Y_{2m}(\omega)\}$, $D_m^{(i)} = A_m^{(i)} \cap (\{\omega : X_m(\omega) < Y_{1m}(\omega)\} \cup \{\omega : X_m(\omega) > Y_{2m}(\omega)\})$, $i = 1, ..., k$, $m = 1, ..., n$. Пусть $\Delta_m^{(i)} = I(D_m^{(i)})$, $i = 0, 1, ..., k$. Тогда $\Delta_m^{(0)} = 1 - w_m$, $\Delta_m^{(i)} = w_m \delta_m^{(i)}$, $i = 1, ...k$; $m = 1, ..., n$ и следовательно с.в. X_m и события $A_m^{(i)}$ наблюдаемы лишь в случае $w_m = 1$ и $\delta_m^{(i)} = 1$. Отметим, что при $w_m = 0$ наблюдается только сам интервал $[Y_{1m}, Y_{2m}]$. Через $p_m(Z^{(n)};\theta)$ плотность вектора $Z^{(n)} = (Z_1, ..., Z_n)$ без учета совместного распределения $G_m(u,v)$ вектора (Y_{1m}, Y_{2m}). Тогда усеченная версия совместного распределения вектора $\left\{\left(Z_m;\ \Delta_m^{(0)}, \Delta_m^{(1)}, ..., \Delta_m^{(k)}\right),\, m = 1, ..., n\right\}$, определяется формулой

$$\frac{d\widetilde{Q}_\theta^{(n)}}{d\theta} = \prod_{m=1}^{n}\left\{\left[\prod_{i=1}^{k}\left[h^{(i)}(X_m;\theta)\right]^{\delta_m^{(i)}}\right]^{w_m}[H(Y_{2m};\theta) - H(Y_{1m};\theta)]^{1-w_m}\right\}.$$

Пусть при каждом $u \in R^1$, $\theta_0 + n^{-1/2}u = \Psi_n(u;\theta_0) \in \Theta$. Тогда статистика отношения правдоподобия (СОП) модели

$$L_{n,\theta_0}(u) = \frac{d\widetilde{Q}^{(n)}_{\Psi_n(u;\theta_0)}(Z^{(n)})}{d\widetilde{Q}^{(n)}_{\theta_0}(Z^{(n)})} = \prod_{m=1}^{n}\left\{\left[\prod_{i=1}^{k}\left[\frac{h_m^{(i)}(X_m;\Psi_n(u;\theta_0))}{h_m^{(i)}(X_m;\theta_0)}\right]^{\delta_m^{(i)}}\right]^{w_m}\cdot\right.$$

$$\left.\cdot\left[\frac{H_m(Y_{2m};\Psi_n(u;\theta_0)) - H_m(Y_{1m};\Psi_n(u;\theta_0)}{H_m(Y_{2m};\theta_0) - H_m(Y_{1m};\theta_0)}\right]^{1-w_m}\right\}, \theta_0 \in \Theta. \qquad (1)$$

Локальная асимптотическая нормальность СОП $L_{n,\theta}(u)$ в точке θ_0 будет установлена при справедливости следующих условий регулярности. Пусть $\chi_{n,\theta}(u) = \log L_{n,\theta}(u)$, $N_m^{(i)} = \{x : h_m^{(i)}(x;\theta) > 0\}$ и $N = \bigcap_{m=1}^{n}\bigcap_{i=1}^{k} N_m^{(i)}$. Пусть

(C1) Множество N не зависит от θ, $\theta \in \Theta$;

(C2) Существуют производные $\frac{\partial^l h_m^{(i)}(x;\theta)}{\partial \theta_j^l}$, $l = 1, 2$; $i = 1, ..., k$; $j = 1, ..., s$, $m = 1, ..., n$ для всех $\theta \in \Theta$;

(C3) $\int\limits_{-\infty}^{\infty} \left| \frac{\partial^l h_m^{(i)}(x;\theta)}{\partial \theta_j^l} \right| \mu_m(dx) < \infty$, $l = 1,2$; $i = 1,...,k$; $j = 1,...,s$; $m = 1,...,n$, для всех $\theta \in \Theta$, где μ_m - σ конечная мера в R^1;

(C4) Интегралы $I_{ljm}^{(i)}(\theta) = E_\theta \left[\frac{\partial \log h_m^{(i)}(X_m;\theta)}{\partial \theta_l} \frac{\partial \log h_m^{(i)}(X_m;\theta)}{\partial \theta_j} \right]$, $l,j = 1,...,s$; $i = 1,...,k$; $m = 1,...,n$; существуют и $\sum\limits_{m=1}^{n} \sum\limits_{i=1}^{k} \left\| I_{ljm}^{(i)}(\theta) \right\|_{l,j=\overline{1,s}} = \sum\limits_{m=1}^{n} I_m(\theta)$ положительны и конечны для всех $\theta \in \Theta$.

1. Локальная асимптотическая нормальность статистики отношения правдоподобия

Пусть $S_n(Z^{(n)};\theta) = \frac{\partial \log \widetilde{Q}_\theta^{(n)}(Z^{(n)})}{\partial \theta} = \sum\limits_{m=1}^{n} l_{\theta m}(X_m, Y_{1m}, Y_{2m}, w_m)$, где $l_{\theta m}(X_m, Y_{1m}, Y_{2m}, w_m) = w_m \sum\limits_{i=1}^{k} \delta_m^{(i)} \frac{\partial \log h_m^{(i)}(X_m;\theta)}{\partial \theta} + (1 - w_m) \frac{\partial \log(H_m(Y_{2m};\theta) - H_m(Y_{1m};\theta))}{\partial \theta}$ вклад выборки рассматриваемой модели. Введем матрицу $\sum\limits_{m=1}^{n} \mathbb{J}_m(\theta) = \sum\limits_{m=1}^{n} \mathbb{J}_{1m}(\theta) + \mathbb{J}_{2m}(\theta)$, где

$$\mathbb{J}_{1m}(\theta) = \sum_{i=1}^{k} \int\limits_{-\infty}^{\infty} \int\limits_{y_1}^{\infty} \left[\int\limits_{-\infty}^{y_1} \frac{\partial \log h_m^{(i)}(x;\theta)}{\partial \theta} \left(\frac{\partial \log h_m^{(i)}(x;\theta)}{\partial \theta} \right)^T dH_m^{(i)}(x;\theta) + \right.$$

$$\left. + \int\limits_{y_2}^{\infty} \frac{\partial \log h_m^{(i)}(x;\theta)}{\partial \theta} \left(\frac{\partial \log h_m^{(i)}(x;\theta)}{\partial \theta} \right)^T dH_m^{(i)}(x;\theta) \right] dG_m(y_1, y_2),$$

$$\mathbb{J}_{2m}(\theta) = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{y_1} \frac{\partial \log(H_m(y_2;\theta) - H_m(y_1;\theta))}{\partial \theta} \left(\frac{\partial \log(H_m(y_2;\theta) - H_m(y_1;\theta))}{\partial \theta} \right)^T \cdot$$

$$\cdot (H_m(y_2;\theta) - H_m(y_1;\theta)) dG_m(y_1, y_2).$$

Пусть (u,v) скалярное произведение для векторов $u, v \in R^s$. Докажем асимптотическое представление для СОП, откуда следует и ЛАН в точке $\theta_0 \in \Theta$. Пусть

Теорема. Пусть справедливы условия регулярности (C1)-(C4) и $\det \left\| \sum\limits_{m=1}^{n} \mathbb{J}_m(\theta) \right\| \neq 0$ для $\theta \in \Theta$. Кроме того, выполнено условия Линдберга: $\varepsilon > 0$, $u \in R^s$:

$$\lim_{n\to\infty} \sum_{m=1}^{n} \sum_{i=1}^{k} E_\theta \left\{ \Psi^{-1}(n;\theta)u, \left(\frac{\partial \log h_m^{(i)}(X_m;\theta)}{\partial \theta} \right)^2 \cdot I \left(\left| \Psi^{-1}(n;\theta)u, \left(\frac{\partial \log h_m^{(i)}(X_m;\theta)}{\partial \theta} \right) \right| > \varepsilon \right) \right\} = 0.$$

где $\Psi^2(n;\theta) = \sum\limits_{m=1}^{n} \mathbb{J}_m(\theta)$. Тогда справедливо представление

$$L_{n,\theta}(u) = \exp \left\{ n^{-1/2} \sum_{m=1}^{n} (l_{\theta_0 m}(X_m, Y_{1m}, Y_{2m}, w_m); u) - \frac{1}{2} \left(\Psi^2(n;\theta_0)u; u \right) + R_n(u;\theta_0) \right\}, \quad (2)$$

где при $n \to \infty$, $R_n(u;\theta_0) \overset{\widetilde{Q}_{\theta_0}^{(n)}}{\to} 0$ и

$$L \left(n^{-1/2} \sum_{j=1}^{n} l_{\theta_0 m}(X_m, Y_{1m}, Y_{2m}, w_m) / \widetilde{Q}_{\theta_0}^{(n)} \right) \to \mathbb{N}_s(0; \sum_{m=1}^{n} \mathbb{J}_m(\theta)). \quad (3)$$

Из (1) следует, что СОП $L_{n,\theta_0}(u)$ аппроксимируется экспоненциальной плотностью, а $\chi_{n,\theta_0}(u)$ имеет асимптотическое s-мерное нормальное распределение.

Для доказательства теоремы нам понадобятся несколько лемм.

Пусть $\left\{l_{im}(x) = (l_{im}^{(1)}(x), ..., l_{im}^{(s)}(x)),\ i = 1, ..., k\right\}$ и $l_{0m}(u, v) = (l_{0m}^{(1)}(u, v), ..., l_{0m}^{(s)}(u, v))$ зависящие от θ интегрируемые функции и $l(x, u, v, w) = w \sum\limits_{i=1}^{k} \delta^{(i)} l_{im}(x) + (1 - w) l_{0m}(y_1, y_2)$, $m = 1, ..., n$.

Лемма 1. Пусть выполнены условия (A) $E_\theta \left[\delta_m^{(i)} \left| l_{im}^j(X_m)\right|\right] < \infty$ для всех $i = 1, ..., k;\ j = 1, ..., s;\ m = 1, ..., n$ и $\theta \in \Theta$; (B) $E_\theta \left[(1 - w_m) \left| l_{0m}^{(j)}(Y_{1m}, Y_{2m})\right|\right] < \infty$ для всех $j = 1, ..., s;$ $m = 1, ..., n$ и $\theta \in \Theta$; Тогда для любого $\theta \in \Theta$ при $m = 1, ..., n$

$$E_\theta l_m(X_m, Y_{1m}, Y_{2m}, w_m) = \int\limits_{-\infty}^{\infty} \int\limits_{y_1}^{\infty} \sum_{i=1}^{k} \left(\int\limits_{-\infty}^{y_1} l_{im}(x) dH_m^{(i)}(x; \theta) + \right.$$

$$\left. + \int\limits_{y_2}^{\infty} l_{im}(x) dH_m^{(i)}(x; \theta) \right) + \int\limits_{-\infty}^{\infty} \int\limits_{y_1}^{\infty} l_{0m}(y_1, y_2)(H_m(y_2; \theta) - H_m(y_1; \theta))\ dG_m(y_1, y_2).$$

Доказательство леммы 1. Легко видеть, что для всех $\theta \in \Theta$ и $m = 1, ..., n$ в условиях леммы 1

$$E_\theta \left[|l_m(X_m, Y_{1m}, Y_{2m}, w_m)|\right] \le k \max_{1 \le i \le k} \left\{E_\theta \left[\delta_m^{(i)} |l_{im}(X_m)|\right]\right\} + E_\theta \left[(1 - w_m) |l_{0m}(Y_{1m}, Y_{2m})|\right] < \infty.$$

Тогда вычислив математические ожидания по событиям $\{w_m = 1\}$ и $\{w_m = 0\}$, имеем

$$E_\theta \left[w_m l_m(X_m, Y_{1m}, Y_{2m}, w_m)\right] = E_\theta \left\{ \sum_{i=1}^{k} \left[E_\theta \left[\delta_m^{(i)} l_{im}(X_m) I(X_m < Y_{1m})/Y_{1m}\right] I(Y_{1m} \le Y_{2m})\right] + \right.$$

$$\left. + E_\theta \left[\delta_m^{(i)} l_{im}(X_m) I(X_m > Y_{2m})/Y_{2m}\right] I(Y_{1m} \le Y_{2m}) \right\} =$$

$$= \int\limits_{-\infty}^{\infty} \int\limits_{y_1}^{\infty} \sum_{i=1}^{k} \left(\int\limits_{-\infty}^{y_1} l_{im}(x) dH_m^{(i)}(x; \theta) + \int\limits_{y_2}^{\infty} l_{im}(x) dH_m^{(i)}(x; \theta) \right) dG_m(y_1, y_2)$$

а также

$$E_\theta \left[(1 - w_m) l_m(X_m, Y_{1m}, Y_{2m}, w_m)\right] =$$

$$= E_\theta \left\{E_\theta \left[l_{0m}(Y_{1m}, Y_{2m}) I(Y_{1m} \le X_m \le Y_{2m})/(Y_{1m}, Y_{2m})\right] I(Y_{1m} \le Y_{2m})\right\} =$$

$$= E_\theta \left\{l_{0m}(Y_{1m}, Y_{2m}) E_\theta \left[I(Y_{1m} \le X_m \le Y_{2m})/(Y_{1m}, Y_{2m})\right] I(Y_{1m} \le Y_{2m})\right\} =$$

$$= E_\theta \left\{l_{0m}(Y_{1m}, Y_{2m}) \left(H_m(Y_{2m}; \theta) - H_m(Y_{1m}; \theta)\right) I(Y_{1m} \le Y_{2m})\right\} =$$

$$= \int\limits_{-\infty}^{\infty} \int\limits_{y_1}^{\infty} l_{0m}(y_1, y_2) \left(H_m(y_2; \theta) - H_m(y_1; \theta)\right) dG_m(y_1, y_2).$$

Теперь складывая эти формулы, получаем утверждение леммы 1. Лемма 1 доказана.

Лемма 2. Пусть выполнены условия регулярности (C1)-(C4). Тогда $E_\theta \left[\frac{\partial \log}{\partial \theta} \left\{\frac{d\widetilde{Q}_\theta^{(n)}(Z^{(n)})}{dZ^{(n)}}\right\}\right] = 0$ для всех $\theta \in \Theta$.

Доказательство леммы 2. Имеем

$$\frac{\partial}{\partial \theta} \left\{\log \left[\frac{d\widetilde{Q}_\theta^{(n)}(Z^{(n)})}{dZ^{(n)}}\right]\right\} = \sum_{m=1}^{n} l_{\theta m}\left(X_m, Y_{1m}, Y_{2m}, w_m\right), \tag{5}$$

где $l_{\theta m}\left(X_m, Y_{1m}, Y_{2m}, w_m\right) = w_m \sum\limits_{i=1}^{k} l_{im}(X_m) + (1-w_m) l_{0m}\left(Y_{1m}, Y_{2m}\right)$ вектор – функция из леммы 1 при

$$l_{im}(x;\theta) = \frac{\partial}{\partial\theta}\log h_m^{(i)}(x;\theta), \quad l_{0m}(y_1,y_2) = \frac{\partial \log(H_m(y_2;\theta) - H_m(y_1;\theta))}{\partial\theta}.$$

Справедливость соотношений (A) и (B) леммы 1 следует из следующих:

$$E_\theta\left[\delta_m^{(i)}\left|l_{im}^{(j)}(X_m)\right|\right] = \int\limits_{-\infty}^{\infty}\left|\frac{\partial h_m^{(i)}(x;\theta)}{\partial\theta_j}\right|\mu_m(dx) < \infty$$

$$\text{для всех } \ i=1,...,k;\ \ j=1,...,s;\ \ m=1,...,n \ \text{ и } \ \theta\in\Theta, \tag{6}$$

а также

$$E_\theta\left[(1-w_m)\left|l_{0m}^{(j)}(Y_{1m},Y_{2m})\right|\right] = \int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left|\frac{\partial}{\partial\theta_j}\left(H_m(y_2;\theta)-H_m(y_1;\theta)\right)\right| dG_m(y_1,y_2) =$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left|\int\limits_{y_1}^{y_2}\left[\frac{\partial\log h_m(x;\theta)}{\partial\theta_j}\sqrt{h_m(x;\theta)}\right]\sqrt{h_m(x;\theta)}\mu_m(dx)\right| dG_m(y_1,y_2) \le$$

$$\le \int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left|\int\limits_{y_1}^{y_2}\left[\left(\frac{\partial\log h_m(x;\theta)}{\partial\theta_j}\right)^2 dH_m(x;\theta)\right]^{1/2}\left[H_m(y_2;\theta)-H_m(y_1;\theta)\right]^{1/2}\right| dG_m(y_1,y_2) \le$$

$$\le \left(I_{jjm}(\theta)\right)^{1/2}\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left[H_m(y_2;\theta)-H_m(y_1;\theta)\right]^{1/2} dG_m(y_1,y_2) \le \left(I_{jjm}(\theta)\right)^{1/2}, \tag{7}$$

где использованы условия регулярности (C1)-(C4) а в (7) также и неравенство Коши-Буньяковского. Таким образом, в силу (6) и (7) математическое ожидание в (5) существует и

$$E_\theta\left[\frac{\partial}{\partial\theta}\left\{\log\left[\frac{d\widetilde{Q}_\theta^{(n)}(Z^{(n)})}{dZ^{(n)}}\right]\right\}\right] = \sum_{m=1}^{n} E_\theta l_m\left(X_m, Y_{1m}, Y_{2m}, w_m\right). \tag{8}$$

Здесь из леммы 1, для любого $\theta\in\Theta$, имеем

$$E_\theta l_m\left(X_m, Y_{1m}, Y_{2m}, w_m\right) =$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\sum_{i=1}^{k}\left(\int\limits_{-\infty}^{y_1}\frac{\partial\log h_m^{(i)}(x;\theta)}{\partial\theta}dH_m^{(i)}(x;\theta) + \int\limits_{y_2}^{\infty}\frac{\partial\log h_m^{(i)}(x;\theta)}{\partial\theta}dH_m^{(i)}(x;\theta)\right) dG_m(y_1,y_2) +$$

$$+ \int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\frac{\partial\log(H_m(y_2;\theta)-H_m(y_1;\theta))}{\partial\theta}(H_m(y_2;\theta)-H_m(y_1;\theta))dG_m(y_1,y_2) =$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\sum_{i=1}^{k}\left(\int\limits_{-\infty}^{y_1}\frac{\partial h_m^{(i)}(x;\theta)}{\partial\theta}\mu_m(dx) + \int\limits_{y_2}^{\infty}\frac{\partial h_m^{(i)}(x;\theta)}{\partial\theta}\mu_m(dx)\right) + \frac{\partial}{\partial\theta}(H_m(y_2;\theta)-H_m(y_1;\theta))dG_m(y_1,y_2) =$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left[-\frac{\partial}{\partial\theta}(H_m(y_2;\theta)-H_m(y_1;\theta)) + \frac{\partial}{\partial\theta}(H_m(y_2;\theta)-H_m(y_1;\theta))\right] dG_m(y_1,y_2) = 0 \tag{9}$$

для всех $\theta\in\Theta$. Из (8) и (9) следует (5). Лемма 2 доказана.

Лемма 3. В условиях (C1)-(C4) для всех $\theta \in \Theta$

$$E_\theta\left[\frac{\partial}{\partial\theta}\left\{\log\left[\frac{d\widetilde{Q}_\theta^{(n)}(Z^{(n)})}{dZ^{(n)}}\right]\right\}\left(\frac{\partial}{\partial\theta}\left\{\log\left[\frac{d\widetilde{Q}_\theta^{(n)}(Z^{(n)})}{dZ^{(n)}}\right]\right\}\right)^T\right] =$$

$$= -\left\|E_\theta\left(\frac{\partial^2}{\partial\theta_l\partial\theta_j}\left\{\frac{d\widetilde{Q}_\theta^{(n)}(Z^{(n)})}{dZ^{(n)}}\right\}\right)\right\|_{l,j=1}^{s}. \tag{10}$$

Доказательство леммы 3. Поскольку $l_m \cdot {l_m}^T = \|l_{mj} \cdot l_{ml}\|_{l,j=1}^s$, где

$$l_{mj}\left(X_m, Y_{1m}, Y_{2m}, w_m\right) l_{ml}\left(X_m, Y_{1m}, Y_{2m}, w_m\right) = w_m \sum_{i=1}^{k} \delta_m^{(i)} \frac{\partial \log h_m^{(i)}(X_m;\theta)}{\partial\theta_j} \frac{\partial \log h_m^{(i)}(X_m;\theta)}{\partial\theta_l} +$$

$$+(1-w_m)\frac{\partial \log(H_m(y_2;\theta)-H_m(y_1;\theta))}{\partial\theta_j}\frac{\partial \log(H_m(y_2;\theta)-H_m(y_1;\theta))}{\partial\theta_l},$$

то в силу лемм 1 и 2

$$E_\theta\left[\frac{\partial \log}{\partial\theta}\left\{\frac{d\widetilde{Q}_\theta^{(n)}(Z^{(n)})}{dZ^{(n)}}\right\}\left(\frac{\partial \log}{\partial\theta}\left\{\frac{d\widetilde{Q}_\theta^{(n)}(Z^{(n)})}{dZ^{(n)}}\right\}\right)^T\right] = \sum_{m=1}^{n}\mathbb{J}_m(\theta),$$

где $\mathbb{J}_m(\theta)$ матрица с элементами

$$J_{jlm}^{(i)}(\theta) = \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\sum_{i=1}^{k}\left(\int_{-\infty}^{y_1}\frac{\partial \log h_m^{(i)}(x;\theta)}{\partial\theta_j}\frac{\partial \log h_m^{(i)}(x;\theta)}{\partial\theta_l}dH_m^{(i)}(x;\theta)+\right.$$

$$\left.+\int_{y_1}^{\infty}\frac{\partial \log h_m^{(i)}(x;\theta)}{\partial\theta_j}\frac{\partial \log h_m^{(i)}(x;\theta)}{\partial\theta_l}\right)dG_m(y_1,y_2)+$$

$$+\int_{-\infty}^{\infty}\int_{y_1}^{\infty}\frac{\partial \log(H_m(y_2;\theta)-H_m(y_1;\theta))}{\partial\theta_j}\frac{\partial \log(H_m(y_2;\theta)-H_m(y_1;\theta))}{\partial\theta_l}dG_m(y_1,y_2) = J_{1jlm}^{(i)}(\theta)+J_{2jlm}^{(i)}(\theta). \tag{11}$$

Легко видеть, что первое слагаемое в (11) оценивается для всех $\theta \in \Theta$ как

$$\left|J_{1jlm}^{(i)}(\theta)\right| \le 2\sum_{i=1}^{k}\int_{-\infty}^{\infty}\left|\frac{\partial \log h_m^{(i)}(x;\theta)}{\partial\theta_j}\right|\left|\frac{\partial \log h_m^{(i)}(x;\theta)}{\partial\theta_l}\right|dH_m^{(i)}(x;\theta) =$$

$$= 2\sum_{i=1}^{k}\left[\int_{-\infty}^{\infty}\left(\frac{\partial \log h_m^{(i)}(x;\theta)}{\partial\theta_j}\right)^2 dH_m^{(i)}(x;\theta)\right]^{1/2}\cdot\left[\int_{-\infty}^{\infty}\left(\frac{\partial \log h_m^{(i)}(x;\theta)}{\partial\theta_l}\right)^2 dH_m^{(i)}(x;\theta)\right]^{1/2} =$$

$$= 2\sum_{i=1}^{k}\left[I_{jjm}^{(i)}(\theta)I_{llm}^{(i)}(\theta)\right]^{1/2} < \infty, \tag{12}$$

где использовано условие (C4) и неравенство Коши-Буньяковского. Аналогично оценим и $J_{2jlm}^{(2)}(\theta)$ для всех $\theta \in \Theta$ и $i = 1, ..., k;\ l, j = 1, ..., s;\ m = 1, ..., n$ и $\theta \in \Theta$:

$$\left|J_{2jlm}^{(i)}(\theta)\right| \le \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\left|\int_{y_1}^{y_2}\frac{\partial h_m(x;\theta)}{\partial\theta_j}\mu_m(dx)\right|\left|\int_{y_1}^{y_2}\frac{\partial h_m(x;\theta)}{\partial\theta_l}\mu_m(dx)\right|.$$

$$\cdot \int_{-\infty}^{\infty} \int_{y_1}^{\infty} \left[\int_{y_1}^{y_2} \left(\frac{\partial \log h_m(x;\theta)}{\partial \theta_j} \sqrt{h_m(x;\theta)} \right) \sqrt{h_m(x;\theta)} \mu_m(dx) \right] \cdot$$

$$\int_{-\infty}^{\infty} \int_{y_1}^{\infty} \left[\int_{y_1}^{y_2} \left(\frac{\partial \log h_m(x;\theta)}{\partial \theta_j} \sqrt{h_m(x;\theta)} \right) \sqrt{h_m(x;\theta)} \mu_m(dx) \right] \cdot$$

$$\cdot \left[\int_{y_1}^{y_2} \left(\frac{\partial \log h_m(x;\theta)}{\partial \theta_l} \sqrt{h_m(x;\theta)} \right) \sqrt{h_m(x;\theta)} \mu_m(dx) \right] \frac{dG_m(y_1,y_2)}{(H_m(y_2;\theta) - H_m(y_1;\theta))} \leq$$

$$\leq \int_{-\infty}^{\infty} \int_{y_1}^{\infty} \left[\int_{y_1}^{y_2} \left(\frac{\partial \log h_m(x;\theta)}{\partial \theta_j} \right)^2 dH_m(x;\theta) \right]^{1/2} \int_{y_1}^{y_2} h(x;\theta) \mu_m(dx) \cdot$$

$$\cdot \left[\int_{y_1}^{y_2} \left(\frac{\partial \log h_m(x;\theta)}{\partial \theta_l} \right)^2 dH_m(x;\theta) \right] \frac{dG_m(y_1,y_2)}{(H_m(y_2;\theta) - H_m(y_1;\theta))} \leq \left[I_{jjm}(\theta) I_{llm}(\theta) \right]^{1/2} < \infty. \quad (13)$$

Теперь из (12) и (13) следует существование элементов матриц $J^X_{jlm}(\theta)$. Для доказательства равенства (10) сперва заметим, что для всех $i = 1, ..., k;\ l, j = 1, ..., s;\ m = 1, ..., n$ и $\theta \in \Theta$:

$$E_\theta \left(\frac{\partial^2}{\partial \theta_l \partial \theta_j} \left\{ \frac{d\widetilde{Q}^{(n)}_\theta(Z^{(n)})}{dZ^{(n)}} \right\} \right) = \sum_{m=1}^{n} \frac{\partial^2 l_m \left(X_m, Y_{1m}, Y_{2m}, w_m \right)}{\partial \theta_j \partial \theta_l}, \quad (14)$$

где в силу леммы 1 имеем цепочку равенств

$$E_\theta \left[\frac{\partial^2 l_m \left(X_m, Y_{1m}, Y_{2m}, w_m \right)}{\partial \theta_j \partial \theta_l} \right] = \int_{-\infty}^{\infty} \int_{y_1}^{\infty} \sum_{i=1}^{k} \left(\int_{-\infty}^{y_1} \frac{\partial^2 \log h^{(i)}_m(x;\theta)}{\partial \theta_j \partial \theta_l} dH^{(i)}_m(x;\theta) + \right.$$

$$\left. + \int_{y_2}^{\infty} \frac{\partial^2 \log h^{(i)}_m(x;\theta)}{\partial \theta_j \partial \theta_l} dH^{(i)}_m(x;\theta) \right) dG_m(y_1,y_2) +$$

$$+ \int_{-\infty}^{\infty} \int_{y_1}^{\infty} \frac{\partial^2 \log(H_m(y_2;\theta) - H_m(y_1;\theta))}{\partial \theta_j \partial \theta_l} (H_m(y_2;\theta) - H_m(y_1;\theta)) dG_m(y_1,y_2) =$$

$$= \int_{-\infty}^{\infty} \int_{y_1}^{\infty} \sum_{i=1}^{k} \left\{ \int_{-\infty}^{y_1} \left[\frac{\partial^2 h^{(i)}_m(x;\theta)}{\partial \theta_j \partial \theta_l} h^{(i)}_m(x;\theta) + \frac{\partial h^{(i)}_m(x;\theta)}{\partial \theta_l} \frac{\partial h^{(i)}_m(x;\theta)}{\partial \theta_j} \right] \frac{\mu_m(dx)}{h^{(i)}_m(x;\theta)} + \right.$$

$$\left. + \int_{y_2}^{\infty} \left[\frac{\partial^2 h^{(i)}_m(x;\theta)}{\partial \theta_j \partial \theta_l} h^{(i)}_m(x;\theta) - \frac{\partial h^{(i)}_m(x;\theta)}{\partial \theta_l} \frac{\partial h^{(i)}_m(x;\theta)}{\partial \theta_j} \right] \frac{\mu_m(dx)}{h^{(i)}_m(x;\theta)} \right\} dG_m(y_1,y_2) +$$

$$+ \int_{-\infty}^{\infty} \int_{y_1}^{\infty} \left[\frac{\partial^2 (H_m(y_2;\theta) - H_m(y_1;\theta))}{\partial \theta_j \partial \theta_l} (H_m(y_2;\theta) - H_m(y_1;\theta)) - \right.$$

$$\left. - \frac{\partial (H_m(y_2;\theta) - H_m(y_1;\theta))}{\partial \theta_l} \frac{\partial (H_m(y_2;\theta) - H_m(y_1;\theta))}{\partial \theta_j} \right] \frac{dG_m(y_1,y_2)}{(H_m(y_2;\theta) - H_m(y_1;\theta))} =$$

$$= \int_{-\infty}^{\infty} \int_{y_1}^{\infty} \sum_{i=1}^{k} \left(\int_{-\infty}^{y_1} \frac{\partial^2 h^{(i)}_m(x;\theta)}{\partial \theta_l \partial \theta_j} \mu_m(dx) + \int_{y_2}^{\infty} \frac{\partial^2 h^{(i)}_m(x;\theta)}{\partial \theta_l \partial \theta_j} \mu_m(dx) \right) dG_m(y_1,y_2) +$$

$$+\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\frac{\partial^2(H_m(y_2;\theta)-H_m(y_1;\theta))}{\partial\theta_l\partial\theta_j}dG_m(y_1,y_2)-J_{ljm}(\theta)=$$

$$+\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left[-\frac{\partial^2(H_m(y_2;\theta)-H_m(y_1;\theta))}{\partial\theta_l\partial\theta_j}+\frac{\partial^2(H_m(y_2;\theta)-H_m(y_1;\theta))}{\partial\theta_l\partial\theta_j}\right]dG_m(y_1,y_2)-J_{ljm}(\theta)=$$

$$=-J_{ljm}(\theta). \tag{15}$$

Равенство (10) теперь следует из (14) и (15). Лемма 3 доказана.

Обозначим $g_m^{(i)}(x;\theta) = \sqrt{h_m^{(i)}(x;\theta)}$, $\xi_{nm}^{(i)}(x;u) = \frac{g_m^{(i)}(x;\theta+u)}{g_m^{(i)}(x;\theta)} - 1$, $q_m(y_1,y_2;\theta) = \sqrt{H_m(y_2;\theta)-H_m(y_1;\theta)}$ и $\eta_{nm}(y_1,y_2;u) = \frac{q_m(y_1,y_2;\theta+u)}{q_m(y_1,y_2;\theta)} - 1$.

Лемма 4. Пусть выполнены условия регулярности (C1)-(C5). Тогда при $|u|\to 0$ справедливы соотношения:

$$E_\theta\left[w_m\sum_{i=1}^{k}\delta_m^{(i)}\,{\xi_{ni}^{(i)}}^2(X_m;u)\right]-\frac{1}{4}\left(\mathbb{J}_1(\theta)u;u\right)=o\left(|u|^2\right), \tag{16}$$

$$E_\theta\left[(1-w_m)\eta_{nm}^2(Y_{1m},Y_{2m};u)\right]-\frac{1}{4}\left(\mathbb{J}_2(\theta)u;u\right)=o\left(|u|^2\right), \tag{17}$$

$$E_\theta\left|w_m\sum_{i=1}^{k}\delta_m^{(i)}\left[{\xi_{ni}^{(i)}}^2(X_m;u)-\left(u;\,\frac{\partial g_m^{(i)}(X_m;\theta)}{\partial\theta}\right)^2\right]\right|=o\left(|u|^2\right), \tag{18}$$

$$E_\theta\left|(1-w_m)\left[\eta_{nm}^2(Y_{1m},Y_{2m};u)-\left(u;\,\frac{\partial q_m(Y_{1m},Y_{2m};\theta)}{\partial\theta}\right)^2\right]\right|=o\left(|u|^2\right), \tag{19}$$

$$P_\theta\left(\left|w_m\sum_{i=1}^{k}\delta_m^{(i)}\xi_{nm}(X_m;u)\right|>\varepsilon\right)=o\left(|u|^2\right), \tag{20}$$

$$P_\theta\left(|(1-w_m)\eta_{nm}(Y_{1m},Y_{2m};u)|>\varepsilon\right)=o\left(|u|^2\right), \tag{21}$$

$$E_\theta\left[w_m\sum_{i=1}^{k}\delta_m^{(i)}\xi_{nm}^{(i)}(X_m;u)\right]+\frac{1}{8}\left(\mathbb{J}_1(\theta)u;u\right)=o\left(|u|^2\right), \tag{22}$$

$$E_\theta\left[(1-w_m)\eta_{nm}(Y_{1m},Y_{2m};u)\right]+\frac{1}{8}\left(\mathbb{J}_2(\theta)u;u\right)=o\left(|u|^2\right). \tag{23}$$

Доказательство леммы 4. В условиях регулярности леммы

$$\sum_{i=1}^{k}\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left(\int\limits_{\overline{N}_m^{(i)}\cap(-\infty,y_1)}h_m^{(i)}(x;\theta+u)\mu_m(dx)+\int\limits_{\overline{N}_m^{(i)}\cap(y_2,\infty)}h_m^{(i)}(x;\theta+u)\mu_m(dx)\right)dG_m(y_1,y_2)=$$

$$=\frac{1}{2}\left(\nabla_{1m}^{(2)}u;u\right)=o\left(|u|^2\right), \tag{24}$$

где

$$\nabla_{1m}^{(2)}=\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left(\int\limits_{\overline{N}_m\cap(-\infty,y_1)}\frac{\partial^2h_m(x;\theta*)}{\partial\theta^2}\mu_m(dx)+\int\limits_{\overline{N}_m\cap(y_2,\infty)}\frac{\partial^2h_m(x;\theta*)}{\partial\theta^2}\mu_m(dx)\right)dG_m(y_1,y_2)$$

$\theta*$ лежит между θ и $\theta+u$. Легко проверить, что при $|u|\to 0$

$$\frac{\sqrt{h_m^{(i)}(x;\theta+u)}-\sqrt{h_m^{(i)}(x;\theta)}}{\sqrt{h_m^{(i)}(x;\theta)}}-\frac{1}{2}\left(u;\frac{\partial\log h_m^{(i)}(x;\theta)}{\partial\theta}\right)=o\left(|u|\right). \tag{25}$$

Следовательно, при $|u|\to 0$

$$E_\theta\left\{w_m\sum_{i=1}^{k}\delta_m^{(i)}\left[\xi_{ni}(X_m;u)-\frac{1}{2}\left(u;\frac{\partial\log h_m^{(i)}(X_m;\theta)}{\partial\theta}\right)\right]\right\}^2=o\left(|u|^2\right). \tag{26}$$

Аналогично (24)-(26), имеем

$$\int\limits_{-\infty}^{\infty}\int\limits_{\overline{N}^{(i)}\cap(y_1,\infty)}\left(H_m(y_2;\theta+u)-H_m(y_1;\theta+u)\right)dG_m(y_1,y_2)=\frac{1}{2}\left(\nabla_{2m}^{(2)}u;u\right)=o\left(|u|^2\right), \tag{27}$$

где

$$\nabla_{2m}^{(2)}=\int\limits_{-\infty}^{\infty}\int\limits_{\overline{N}^{(i)}\cap(-\infty,y_1)}\left[\frac{\partial^2}{\partial\theta^2}\left(H_m(y_2;\theta*)-H_m(y_1;\theta*)\right)\right]dG_m(y_1,y_2),$$

а также

$$\frac{\sqrt{H_m(y_2;\theta+u)-H_m(y_1;\theta+u)}-\sqrt{H_m(y_2;\theta)-H_m(y_1;\theta)}}{\sqrt{H_m(y_2;\theta)-H_m(y_1;\theta)}}-$$
$$-\frac{1}{2}\left(u;\frac{\partial\log\left(H_m(y_2;\theta)-H_m(y_1;\theta)\right)}{\partial\theta}\right)=o\left(|u|^2\right). \tag{28}$$

Из этого следует, что

$$E_\theta\left\{(1-w_m)\left[\eta_{nm}(Y_{1m},Y_{2m};u)-\frac{1}{2}\left(u;\frac{\partial\log\left(H_m(Y_2;\theta)-H_m(Y_1;\theta)\right)}{\partial\theta}\right)\right]\right\}^2=o\left(|u|^2\right). \tag{29}$$

В силу (24)

$$E_\theta\left[w_m\sum_{i=1}^{k}\delta_m^{(i)}\left[\xi_{ni}^{(i)\,2}(X_m;u)\right]\right]=\sum_{i=1}^{k}\left[\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left(\int\limits_{\overline{N}^{(i)}\cap(-\infty,y_1)}\left(\sqrt{h_m^{(i)}(x;\theta+u)}-\sqrt{h_m^{(i)}(x;\theta)}\right)^2\mu_m(dx)+\right.\right.$$
$$\left.\left.+\int\limits_{\overline{N}^{(i)}\cap(y_2,\infty)}\left(\sqrt{h_m^{(i)}(x;\theta+u)}-\sqrt{h_m^{(i)}(x;\theta)}\right)^2\mu_m(dx)\right)dG_m(y_1,y_2)\right]=$$
$$=2\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left(\int\limits_{-\infty}^{y_1}h_m(x;\theta)\mu_m(dx)+\int\limits_{y_2}^{\infty}h_m(x;\theta)\mu_m(dx)\right)dG_m(y_1,y_2)+o\left(|u|^2\right)-$$
$$-2\sum_{i=1}^{k}\left[\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left(\int\limits_{-\infty}^{y_1}\sqrt{h_m^{(i)}(x;\theta+u)}\left(\sqrt{h_m^{(i)}(x;\theta)}\right)^{-1}\mu_m(dx)+\right.\right.$$
$$\left.\left.+\int\limits_{y_2}^{\infty}\sqrt{h_m^{(i)}(x;\theta+u)}\left(\sqrt{h_m^{(i)}(x;\theta)}\right)^{-1}\mu_m(dx)\right)\right]dG_m(y_1,y_2)=$$
$$=2\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left[1-\left(H_m(y_2;\theta)-H_m(y_1;\theta)\right)\right]dG_m(y_1,y_2)+o\left(|u|^2\right)-$$

$$= 2\sum_{i=1}^{k}\left[\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left(\int\limits_{-\infty}^{y_1}(\xi_{nm}(x;u)+1)\,dH_m^{(i)}(x;\theta)+\int\limits_{y_2}^{\infty}(\xi_{nm}(x;u)+1)\,dH_m^{(i)}(x;\theta)\right)dG_m(y_1,y_2)\right]=$$

$$-2\sum_{i=1}^{k}\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left(\int\limits_{-\infty}^{y_1}\xi_{nm}^{(i)}(x;u)dH_m^{(i)}(x;\theta)+\int\limits_{y_2}^{\infty}\xi_{nm}^{(i)}(x;u)dH_m^{(i)}(x;\theta)\right)dG_m(y_1,y_2)+$$

$$+o\left(|u|^2\right)=2E_\theta\left[w_m\sum_{i=1}^{k}\delta_m^{(i)}\xi_{nm}^{(i)}(X_m;u)\right]+o\left(|u|^2\right). \tag{30}$$

Из (26) следует (16) и (18), а также из (16) следует (17) и (19). Теперь из (16) и (30) получается (22). С другой стороны, в силу (27)

$$E_\theta\left[(1-w_m)\eta_{nm}^2(Y_{1m},Y_{2m};u)\right]=$$

$$=\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left(\sqrt{H_m(y_2;\theta+u)-H_m(y_1;\theta+u)}-\sqrt{H_m(y_2;\theta)-H_m(y_1;\theta)}\right)^2dG_m(y_1,y_2)=$$

$$=2\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left(H_m(y_2;\theta)-H_m(y_1;\theta)\right)dG_m(y_1,y_2)+o\left(|u|^2\right)-$$

$$-2\int\limits_{-\infty}^{\infty}\int\limits_{y_1}^{\infty}\left(\eta_{nm}(y_1,y_2;u)+1\right)\left(H_m(y_2;\theta)-H_m(y_1;\theta)\right)dG_m(y_1,y_2)=$$

$$=2E_\theta\left[(1-w_m)\eta_{nm}(Y_{1m},Y_{2m};u)\right]+o\left(|u|^2\right). \tag{31}$$

Теперь (23) следует из (17) и (31). Прежде всего установить (20) и (21), сперва заметим, что из (26) и (29) соответственно имеем

$$E_\theta\left\{w_m\sum_{i=1}^{k}\delta_m^{(i)}\left[\xi_{nm}^{(i)}(X_m;u)-\frac{1}{2}\left(u;\frac{\partial\log h_m^{(i)}(X_m;\theta)}{\partial\theta}\right)\right]\right\}^2=o\left(|u|^2\right), \tag{32}$$

$$E_\theta\left\{(1-w_m)\left[\eta_{nm}(Y_{1m},Y_{2m};u)-\frac{1}{2}\left(u;\frac{\partial\log\left(H_m(Y_{2m};\theta)-H_m(Y_{1m};\theta)\right)}{\partial\theta}\right)\right]\right\}^2=o\left(|u|^2\right). \tag{33}$$

В силу (32),

$$P_\theta\left[\left|w_m\sum_{i=1}^{k}\delta_m^{(i)}\xi_{nm}(X_m;u)\right|>\varepsilon\right]\le$$

$$\le P_\theta\left[\left|w_m\sum_{i=1}^{k}\delta_m^{(i)}\left[\xi_{nm}^{(i)}(X_m;u)-\frac{1}{2}\left(u;\frac{\partial\log h_m^{(i)}(X_m;\theta)}{\partial\theta}\right)\right]\right|>\frac{\varepsilon}{2}\right]+$$

$$\le P_\theta\left[\left|w_m\sum_{i=1}^{k}\delta_m^{(i)}\left(u;\frac{\partial\log h_m^{(i)}(X_m;\theta)}{\partial\theta}\right)^2\right|>\varepsilon^2\right]\le$$

$$\le\frac{4}{\varepsilon^4}E_\theta\left\{w_m\sum_{i=1}^{k}\delta_m^{(i)}\left[\xi_{nm}^{(i)}(X_m;u)-\frac{1}{2}\left(u;\frac{\partial\log h_m^{(i)}(X_m;\theta)}{\partial\theta}\right)\right]\right\}^2+$$

$$+\frac{1}{\varepsilon^2}\int\limits_{-\infty}^{\infty}\sum_{i=1}^{k}\int\limits_{y_1}^{\infty}\left(\int\limits_{(-\infty,y_1)\cap\left\{x:\left|\frac{\partial\log h_m^{(i)}(x;\theta)}{\partial\theta}\right|>\frac{\varepsilon}{|u|}\right\}}\left(u;\frac{\partial\log h_m^{(i)}(x;\theta)}{\partial\theta}\right)^2 dH_m^{(i)}(x;\theta)+\right.$$

$$\left.+\int\limits_{(y_2,\infty)\cap\left\{x:\left|\frac{\partial\log h_m^{(i)}(x;\theta)}{\partial\theta}\right|>\frac{\varepsilon}{|u|}\right\}}\left(u;\frac{\partial\log h_m^{(i)}(x;\theta)}{\partial\theta}\right)^2 dH_m^{(i)}(x;\theta)\right) dG_m(y_1,y_2)=o\left(|u|^2\right), \quad (34)$$

где первое слагаемое в правой части неравенства (34) в силу (26) есть $o\left(|u|^2\right)$, а второе в силу сходимости интеграла $\mathbb{J}_1(\theta)$, также имеет порядок $o\left(|u|^2\right)$. Вполне аналогично, с учётом (33) имеем

$$P_\theta\left(|(1-w_m)\eta_{nm}(Y_{1m},Y_{2m};u)|>\varepsilon\right)\le$$

$$\le P_\theta\left(\left|(1-w_m)\eta_{nm}(Y_{1m},Y_{2m};u)-\frac{1}{2}\left(u;\frac{\partial\log(H_m(Y_{2m};\theta)-H_m(Y_{1m};\theta))}{\partial\theta}\right)\right|>\frac{\varepsilon}{2}\right)+$$

$$\le P_\theta\left((1-w_m)\eta_{nm}^2(Y_{1m},Y_{2m};u)\ >\varepsilon^2\right)=o\left(|u|^2\right). \quad (60)$$

Теперь из (34) и (35) следуют соответственно (20) и (21). Лемма 4 доказана.

Доказательство теоремы. Согласно (8)

$$L_{n,\theta_0}(u)=\exp\left\{\chi_{n,\theta_0}^{(1)}(u)+\chi_{n,\theta_0}^{(2)}(u)\right\}, \quad (36)$$

где

$$\chi_{n,\theta_0}^{(1)}(u)=\sum_{m=1}^{n}w_m\sum_{i=1}^{k}\delta_m^{(i)}\log\left[\frac{h_m^{(i)}(X_m;\,\theta_0+n^{-1/2}u)}{h_m^{(i)}(X_m;\theta_0)}\right]=2\sum_{m=1}^{n}\sum_{i=1}^{k}\delta_m^{(i)}\log\left(1+\xi_{nm}^{(i)}(X_m;n^{-1/2}u)\right), \quad (37)$$

$$\chi_{n,\theta_0}^{(2)}(u)=\sum_{m=1}^{n}(1-w_m)\log\left[\frac{H_m(Y_{2m};\,\theta_0+n^{-1/2}u)-H_m(Y_{1m};\,\theta_0+n^{-1/2}u)}{H_m(Y_{2m};\theta)-H_m(Y_{1m};\theta)}\right]=$$

$$=2\sum_{m=1}^{n}(1-w_m)\log\left(1+\eta_{nm}(Y_{1m},Y_{2m};n^{-1/2}u)\right), \quad (38)$$

и

$$\chi_{n,\theta_0}(n^{-1/2}u)=\chi_{n,\theta_0}^{(1)}(u)+\chi_{n,\theta_0}^{(2)}(u). \quad (39)$$

Определим события $A_n=\left\{\max\limits_{1\le m\le n}\max\limits_{1\le i\le k}\left|\xi_{nm}^{(i)}(X_m;n^{-1/2}u)\right|<\varepsilon\right\}$ и $B_n=\left\{\max\limits_{1\le m\le n}\left|\eta_{nm}(Y_{1m},Y_{2m};n^{-1/2}u)\right|<\varepsilon\right\}$. Тогда при справедливости этих событий в силу формулы Тейлора при некоторых $\left|\alpha_{mn}^{(i)}\right|<1$ и $|\beta_{mn}|<1$ имеем

$$\chi_{n,\theta_0}^{(1)}(u)=2\sum_{m=1}^{n}w_m\sum_{i=1}^{k}\delta_m^{(i)}\xi_{nm}(X_m;n^{-1/2}u)-\sum_{m=1}^{n}w_m\sum_{i=1}^{k}\delta_m^{(i)}\xi_{nm}^2(X_m;n^{-1/2}u)+$$

$$+\sum_{m=1}^{n}w_m\sum_{i=1}^{k}\delta_m^{(i)}\alpha_{mn}^{(i)}\left|\xi_{nm}(X_m;n^{-1/2}u)\right|^3, \quad (40)$$

и

$$\chi_{n,\theta_0}^{(2)}(u)=2\sum_{m=1}^{n}(1-w_m)\eta_{nm}(Y_{1m},Y_{2m};n^{-1/2}u)-\sum_{m=1}^{n}(1-w_m)\eta_{nm}^2(Y_{1m},Y_{2m};n^{-1/2}u)+$$

$$+\sum_{m=1}^{n}(1-w_m)\beta_{mn}\left|\eta_{nm}(Y_{1m},Y_{2m};n^{-1/2}u)\right|^3. \tag{41}$$

Так как условие (2) обеспечивает асимптотическую нормальность последовательности $\frac{1}{\Psi(n,\theta)}\sum\limits_{m=1}^{n}\sum\limits_{i=1}^{k}\frac{\partial\log h_m^{(i)}(X_m;\theta)}{\partial\theta}$ по теореме Линдеберга, то нам остается доказывать справедливость следующих соотношений при $n\to\infty$:

$$P_\theta(\overline{A}_n)\to 0,\ \ P_\theta(\overline{B}_n)\to 0, \tag{42}$$

$$P_{\theta_0}\left(\left|2\sum_{m=1}^{n}w_m\sum_{i=1}^{k}\delta_m^{(i)}\xi_{nm}^{(i)}(X_m;n^{-1/2}u)-n^{-1/2}\sum_{m=1}^{n}w_m\sum_{i=1}^{k}\delta_{nm}^{(i)}\left(\frac{\partial\log h_m^{(i)}(X_m;\theta)}{\partial\theta};u\right)+\frac{1}{4}\left(\mathbb{J}_{1m}(\theta_0)u;u\right)\right|>\varepsilon\right) \tag{43}$$

$$P_{\theta_0}\left(2\sum_{m=1}^{n}(1-w_m)\eta_{nm}(Y_{1m},Y_{2m};n^{-1/2}u)-n^{-1/2}\sum_{m=1}^{n}(1-w_m)\left(\frac{\partial\log\left(H_m(Y_{2m};\,\theta)-H_m(Y_{1m};\,\theta)\right)}{\partial\theta};u\right)+\right.$$

$$\left.+\frac{1}{4}\left(\mathbb{J}_{2m}(\theta)u;u\right)\right|>\varepsilon\right)\to 0, \tag{44}$$

$$P_{\theta_0}\left(\left|\sum_{m=1}^{n}w_m\sum_{i=1}^{k}\delta_m^{(i)}\xi_{nm}^2(X_m;n^{-1/2}u)-\frac{1}{4}\left(\mathbb{J}_{1m}(\theta)u;u\right)\right|>\varepsilon\right)\to 0, \tag{45}$$

$$P_{\theta_0}\left(\left|\sum_{m=1}^{n}(1-w_m)\eta_{nm}^2(Y_{1m},Y_{2m};n^{-1/2}u)-\frac{1}{4}\left(\mathbb{J}_{2m}(\theta)u;u\right)\right|>\varepsilon\right)\to 0, \tag{46}$$

$$P_{\theta_0}\left(\sum_{m=1}^{n}w_m\sum_{i=1}^{k}\delta_m^{(i)}\left|\xi_{nm}^{(i)}(X_m;n^{-1/2}u)\right|^3>\varepsilon\right)\to 0, \tag{47}$$

$$P_{\theta_0}\left(\sum_{m=1}^{n}(1-w_m)\left|\eta_{nm}(Y_{1m},Y_{2m};n^{-1/2}u)\right|^3>\varepsilon\right)\to 0. \tag{48}$$

В силу (20), при $n\to\infty$

$$P_{\theta_0}(\overline{A}_n)\le\sum_{m=1}^{n}P_{\theta_0}\left(\left|w_m\sum_{i=1}^{k}\delta_m^{(i)}\xi_{nm}^{(i)}(X_m;n^{-1/2}u)\right|>\varepsilon\right)=$$

$$=nP_{\theta_0}\left(\left|w_m\sum_{i=1}^{k}\delta_m^{(i)}\xi_{nm}^{(i)}(X_m;n^{-1/2}u)\right|>\varepsilon\right)=o(1).$$

Аналогично доказывается и вторая сходимость в (42) с использованием (21). Согласно (18) и неравенства Маркова при $n\to\infty$

$$P_{\theta_0}\left(\left|\sum_{m=1}^{n}w_m\sum_{i=1}^{k}\delta_m^{(i)}{\xi_{ni}^{(i)}}^2(X_m;n^{-1/2}u)-\frac{1}{n}\sum_{m=1}^{n}w_m\sum_{i=1}^{k}\delta_m^{(i)}\left(u;\frac{\partial g_m^{(i)}(X_m;\theta)}{\partial\theta}\right)^2\right|>\varepsilon\right)\le$$

$$\le\frac{1}{\varepsilon}E_\theta\left\{\sum_{m=1}^{n}w_m\sum_{i=1}^{k}\delta_m^{(i)}\left[{\xi_{ni}^{(i)}}^2(X_m;n^{-1/2}u)-\frac{1}{n}\left(u;\frac{\partial g_m^{(i)}(X_j;\theta)}{\partial\theta}\right)^2\right]\right\}\le$$

$$\le\frac{1}{\varepsilon}E_\theta\left|\sum_{m=1}^{n}w_m\sum_{i=1}^{k}\delta_m^{(i)}\left[{\xi_{ni}^{(i)}}^2(X_m;n^{-1/2}u)-\left(n^{-1/2}u;\frac{\partial g_m^{(i)}(X_m;\theta_0)}{\partial\theta}\right)^2\right]\right|=o(1). \tag{49}$$

С другой стороны, согласно лемме 1 и закону больших чисел при $n \to \infty$

$$\frac{1}{n}\sum_{m=1}^{n} w_m \sum_{i=1}^{k} \delta_m^{(i)} \left(u; \frac{\partial g_m^{(i)}(X_m;\theta)}{\partial \theta} \right)^2 =$$

$$= u^T \frac{1}{n}\sum_{m=1}^{n} w_m \sum_{i=1}^{k} \frac{\partial g_m^{(i)}(X_m;\theta_0)}{\partial \theta} \left(\frac{\partial g_m^{(i)}(X_m;\theta_0)}{\partial \theta} \right)^T u \; - \; \frac{1}{4} \left(\sum_{m=1}^{n} \mathbb{J}_{1m}(\theta_0) u; u \right) \overset{\widetilde{Q}_\theta^{(n)}}{\to} 0. \qquad (50)$$

Из (49) и (50) следует (45). Аналогично доказывается и (46). В силу (42) и (45) получаем (47). Аналогично доказывается (48). Докажем (43) и (44). Согласно (22), при $n \to \infty$

$$E_\theta \left(w_m \sum_{i=1}^{k} \delta_m^{(i)} \xi_{nm}^{(i)}(X_m; n^{-1/2}u) \right) = -\frac{1}{8} \left(\mathbb{J}_1(\theta_0) u; u \right) + o\left(\frac{1}{n} \right). \qquad (51)$$

Ввиду этого при $n \to \infty$ (43) становится эквивалентным соотношению

$$\lambda(u) = P_{\theta_0} \left(2 \left| \sum_{j=1}^{n} \left\{ \; w_m \sum_{i=1}^{k} \delta_m^{(i)} \xi_{nm}^{(i)}(X_m; n^{-1/2}u) - E_\theta \left[w_m \sum_{i=1}^{k} \delta_m^{(i)} \xi_{nm}^{(i)}(X_m; n^{-1/2}u) \right] - \right. \right. \right.$$

$$\left. \left. \left. - \; \frac{n^{-1/2}}{2} w_m \sum_{i=1}^{k} \delta_m^{(i)} \left(\frac{\partial \log h_m^{(i)}(X_m;\theta_0)}{\partial \theta}; u \right) \right\} \right| > \varepsilon \right) \to 0.$$

Теперь учитывая независимость слагаемых

$$\lambda(u) \le \frac{4}{\varepsilon^2} E_{\theta_0} \left\{ \sum_{m=1}^{n} w_m \left[\sum_{i=1}^{k} \delta_m^{(i)} \xi_{nm}^{(i)}(X_m; n^{-1/2}u) - E_\theta \left[w_m \sum_{i=1}^{k} \delta_m^{(i)} \xi_{nm}^{(i)}(X_m; n^{-1/2}u) \right] - \right. \right.$$

$$\left. \left. - \; \frac{n^{-1/2}}{2} w_m \sum_{i=1}^{k} \delta_m^{(i)} \left(\frac{\partial \log h_m^{(i)}(X_m;\theta)}{\partial \theta}; u \right) \right] \right\}^2 = \frac{4n}{\varepsilon^2} E_\theta \left\{ w_m \sum_{i=1}^{k} \delta_m^{(i)} \xi_{nm}^{(i)}(X_m; n^{-1/2}u) - \right.$$

$$\left. - E_{\theta_0} \left[w_m \sum_{i=1}^{k} \delta_m^{(i)} \xi_{nm}^{(i)}(X_m; n^{-1/2}u) \right] - \frac{1}{2} w_m \sum_{i=1}^{k} \delta_m^{(i)} \left(\frac{\partial \log h_m^{(i)}(X_m;\theta)}{\partial \theta}; n^{-1/2}u \right) \right\}^2 =$$

$$= \frac{4n}{\varepsilon^2} E_{\theta_0} \left\{ w_m \left[\sum_{i=1}^{k} \delta_m^{(i)} \xi_{nm}^{(i)}(X_m; n^{-1/2}u) - \frac{1}{2} w_m \sum_{i=1}^{k} \delta_m^{(i)} \left(\frac{\partial \log h_m^{(i)}(X_m;\theta)}{\partial \theta}; n^{-1/2}u \right) \right] \right\}^2 -$$

$$- \frac{8n}{\varepsilon^2} E_{\theta_0} \left[w_m \sum_{i=1}^{k} \delta_m^{(i)} \xi_{nm}^{(i)}(X_m; n^{-1/2}u) - \frac{1}{2} w_m \sum_{i=1}^{k} \delta_m^{(i)} \left(\frac{\partial \log h_m^{(i)}(X_m;\theta)}{\partial \theta}; n^{-1/2}u \right) \right] \cdot$$

$$\cdot E_{\theta_0} \left[w_m \sum_{i=1}^{k} \delta_m^{(i)} \xi_{nm}^{(i)}(X_m; n^{-1/2}u) \right] + \frac{4n}{\varepsilon^2} \left\{ E_{\theta_0} \left[w_m \sum_{i=1}^{k} \delta_m^{(i)} \xi_{nm}^{(i)}(X_m; n^{-1/2}u) \right] \right\}^2 =$$

$$= \frac{4n}{\varepsilon^2} \left\{ E_{\theta_0} \left[w_m \sum_{i=1}^{k} \delta_m^{(i)} \xi_{nm}^{(i)}(X_m; n^{-1/2}u) - \frac{1}{2} w_m \sum_{i=1}^{k} \delta_m^{(i)} \left(\frac{\partial \log h_m^{(i)}(X_m;\theta)}{\partial \theta}; n^{-1/2}u \right) \right]^2 - \right.$$

$$\left. - \left[E_{\theta_0} \left[w_m \sum_{i=1}^{k} \delta_m^{(i)} \xi_{nm}^{(i)}(X_m; n^{-1/2}u) \right] \right]^2 \right\} - \frac{4n^{-1/2}}{\varepsilon^2} E_{\theta_0} \left[\sum_{m=1}^{n} w_m \sum_{i=1}^{k} \delta_m^{(i)} \left(\frac{\partial \log h_m^{(i)}(X_m;\theta)}{\partial \theta}; u \right) \right] \cdot$$

$$\cdot E_{\theta_0} \left[\sum_{m=1}^{n} w_m \sum_{i=1}^{k} \delta_m^{(i)} \xi_{nm}^{(i)}(X_m; n^{-1/2}u) \right]. \qquad (52)$$

Первое слагаемое в правой части последнего неравенства стремится к нулю при $n \to \infty$ в силу (26) из доказательства леммы 4 и (51), а второе слагаемое также в силу (51).

Следовательно, (43) справедливо. Вполне аналогично устанавливается и (44). Теперь ЛАН следует из соотношений (42)-(48) и многомерной ЦПТ. Теорема доказана.

Список литературы

[1] Ibragimov I., Khas'minskii R., *Statistical Estimation: Asymptotic Theory*, New York: Springer-Verlag, 1981. 795 p.

[2] Leman E., *Testing statistical hypothesis*, New York: Springer-Verlag, 2005. 785 p.

[3] Rusas J., *Contiguity of probability measures*, M.: Mir, 1975. 258 p.

[4] Bilingsley P., *Contiguity of probability measures*, M.: Nauka, 1977. 352 p.

[5] Le Cam L., Lo Yang G., *Asyrnptotics in Statistics: some basic concepts*, New York: Springer-Verlag, 2000. 285 p.

[6] Voinov V., Nikulin M., Balakrishnan N., *Chi-squared goodness of fit tests with application*, USA: Elsevier, 2013. 226 p.

[7] A. W. van der Vaart, *Asymptotic Statistics*, USA: Cambridge University Press, 1998. 903 p.

[8] Abdushukurov A., Nurmukhamedova N., Locally asymptotically normality of the family of distributions by incomplete observations, *Journal of Siberian Federal University. Mathematics & Physics*, Vol.7, 2014, pp.141-154.

[9] Abdushukurov A., Nurmukhamedova N., Asymptotic properties of Bayesian – type estimates in the competing risks model under random censoring, *Journal of Mathematical Sciences (United States), Springer*, Vol.245, No.3, 2020, pp.341-349.

[10] Abdushukurov A., Nurmukhamedova N., Asymptotic properties of likelihood ratio statistics in competing risks model under interval random censoring, *Lobachevskii Journal of Mathematics*, Vol.42, No.12, 2021, pp. 2687-2696.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ҚАРЖЫ-ЭКОНОМИКАЛЫҚ ЖҮЙЕЛЕРДІ ЫҚТИМАЛДЫҚ МОДЕЛЬДЕУДЕ ГРАФТАРДЫ ҚОЛДАНУ

АЙГУЛЬ МЫРЗАШЕВА[1], НУРГУЛЬ ШАЖДЕКЕЕВА[2], АЙНАГУЛЬ АДИЕВА[3]

[1,2,3]Х. Досмұхамедов атындағы Атырау университеті , Атырау, Қазақстан
e-mail: [1]aigul_mn@mail.ru, [2]n.shazhdekeeva@mail.ru, [3]a.aina70@mail.ru

Кездейсоқ процестер теориясы – ықтималдықтар теориясының физикада, биологияда, техника мен медицинада, қаржы – экономикалық жүйелерде және білімнің басқа салаларында да қарқынды дамып отырған бөлімі. Кездейсоқ процесс уақытқа байланысты жүріп отыратын кездейсоқ құбылыстарды сипаттау үшін қолданылатын математикалық ықтималды модель болып табылады. Қазіргі кезде кездейсоқ процестер теориясының стационар және гаустық, марковтық процестері негізінен радио және электротехникада, кибернетикада қолданылады. Математикалық экономика, математикалық биологияда әртүрлі марковтық процестер, газдардың молекулярлық теориясында броундық қозғалыс процесі, космостық бөлшектер каскады теориясында тәуелсіз өсімшелер процестері мен марковтық процестер қолданылады. Соңғы кезде жаратылыстану ғылымдарының қандайы да өз салаларында кездейсоқ процестер теориясын пайдаланып отырады. Әрбір білім алушы бұл теорияның әдістерін меңгеру үшін ықтималдықтар теориясы мен математикалық статистиканың дәстүрлі бөлімдері мен жоғары математика, графтар теориясы дифференциалдық теңдеулер, математикалық физика теңдеулері курстарын білгендері дұрыс. Мақала экономикалық-математикалық модельдеудің негізгі бөлімдерінің бірі болып табылатын кездейсоқ дискретті марковтық процестерге, оларды зерттеуде графтардың, өтпелі матрицалардың қолданылуына арналған.

Марковтық процестер кездейсоқ процестердің жеке жағдайы болып табылады. Марковтық процестердің басқа кездейсоқ процестерден ерекшелігі мынада:

- оның көптеген қолданбалы есептерді шешуге мүмкіндік беретін математикалық аппаратының болуы;

- марковтық процестер көмегімен күрделі жүйелердің жағдайын сипаттауға болатындығы.

Бұл модельдеу көмегімен қаржы, экономикалық теорияның басқа әдістерді пайдаланып шешкенде барынша күрделі болатын банк жүйесіндегі немесе товар айналымы процесіндегі басқару жүйесін тиімді реттеу, шешім қабылдау, болжам жасау, жоспарлау есептерін зерттеуге болады. Ал, бұл жағдайлар бүгінгі күннің өзекті мәселелерінің бірі.

Анықтама 1. Тәжірибе жүргізілетін уақыт аралығының әрбір t уақыт мезетіне бір ғана $S(t)$ кездейсоқ шаманы сәйкестендіретін функция кездейсоқ үдеріс (процесс) немесе $S(t)$ кездейсоқ функция деп аталады [1, 2].

Сонымен, кездейсоқ функцияның аргументі-уақыт, ал оның мәні-кездейсоқ шама болып табылады. Осылайша, кездейсоқ функция тәжірибе барысында кездейсоқ шаманың уақыт бойынша өзгерісін сипаттайды.

Жалпылама түрде, жүйе түсінігіне келесі түрде анықтама беруге болады.

Анықтама 2. Тәуелсіз ішкі бөлік жиындарға бөлуге болмайтын, өзара байланысқан элементтердің тұтас жиынтығын S жүйесі ретінде түсінуге болады.

Жүйе элементтері мен олардың арасындағы байланыстар уақыт бойынша өзгеріп отырады және әрбір t уақыт мезетінде S жүйенің $S(t)$ күйін сипаттайды.

Анықтама 3. Егер S жүйе t уақыт өту барысында өзінің $S(t)$ күйін кездейсоқ түрде өзгертіп отыратын болса, онда S жүйеде кездейсоқ үдеріс өтеді (жүреді) делінеді.

$s_1, s_2, ..., s_n, ...-S$ жүйесінің мүмкін жағдайлары (күйлері) болсын. Әдетте, берілген жағдайлар, кез келген уақыт мезетінде жүйе олардың тек біреуінде ғана болатындай (сапалық түрде) анықталады деп есептеледі, яғни кез келген tуақыт мезеті үшін $S(t) = s_i$ орындалатындай бір ғана s_i жағдайы табылады.

Егер жағдайлар жиынтығы есептік санынан аспайтын, яғни ақырлы $\{s_1, s_2, ..., s_n\}$ немесе есептік $\{s_1, s_2, ..., s_n, ...\}$ болса, онда ол дискреттік. Егер күйлердің жиынтығы есептіктен артық болса (мысалы, континуум қуаты бар болса), онда ол үздіксіз болады.

Жағдайлар жиынтығы дискретті болғанда жүйе өзінің жағдайын кенет (лезде) секірмелі түрде өзгертіп отырады. Ал, жағдайлар жиынтығы үздіксіз болғанда жүйенің бір жағдайдан екіншісіне өтуі үздіксіз, біртіндеп, қалыпты түрде жүзеге асырылады.

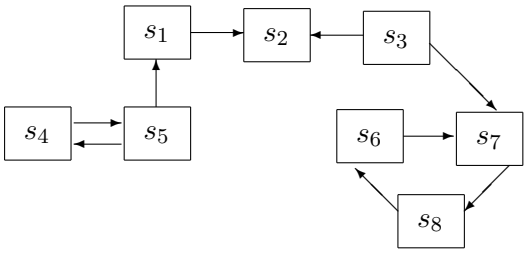
Жүйе элементтері арасындағы бір немесе екі жақтағы байланыстар әрі тікелей, әрі жанама болуы мүмкін.

Жүйеде жүретін дискретті кездейсоқ үдерістерді талдау үшін оның жағдайларының графтарын пайдаланған ыңғайлы.

Анықтама 4. Жүйенің жағдайларының графы ретінде ішкі жағында жағдайлардың белгіленуі және аралықтарында бір жағдайдан келесісіне мүмкін тікелей ауысуларды білдіретін бағытталған сызықтардың (стрелкалардың) жиынтығы орналасатын, жағдайларды шартты түрде кескіндейтін квадраттар жиынтығын (олардың орнына тіктөртбұрыштарды немесе дөңгелектерді де алуға болады) түсінеміз [3, 4].

Мысал 1. 1-суретте $s_1 \to s_2,\ s_3 \to s_2,\ s_3 \to s_7,\ s_4 \to s_5,\ s_5 \to s_4,\ s_5 \to s_1,\ s_6 \to s_7,\ s_7 \to s_8,\ s_8 \to s_6$ мүмкін тікелей ауысуларымен $s_1 - s_8$ сегіз жағдайы бар S жүйесінің жағдай графы бейнеленген.

Мысалы, s_3 жағдайдан s_8 жағдайға өту тек s_7 жағдайы арқылы мүмкін болады, сондықтан ол жанама-тікелей емес болып табылады; $s_3 \to s_8$ тікелей ауысуы мүмкін емес, себебі жағдай графында сәйкесті стрелка көрсетілмеген.



1 сурет - $s_1 - s_8$ жағдайларымен берілген S жүйесінің жағдайлар графы

S жүйесінде өтетін Марковтық кездейсоқ дискретті процесс жүйе кездейсоқ келетін мүмкін жағдайлармен ғана емес, сонымен қатар оның бір жағдайдан келесі бір жағдайға өтетін уақыт мезетімен де сипатталады.

Анықтама 5. Егер жүйенің бір жағдайдан келесі жағдайға өтуі осы процестің кезеңдері (кадамдары немесе адымдары) деп аталатын алдын-ала анықталған $\{t_1, t_2, ..., t_k, ...\}$ уақыт мезеттерінде іске асырылатын болса, онда жүйеде жүретін кездейсоқ процесс дискретті уақытқа тәуелді процесс деп аталады.

Анықтама 6. Егер жүйенің жағдайдан жағдайға кез келген, алдын ала белгісіз, кездейсоқ уақыт мезетінде өтуі мүмкін болса, онда жүйеде жүретін кездейсоқ процесс үзілісті уақытқа тәуелді процесс деп аталады.

Жұмыста экономикалық және қаржылық жүйелердегі дискретті уақытқа тәуелді марковтық дискретті процестерді қарастырамыз.

$s_1, s_2, ..., s_n, ...$ – S жүйесінің барлық мүмкін жағдайлары болсын, осы жағдайлардың біреуінен келесісіне тек қана $t_1, t_2, ..., t_k, ...$ уақыт мезеттерінде ғана секіріп өтеді. $t \in [t_k, t_{k+1}),\ k = 1, 2, 3, ...$ уақыт мезетінде S жүйесі $S(t) = S(t_k)$ жағдайда болатындықтан, берілген процесті t_k кезеңдердің (адымдардың немесе қадамдардың) кездейсоқ функциясы ретінде қарастыруға болады, немесе, оның k номерлеріне тепе-тең деп есептеуге болады. Сонымен, бұл процесс $k = 1, 2, 3, ...$ натурал аргументтің кездейсоқ функциясы, яғни кездейсоқ тізбек болып табылады. Егер S жүйесінің k кезеңнен $k+1$ кезеңге дейінгі аралықта s_i жағдайында болатындығын білдіретін оқиғаны $S_i(k)$ деп белгілесек, яғни $S_i(k)$ оқиғасы « $t \in [t_k, t_{k+1})$ болғанда $S(t) = s_i$ болады» дегенді білдіреді.

Анықтама 7. Егер әрбір кезең үшін кез келген s_i жағдайдан кез келген s_j жағдайына өту ықтималдығы S жүйесінің s_i жағдайда қалай және қашан болғандығына тәуелді болмаса, онда кездейсоқ тізбек марковтық тізбек деп аталады.

Анықтама 8. t_k уақыт мезетінде S жүйесінің s_i жағдайдан s_j жағдайға тікелей өту ықтималдығы k кезең үшін i жағдайдан j жағдайға өтетін өтпелі ықтималдығы деп аталады да, $p_{ij}(k)$ деп белгіленеді.

Егер $i = j$ болса, онда өтпелі ықтималдық $p_{ij}(k) = p_{ii}(k)$ болады, бұл ықтималдық S жүйесінің s_i жағдайда кідіріс ықтималдығы деп аталады. Егер k кезеңде s_i жағдайдан келесі s_j $(i \neq j)$ жағдайға тікелей өту мүмкін болмаса немесе i жағдайда кідіріс $(i = j)$ мүмкін болмаса, онда $p_{ij}(k) = 0$ болады.

Анықтама 9. Егер өтпелі ықтималдықтар k кезеңге байланысты болмаса, онда марковтық тізбек біртекті деп аталады.

Әрбір $k = 1, 2, 3, ...$ кезеңдерінде $S_1(k), S_2(k), ..., S_n(k)$ оқиғалары үйлесімсіз және толық топ құрайтын болғандықтан, ықтималдықтар теориясындағы қосу теоремасы бойынша бұл оқиғалардың ықтималдықтарының қосындысы 1-ге тең болады:

$$\sum_{i=1}^{n} p_i(k) = 1, \quad k = 1, 2, 3, ... \tag{1}$$

Бұл бөлімде біртекті марковтық тізбектер қарастырылады. Өтпелі ықтималдықтарды мына түрдегі n ретті квадраттық матрицалар түрінде жазамыз:

$$P = \left(p_{ij}\right)_{i,j=1}^{n} = \begin{pmatrix} p_{11} & p_{12} & ... & p_{1n} \\ p_{21} & p_{22} & ... & p_{2n} \\ ... & ... & ... & ... \\ p_{n1} & p_{n2} & ... & p_{nn} \end{pmatrix} \tag{2}$$

P матрицасы k кезеңнің номеріне байланысты емес; оның n реті жүйедегі жағдайлар санына сәйкес келеді; матрицаның бас диагоналы бойында кідірістер ықтималдықтары орналасқан [3,4].

p_{ij} өтпелі ықтималдықты $S_j(k)$ (S жүйесі k кезеңнен $(k+1)$ кезеңге дейін s_j жағдайында болады) оқиғасының $S_i(k-1)$ оқиғасы пайда болғаннан кейінгі $p_{ij} = p_{S_i(k-1)}(S_j(k))$ шартты ықтималдығы ретінде түсіндіруге болады, мұндағы $S_i(k-1)$ оқиғасы S жүйесі $(k-1)$ кезеңнен k кезеңге дейін s_i жағдайында болды дегенді білдіреді. Ендеше, $S_1(k), S_2(k), ..., S_n(k)$ оқиғаларының үйлесімсіз және толық топ құрайтындығынан олардың ықтималдықтарының қосындысы 1-ге тең болады:

$$\sum_{j=1}^{n} p_{ij} = 1, \quad i = 1, ..., n, \tag{3}$$

яғни, осыған сәйкес өтпелі ықтималдықтар матрицасының әрбір жатық жолының элементтерінің қосындысы 1-ге тең болады екен. Ендеше, өтпелі ықтималдықтар матрицасы стохастикалық матрица болады.

Теорема. Біртекті марковтық тізбек үшін $k-$ дан $(k+1)$ кезеңге дейінгі жағдай ықтималдықтарының жол-векторы $(k-1)-$ден k кезеңге дейінгі жағдай ықтималдықтарының жол-векторы матрицасы мен өтпелі ықтималдықтар матрицасының көбейтіндісіне тең болады:

$$\big(p_1(k), p_2(k), ..., p_n(k)\big) = \big(p_1(k-1), p_2(k-1), ..., p_n(k-1)\big) \cdot P \tag{4}$$

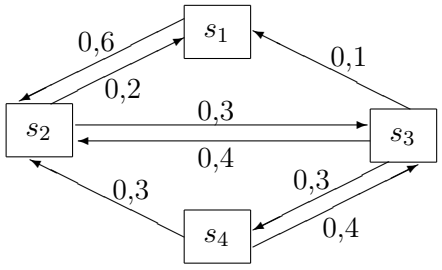
Салдар. Біртекті марковтық тізбек үшін төмендегі формула орындалады:

$$\big(p_1(k), p_2(k), ..., p_n(k)\big) = \big(p_1(0), p_2(0), ..., p_n(0)\big) \cdot P^k, \quad k = 1, 2, ... \tag{5}$$

Осы теорема мен салдар көмегімен банк жүйесіндегі шешім қабылдау, талдау, болжам жасау жұмысында пайдалы болатын есепті қарастыруға болады.

Мысал 2. S_1, S_2, S_3, S_4 жағдайлары сәйкесінше 3%, 4%, 5% және 6% пайыздық мөлшерлемелермен сипатталатын банк жағдайын қарастырайық. Бұл пайыздық мөлшерлемелер әр айдың басында тағайындалады және ай бойына өзгеріссіз қалады. Алдыңғы уақыт аралықтарындағы банк жұмысын талдау, бақылау нәтижесі тоқсан барысында өтпелі ықтималдықтардың өзгерісінің елемеуге болатындай, аз шамада екендігі анықталған, ендеше оларды тұрақты деп есептеуге болады. Егер өткен тоқсан аяғында банктің пайыздық мөлшерлемесі 5%-ды құраған болса және сәйкесті ықтималдықтар $p_{12} = 0,6; p_{21} = 0,2; p_{23} = 0,3; p_{31} = 0,1; p_{32} = 0,4; p_{34} = 0,3; p_{42} = 0,3; p_{43} = 0,4$ екендігі белгілі болса, тоқсан аяғындағы банктің жағдай ықтималдықтарын анықтау керек болсын [5].

Шешуі. S жүйесі ретінде қарастырылып отырған банк жүйесін алып, берілген ықтималдықтарға сәйкес таңбаланған жағдай графы тұрғызылады, ол 2-суретте келтірілген. Жүйе әрбір уақыт мезетінде төрт жағдайдың тек біреуінде ғана бола алады. Келтірілген анықтамаларға сәйкес банк жүйесінде біртекті марковтық дискретті уақытқа байланысты (тәуелді) дискретті кездейсоқ процесс жүреді, яғни біртекті марковтық тізбек болады.



2 сурет - 2 мысалға сәйкесті жүйенің жағдай графы

Алдымен есепті шешуді (5) формуласы бойынша жүргізейік. 2 суреттегі таңбаланған граф бойынша p_{ij} өтпелі ықтималдықтардың мәндерін жазамыз: $p_{12} = 0,6; p_{13} = 0; p_{14} = 0$, онда $i = 1$ болғанда, бұл оқиғалар толық топ құрайтын болғандықтан $p_{ii} = 1 - \sum\limits_{j=1, j\neq i}^{n} p_{ij}$, $i = 1, ..., n$ формуласы бойынша $p_{11} = 1 - (p_{12} + p_{13} + p_{14}) = 1 - 0,6 = 0,4$ болады.

Сол сияқты,

$$p_{21} = 0,2; \ p_{23} = 0,3; \ p_{24} = 0; \ p_{22} = 1 - (p_{21} + p_{23} + p_{24}) = 1 - (0,2 + 0,3) = 0,5$$

$p_{31} = 0,1;\ p_{32} = 0,4;\ p_{34} = 0,3;\ p_{33} = 1 - (p_{31} + p_{32} + p_{34}) = 1 - (0,1 + +0,4 + 0,3) = 0,2.$

Осы мәндер бойынша өтпелі ықтималдықтар матрицасы құрылады, дұрыс құрылған матрицаның әрбір жатық жолындағы элементтер қосындысы 1-ге тең болуы керек:

$$P = \begin{pmatrix} 0,4 & 0,6 & 0 & 0 \\ 0,2 & 0,5 & 0,3 & 0 \\ 0,1 & 0,4 & 0,2 & 0,3 \\ 0 & 0,3 & 0,4 & 0,3 \end{pmatrix}.$$

$t = 0$ бастапқы уақыт мезетінде S жүйесі s_3 жағдайда болғандықтан, ықтималдықтың бастапқы таралу матрицасы мынадай болады:

$$\begin{pmatrix} p_1(0) & p_2(0) & p_3(0) & p_4(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}.$$

Үш ай өткеннен кейінгі тоқсан аяғында банктің жағдай ықтималдығын $n = 4$ және $k = 3$ мәндері үшін (5) формуласы бойынша табуға болады. Осы жағдай үшін бұл формула былай жазылады:

$$\begin{pmatrix} p_1(3) & p_2(3) & p_3(3) & p_4(3) \end{pmatrix} = \begin{pmatrix} p_1(0) & p_2(0) & p_3(0) & p_4(0) \end{pmatrix} \cdot P^3.$$

P^3 матрицасын тауып алу керек, ол үшін алдымен P^2 табамыз:

$$P^2 = \begin{pmatrix} 0,4 & 0,6 & 0 & 0 \\ 0,2 & 0,5 & 0,3 & 0 \\ 0,1 & 0,4 & 0,2 & 0,3 \\ 0 & 0,3 & 0,4 & 0,3 \end{pmatrix} \cdot \begin{pmatrix} 0,4 & 0,6 & 0 & 0 \\ 0,2 & 0,5 & 0,3 & 0 \\ 0,1 & 0,4 & 0,2 & 0,3 \\ 0 & 0,3 & 0,4 & 0,3 \end{pmatrix} =$$

$$\begin{pmatrix} 0,28 & 0,54 & 0,18 & 0 \\ 0,21 & 0,49 & 0,21 & 0,09 \\ 0,14 & 0,43 & 0,28 & 0,15 \\ 0,1 & 0,4 & 0,29 & 0,21 \end{pmatrix}.$$

$$P^3 = \begin{pmatrix} 0,28 & 0,54 & 0,18 & 0 \\ 0,21 & 0,49 & 0,21 & 0,09 \\ 0,14 & 0,43 & 0,28 & 0,15 \\ 0,1 & 0,4 & 0,29 & 0,21 \end{pmatrix} \cdot \begin{pmatrix} 0,4 & 0,6 & 0 & 0 \\ 0,2 & 0,5 & 0,3 & 0 \\ 0,1 & 0,4 & 0,2 & 0,3 \\ 0 & 0,3 & 0,4 & 0,3 \end{pmatrix} =$$

$$\begin{pmatrix} 0,238 & 0,51 & 0,198 & 0,054 \\ 0,203 & 0,482 & 0,225 & 0,09 \\ 0,17 & 0,456 & 0,245 & 0,129 \\ 0,149 & 0,439 & 0,262 & 0,15 \end{pmatrix}.$$

Сонда, ізделінді ықтималдық (үш ай өткеннен кейінгі тоқсан аяғында банктің жағдай ықтималдығы):

$$\begin{pmatrix} p_1(3) & p_2(3) & p_3(3) & p_4(3) \end{pmatrix} = \begin{pmatrix} p_1(0) & p_2(0) & p_3(0) & p_4(0) \end{pmatrix} \cdot P^3 =$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \cdot P^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0,238 & 0,51 & 0,198 & 0,054 \\ 0,203 & 0,482 & 0,225 & 0,09 \\ 0,17 & 0,456 & 0,245 & 0,129 \\ 0,149 & 0,439 & 0,262 & 0,15 \end{pmatrix} =$$

$$= \begin{pmatrix} 0,17 & 0,456 & 0,245 & 0,129 \end{pmatrix}.$$

Бұдан

$$p_1(3) = 0,17;\ p_2(3) = 0,456;\ p_3(3) = 0,245;\ p_4(3) = 0,129,$$

яғни тоқсан аяғында 3%, 4%, 5% және 6% пайыздық мөлшерлемелердің ықтималдығы сәйкесінше 0,17; 0,456; 0,245; 0,129 болады екен. Нақтырақ айтқанда, өткен тоқсан аяғында

5% пайыздық мөлшерлеме болса, бұл тоқсан аяғында басқа пайыздық мөлшерлемелерге қарағанда 4% пайыздық мөлшерлеме болуы ықтималды болады екен.

Енді қажетті ықтималдықтарды біртекті марковтық тізбек үшін k-дан $(k+1)$ кезеңге дейінгі жағдай ықтималдықтарының жол-векторының немесе жол матрицасының (4) формуласы бойынша тауып көрелік. $t=0$ бастапқы уақыт мезетінде S жүйесі s_3 жағдайда болғандықтан, ықтималдықтың бастапқы таралу матрицасы мынадай болады [5].

Қалған ықтималдықтарды біртіндеп осы формула бойынша рекуррентті түрде таба берсек

$$\begin{pmatrix} p_1(1) & p_2(1) & p_3(1) & p_4(1) \end{pmatrix} = \begin{pmatrix} p_1(0) & p_2(0) & p_3(0) & p_4(0) \end{pmatrix} \cdot P =$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0,4 & 0,6 & 0 & 0 \\ 0,2 & 0,5 & 0,3 & 0 \\ 0,1 & 0,4 & 0,2 & 0,3 \\ 0 & 0,3 & 0,4 & 0,3 \end{pmatrix} = \begin{pmatrix} 0,1 & 0,4 & 0,2 & 0,3 \end{pmatrix}.$$

$$\begin{pmatrix} p_1(2) & p_2(2) & p_3(2) & p_4(2) \end{pmatrix} = \begin{pmatrix} p_1(1) & p_2(1) & p_3(1) & p_4(1) \end{pmatrix} \cdot P =$$

$$= \begin{pmatrix} 0,1 & 0,4 & 0,2 & 0,3 \end{pmatrix} \cdot \begin{pmatrix} 0,4 & 0,6 & 0 & 0 \\ 0,2 & 0,5 & 0,3 & 0 \\ 0,1 & 0,4 & 0,2 & 0,3 \\ 0 & 0,3 & 0,4 & 0,3 \end{pmatrix} = \begin{pmatrix} 0,14 & 0,43 & 0,28 & 0,15 \end{pmatrix}.$$

$$\begin{pmatrix} p_1(3) & p_2(3) & p_3(3) & p_4(3) \end{pmatrix} = \begin{pmatrix} p_1(2) & p_2(2) & p_3(2) & p_4(2) \end{pmatrix} \cdot P =$$

$$= \begin{pmatrix} 0,14 & 0,43 & 0,28 & 0,15 \end{pmatrix} \cdot \begin{pmatrix} 0,4 & 0,6 & 0 & 0 \\ 0,2 & 0,5 & 0,3 & 0 \\ 0,1 & 0,4 & 0,2 & 0,3 \\ 0 & 0,3 & 0,4 & 0,3 \end{pmatrix} = \begin{pmatrix} 0,17 & 0,456 & 0,245 & 0,129 \end{pmatrix}.$$

Бұдан алдыңғы шешіммен бірдей жағдай ықтималдықтың мынадай нәтижелерін аламыз:

$$p_1(3) = 0,17;\ p_2(3) = 0,456;\ p_3(3) = 0,245;\ p_4(3) = 0,129.$$

Олай болса, бұл әдіспен есептеу жүргізгенде де 4% пайыздық мөлшерлеме болуы тиімді болады екен.

Осындай есептерді шығару барысында есептеулердің дұрыстығына көз жеткізу үшін берілген мәліметтер бойынша жүйенің жағдай графын тұрғызып, граф бойынша белгісіз ықтималдықтарды тауып, алынған матрицаларды стохастикалыққа тексеріп отыру керек. Аталған сипаттағы есептер шешімі қаржы, экономикалық жүйелер жұмысының тиімділігін арттыруда пайдасы мол. Сонымен қатар, келтірілген материалды меңгеру білім алушылардың математикалық, ақпараттық мәдениетін қалыптастырып, дамытуға, қаржы-экономикалық жүйелердің қызметін талдап, болжам жасау, тиімді шешім қабылдау, қаржылық жоспарлау және басқаруды жетілдіру, қаржылық тұрақтылық жағдайын қамтамасыз ету жұмыстарын тиянақты да сауатты орындауларына игі әсер етері анық.

Кілттік сөздер: Кездейсоқ шама, кездейсоқ үдеріс, кездейсоқ функция, жүйе, жүйенің жағдайы, жүйеде өтетін кездейсоқ үдеріс, марковтік үдерістер, жүйенің жағдай графы.

AMS пәндік классификация: 60

Әдебиеттер тізімі

[1] Гмурман В.Е., *Теория вероятностей и математическая статистика*, М.: Высшая школа, 2003, 479 с.

[2] Мырзашева А.Н., *Ықтималдықтар теориясы және математикалық статистика есептері*, АГУ баспасы, 2008, 80 б.

[3] Кремер Н.Ш., и др. *Исследование операции в экономике* , М.: ЮНИТИ, 2002, 407 с.

[4] Бережная Е.В., Бережной В.И., *Математические методы моделирования экономических систем*, М.: Финансы и статистика, 2006, 432 с.

[5] Лабскер Л.Г., *Вероятностное моделирование в финансово-экономической области*, М.: Альпина Паблишер, 2002, 224 с.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ПАРАБОЛИЧЕСКАЯ ИНТЕРПОЛЯЦИЯ ТАБЛИЦ СМЕРТНОСТИ ДЛЯ ДРОБНЫХ ВОЗРАСТОВ

ФАРХАТ НАЗАРБАЕВ[1]

[1]Кыргызский Национальный Университет им. Ж.Баласагына, Бишкек, Кыргызстан
e-mail: nft8888@gmail.com

Аннотация. В статье исследуется новый метод параболической интерполяции таблиц смертности для дробных возрастов. Практически все предлагаемые формулы интерполяции для дробных возрастов имеют выпуклость вниз или линейны. А предложенная параболическая интерполяция имеет выпуклость вверх, что более соответствует таблицам смертности на примере Кыргызстана [3] в области от года до 80 лет. Также исследуем показатели таблиц смертности.

Ключевые слова: актуарная математика, таблицы смертности, интерполяция таблиц смертности, таблицы смертности для дробных возрастов, страхование жизни.

Предметная классификация AMS: 62P05.

1. Описание параболической интерполяции

Предоставляемые статистическими организациями таблицы смертности обычно имеют шаг в целых годах, что усложняет их применение в приложениях, где важны таблицы смертности для дробных возрастов, таких как страхование жизни, негосударственные пенсионные фонды и другие виды деятельности. Для таких случаев необходимо составить таблицы смертности для дробных возрастов. Формулы интерполяции, предлагаемые в книгах Бауэрса Н., Гербера Х., Джонса Д., Несбитта С., Хикмана Дж. [1], Фалина Г. И., Фалина А. И. [2], [4] и [5] имеют как уже говорилось выпуклость вниз или линейны, но они более легки в вычислительном смысле. Но на сегодняшний день вычислительные процессы легко запрограммировать, и поэтому данный способ интерполяции также легко можно запрограммировать. Приведем определение параболической интерполяции из [6].

Определение 1.1. *Интерполяционная формула для определения функции дожития для дробных возрастов определяемая по формуле*

$$s^2(x+t) = (1-t)s^2(x) + ts^2(x+1) \tag{1}$$

где $0 \le t \le 1, x \in Z^+$ ***назовем параболической интерполяцией.***

Так как функция дожития невозрастающая функция, перепишем интерполяционную формулу в виде

$$s(x+t) = \sqrt{s^2(x) - t \cdot (s^2(x) - s^2(x+1))}. \tag{2}$$

Найдя производную функции (2)

$$s'_t(x+t) = \frac{s^2(x) - s^2(x+1)}{2\sqrt{s^2(x) - t \cdot (s^2(x) - s^2(x+1))}}$$

можно заметить что производная функции отрицательна, т.к. $s(x) > s(x+1)$, следовательно функция дожития имеет выпуклость вверх.

Вычислим вероятности дожития человека в возрасте x до возраста $x+t$, с учетом параболической интерполяции

$$ {}_tp_x = \sqrt{1 - t(1 - p_x^2)} \tag{3} $$

а вероятность смерти человека в возрасте x в течении дробного времени t можно вычислить по формуле

$$ {}_tq_x = 1 - {}_tp_x = 1 - \sqrt{1 - t(1 - p_x^2)} = 1 - \sqrt{1 + t(q_x^2 - 2q_x)} \tag{4} $$

Полученные формулы (3) и (4) для определения вероятности дожития до возраста $x+t$, и вероятности смерти человека в возрасте x в течении времени t будем использовать в дальнейшем.

2. Вычисление показателей таблиц смертности

Одним из важных вероятностных функций таблиц смертности для дробных возрастов является интенсивность смертности. Докажем следующую теорему для интенсивности смертности.

Теорема 2.1. *интенсивность смертности для дробных возрастов в случае параболической интерполяции определяется по формуле*

$$ \mu(x+t) = \frac{1 - p_x^2}{2 \cdot (1 - t(1 - p_x^2))} \tag{5} $$

где $x \in Z^+$ *а* $0 < t < 1$.

Доказательство. Для доказательства теоремы воспользуемся определением формулы интенсивности смертности, которая определяется по формуле $\mu(x) = \frac{-s'(x)}{s(x)}$.

Учитывая формулу (2) для определения функции дожития при параболической интерполяции мы получим следующее

$$ \mu(x+t) = \frac{-s'_t(x+t)}{s(x+t)} = $$

$$ = \frac{-\left(\sqrt{s^2(x) - t(s^2(x) - s^2(x+1))}\right)'_t}{\sqrt{s^2(x) - t(s^2(x) - s^2(x+1))}} = $$

$$ = \frac{s^2(x) - s^2(x+1)}{2 \cdot (s^2(x) - t(s^2(x) - s^2(x+1)))}, \tag{6} $$

учитывая, что $s(x) \neq 0$, поделим числитель и знаменатель последней строки формулы (6) на $s^2(x)$, получим

$$ \mu(x+t) = \frac{1 - \left(\frac{s(x+1)}{s(x)}\right)^2}{2 \cdot \left(1 - t\left(1 - \left(\frac{s(x+1)}{s(x)}\right)^2\right)\right)} $$

так как $\frac{s(x+1)}{s(x)} = p_x$ мы и получим нашу искомую формулу (5), т.е.

$$ \mu(x+t) = \frac{1 - p_x^2}{2 \cdot (1 - t(1 - p_x^2))} $$

Что и требовалось доказать. □

Теперь вычислим другие вероятностные функции для дробных возрастов:

1. Вычислим ${}_{1-t}q_{x+t}$ – вероятность того, что человек в возрасте $x+t$ умрет в течении $1-t$ лет

$$
{}_{1-t}q_{x+t} = 1 - \frac{s(x+1)}{s(x+t)} = 1 - \frac{\frac{s(x+1)}{s(x)}}{\frac{s(x+t)}{s(x)}} = 1 - \frac{p_x}{{}_tp_x} =
$$

$$
= 1 - \frac{1-q_x}{1-{}_tq_x} = 1 - \frac{1-q_x}{\sqrt{1+t(q_x^2-2q_x)}} \quad (7)
$$

2. Вычислим ${}_yq_{x+t}$ – вероятность того, что человек в возрасте $x+t$ умрет в течении y лет, при условии $0 < y < 1$ и $0 < y+t \leq 1$

$$
{}_yq_{x+t} = 1 - \frac{s(x+t+y)}{s(x+t)} = 1 - \frac{\frac{s(x+t+y)}{s(x)}}{\frac{s(x+t)}{s(x)}} = 1 - \frac{{}_{t+y}p_x}{{}_tp_x} =
$$

$$
= 1 - \frac{1-{}_{t+y}q_x}{1-{}_tq_x} = 1 - \sqrt{\frac{1+(t+y)(q_x^2-2q_x)}{1+t(q_x^2-2q_x)}} =
$$

$$
= 1 - \sqrt{1+\frac{y(q_x^2-2q_x)}{1+t(q_x^2-2q_x)}} \quad (8)
$$

учитывая выведенные выше формулы (3), (4), (7) и (8) можно сформулировать одну большую теорему.

Теорема 2.2. *В случае параболической интерполяции дробных возрастов, имеют место следующие формулы для определения вероятностных функций дробных возрастов:*

(1) ${}_tp_x = \sqrt{1-t(1-p_x^2)}$

(2) ${}_tq_x = 1-{}_tp_x = 1-\sqrt{1-t(1-p_x^2)} = 1-\sqrt{1+t(q_x^2-2q_x)}$

(3) ${}_{1-t}q_{x+t} = 1 - \frac{1-q_x}{\sqrt{1+t(q_x^2-2q_x)}}$

(4) ${}_yq_{x+t} = 1-\sqrt{1+\frac{y(q_x^2-2q_x)}{1+t(q_x^2-2q_x)}}$

Доказательсво каждого из пунктов данной теоремы уже приведены выше в формулах (3), (4), (7) и (8).

3. Заключение

Данный метод интерполяции можно применять в случаях расчета различных показателей страхования жизни и страхования на дожитие. Также данный метод интерполяции можно использовать в образовательных целях при подготовке актуариев, создавая задачи с учетом данного метода интерполирования для дробных возрастов.

Но следует заметить что данный метод лучше применять в области $(5; 80)$ т.е. в случаях когда необходимо сведения для возрастов в промежутке от 5 до 80 лет. Т.к. именно в данной области график функции в [6] из таблицы смертности [3] имеет выпуклость вверх.

Список литературы

[1] Бауэрс Н., Гербер Х., Джонс Д., Несбитт С., Хикман Дж. Актуарная математика. Перев. С англ. / Под ред. В.К. Малиновского. – М.: Янус-К, 2001. – 656 с., илл. стр. 63-89.

[2] Фалин Г.И. Математические основы теории страхования жизни и пенсионных схем. — Издание 2-е, переработанное и дополненное. — М.: Анкил, 2002 г. 262 стр. ISBN 5-86476-194-X. стр. 47-57.

[3] Демографический ежегодник Кыргызской Республики: 2016-2020.-Б: Нацстатком Кырг. Респ., 2021:-312с. ISBN 978-9967-26-837-1. стр. 161-163.

[4] Фалин Г. И., Фалин А. И. Актуарная математика в задачах. — 2-е изд., перераб. и доп. — М.: ФИЗМАТЛИТ, 2003. — 192 с. — ISBN 5-9221-0451-9. стр. 35-44.

[5] А.Г.Фалин, Г.И.Фалин. Введение в математику финансов и инвестиций для актуариев: Учебное пособие. – Изд. 2-е, перераб. и доп.– М.: МАКС Пресс, 2019 – 359 с. ISBN 978-5-317-06167-8

[6] Назарбаев Ф.Т., Доолбекова А.У. Об одном методе интерполяции таблиц смертности для дробных возрастов // Международный журнал гуманитарных и естественных наук. 6-3 (69), июнь 2022, ISSN 2500-1000 стр. 103-107.

Кері және қисынсыз есептер

Inverse and ill-posed problems

Обратные и некорректные задачи

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON THE REGULARIZATION OF THE SOLUTION OF A NONLINEAR OPERATOR EQUATION OF THE FIRST KIND IN HILBERT SPACE

ASKERBEK SAADABAEV[1], IZAT USENOV[1]

[1] KNU named after J.Balasagyn, Kyrgyzstan, Bishkek
e-mail: iausen@mail.ru

Abstract. Many applied problems of physics and biophysics are reduced to the operator equations of the first kind. These equations are reduced as inverse problems of mathematical physics in cases where the expression for the Green's function is unknown. The inverse problem of electric logging of wells defining deposits and reserve calculation of mineral resources is an example of such problems. The above application tasks are urgent tasks of modern science, these tasks can discover new facets of the current state of development of mankind. In this regard, it stresses the importance of studying ill-posed problems. In [1], MM Lavrent'ev proposed regularization method for solving linear operator equations of the first kind in Hilbert space with the replacement of the original equation close to him, in a sense, for which the task of finding a solution is resistant to small changes in the right-hand side and is solvable for any right-hand side, ie, the equation is replaced by an equation

$$\alpha z + Az = u,$$

where A – linear operator - regularization parameter. In [2] Newton's method of approximate solution of the equations was distributed L.V.Kantorovich on functional equations

$$K(z) = 0$$

where $K(z)$- non-linear, twice Frechet differentiable operator acting from one Banach space into another.In this paper, we propose a combined method of regularization of a new type, combining the idea of the method of M. M. Lavrent'ev [1], Newton-Kantorovich method [2] for the regularization of solutions of nonlinear operator equations of the first kind in Hilbert space.

Keywords: nonlinear operator, regularization, Hilbert space, the Frechet differential, linear operator, bounded operator, Lipschitz condition.

AMS Subject Classification: 517.94

1. Statement of objectives

Consider the nonlinear operator equation of the first kind type

$$K(z) = u \tag{1}$$

where $K : H \to H$ the non-linear operator, Frechet differentiable.

In the space we use the notation of the ball $S(z_0, r) = \{z : \|z - z_0\| \leq r_z\}$.

We assume that

1) there $u = u^*$exists a unique solution z^* of equation (1), ie, we have the identity

$$K(z^*) \equiv u^* \tag{2}$$

2) the operator K is continuous in the ball $S(z_0, r_z)$, and has a Frechet K', derivative continuous in point z_0.

3) linear non-self-adjoint operator $K'(z_0)$ is reversible, but unlimited, ie $\left\|[K'(z_0)]^{-1}\right\| = \infty$;

4) operator satisfies the Holder condition

$$\left\|K'(z_1) - K'(z_2)\right\| \leq N\|z_1 - z_2\|^{\beta} \tag{3}$$

where $N$and β are constants $0 < \beta \leq 1$,

5) element u^* is unknown to us, but instead is known element u_δ such that

$$\|u_\delta - u^*\| \leq \delta \tag{4}$$

where $\delta > 0$- numeric parameter.

Thus, equation (1) in the ball $S(z_0, r_z)$ is ill-posed problem.

With $K'^*(z_0)$ we denote the operator, which is conjugate to the operator $K'(z_0)$.

Consider the nonlinear operator equation of the first kind type

$$K'^*(z_0)\,K(z) = K'^*(z_0)\,u \tag{1*}$$

Equations (1) and (1*) are equivalent.

2. Regularization of the solution of the problem

Along with the equation (1*) we consider the equation

$$\alpha z^{\alpha} + K'^*(z_\alpha)\,K(z^{\alpha}) = K'^*(z_\alpha)\,u \tag{5}$$

where $\alpha > 0$ the small regularization parameter.

Lemma 2.1. *If $K'^*(z_\alpha)\,K(z)$ – Fréchet differentiable operator from H in H and performed the $K'^*(z)K'(z)$ weakened condition of the Holder with a fixed element $z_2 \equiv z_\alpha$, then*

$$\left\|K'^*(z_\alpha)\right\|\left\|K(z) - K(z_\alpha) - K'(z_\alpha)(z - z_\alpha)\right\| \leq \frac{NM}{1+\beta}\|z - z_\alpha\|^{1+\beta} \tag{6}$$

It is sufficient to fulfill the conditions of the Holder (3) on the segment $z_\tau = z_\alpha + \tau(z - z_\alpha)$, $0 \leq \tau \leq 1$

Proof. The lemma follows from the identity

$$K(z) - K(z_\alpha) - K'(z_\alpha)(z - z_\alpha) = \int_0^1 \left\{K'(z_\alpha + \tau(z - z_\alpha)) - K'(z_\alpha)\right\}(z - z_\alpha)d\tau \tag{7}$$

To construct a regularizing operator inside the ball $S(z_0, r_z)$ picking the number z_α form the ball $S(z_\alpha, r_z(\alpha))$, where the center of the ball at $\alpha \to 0$ the center of the ball $S(z_0, r_z)$approaches, ie $z_\alpha(u_0) \to z_0$, consequently $r_z(\alpha) \to 0$.

Elements ball $S(z_\alpha, r_z(\alpha))$ satisfy the inequality $\|z - z_\alpha\| \leq r_z(\alpha)$, then between the elements z_α and z^* have the estimate $\|z^* - z_\alpha\| \leq \gamma r_z(\alpha)$ where $0 < \gamma < {}^1/_2$. □

Theorem 2.2. *Suppose that 1) the condition 2), and 2) the linear operator $K'^*(z_\alpha)K'(z_\alpha)$ is continuous positive self-adjoint operator; 3) linear operator $K'^*(z_\alpha)$ in the ball approaches the inequality*

$$\left\|K'^*(z_\alpha)\right\| \leq M \tag{8}$$

4) the operator $K'(z)$ holds the weakened condition of the Holder with a fixed element $z_2 \equiv z_\alpha$

$$\left\|K'(z) - K'(z_\alpha)\right\| \leq N\|z - z_\alpha\|^{\beta} \text{ at } S(z_\alpha, r_z(\alpha)) \tag{9}$$

5) there is a limit relation

$$\lim_{r_z(\alpha)\to 0} \frac{r_z(\alpha)}{\alpha(r_z(\alpha))} = 0, \alpha(r_z(\alpha)) = \sigma r_z^{\sigma\beta}(\alpha),\ 0 < \sigma < \left(\frac{1}{2}\right)^{\frac{\beta(1-\sigma)}{2-\beta-\beta\sigma}}\left(\frac{NM}{1+\beta}\right)^{\frac{1-\beta}{2-\beta-\beta\sigma}} \tag{10}$$

then there is a number $\tilde{r} > 0$, that the $r_z(\alpha) < \tilde{r}$ operator

$$I_1[\tau] = \left(\alpha E + K'^*(z_\alpha)K'(z_\alpha)\right)$$

$$\left(E+\left(\alpha E+K'^{*}(z_\alpha)\quad K'(z_\alpha)\right)^{-1}K'^{*}(z_\alpha)\left(\int_0^1\left\{K'(z_\alpha+\tau(z-z_\alpha))-K'(z_\alpha)\right\}d\tau\right)\right)$$

(where $0\le\tau\le 1$, z- fixed element) in a ball $S\left(z_\alpha,r_z\left(\alpha\right)\right)$ has an inverse bounded operator.

Proof. We make an assessment

$$\left\|\left(\alpha E+K'^{*}(z_\alpha)K'(z_\alpha)\right)^{-1}K'^{*}(z_\alpha)\left(\int_0^1\left\{K'(z_\alpha+\tau(z-z_\alpha))-K'(z_\alpha)\right\}d\tau\right)\right\|\le\frac{1}{\alpha}\frac{NM}{1+\beta}\|z-z_\alpha\|^{\beta}\le$$

$\frac{NMr_z^{\beta}(\alpha)}{(1+\beta)\alpha}$,

where $\left\|\left(\alpha E+K'^{*}(z_\alpha)K'(z_\alpha)\right)^{-1}\right\|\le\frac{1}{\alpha}$, [1].

From (4) it follows that there exists $\tilde{r}>0$ such that

$$\frac{NMr_z^{\beta}\left(\alpha\right)}{\left(1+\beta\right)\alpha(r_z\left(\alpha\right))}<\frac{NM\tilde{r}}{\left(1+\beta\right)\alpha(\tilde{r})}=q<\sigma\ \text{at}\ \ r_z\left(\alpha\right)<\tilde{r},\ 0<\sigma<\left(\frac{1}{2}\right)^{\frac{\beta(1-\sigma)}{2-\beta-\beta\sigma}}\left(\frac{NM}{1+\beta}\right)^{\frac{1-\beta}{2-\beta-\beta\sigma}}\tag{11}$$

When selecting $\alpha(r_z\left(\alpha\right))=\sigma r_z^{\sigma\beta}\left(\alpha\right)$, the condition (4) is satisfied, and choose a specific number $\tilde{r}>0$, that

$$r_z\left(\alpha\right)<\left(\frac{\left(1+\beta\right)\sigma^2}{NM}\right)^{1/\beta(1-\sigma)}=\tilde{r}\ and\ \alpha(\tilde{r})=\sigma\left(\frac{\left(1+\beta\right)\sigma^2}{NM}\right)^{\sigma/1-\sigma}\tag{12}$$

Consequently,

$$q_1=\frac{NM\tilde{r}}{\left(1+\beta\right)\alpha(\tilde{r})}=\frac{\left(^{(1+\beta)\sigma^2}/_{NM}\right)^{1/\beta(1-\sigma)}NM}{\left(1+\beta\right)\sigma\left(^{(1+\beta)\sigma^2}/_{NM}\right)^{\sigma/1-\sigma}}=\left(\frac{\left(1+\beta\right)}{NM}\right)^{1-\beta/\beta(1-\sigma)}\sigma^{\frac{2-\beta-\beta\sigma}{\beta(1-\sigma)}}<\frac{1}{2}\tag{13}$$

By virtue of the Banach theorem [3] in the ball $S\left(z_\alpha,\ r\left(\alpha\right)\right)$ of the operator

$$\varphi\left[\tau\right]\equiv E+\left(\alpha E+K'^{*}(z_\alpha)K'(z_\alpha)\right)^{-1}K'^{*}(z_\alpha)\left(\int_0^1\left\{K'(z_\alpha+\tau(z-z_\alpha))-K'(z_\alpha)\right\}d\tau\right)$$

reversible and the estimate

$$\left\|\left(\varphi\left[\tau\right]\right)^{-1}\right\|=$$

$$\left\|\left(E+\left(\alpha E+K'^{*}(z_\alpha)K'(z_\alpha)\right)^{-1}K'^{*}(z_\alpha)\left(\int_0^1\left\{K'(z_\alpha+\tau(z-z_\alpha))-K'(z_\alpha)\right\}d\tau\right)\right)^{-1}\right\|\le$$

$$\frac{1}{1-q_1}$$

Thus, the operator is reversible $I_1\left[\tau\right]$, and the inverse has the form

$$\left(I_1\left[\tau\right]\right)^{-1}=\left(\varphi\left[\tau\right]\right)^{-1}\left(\alpha E+K'^{*}(z_\alpha)K'(z_\alpha)\right)^{-1}$$

and the estimate

$$\left\|\left(I_1\left[\tau\right]\right)^{-1}\right\|\le\frac{1}{1-q_1}\frac{1}{\alpha}\tag{14}$$

QED. □

Equation (4) is equivalent to write in the form

$$z^{\alpha} = z^{\alpha} - (I_1[\tau])^{-1}\left(\alpha z^{\alpha} + K'^{*}(z_{\alpha})K(z^{\alpha}) - K'^{*}(z_{\alpha})u\right) \tag{15}$$

Theorem 2.3. *Suppose that the equation (5) the following conditions: 1) at the point z_{α} linear operator*

$$K'^{*}(z_{\alpha})K'(z_{\alpha})$$

is continuous positive self-adjoint; 2) the operator

$$(I_1[\tau])^{-1}\left(\alpha z_{\alpha} + K'^{*}(z_{\alpha})K(z_{\alpha}) - K'^{*}(z_{\alpha})u\right)$$

satisfies the condition

$$\left\|(I_1[\tau])^{-1}\left(\alpha z_{\alpha} + K'^{*}(z_{\alpha})K(z_{\alpha}) - K'^{*}(z_{\alpha})u\right)\right\| \leq \frac{NM}{(1-q_1)\sigma^{1/\sigma}}\alpha^{\frac{1-\sigma}{\sigma}} = \eta$$

3) for the operator $K'(z_{\alpha})$ is satisfied Holder (4) with a fixed element $z_2 \equiv z_{\alpha}$ and for any z_1 region of

$$\|z - z_{\alpha}\| \leq t\eta, \text{ where } 1 < t \leq \frac{2+\beta}{\beta} \tag{16}$$

4) permanent $\frac{1}{(1-q_1)\alpha}$, η, N, M, β and t in conditions 1), 2) and 3) are such that

$$h = \frac{(NM)^{1+\beta}}{(1-q)^{1+\beta}\sigma^{\beta/\sigma}}\alpha^{\frac{\beta-(1+\beta)\sigma}{\sigma}} \leq \frac{1+\beta}{2}\frac{t-1}{t^{1+\beta}} \tag{17}$$

then in a bowl (17), equation (5) has a unique solution $z^{*,\alpha}$.

Proof. Evidence. Based on the initial approximation z_{α}, we construct a sequence of elements $\{z_n\}_{n=0}^{\infty}$

$$z_{n+1} = z_n - (I_1[\tau])^{-1}\left(\alpha z_n + K'^{*}(z_{\alpha})K(z_n) - K'^{*}(z_{\alpha})u\right) \tag{18}$$

Firstly, by induction show that all z_n are in the ball (17). Condition 2) we have

$$\|z_1 - z_{\alpha}\| = \left\|(I_1[\tau])^{-1}\left(\alpha z_{\alpha} + K'^{*}(z_{\alpha})K(z_{\alpha}) - K'^{*}(z_{\alpha})u\right)\right\| \leq \eta < t\eta$$

, and z_1 enters into a ball (17). Suppose that for some natural n executed $\|z_n - z_{\alpha}\| \leq t\eta$. With this statement, we show that z_{n+1} part of the ball (17). From (19) we have

$$\begin{aligned} z_{n+1} - z_{\alpha} = z_n - z_{\alpha} - (I_1[\tau])^{-1}\left(\alpha z_n + K'^{*}(z_{\alpha})K(z_n) - K'^{*}(z_{\alpha})u\right) = \\ = -(I[\tau])^{-1}K'^{*}(z_{\alpha})\left(K(z_n) - K(z_{\alpha}) - K'(z_{\alpha})(z_n - z_{\alpha})\right) - \\ - (I[\tau])^{-1}\left(\alpha z_{\alpha} + K'^{*}(z_{\alpha})K(z_{\alpha}) - K'^{*}(z_{\alpha})u\right) + \\ + (I[\tau])^{-1}K'^{*}(z_{\alpha})\left(\int_0^1 \left\{K'(z_{\alpha} + \tau(z - z_{\alpha})) - K'(z_{\alpha})\right\}d\tau\,(z_n - z_{\alpha})\right) \end{aligned} \tag{19}$$

By (6), we have given the condition 2), (3) and (4) of theorem 2:

$$\|z_{n+1} - z_{\alpha}\| \leq t\eta, \tag{20}$$

and, therefore, z_{n+1} is a ball (17).

We now estimate the difference between two successive iterations $\|z_{n+1}-z_n\|$. From (19) we have

$$z_{n+1}-z_n = z_n - z_{n-1} - (I_1[\tau])^{-1}\left(\alpha z_n + K'^*(z_\alpha)K(z_n) - K'^*(z_\alpha)u\right) + \\ + (I[\tau])^{-1}\left(\alpha z_{n-1} + K'^*(z_\alpha)K(z_{n-1}) - K'^*(z_\alpha)u\right) = \\ = (I[\tau])^{-1}\left(\alpha(z_n - z_{n-1}) + K'^*(z_\alpha)K'(z_\alpha)(z_n - z_{n-1}) + \right. \\ \left. + K'^*(z_\alpha)\int_0^1 \left\{K'(z_\alpha + \tau(z-z_\alpha)) - K'(z_\alpha)\right\}d\tau\,(z_n - z_{n-1}) - \right. \\ \left. -\alpha z_n - K'^*(z_\alpha)K(z_n) - \alpha z_{n-1} - K'^*(z_\alpha)K(z_{n-1})\right).$$

We introduce the operator $B_\alpha(z)$

$$B_\alpha(z) = z - (I_1[\tau])^{-1}\left(\alpha z + K'^*(z_\alpha)K(z) - K'^*(z_\alpha)u\right) \tag{21}$$

Transforming the operator $B_\alpha(z)$ have

$$B_\alpha(z) = (I_1[\tau])^{-1}\left(\left(\alpha E + K'^*(z_\alpha)K'(z_\alpha)\right)\varphi[\tau]\,z - \alpha z - K'^*(z_\alpha)K(z) + K'^*(z_\alpha)u\right) \tag{22}$$

The derivative operator $B_\alpha(z)$ at the point $\forall z \in S(z_\alpha,\ r(\alpha))$ has the form

$$B'_\alpha(z) = (I_1[\tau])^{-1}K'^*(z_\alpha)\left(K'(z_\alpha) - K'(z) + \int_0^1 \left\{K'(z_\alpha + \tau(z-z_\alpha)) - K'(z_\alpha)\right\}d\tau\right). \tag{23}$$

We estimate the norm of the operator $B'_\alpha(z)$

$$\left\|B'_\alpha(z)\right\| \le \frac{2+\beta}{1+\beta}\frac{1}{1-q_1}\frac{NM}{\alpha}\|z-z_\alpha\|^\beta \le \frac{2+\beta}{1+\beta}ht^\beta. \tag{24}$$

The estimate (25) implies that $\forall z_1, z_2 \in S(z_\alpha,\ r(\alpha))$ the operator $B_\alpha(z)$ is Lipschitz

$$\|B_\alpha(z_1) - B_\alpha(z_2)\| \le \frac{2+\beta}{1+\beta}ht^\beta\|z_1 - z_2\| \tag{25}$$

We now estimate $\|z_{n+1}-z_n\|$. Using (26) we have

$$\|z_{n+1}-z_n\| = \|B_\alpha(z_n) - B_\alpha(z_{n-1})\| \le \frac{2+\beta}{1+\beta}ht^\beta\|z_n - z_{n-1}\|,\ q_2 = \frac{2+\beta}{1+\beta}ht^\beta < 1 \tag{26}$$

where

$$q_2 = \frac{2+\beta}{1+\beta}ht^\beta \le \frac{2+\beta}{2}\frac{t-1}{t} < 1$$

, because when $t < \frac{2+\beta}{\beta}$

Continuing evaluation (27), we obtain

$$\|z_{n+1}-z_n\| \le q_2^n\|z_1 - z_\alpha\| \le q_2^n\eta \tag{27}$$

Next we show that the sequence $\{z_n\}_{n=0}^\infty$ is a fundamental sequence. Valid for all, n and p we have, using the triangle inequality and (28)

$$\|z_{n+p}-z_n\| \le \|z_{n+p}-z_{n+p-1}\| + \|z_{n+p-1}-z_{n+p-2}\| + ... + \|z_{n+1}-z_n\| \le q_2^{n+p-1}\|z_1 - z_\alpha\| + \\ + q_2^{n+p-2}\|z_1 - z_\alpha\| + ... + q_2^n\|z_1 - z_\alpha\| = \left(q_2^{n+p-1} + q_2^{n+p-2} + ... + q_2^n\right)\|z_1 - z_\alpha\|. \tag{28}$$

Inequality (29) using the sum of a geometric progression, we have by $q_1 < 1$

$$\|z_{n+p}-z_n\| \le \|z_1 - z_\alpha\|\,q_2^n\sum_{i=1}^p q_2^{i-1} \le q_2^n\frac{q_2^p}{1-q_2}\|z_1 - z_\alpha\|. \tag{29}$$

In view of the completeness of H there

$$\lim_{n\to\infty} z_n = z^{\alpha} \tag{30}$$

We fix in (30), and the index n tends p to infinity. Then we obtain the estimate

$$\|z_n - z^*\| \le \frac{q_2^n}{1-q_2}\eta,\ q_2 = \frac{2+\beta}{1+\beta}ht^{\beta} < 1 \tag{31}$$

This estimate (32) is the speed of convergence of the method. This is because when $n \to \infty$ error decreases as q^n.

Thus, for the solution of the equation (5) in the bowl (17) we have the estimate

$$\|z^{\alpha} - z_{\alpha}\| \le \frac{(2+\beta)\,NM}{\beta\,(1-q_1)\,\sigma^{1/\sigma}}\alpha^{\frac{1-\sigma}{\sigma}} \tag{32}$$

Theorem 2 is proved. □

Theorem 2.4. *Suppose that 1) all the conditions of Theorem 2; 2) between the elements $z_{\alpha}$and z^* have the estimate $\|z^* - z_{\alpha}\| \le \gamma r_z(\alpha)$, where $0 < \gamma < 1/2$.*

Then a unique continuous solution $z^{\alpha,*}$ of equation (16) $u = u^*$ converges in the norm of the exact H solution z^* of the equation (1) $\alpha \to 0$.

Proof. Evidence. Using the triangle inequality, the inequality (33) for $u = u^*$ the second condition of Theorem 3, estimating the norm $\|z^{\alpha,*} - z^*\|$, we have

$$\|z^{\alpha,*} - z^*\| \le \|z^{\alpha,*} - z_{\alpha}\| + \|z^* - z_{\alpha}\| \le (1+\gamma)\,\frac{(2+\beta)\,NM}{\beta\,(1-q_1)\,\sigma^{1/\sigma}}\alpha^{\frac{1-\sigma}{\sigma}} \tag{33}$$

Inequality (34) implies that $z^{\alpha,*} \to z^*$ *at* $\alpha \to 0$ the norm of H. Thus, the solution $z^{\alpha,*}$of equation (16) $u = u^*$ is an approximate solution of equation (1).

QED.

The main problem to be examined is the construction of the approximate solutions according $\{u_{\delta}\}$ to the following sequence $z^{\alpha,\delta}$, which converges in the space $H$to the exact solution z^* of equation (1), provided that the initial convergence $u_{\delta} \to u^*$ of at $\delta \to 0$. To do this, we estimate the difference $z^{\alpha,\delta} - z^*$, where $z^{\alpha,\delta}$- the decision of the equation (16) $u = u_{\delta}$. It is represented in the form

$$z^{\alpha,\delta} = z^{\alpha,\delta} - (I_1[\tau])^{-1}\left(\alpha z^{\alpha,\delta} + K'^*(z_{\alpha})\,K(z^{\alpha,\delta}) - K'^*(z_{\alpha})\,u_{\delta}\right) \tag{34}$$

Using the triangle inequality, we have

$$\left\|z^{\alpha,\delta} - z^*\right\| \le \left\|z^{\alpha,\delta} - z^{\alpha,*}\right\| + \|z^{\alpha,*} - z^*\| \tag{35}$$

where $z^{\alpha,*} = z^{\alpha,*} - (I_1[\tau])^{-1}\left(\alpha z^{\alpha,*} + K'^*(z_{\alpha})\,K(z^{\alpha,*}) - K'^*(z_{\alpha})\,u^*\right)$ the solution of the equation (16) $u = u^*$

Consider the difference $z^{\alpha,\delta} - z^{\alpha,*}$

$$\begin{aligned} z^{\alpha,\delta} - z^{\alpha,*} &= -(I_1[\tau])^{-1}K'^*(z_{\alpha})\left(K'(z^{\alpha,*}) - K'(z_{\alpha})\right)\left(z^{\alpha,\delta} - z^{\alpha,*}\right) - \\ &- (I_1[\tau])^{-1}K'^*(z_{\alpha})\int_0^1 \left\{\left(K'(z^{\alpha,*} + \tau\left(z^{\alpha,\delta} - z^{\alpha,*}\right)) - K'(z^{\alpha,*})\right)\right\}\left(z^{\alpha,\delta} - z^{\alpha,*}\right)d\tau + \\ &+ (I_1[\tau])^{-1}K'^*(z_{\alpha})\int_0^1 \left\{\left(K'(z_{\alpha} + \tau\,(z - z_{\alpha})) - K'(z_{\alpha})\right)\right\}\left(z^{\alpha,\delta} - z^{\alpha,*}\right)d\tau + (I_1[\tau])^{-1}K'^*(z_{\alpha})\,(u_{\delta} - u^*). \end{aligned} \tag{36}$$

Assessing the converted difference, we have

$$\left\|z^{\alpha,\delta}-z^{\alpha,*}\right\|\leq\frac{\left(2+2^{\beta}+\beta\right)NM}{\left(1-q_1\right)\left(1+\beta\right)}\cdot\frac{r_z^{\beta}(\alpha)}{\alpha}\left\|z^{\alpha,\delta}-z^{\alpha,*}\right\|+\frac{1}{1-q_1}\cdot\frac{\delta}{\alpha}. \tag{37}$$

We show that when choosing $\alpha\left(r_z\left(\alpha\right)\right)=\sigma_1 r_z^{\beta\sigma_1}\left(\alpha\right)$ where $\sigma<\sigma_1$ there exists $\tilde{\tilde{r}}>0$ such that

$$\frac{\left(2+2^{\beta}+\beta\right)NM}{\left(1-q_1\right)\left(1+\beta\right)}\cdot\frac{r_z^{\beta}(\alpha)}{\alpha}<\frac{\left(2+2^{\beta}+\beta\right)NM}{\left(1-q_1\right)\left(1+\beta\right)}\cdot\frac{\tilde{\tilde{r}}}{\alpha\left(\tilde{\tilde{r}}\right)}=q_3\leq\sigma_1 \tag{38}$$

when $r_z<\tilde{\tilde{r}}$, where $0<\sigma_1<\left(\frac{1}{2}\right)^{\frac{\beta(1-\sigma_1)}{2-\beta-\beta\sigma_1}}\left(\frac{\left(2+2^{\beta}+\beta\right)NM}{(1-q_1)(1+\beta)}\right)^{\frac{1-\beta}{\beta(1-\sigma_1)}}$.

Further, from (41) we have

$$r_z<\left(\frac{\sigma_1^2\left(1-q_1\right)\left(1+\beta\right)}{\left(2+2^{\beta}+\beta\right)NM}\right)^{\frac{1}{\beta(1-\sigma_1)}}\equiv\tilde{\tilde{r}}\ \ end\quad\alpha\left(\tilde{\tilde{r}}\right)=\sigma_1\left(\frac{\sigma_1^2\left(1-q_1\right)\left(1+\beta\right)}{\left(2+2^{\beta}+\beta\right)NM}\right)^{\frac{\sigma_1}{1-\sigma_1}} \tag{39}$$

Under these conditions (40), $q_3<\frac{1}{2}$.

Then from (38) we have

$$\left\|z^{\alpha,\delta}-z^{\alpha,*}\right\|\leq C_1\frac{\delta}{\alpha},\ C_1=\frac{1}{\left(1-q_3\right)\left(1-q_1\right)} \tag{40}$$

In view of (34) and (41) from (36) we have

$$\left\|z^{\alpha,\delta}-z^{*}\right\|\leq\left\|z^{\alpha,\delta}-z^{\alpha,*}\right\|+\left\|z^{\alpha,*}-z^{*}\right\|\leq C_2\alpha^{\frac{1-\sigma}{\sigma}}+C_1\frac{\delta}{\alpha},$$

$$C_2=\frac{\left(1+\gamma\right)\left(2+\beta\right)NM}{\beta\left(1-q_1\right)\sigma^{1/\sigma}}. \tag{41}$$

Minimizing the right-hand side of inequality (44) we obtain the dependence of the regularization α parameter on the right side of the parameter error δ

$$\alpha\left(\delta\right)=\left(\frac{C_1}{C_2}\frac{\sigma}{1-\sigma}\delta\right)^{\sigma}. \tag{42}$$

Then the estimate (42) has the form

$$\left\|z^{\alpha,\delta}-z^{*}\right\|\leq C_1^{1-\sigma}C_2^{\sigma}(1-\sigma)^{1-\sigma}\left(\frac{1}{\sigma}\right)^{\sigma}\delta^{1-\sigma} \tag{43}$$

From (44) it follows that $z^{\alpha,\delta}\to z^{*}\ \delta\to 0$ in the norm H. Thus, it is proved. □

Theorem 2.5. *Suppose that: 1) all the conditions of Theorem 3; 2) the element satisfies condition (3); 3)* $\alpha\left(r_z\left(\alpha\right)\right)=\sigma_1 r_z^{\beta\sigma_1}\left(\alpha\right)$ *where* $\sigma<\sigma_1$ *it performed condition* $\frac{\left(2+2^{\beta}+\beta\right)NM}{(1-q_1)(1+\beta)}\cdot\frac{r_z^{\beta}(\alpha)}{\alpha}<\frac{\left(2+2^{\beta}+\beta\right)NM}{(1-q_1)(1+\beta)}\cdot\frac{\tilde{\tilde{r}}}{\alpha\left(\tilde{\tilde{r}}\right)}=q_3\leq\sigma_1$ *when* $r_z<\tilde{\tilde{r}}$, *where* $0<\sigma_1<\left(\frac{1}{2}\right)^{\frac{\beta(1-\sigma_1)}{2-\beta-\beta\sigma_1}}\left(\frac{\left(2+2^{\beta}+\beta\right)NM}{(1-q_1)(1+\beta)}\right)^{\frac{1-\beta}{\beta(1-\sigma_1)}}$; *4) the regularization parameter* α *satisfies the condition (43). Then the solution of equation (16)* $u=u_\delta$, *converges* $\delta\to 0$ *to the exact solution* z^* *of equation (1). The rate of convergence satisfies (44).*

3. Conclusion

In this paper we show that, based on the combination of the regularization method for solving operator equations in Hilbert space is regularizability.

Justification of the new combined regularization method proposed in this paper consists of the following study results:

1. regularizing operator is constructed for the solution of operator equations of the first kind in Hilbert space;

2. Prove the convergence of regularized solutions to the exact solution of the original equation;
3. Get prametra regularization choice depending on the error, right side;
4. estimates of the rate of convergence of the regularized solutions to the exact solution.

For the first time, based on the combined method of regularization regularizing operator is constructed for the solution of operator equations of the first kind in Hilbert space. Established convergence and convergence rate of regularized solutions to the exact solution.

This work has a theoretical orientation. The result can be used to solve applied problems of geophysics, thermal physics, filtering optics, medical imaging itp

References

[1] Lavrentiev M.M. Some ill-posed problems of mathematical physics. - Novosibirsk: SB RAS, 1962.
[2] Krasnosel'skii. M.A. and others. An approximate solution of operator equations. M-1969.
[3] Trenogin V.A. Functional analysis. - M.: FIZMATLIT, 3rd ed., Rev. - 2002.
[4] A. Saadabaev Convergence of Newton's method in nonlinear ill-posed problems // News of high schools № 1-2, Bishkek-2003.
[5] Usenov I.A. Regularization of the solution of implicit operator equations of the first kind // "Functional analysis and its applications," Astana-2012.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

REGULARIZATION AND STABILITY OF SOLUTIONS OF SYSTEM OF LINEAR INTEGRAL FREDHOLM EQUATIONS OF THE FIRST KIND ON A SEMI-AXIS

AVYT ASANOV[1], ZUURAKAN KADENOVA[2], DAMIRA BEKESHOVA[3]

[1] Department of Mathematics, Kyrgyz-Turkish Manas University Kyrgyz Republic, Bishkek, Kyrgyzstan
[2,3] Institute of Mathematics of NAS of the Kyrgyz Republic Kyrgyzstan
e-mail: avyt.asanov@manas.egu.kg, kadenova71@mail.ru

Abstract. In this article, regularizations and stability of solutions of system linear integral Fredholm equations of the first kind are obtained by the methods of functional analysis.

Keywords: linear, inteqral equations, first kind, regularization.

1. Introduction

Inverse and ill-posed problems are currently attracting great interest. The theory and numerical methods for solving inverse and ill-posed problems were studied in [1-17].The notion of correctness in the works of A.N. Tikhonov [1], M.M. Lavrent'ev [2] and V.K. Ivanov [3], different from the classical one, provided a means for studying ill-posed problems and stimulated interest in integral equations with large applied value.

The fundamental results for Fredholm integral equations of the first kind were obtained by M.M. Lavrentiev in [4], [5], where the regularizing operators in the sense of M.M. Lavrentiev.

In [10]-[12], uniqueness theorems were proved and regularization operators in the sense of Lavrent'ev were constructed for systems of linear Volterra and Fredholm integral equations of the third kind.

In [14]-[16] problems of uniqueness and stability of solutions for linear Fredholm integral equations of the first kind were investigated.

In this work, we apply the method of integral transformation to prove regularizations and stability of solutions of system of linear integral Fredholm equations of the first kind in the semi-axis.

Consider of the system of Fredholm linear integral equations

$$Ku \equiv \int_{-\infty}^{a} K(t,s)u(s)ds = f(t), t \in (-\infty, a], \tag{1}$$

where

$$K(t,s) = \begin{cases} A(t,s), & -\infty < s \le t \le a, \\ B(t,s), & -\infty < t \le s \le a, \end{cases} \tag{2}$$

$$A(t,s) = (a_{ij}(t,s)) = \begin{pmatrix} a_{11}(t,s) & a_{12}(t,s) & \dots & a_{1n}(t,s) \\ a_{21}(t,s) & a_{22}(t,s) & \dots & a_{2n}(t,s) \\ \dots & \dots & \dots & \dots \\ a_{n1}(t,s) & a_{n2}(t,s) & \dots & a_{nn}(t,s) \end{pmatrix},$$

$$B(t,s)=(b_{ij}(t,s))=\begin{pmatrix} b_{11}(t,s) & b_{12}(t,s) & \dots & b_{1n}(t,s) \\ b_{21}(t,s) & b_{22}(t,s) & \dots & b_{2n}(t,s) \\ \dots & \dots & \dots & \dots \\ b_{n1}(t,s) & b_{n2}(t,s) & \dots & b_{nn}(t,s) \end{pmatrix}$$

$$f(t)=(f_i(t))=(f_1(t)...f_n(t))^T, u(t)=(u_i(t))=(u_1(t)...u_n(t))^T.$$

$B(t,s)$ and $A(t,s)$ are the known matrix functions, $f(t)$ is known vector function, $u(t)$ is unknown vector function.

We introduce possible notation:

1. For vectors $u=(u_1,u_2,...,u_n), v=(\vartheta_1,\vartheta_2,...,\vartheta_n)\in R^n$ we define the scalar product by the equality

$$\langle u,\vartheta\rangle=\sum_{i=1}^{n}u_iv_i.$$

2. For $A=(a_{ij})-n\times n$ - square matrix and $u=(u_1,u_2,...,u_n)\in R^n$ we define the norm

$$\|A\|=\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}},$$

$$\|u\|=\left(\sum_{i=1}^{n}|u_i|^2\right)^{\frac{1}{2}}.$$

3. Through $L_{2,n}(-\infty,a]$ denote the space of all $n-$ dimensional vector functions $u(t)=(u_1(t),...,u_n(t))$ satisfying the condition $u_i(t)\in L_2(-\infty,a]$ for all $i=1,2,...,n$.

For $u(t)=(u_1(t),...,u_n(t))\in L_{2,n}(-\infty,a]$ define the norm

$$\|u(t)\|_{L_{2,n}}=\left(\int_a^b\|u(t)\|^2dt\right)^{\frac{1}{2}}.$$

We will assume that

$$\|K(t,s)\|\in L_2((-\infty,a]\times(-\infty,a]),\|f(t)\|\in L_2(-\infty,a].$$

System of equations (1) by virtue of relation (2) can be expressed as

$$\int_{-\infty}^{t}A(t,s)u(s)ds+\int_t^aB(t,s)u(s)ds=f(t). \tag{3}$$

Both parts of (3) are scalarly multiplied by the $u(t)-$ vector function and, integrating the results on $-\infty<t\le a$, we obtain

$$\int_{-\infty}^{a}\int_{-\infty}^{t}\langle A(t,s)u(s),u(t)\rangle\,dsdt+\int_{-\infty}^{a}\int_t^a\langle B(t,s)u(s),u(t)\rangle\,dsdt=\int_{-\infty}^{a}\langle f(t),u(t)\rangle\,dt, \tag{4}$$

$$\int_{-\infty}^{a}\int_{-\infty}^{t}\Big\langle A(t,s)u(s),u(t)dsdt+\int_{-\infty}^{a}\int_s^a\Big\langle B^*(s,t)u(s),u(t)dtds=\int_{-\infty}^{a}\langle f(t),u(t)\rangle\,dt,$$

where $B^*(s,t)$ is the transposed matrix to the matrix $B(s,t)$.

Integrating by parts and using the Dirichlet formula we have

$$\int_{-\infty}^{a}\int_s^a\langle B(s,t)u(t),u(s)\rangle\,dtds=$$

$$\int_{-\infty}^{a}\int_s^a\langle B^*(s,t)u(s),u(t)\rangle\,dtds=\int_{-\infty}^{a}\int_{-\infty}^{t}\langle B^*(s,t)u(s),u(t)\rangle\,dsdt.$$

Then from (4) we obtain

$$\int_{-\infty}^{a}\int_{-\infty}^{t}\langle[A(t,s)+B^{*}(s,t)]u(s),u(t)\rangle\,dsdt=\int_{-\infty}^{a}\langle f(t),u(t)\rangle\,dt \tag{5}$$

Denote

$$H(t,s)=\frac{1}{2}(A(t,s)+B^{*}(s,t)),(t,s)\in G=-\infty<S\leq t\leq a$$

Then from (5) we obtain

$$2\int_{-\infty}^{a}\int_{-\infty}^{t}\langle H(t,s)u(s),u(t)\rangle\,dsdt=\int_{-\infty}^{a}\langle f(t),u(t)\rangle\,dt, \tag{6}$$

where

$$\int_{-\infty}^{a}\int_{-\infty}^{t}H^{2}(t,s)dsdt<+\infty.$$

We introduce a new matrix function $M(t,s)$ as follows

$$M(t,s)=\begin{cases}H(t,s), & -\infty<s\leq t\leq a,\\ H^{*}(s,t), & -\infty<t\leq s\leq a.\end{cases} \tag{7}$$

It's clear that $M(t,s)=M^{*}(s,t),(t,s)\in(-\infty,a]\times(-\infty,a]$.
It is easy to verify the validity of the equality

$$\int_{-\infty}^{a}\int_{-\infty}^{a}\|M(t,s)\|^{2}\,dsdt<+\infty.$$

Then, it is known that

$$M(t,s)=\sum_{i=1}^{\infty}\frac{\varphi_{i}(t)\varphi_{i}(s)}{\lambda_{i}}, \tag{8}$$

where, λ_i - the characteristic numbers of the matrix kernel $M(t,s)$, which are arranged in ascending order
of their modules, $|\lambda_1|\leq|\lambda_2|\leq ...$ and $\varphi_1(t),...,\varphi_n(t)$ the
corresponding orthonormal eigen vector-functions. It is assumed that $M(t,s)$ is a complete kernel and $0<\lambda_1\leq\lambda_2\leq ....$
In this case, the solution of the system (1) will be unique in $L_{2,n}(-\infty,a]$.

In what follows, we will assume that all characteristic numbers of the matrix kernel are positive.

For $u(t)=(u_i(t))\in L_{2,n}(-\infty,a]$ we obtain

$$\|u(t)\|_{2,n}=\left(\sum_{i=1}^{n}\int_{-\infty}^{a}|u_{i}(t)|^{2}dt\right)^{\frac{1}{2}}=\left(\int_{-\infty}^{a}\|u(t)\|^{2}\,dt\right)^{\frac{1}{2}}=\left(\sum_{\nu=1}^{\infty}\left|u^{(\nu)}\right|^{2}\right)^{\frac{1}{2}},$$

where

$$u^{(\nu)}=\int_{-\infty}^{a}\left\langle u(t),\varphi^{(\nu)}(t)\right\rangle dt=\int_{-\infty}^{a}\left(\sum_{i=1}^{n}u_{i}(t)\varphi_{i}^{(\nu)}(t)\right)dt,(\nu=1,2,...).$$

We distinguish the family of correctness sets depending on the parameter as follows:

$$N_{\alpha}=\left\{u(t)\in L_{2,n}(-\infty,a]:\sum_{\nu=1}^{\infty}\lambda_{\nu}^{-\alpha}\left|u^{(\nu)}\right|^{2}\leq c\right\},where\ c>0,\ 0<\alpha<\infty,$$

$$u^{(\nu)}=\int_{-\infty}^{a}\langle u(t),\varphi^{\nu}(t)\rangle\,dt,\ (\nu=1,2,...). \tag{9}$$

Let $u(t)=(u_i(t))\in N_\alpha$. Then

$$2\int_{-\infty}^{\alpha}\int_{\alpha}^{t}\sum_{\nu=1}^{\infty}\lambda_\nu\left\langle\begin{pmatrix}\varphi_1^{(\nu)}(t)\\ \dots\\ \varphi_n^{(\nu)}(t)\end{pmatrix}(\varphi_1^{(\nu)}(s)...\varphi_1^{(\nu)}(s))\begin{pmatrix}u_1(s)\\ \dots\\ u_n(s)\end{pmatrix}u(t)\right\rangle dsdt=$$

$$\int_{-\infty}^{\alpha}\left[\sum_{i=1}^{n}f_i(t)u_i(t)\right]dt,$$

$$\|u(t)\|_{2,n}^2=\sum_{i=1}^{\infty}\left|u^{(\nu)}\right|^2=\sum_{i=1}^{\infty}\lambda_\nu^{\alpha}\lambda_\nu^{-\alpha}\left|u^{(\nu)}\right|^2=\lambda_1^{\alpha}\left(\sum_{\nu=1}^{\infty}\lambda_\nu^{-\alpha}\left|u^{(\nu)}\right|^2\right)\le c\lambda_1^{\alpha},$$

$$\|u(t)\|_{2,n}^2\le c\lambda_1^{\alpha}. \tag{10}$$

We will assume that $f(t)\in K(N_\alpha)$. Then the system (1) has a solution $(u_i)\in N_\alpha$ and by virtue of (6), (7)and (8) we have:

$$2\int_{-\infty}^{\alpha}\int_{\alpha}^{t}\sum_{\nu=1}^{\infty}\lambda_\nu\left\langle\begin{pmatrix}\varphi_1^{(\nu)}(t)\\ \dots\\ \varphi_n^{(\nu)}(t)\end{pmatrix}(\varphi_1^{(\nu)}(s)...\varphi_1^{(\nu)}(s))\begin{pmatrix}u_1(s)\\ \dots\\ u_n(s)\end{pmatrix}u(t)\right\rangle dsdt=$$

$$\int_{-\infty}^{\alpha}\left[\sum_{i=1}^{n}f_i(t)u_i(t)\right]dt,$$

$$\sum_{\nu=1}^{\infty}2\int_{-\infty}^{\alpha}\left[\int_{\alpha}^{t}\left\langle u(s),\varphi^{(\nu)}(s)\right\rangle ds\right]\left\langle u(t),\varphi^{(\nu)}(t)\right\rangle dt=\int_{-\infty}^{\alpha}\langle f(t),u(t)\rangle\,dt,$$

$$\sum_{\nu=1}^{\infty}\lambda_\nu\left|\int_{-\infty}^{\alpha}\left\langle u(t),\varphi^{(\nu)}(t)\right\rangle dt\right|^2=\int_{-\infty}^{\alpha}\langle f(t),u(t)\rangle\,dt,$$

$$\sum_{\nu=1}^{\infty}\lambda_\nu\left|u^{(\nu)}\right|^2=\int_{-\infty}^{\alpha}\langle f(t),u(t)\rangle\,dt$$

Hence, using Holder's inequalities, we have

$$\sum_{\nu=1}^{\infty}\lambda_\nu\left|u^{(\nu)}\right|\le\|f(t)\|_{2,n}\|u(t)\|_{2,n} \tag{11}$$

On the other side:

$$\|u(t)\|_{2,n}^2=\sum_{\nu=1}^{\infty}\frac{\left|u^{(\nu)}\right|^{\frac{2\alpha}{1+\alpha}}}{\lambda_\nu^{\frac{\alpha}{1+\alpha}}}\lambda_\nu^{\frac{\alpha}{1+\alpha}}\left|u^{(\nu)}\right|^{\frac{2}{1+\alpha}}\le\left(\sum_{\nu=1}^{\infty}\frac{|u_\nu|^2}{\lambda_\nu}\right)^{\frac{\alpha}{1+\alpha}}\left(\sum_{\nu=1}^{\infty}\lambda_\nu^{-\alpha}\left|u^{(\nu)}\right|^2\right)^{\frac{1}{1+\alpha}}.$$

Here we have applied Holder's inequality for $p=1+\alpha, q=\frac{(1+\alpha)}{\alpha}$. Taking into account $u(t)\in N_\alpha$ and (11), from the last inequality we have

$$\|u(t)\|_{2,n}^2\le c^{\frac{1}{1+\alpha}}\left(\|f(t)\|_{2,n}\|u(t)\|_{2,n}\right)^{\frac{\alpha}{1+\alpha}}.$$

Hence we obtain the following stability estimate

$$\|u(t)\|_{2,n}\le c^{\frac{1}{2+\alpha}}\|f(t)\|_{2,n}^{\frac{\alpha}{2+\alpha}},0<\alpha<\infty. \tag{12}$$

Thus, the following theorem has been proved.

Theorem 1. Let the operator M generated by the matrix kernel $M(t,s)$ be positive, where it is defined $M(t,s)$ by formula (7)Then on the set $K(N_\alpha)$ ($K(N_\alpha)$ image when N_α displayed by the operator K) the operator K^{-1}, the inverse of K, is uniformly continuous with the Holder

exponent $\frac{\alpha}{2+\alpha}$ i.e. estimate (12) is valid.

Let us show that the solution of the system of equations

$$\epsilon u(t,\epsilon)+\int_{-\infty}^{\alpha}K(t,s)u(s,\epsilon)ds=f(t),\ t\in(-\infty,\alpha]),\ \epsilon>0 \tag{13}$$

will be regularizing for equation (1) on the set N_α. Indeed, by making the following substitution into the in system equations (13) $u(t,\epsilon)=u(t)+\xi(t,\epsilon)$, where $u(t)\in N_\alpha-$ solution of systems of equations (1), we obtain

$$\epsilon\xi(t,\epsilon)+\int_{-\infty}^{\alpha}K(t,s)\xi(s,\epsilon)ds=-\epsilon(t)$$

Scalarly multiplying the last system of equations by $\xi(t,\epsilon)$ and integrating from $-\infty$ to a, taking into account (2) and (8), we have:

$$\epsilon\left\|\xi(t,\epsilon)\right\|_{2,n}^2+\sum_{\nu=1}^{\infty}\lambda_\nu^{-1}\left|\xi_\nu(\epsilon)\right|^2\le\epsilon\sum_{\nu=1}^{\infty}\left|u^{(\nu)}\right|\left|\xi_\nu(\epsilon)\right|, \tag{14}$$

where $\xi_i(\epsilon)-$ are the Fourier coefficients for the function $\xi(t,\epsilon)$, according to the orthonormal system $\varphi^{(\nu)}(t)=\left\{\varphi_i^{(\nu)}(t)\right\}$ that is

$$\xi_\nu(\epsilon)=\int_{\alpha}^{\infty}\left\langle\xi(t,\epsilon),\varphi^{(\nu)}(t)\right\rangle dt,(\nu=1,2,...).$$

Applying the Holder inequality for $p=q=\frac{1}{2}$, from (14) we obtain

$$\left\|\xi(t,\epsilon)\right\|_{2,n}\le\left\|u(t)\right\|_{2,n}, \tag{15}$$

$$\sum_{\nu=1}^{\infty}\lambda_\nu\left|\xi_\nu(\epsilon)\right|^2\le\epsilon\left\|u(t)\right\|_{2,n}^2\le\epsilon c\lambda_1^{\alpha},\epsilon>0. \tag{16}$$

On the other side

$$\sum_{\nu=1}^{\infty}\left|u^{(\nu)}\right|\left|\xi_\nu(\epsilon)\right|=\sum_{\nu=1}^{\infty}\frac{\left|\xi_\nu(\epsilon)\right|^{\frac{\alpha}{1+\alpha}}}{\lambda_\nu^{\frac{\alpha}{2(1+\alpha)}}}\lambda_\nu^{\frac{\alpha}{2(1+\alpha)}}\left|u^{(\nu)}\right|^{\frac{1}{1+\alpha}}\left|\xi_\nu(t,\epsilon)\right|^{\frac{1}{1+\alpha}}\left|u^{(\nu)}\right|^{\frac{1}{1+\alpha}}.$$

Therefore, after applying the generalized Holder inequality to the right-hand side for

$$p=\frac{2(1+\alpha)}{\alpha},q=2(1+\alpha),m=2(1+\alpha),n=\frac{2(1+\alpha)}{\alpha},$$

we have

$$\sum_{\nu=1}^{\infty}\left|u^{(\nu)}\right|\left|\xi_\nu(\epsilon)\right|\le\left(\sum_{\nu=1}^{\infty}\lambda_\nu\left|\xi_\nu(\epsilon)\right|^2\right)^{\frac{1}{p}}\left(\sum_{\nu=1}^{\infty}\frac{\left|u^{(\nu)}\right|^2}{\lambda_\nu^{\alpha}}\right)^{\frac{1}{q}}\left\|\xi(t,\epsilon)\right\|^{\frac{2}{q}}\left\|u(t)\right\|^{\frac{2}{p}},$$

$$\sum_{\nu=1}^{\infty}\left|u^{(\nu)}\right|\left|\xi_\nu(\epsilon)\right|\le\left(\sum_{\nu=1}^{\infty}\left|\xi_\nu(\epsilon)\right|^2\lambda_\nu^{-1}\right)^{\frac{\alpha}{(1+\alpha)2}}\left(\sum_{\nu=1}^{\infty}\lambda_\nu^{\alpha}\left|u^{(\nu)}\right|^2\right)^{\frac{1}{(1+\alpha)2}}\left\|\xi(t,\epsilon)\right\|^{\frac{1}{1+\alpha}}\left\|u(t)\right\|^{\frac{\alpha}{1+\alpha}},$$

$$\sum_{\nu=1}^{\infty}\left|u^{(\nu)}\right|\left|\xi_\nu(\epsilon)\right|\le\left(\sum_{\nu=1}^{\infty}\left|\xi_\nu(\epsilon)\right|^2\lambda_\nu^{-1}\right)^{\frac{1}{p}}\left(\sum_{\nu=1}^{\infty}\lambda_\nu^{\alpha}\left|u^{(\nu)}\right|^2\right)^{\frac{1}{q}}\left\|\xi(t,\epsilon)\right\|^{\frac{2}{q}}\left\|u(t)\right\|^{\frac{2}{p}}.$$

Further in force $u(t)\in N_\alpha$, (15) and (16) from the last inequality we have

$$\sum_{\nu=1}^{\infty}\left|u^{(\nu)}\right|\left|\xi_\nu(\epsilon)\right|\le(\epsilon c\lambda_1^{\alpha})^{\frac{1}{p}}c^{\frac{1}{q}}(c\lambda_1^{\alpha})^{\frac{p+q}{pq}}.$$

Hence, substituting $p = \frac{2(1+\alpha)}{\alpha}, q = 2(1+\alpha)$ we get

$$\sum_{\nu=1}^{\infty} \left| u^{(\nu)} \right| |\xi_\nu(\epsilon)| \leq c^{\frac{1}{2(1+\alpha)}} (c\lambda_1^\alpha)^{\frac{1}{2}} (\epsilon c \lambda_1^\alpha)^{\frac{\alpha}{2(1+\alpha)}}, \tag{17}$$

$$\sum_{\nu=1}^{\infty} \left| u^{(\nu)} \right| |\xi_\nu(\epsilon)| \leq c\lambda_1^{\frac{1}{2(1+\alpha)}} c^{\frac{1}{2}} c^{\frac{\alpha}{2(1+\alpha)}} \lambda_1^{\frac{\alpha}{2}} \lambda_1^{\frac{\alpha^2}{2(1+\alpha)}} \epsilon^{\frac{\alpha}{2(1+\alpha)}}$$

$$\sum_{\nu=1}^{\infty} \left| u^{(\nu)} \right| |\xi_\nu(\epsilon)| \leq c\lambda_1^{\frac{\alpha(2\alpha+1)}{2(1+\alpha)}} \epsilon^{\frac{\alpha}{2(1+\alpha)}}. \tag{18}$$

Taking into account (18), from (14) we have

$$\|u(t,\epsilon) - u(t)\|_{2,n} \leq c^{\frac{1}{2}} \lambda_1^{\frac{\alpha(2\alpha+1)}{4(1+\alpha)}} \epsilon^{\frac{\alpha}{4(1+\alpha)}}, 0 < \alpha < \infty \tag{19}$$

Thus proven.

Theorem 2. Let the operator M generated by the matrix kernel $M(t,s)$ be positive and $f(t) \in K(N_\alpha)$. Then estimate (19) is valid, where $u(t,\epsilon)-$ is the solution of the system (13), $u(t)$ is the solution of the system (1) $M(t,s)$ and is determined by formula (8).

References

[1] Tikhonov A.N. On methods for solving ill-posed problems. // In the book: Abstracts of reports, International Congress of Mathematicians. - M., 1966.(russ).

[2] Lavrent'ev M. M. On some ill-posed problems of mathematical physics. - Novosibirsk: Publishing House of Sib. Department of the Academy of Sciences of the USSR, 1962. - 92 p.(russ).

[3] Ivanov V.K. About ill-posed tasks. Differential Equations.-1968.- No. 2.-P.61.

[4] Lavrent'ev M. M. On integral equations of the first kind Doklady of the Academy of Sciences of the USSR. 1959. V. 127. No. 1. S. 31.(russ).

[5] Lavrent'ev M. M., Romanov V. G., Shishatskii S. P. Ill-Posed Problems of Mathematical Physics and Analysis (Providence, RI: American Mathematical Society). – 1986.

[6] Apartsyn, A. S. Nonclassical Linear Volterra Equations of the First Kind / A. S. Apartsyn. – Utrecht-Boston: VSP, 2003. – 168 p. – ISBN 90-6764-375-0. – EDN RUCKLT.

[7] Magnitskii N. A. Linear VolterrA Integral Equations of the First and Third Kind // Doklady Akademii Nauk. - 1991. - Vol. 317, No 2.-Pp. 330-333.

[8] Bukhgeim A.L. Volterra Equations and Inverse Problems, VSP, Utrecht, The Netherlands, 1999.

[9] A.Asanov, Regularization, Uniqueness and Existence of Solutions of Volterra Equations of the First Kind. VSP, Utrecht, p.272. (1998).

[10] Imanaliev, M. I., Asanov A., Solutions of system of nonlinear Volterra integral equations of the first kind, Soviet Math. Dokl. 309, (1989) 1052–1055.

[11] Imanaliev, M. I., Asanov A., Asanov R. A. On a class of systems of linear and nonlinear Fredholm integral equations of the third kind with multipoint singularities // Differential Equations. - 2018. - T. 54. - No. 3. - P. 387. – DOI 10.1134/S037406411803010X. – EDN YQYHAP. (Russ).

[12] Imanaliev, M. I., Asanov A. Regularization and uniqueness of solutions to systems of nonlinear Volterra integral equations of the third kind // Doklady Mathematics. – 2007. – Vol. 415. – No 1. – P. 14-17. – DOI 10.1134/S1064562407040035. – EDN XLWHPB.(Russ)

[13] Imanaliev M. I., Asanov A., Asanov R. A. A class of systems of linear Fredholm integral equations of the third kind // Doklady Mathematics. – 2011. – Vol. 83. – No 2. – P. 227-231. – DOI 10.1134/S1064562411020293. – EDN XKKSFX.

[14] Asanov A., Kadenova Z. A. Regularization and Stability of Systems of Linear Integral Fredholm Equations of the First Kind, Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 38, Samara State Technical University, Samara, 2005, Pp.11–14. (In Russ.), http://mi.mathnet.ru/eng/vsgtu/v38/p11, DOI: https://doi.org/10.14498/vsgtu363.(Russ).

[15] Imanaliev M. I., Asanov A., Kadenova Z. A., A class of linear integral equations of the first kind with two independent variables.// Doklady Mathematics. – 2014. – Vol. 89. – No 1. – P. 98-102. – DOI 10.1134/S1064562414010281. – EDN XLBERD.

[16] Asanov A., M. H. Chelik, Kadenova Z. A., Uniqueness and stability of solutions of linear intergral equations of the first kind with two variables. // International Journal of Mathematical Analysis. – 2013. – Vol. 7. – No 17-20. – P. 907-914. – DOI 10.12988/ijma.2013.13088. – EDN XKWECX.

[17] Asanov A., Kadenova Z.A., Bekeshova D. On the uniqueness of solutions of Fredholm linear integral equations of the first kind on a semi-axis, // Herald of Institute Mathematics of the National Academy of Sciences of the Kyrgyz Republic. – 2022. – No 1. – P. 82-87. – DOI 10.52448/16948173_2022_1_82. – EDN FXALAA.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

INTEGRATION OF THE PERIODIC HUNTER-SAXTON EQUATION WITH A SELF-CONSISTENT SOURCE

BAZAR BABAJANOV[1,2], DILSHOD ATAJONOV[1]

[1]Urgench State University, Urgench, Uzbekistan
[2]Khorezm branch of Uzbekistan Academy of Sciences V.I.Romanovskiy Institute of Mathematics, Urgench, Uzbekistan
e-mail: a.murod@mail.ru, diwa_4848@mail.ru

1. Introduction

The scale-invariant nondispersive wave equation

$$u_{xxt} + uu_{xxx} + 2u_x u_{xx} = 0 \tag{1}$$

was introduced by J.Hunter and R.Saxton [1] in connection with a model of nematic liquid crystals. J.Hunter and Y.Zheng [2],[3] have given a very beautiful and thorough treatment of equation (1), including the introduction of a Hamiltonian structure and proof of complete integrability. In [4], the inverse scattering problem is used to obtain explicit solutions for finite-dimensional flows in both compact and non-compact cases. The destruction phenomena and the global existence of strong solutions were discussed in [5, 6]. The Hunter-Saxton equation also appears in a different physical context as the high-frequency limit of the Camassa-Holm equation, a model equation for shallow water [7, 8, 9]. The hydrodynamic relevance of this model was explored in the article [10] and the reexpression of the geodesic flow on the diffeomorphism of the circle group [11].

The periodic Cauchy problem for the Hunter-Saxton equation is first considered in [12]. They proved the local existence of strong solutions to the periodic Hunter-Saxton equation and showed that all space-independent strong solutions collapse in a finite time.

In the work [13] V.K. Melnikov integrated the KdV equation with a self-consistent source in the class of rapidly decreasing functions and [14] studied the KdV equation with a self-consistent source in the class of periodic functions. Similar papers on integrating non-linear evolution equations with sources can be found in [16]-[29].

The paper is organized as follows. The formulation of the problem under consideration is given in section 2. In section 3, it is given some basic information about the direct and inverse spectral problem for the weighted Sturm-Liouville operator with periodic coefficient. Section 4 is devoted to deriving the evolution of the spectral data corresponding to the problem under consideration.

2. Problem Statement

We consider the Hunter-Saxton equation with a self-consistent source

$$u_{xxt} = -uu_{xxx} - 2u_x u_{xx} + \sum_{k=0}^{\infty} \alpha_k(t) s\left(\pi,\ \lambda_k, t\right) \left[q_x\left(x,t\right)\psi^2\left(x,\lambda_k,t\right) + 2q(x,t)\left(\psi^2\left(x,\lambda_k,t\right)\right)'\right] \tag{2}$$

in the class of real-valued π-periodic on the spatial variable x function $u = u(x,t)$ which satisfy the regularity of assumption

$$u \in C_x^3(t>0) \cap C_t^1(t>0) \cap C(t \geq 0)$$

with the initial condition

$$u(x,0) = u_0(x), x \in R, \tag{3}$$

where $q(x,t) = u_{xx}(x,t)$, $u_0(x) \in C^3(R)$ is the given real-valued π -periodic function and $\psi(x,\lambda_k,t)$ are the Floquet solution (normalized by the condition $\psi(0,\lambda_k,t)=1$) of the weighted Sturm–Liouville equation

$$y'' = \lambda q(x,t)y, \quad x \in R. \tag{4}$$

Here λ_k is zeros of the function $\Delta^2(\lambda) - 4$, where $\Delta(\lambda) = c(\pi,\ \lambda,t) + s'(\pi,\ \lambda,t)$. We denote by $c(x,\ \lambda,t)$ and $s(x,\lambda,t)$ the solutions of equation (4) satisfying the initial conditions $c(0,\lambda,t) = 1, c'(0,\lambda,t) = 0$ and $s(0,\lambda,t) = 0,\ s'(0,\lambda,t) = 1$ respectively. In system (2), the functions $\alpha_k(t)$, $k \in Z$, can be chosen freely within the class of real-valued continuous functions having uniform asymptotic decay $\alpha_k = O\left(\frac{1}{k^2}\right),\ k \to +\infty$. Using the expression for the Floquet solutions, one may derive the identity

$$s(\pi,\lambda_k,t)\psi^2(\tau,\lambda_k,t) = s(\pi,\lambda_k,t,\tau), \tag{5}$$

where $s(x,\lambda,t,\tau)$ is the solution of the Eq. (4) with coefficient $q(x+\tau,t)$ satisfying the initial conditions $s(0,\lambda,t,\tau) = 0$, $s'(0,\lambda,t,\tau) = 1$. Equality (5), the uniform decay condition $\alpha_k(t) = O\left(\frac{1}{k^2}\right)$ and asymptotic formulas ([**?**])

$$s(\pi,\lambda,t,\tau) = O(\lambda^{-\frac{1}{2}}), \quad \frac{\partial s(\pi,\lambda,t,\tau)}{\partial \tau} = O(\lambda^{-\frac{1}{2}})$$

provide uniform convergence of the series in equation (1). In (2) and elsewhere, the prime "$\prime$" means the derivative with respect to the variable x.

The aim of this work is to provide a procedure for constructing the solution $u(x,t)$, $\psi(x,\lambda_k,t)$ of problem (2)-(4) using the inverse spectral theory for the weighted Sturm–Liouville equation (4).

3. Preliminaries

The spectrum of the weighted Sturm–Liouville operator (4) with $q(x,0) < 0,\ x \in R$ is absolutely continuous and coincides with the set

$$E = \{\lambda \in R\colon -2 \leq \triangle(\lambda) \leq 2\} = [\lambda_0,\ \lambda_1] \cup [\lambda_2,\ \lambda_3] \cup \cdots \cup [\lambda_{2n},\ \lambda_{2n+1}] \ldots .$$

The intervals $(-\infty,\ \lambda_0)$, $(\lambda_{2n-1},\ \lambda_{2n})$, $n \geq 1$ are called the gaps or lacunas.

The numbers $\xi_n,\ n \geq 1$ with the signs $\sigma_n = sign\{s'(\pi,\ \xi_n) - c(\pi,\ \xi_n)\}$, $n \geq 1$ are called the spectral parameters of the weighted Sturm–Liouville equation (4) with $q(x,0) < 0,\ x \in R$. Notice that ξ_n coincides with the eigenvalues of the Dirichlet problem for equation (4). Moreover, the inclusions $\xi_n \in [\lambda_{2n-1},\ \lambda_{2n}]$ and the equality

$$s(\pi,\ \lambda) = \pi \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\xi_n}\right) \tag{6}$$

are fulfilled.

The boundaries λ_n of the spectrum and the spectral parameters ξ_n, σ_n are called the spectral data of the weighted Sturm–Liouville equation (4). The determination of spectral data of (4) is called a direct spectral problem and conversely, the restoration of the coefficient q of (4) by spectral data is called an inverse spectral problem.

The spectrum of the weighted Sturm–Liouville operator (4) with the coefficient $q(x+\tau) < 0$ does not depend on the real parameter τ, but the spectral parameters do. The spectral parameters satisfy the system of Dubrovin differential equations

$$\frac{d\xi_n(\tau)}{d\tau}=\frac{\xi_n(\tau)}{\pi}\ \frac{\sqrt{\Delta^2\left(\xi_n(\tau)\right)-4}}{\prod_{m\neq n}\left(1-\frac{\xi_n(\tau)}{\xi_m(\tau)}\right)},\ \xi_n\left(0\right)=\xi_n, \tag{7}$$

where the signature of the radical $\sqrt{\Delta^2\left(\xi_n\right)-4}$ is given by

$$\sqrt{\Delta^2\left(\xi_n\right)-4}=-\triangle\left(\xi_n\right)-\frac{1}{\frac{\partial s(\pi,\xi_n)}{\partial\lambda}}\int_0^\pi q\left(x\right)s^2\left(x,\xi_n\right)dx,\quad n\geq 1.$$

The system of Dubrovin equations (7) and the following trace formulas:

$$q\left(\tau\right)=\frac{1}{2}\sum_{n\geq 1}\left[\frac{1}{\xi_n\left(\tau\right)}-\left(\frac{1}{\xi_n\left(\tau\right)}\right)''\right]-\frac{1}{2}\sum_{k\geq 0}\frac{1}{\lambda_k} \tag{8}$$

provide the method for solving the inverse problem.

4. Main Result

The main result of the paper is stated in the theorem below.

Theorem 1. Let $u(x,t)$ and $\psi(x,\lambda_k,t)$ be solution of the problem (2)-(4). Then the spectrum of the problem (4) does not depend on t, and the spectral parameters $\xi_n=\xi_n\left(t\right),\quad \sigma_n=\sigma_n\left(t\right),\quad n\geq 1$ satisfy the analogue of the system of Dubrovin equations

$$\dot{\xi}_n=\left\{\frac{1}{2\xi_n}+\frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{\xi_j}-\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{\lambda_k}+\sum_{k=0}^{\infty}\ \frac{\xi_n\alpha_k(t)s\left(\pi,\ \lambda_k,\mathrm{t}\right)}{\lambda_k-\xi_n}\right\}h_n\left(\xi\right), \tag{9}$$

where

$$h_n\left(\xi\right)=\frac{\sigma_n\xi_n\sqrt{\left(1-\frac{\xi_n}{\lambda_0}\right)\ \prod_{i=1}^{\infty}\left(1-\frac{\xi_n}{\lambda_{2i-1}}\right)\left(1-\frac{\xi_n}{\lambda_{2i}}\right)}}{\prod\limits_{j\neq n,\,j=1}^{\infty}\left(1-\frac{\xi_n}{\xi_j}\right)}.$$

The sign $\sigma_n(t)=\pm 1$ changes at each collision of the point $\xi_n(t)$ with the boundaries of its gap $[\lambda_{2n-1},\lambda_{2n}]$. Moreover, the following initial conditions are fulfilled:

$$\xi_n(t)|_{t=0}=\xi_n^0,\quad \sigma_n(t)|_{t=0}=\sigma_n^0,\ n\geq 1, \tag{10}$$

where ξ_n^0, σ_n^0, $n\geq 1$ are the spectral parameters of the weighted Sturm–Liouville equations (4) corresponding to the coefficients $q_0\left(x\right)=u_{xx}\left(x,0\right)<0$.

Proof. We rewrite (2) as follows

$$q_t=-uu_{xxx}-2u_xu_{xx}+G\left(x,t\right) \tag{11}$$

where

$$q_t=\frac{\partial q\left(x,t\right)}{\partial t},\quad q=q\left(x,t\right)=u_{xx}\left(x,t\right),$$

$$G(x,t)=\sum_{k=0}^{\infty}\alpha_k(t)s(\pi,\lambda_k,t)[q_x(x,t)\psi^2(x,\lambda_k,t)+2q(x,t)(\psi^2(x,\lambda_k,t))'].$$

Let $y_n=y_n\left(x,t\right)$ be the normalized eigenfunction of the Dirichlet problem for equation (4) corresponding to the eigenvalue $\xi_n=\xi_n\left(t\right)$. It is easy to see that

$$y_n\left(x,t\right)=\frac{1}{\alpha_n\left(t\right)}s\left(x,\xi_n,t\right), \tag{12}$$

$$\alpha_n(t) = \sqrt{\int_0^\pi s^2(x,\xi_n,t)\, q(x,t)\, dx}.$$

Differentiating the identity

$$y_n'' = \xi_n q y_n \tag{13}$$

with respect to t, we get

$$\dot{y}_n'' = \dot{\xi}_n q y_n + \xi_n \dot{q}_t y_n + \xi_n q \dot{y}_n. \tag{14}$$

After multiplying identity (14) by y_n, we integrate over x from 0 to π, we get

$$\dot{\xi}_n \int_0^\pi q y_n^2 dx = \int_0^\pi \left(\dot{y}_n' y_n - y_n{}' \dot{y}_n\right)' dx - \int_0^\pi \xi_n q_t y_n^2 dx$$

Hence, based on Dirichlet boundary condition and normalization

$$\int_0^\pi q(x,t) y_n^2 dx = 1,$$

we have

$$\dot{\xi}_n = -\int_0^\pi \xi_n q_t y_n^2 dx. \tag{15}$$

Substituting Eq. (11) into (15), we obtain

$$\dot{\xi}_n = \xi_n \int_0^\pi \left(u u_{xxx} + 2u_x u_{xx}\right) y_n^2 dx - \int_0^\pi \xi_n G(x,t)\, y_n^2 dx. \tag{16}$$

We seek the antiderivative of the first integrand in (16) as a quadratic form of y_n and y_n', that is

$$\left(a y_n^2 + b y_n y_n' + c y_n'^2\right)' = \xi_n \left(u u_{xxx} + 2u_x u_{xx}\right) y_n^2 \tag{17}$$

where $a = a(x,t,\xi_n)$, $b = b(x,t,\xi_n)$ and $c = c(x,t,\xi_n)$ are independent of y_n and y_n'. Using equality (13), we get from (17)

$$\begin{aligned} y_n^2(a' + \frac{1}{4}b + \xi_n b q) + y_n y_n'(2a + b' + \frac{1}{2}c + 2\xi_n c q) + (y_n')^2(b + c') = \\ = \xi_n \left(u u_{xxx} + 2u_x u_{xx}\right) y_n^2 \end{aligned} \tag{18}$$

comparing the left and right hand sides of (18) we find $b = -c'$, $a = \frac{1}{2}c'' - \frac{1}{4}c - c\xi_n q$

$$\frac{1}{2}c''' - \frac{1}{2}c' - 2c'\xi_n q - c\xi_n(t)\, q' = \xi_n \left(u u_{xxx} + 2u_x u_{xx}\right). \tag{19}$$

It is easy to check that the function $c = -\frac{1}{2\xi_n} - u(x,t)$ satisfies the equality (19). Substituting obtained values of $a(x,t,\xi_n)$, $b(x,t,\xi_n)$ and $c(x,t,\xi_n)$ into (17) we arrive at

$$\begin{aligned} \int_0^\pi \xi_n \left(u u_{xxx} + 2u_x u_{xx}\right) y_n^2 dx = \int_0^\pi \left(a y_n^2 + b y_n y_n' + c y_n'^2\right)' dx = \\ = \left(a y_n^2 + b y_n y_n'\right)|_0^\pi - \left(\frac{1}{2\xi_n} + u(x,t)\right) y_n'^2|_0^\pi = \\ = -\left(\frac{1}{2\xi_n} + u(0,t)\right)\left[y_n'^2(\pi,t) - y_n'^2(0,t)\right] \end{aligned} \tag{20}$$

We now calculate the antiderivative of the second integrand in (16).

$$\int_0^\pi \xi_n G(x,t)\, y_n^2 dx =$$

$$= \sum_{k=0}^{\infty} \alpha_k(t)\xi_n s(\pi,\ \lambda_k, t) \int_0^\pi \left[q_x \psi^2(x,\lambda_k,t) + 2q\left(\psi^2(x,\lambda_k,t)\right)'\right] y_n^2 dx. \tag{21}$$

It is easy to see that

$$I = \int_0^\pi \left[q_x \psi^2(x,\lambda_k,t) + 2q\left(\psi^2(x,\lambda_k,t)\right)'\right] y_n^2 dx =$$

$$= \int_0^\pi \left[2\left(q\psi^2(x,\lambda_k,t)\right)' - q_x\psi^2(x,\lambda_k,t)\right] y_n^2 dx =$$

$$= \int_0^\pi \left(q\psi^2(x,\lambda_k,t)\right)' y_n^2 dx - \int_0^\pi q_x\psi^2(x,\lambda_k,t) y_n^2 dx +$$

$$+ q\psi^2(x,\lambda_k,t)\, y_n^2|_0^\pi - \int_0^\pi q\psi^2(x,\lambda_k,t)\left(y_n^2\right)' dx =$$

$$= \int_0^\pi q\left(\psi^2(x,\lambda_k,t)\right)' y_n^2 dx - \int_0^\pi 2q\psi^2(x,\lambda_k,t) y_n y_n' dx =$$

$$= \int_0^\pi 2q\psi(x,\lambda_k,t)\, y_n\left[\psi'(x,\lambda_k,t)\, y_n - \psi(x,\lambda_k,t)\, y_n'\right] dx =$$

$$= -\int_0^\pi 2q\psi(x,\lambda_k,t)\, y_n W\left\{\psi(x,\lambda_k,t),\ y_n\right\} dx. \tag{22}$$

Using the equality

$$q\psi(x,\lambda_k,t)\, y_n = \frac{\left(\psi(x,\lambda_k,t)\, y_n' - \psi'(x,\lambda_k,t)\, y_n\right)'}{\xi_n - \lambda_k}$$

and

$$\psi(\pi,\lambda_k,t) = \psi(0,\lambda_k,t) = 1,$$

we obtain

$$I = \int_0^\pi \frac{2}{\lambda_k - \xi_n} W\left\{\psi(x,\lambda_k,t),\ y_n\right\} W'\left\{\psi(x,\lambda_k,t),\ y_n\right\} dx =$$

$$= \frac{1}{\lambda_k - \xi_n} W^2\left\{\psi(x,\lambda_k,t), y_n\right\}|_0^\pi =$$

$$= \frac{1}{\lambda_k - \xi_n}\left(y_n'^2(\pi)\,\psi^2(\pi,\lambda_k,t) - y_n'^2(0)\,\psi^2(0,\lambda_k,t)\right) =$$

$$= \frac{1}{\lambda_k - \xi_n}\left(y_n'^2(\pi) - y_n'^2(0)\right). \tag{23}$$

Putting (23) to (21), we get

$$\int_0^\pi \xi_n G(x,t)\, y_n^2 dx = \sum_{k=0}^{\infty} \frac{\xi_n \alpha_k(t) s(\pi,\ \lambda_k, t)}{\lambda_k - \xi_n}\left(y_n'^2(\pi) - y_n'^2(0)\right). \tag{24}$$

Substituting (20) and (24) into (16), we obtain

$$\dot{\xi}_n = -\left\{\frac{1}{2\xi_n} + u(0,t) + \sum_{k=0}^{\infty} \frac{\xi_n \alpha_k(t) s(\pi,\ \lambda_k, t)}{\lambda_k - \xi_n}\right\}\left[y_n'^2(\pi,t) - y_n'^2(0,t)\right]. \tag{25}$$

According to (12) and to the identity

$$\alpha_n^2(t)=\int_0^\pi s^2(x,\xi_n,t)\,q(x,t)\,dx=s'(\pi,\xi_n,t)\,\frac{\partial s(\pi,\lambda,t)}{\partial\lambda}\mid_{\lambda=\xi_n},\tag{26}$$

we derive the equality

$$y_n'^2(\pi,t)-y_n'^2(0,t)=\frac{1}{\alpha_n^2(t)}\left[s^2(\pi,\xi_n,t)-s'^2(0,\xi_n,t)\right]=$$

$$=\frac{1}{s'(\pi,\xi_n,t)\,\frac{\partial s(\pi,\lambda,t)}{\partial\lambda}\mid_{\lambda=\xi_n}}\left[s'^2(\pi,\xi_n,t)-1\right]=$$

$$=\frac{1}{\frac{\partial s(\pi,\lambda,t)}{\partial\lambda}\mid_{\lambda=\xi_n}}\left[s'(\pi,\xi_n,t)-\frac{1}{s'(\pi,\xi_n,t)}\right].\tag{27}$$

Now, substituting the values $x=\pi$ and $\lambda=\xi_n(t)$ into the identity

$$c(x,\lambda,t)\,s\prime(x,\lambda,t)-c\prime(x,\lambda,t)\,s(x,\lambda,t)=1,$$

we find that

$$c(\pi,\xi_n,t)=\frac{1}{s'(\pi,\xi_n,t)}.\tag{28}$$

According to (28) and the identity

$$[c(\pi,\lambda,t)-s'(\pi,\lambda,t)]^2=(\Delta^2(\lambda)-4)-4c'(\pi,\lambda,t)s(\pi,\lambda,t))$$

we obtain the equality

$$s'(\pi,\xi_n,t)-\frac{1}{s'(\pi,\xi_n,t)}=\sigma_n(t)\sqrt{\Delta^2(\xi_n)-4},\tag{29}$$

where

$$\sigma_n(t)=sign\left\{s'(\pi,\xi_n,t)-c(\pi,\xi_n,t)\right\},\ n\geq 1.$$

Using (6) and the expansions

$$\frac{\partial s(\pi,\lambda,t)}{\partial\lambda}\mid_{\lambda=\xi_n}=-\xi_n^{-1}\pi\prod_{\substack{j\neq n\\ j=1}}^{\infty}\left(1-\frac{\xi_n}{\xi_j}\right),\tag{30}$$

$$\Delta^2(\xi_n)-4=\pi^2\left(1-\frac{\xi_n}{\lambda_0}\right)\prod_{i=1}^{\infty}\left(1-\frac{\xi_n}{\lambda_{2i-1}}\right)\left(1-\frac{\xi_n}{\lambda_{2i}}\right),\tag{31}$$

we deduce

$$\frac{s'(\pi,\xi_n,t)-\frac{1}{s'(\pi,\xi_n,t)}}{\frac{\partial s(\pi,\lambda,t)}{\partial\lambda}\mid_{\lambda=\xi_n}}=\frac{\sigma_n\sqrt{\Delta^2(\xi_n)-4}}{\frac{\partial s(\pi,\xi_n,t)}{\partial\lambda}}=$$

$$=-\frac{\sigma_n\sqrt{\left(1-\frac{\xi_n}{\lambda_0}\right)\prod_{i=1}^{\infty}\left(1-\frac{\xi_n}{\lambda_{2i-1}}\right)\left(1-\frac{\xi_n}{\lambda_{2i}}\right)}}{\xi_n^{-1}\prod_{\substack{j\neq n,\\ j=1}}^{\infty}\left(1-\frac{\xi_n}{\xi_j}\right)}.\tag{32}$$

From (25), (32) and trace formulas

$$u(0,t) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\xi_k} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{\lambda_k}$$

we conclude (9).

We notice that if instead of Dirichlet boundary conditions we consider periodic or anti-periodic boundary conditions, then equation (25) remains valid and we can deduce $\dot{\lambda}_n(t) = 0$ for all $n \in Z$. Hence, the spectrum of problem (13) does not depend on the parameter t, and the theorem is proved.

Remark 1. Theorem 1 provides the method for solving problem (2)-(4). It is the following one.

(i) Solving the direct spectral problem for (4) with $q(x+\tau)$, spectral data λ_n, $n \geq 0$, $\xi_n^0(\tau)$, $\sigma_n^0(\tau)$, $n \geq 1$ are obtained.

(ii) Using the result of Theorem 1, we find the solution of the Cauchy problem $\xi_n(\tau,t)|_{t=0} = \xi_n^0(\tau)$, $\sigma_n(\tau,t)|_{t=0} = \sigma_n^0(\tau), n \geq 1$, for the Dubrovin-type system (9).

(iii) Finally, by using trace formulas

$$u(\tau,t) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\xi_k(\tau,t)} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{\lambda_k}$$

we obtain the expressions for $u(\tau,t)$. After that the Floquet solutions $\psi(x,\lambda_k,t)$ of equation (4) can be determined. The uniformly convergence of the last series follows from the asymptotic behavior of the eigenvalues $\xi_k(\tau,t)$ and λ_k.

Conclusion

We present a simple method for constructing a source for the Hunter-Saxton equation and show that the inverse spectral method of the weighted Sturm-Liouville operator with periodic coefficient is applicable to solving the Cauchy problem for the Hunter-Saxton equation with a source in the class of periodic functions. In this approavich, an analogue of the Dubrovin system of differential equations is derived, and then the Hunter-Saxton equation with a self-consistent source in the class of periodic functions is solved in the form of a uniformly converging functional series.

Keywords: Hunter-Saxton equation, self-consistent source, trace formulas, inverse spectral problem, weighted Sturm - Liouville operator.

AMS Subject Classification: 34L25, 35P25, 37K15, 47F05.

References

[1] Hunter J., Saxton R., Dynamics of director fields, *SIAM J. Appl.Math.*, Vol.51, No.6, 1991, pp.1498-1521.

[2] Hunter J., Zheng Y., On a completely integrable nonlinear hyperbolic variational equation, *Physica D.*, Vol.79, No.2, 1994, pp.361-386.

[3] Hunter J., Zheng Y., On a nonlinear hyperbolic variational equation: I, global existence of weak solutions,, *Arch. Ration. Mech. Anal.*, Vol.129, No.4, 1995, pp.305-353.

[4] Beals R., Sattinger D.H., Szmigielski J. Inverse scattering solutions of the Hunter–Saxton equations. *Appl. Anal.*, 2001, Vol.78, pp.255–269.

[5] Helge Holden, Kenneth H. Karlsen, Peter H.C. Pang, The Hunter–Saxton equation with noise. *Journal of Differential Equations*, 2021, Vol.270, pp. 725-786.

[6] Bressan A., Constantin A., Global solutions of the Hunter–Saxton equation. *SIAM J. Math. Anal.*, 2005, Vol.37, pp.996–1026.

[7] Johnson R.S., Camassa–Holm, Korteweg–de Vries and related models for water waves. *J. Fluid. Mech.*, 2002, Vol.455, pp.63–82.

[8] Camassa R., Holm D., An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.*, 1993, Vol.71, pp.1661–1664.

[9] Dai H.H., Pavlov M., Transformations for the Camassa–Holm equation, its high frequency limit and the Sinh-Gordon equation. *J. Phys. Soc. Japan*, 1998, Vol.67, pp.3655–3657.
[10] Constantin A., Lannes D., The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations. *Arch. Ration. Mech. Anal.*, 2009, Vol.192, pp.165–186.
[11] Constantin A., Kolev B., On the geometric approach to the motion of inertial mechanical systems. *J. Phys. A.*, 2002, Vol.35, pp.51–79.
[12] Yin Z., On the structure of solutions to the periodic Hunter–Saxton equation. *SIAM J. Math. Anal.*, 2004, Vol.36, pp272–283.
[13] Mel'nikov V.K., A method for integrating the Korteweg-de Vries equation with a self-consistent source. *Dubna, Joint Institute for Nuclear Research*, 1988, Vol.88
[14] Khasanov A.B., Yakhshimuratov A.B., The Korteweg–de Vries equation with a self-consistent source in the class of periodic functions. *Theoretical and Mathematical Physics*, 2010, Vol.164, pp.1008-1015.
[15] Khasanov, A.B., Hoitmetov, U.A., Integration of the General Loaded Korteweg–de Vries Equation with an Integral Type Source in the Class of Rapidly Decreasing Complex-Valued Functions. *Russian Mathematics*, 2021, Vol.65, pp.43-57.
[16] Yakhshimuratov A.B., Babajanov B.A., Integration of equations of Kaup system kind with self-onsistent source in class of periodic functions. *Ufa Mathematical Journal*, 2020, Vol.12, pp.103-113.
[17] Babadzhanov, B.A., Khasanov, A.B., Yakhshimuratov, A.B., On the inverse problem for a quadratic pencil of sturm-liouville operators with periodic potential. *Differential Equations*, 2005, Vol.41, pp.310-318.
[18] Babajanov B.A., Babadjanova A.K., Azamatov A.S., Integration of the differential-difference sine-Gordon equation with a self-consistent source. *Theor. Math. Phys.*, 2022, Vol.210, pp.327-336.
[19] Leon J., Latifi A., Solution of an initial-boundary value problem for coupled nonlinear waves. *J. Phys. A: Math. Gen.*, 1990, Vol.23, pp.1385-1403.
[20] Claude C., Latifi A., Leon J., Nonlinear resonant scattering and plasma instability: an integrable model. *J. Math. Phys.*, 1990, Vol.32, pp.3321-3330.
[21] Li Q., Zhang J., Chen D., The Eigenfunctions and Exact Solutions of Discrete mKdV Hierarchy with Self-Consistent Sources via the Inverse Scattering Transform. *Advances in Applied Mathematics and Mechanics*, 2015, Vol.7, pp.663-674.
[22] Zeng Y. B., Ma W. X. and Lin R. L., Integration of the soliton hierarchy with self-consistent sources. *Journal of Mathematical Physics*, 2000, Vol.41, pp.5453-5489.
[23] Babajanov B.A., Feckan M., Urazbayev G.U., On the periodic Toda Lattice with self-consistent source. *Commun. Sci. Numer. Simul.*, 2015, Vol.22, pp.1223-1234.
[24] Babajanov B.A., Feckan M., Urazbayev G.U., On the periodic Toda Lattice hierarchy with an integral source. *Commun. Sci. Numer. Simul.*, 2017, Vol.52, pp.110-123.
[25] Babajanov B.A., Khasanov A.B., Periodic Toda chain with an integral source. *Theor Math Phys.*, 2015, Vol.184, pp.1114-1128.
[26] Yakhshimuratov A.B., Kriecherbauer T., Babajanov B.A., On the construction and integration of a hierarchy for the Kaup system with a self-consistent source in the class of periodic functions. *Journal of Mathematical Physics, Analysis, Geometry*, 2021, Vol.17, pp.233-257.
[27] Hasanov A.B., Hasanov M.M., Integration of the nonlinear Schrodinger Equation with an Additional Term in the Class of Periodic Functions. *Theoretical and Mathematical Physics*, 2019, T.199, N.1, C.60–68.
[28] Hasanov A.B., Matyakubov M.M., Integration of the nonlinear Korteweg–de Vries equation with an additional term. *Theoretical and Mathematical Physics*, 2020, T.203, N.2, C.192–204.
[29] Ma W.X., Soliton, positon and negaton solutions to a Schroedinger self-consistent source equation. *Journal of the Physical Society of Japan*, 2003, Vol.72, pp.3017-3019.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

SOLVING OF SOME INVERSE PROBLEM OF EPIDEMIOLOGY BY STOCHASTIC NATURE-LIKE METHODS

SERGEI KABANIKHIN[1], MAKTAGALI BEKTEMESOV[2], ZHOLAMAN BEKTEMESSOV[3], ZHULDYZ ALMATYKYZY[3]

[1] Sobolev Institute of Mathematics, SB, RAS, Novosibirsk, Russian Federation
[2] Abai Kazakh National Pedagogical University, Almaty, Kazakhstan
[3] Al-Farabi Kazakh National University, Almaty, Kazakhstan

e-mail: jolaman252@gmail.com

1. Introduction

Mathematical modeling of the spread of viral infection in Kazakhstan using inverse problem solving methods can be a useful tool for understanding disease dynamics, predicting future spread, and evaluating the effectiveness of various infection control measures. Modeling the spread of infection is usually based on a system of complex differential equations that describe the change in the number of infected, recovered, and affected individuals over time and space. However, in the case of inverse problem solving methods, we work in the opposite direction – based on available data on incidence and other factors, we try to estimate model parameters that best fit the observed data [1].

The basic idea of inverse problem solving methods is to find the optimal values of the model parameters that minimize the difference between the predicted values of the model and the actual incidence data. For this, optimization method such as firefly algorithm is used. Mathematical modeling of the spread of viral infection using inverse problem solving methods is one of the ways to help understand and fight infectious diseases. However, it should be noted that the accuracy of the models depends on the available data, the quality of the model and the assumptions made during the modeling process. Therefore, it is important to use reliable data and take into account the various factors that influence the spread of infection [2].

Mathematical modeling of the spread of coronavirus infection by the SEIR-HCD (Susceptible – Exposed – Infected – Recovered – Hospitalized – Critical - Dead) method using the firefly algorithm is one of the ways to analyze and predict the dynamics of morbidity. The SEIR-HCD method is based on dividing the population into several categories: Susceptible, Exposed, Infected, Recovered, Hospitalized, Critical, and Dead. The model takes into account the transitions between these categories based on certain parameters, for example, the probability of infection, the incubation period, the average duration of the disease, etc.

The firefly algorithm is a metaheuristic optimization method that simulates the behavior of the brightness field in finding the optimal solution. Each firefly represents a possible solution to the problem, and their movement and interaction allows you to find the optimal solution. In the context of modeling coronavirus infection, the firefly algorithm can be used to determine the optimal model parameter values that best fit the observed disease data.

This work was partially supported by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19579325).

The use of the SEIR-HCD method and the firefly algorithm allows modeling the spread of coronavirus infection, taking into account the dynamics of morbidity, the transition between categories and the influence of various factors such as control measures and vaccination. The model can be used to predict future disease trends, evaluate the effectiveness of infection control measures, and inform public health decisions.

However, it should be noted that the accuracy of the model depends on the quality of the data used to parameterize the model and the correct choice of the modeling parameters.

2. Formulation of the problem

Let consider the SEIR-HCD model of the spread of COVID-19 [3].

$$\begin{cases} \frac{dS}{dt} = -\frac{5-a(t-\tau)}{5}\left(\frac{\alpha I(t)S(t)I(t)}{N} + \frac{\alpha E(t)S(t)E(t)}{N}\right), \\ \frac{dE}{dt} = \frac{5-a(t-\tau)}{5}\left(\frac{\alpha I(t)S(t)I(t)}{N} + \frac{\alpha E(t)S(t)E(t)}{N}\right) - \frac{1}{t_{inc}}E(t), \\ \frac{dI}{dt} = \frac{1}{t_{inc}}E(t) - \frac{1}{t_{inf}}I(t), \\ \frac{dR}{dt} = \frac{\beta}{t_{inf}}I(t) - \frac{1-\varepsilon_{HC}}{t_{hosp}}H(t), \\ \frac{dH}{dt} = \frac{1-\beta}{t_{inf}}I(t) + \frac{1-\mu}{t_{crit}}C(t) - \frac{1}{t_{hosp}}H(t), \\ \frac{dC}{dt} = \frac{\varepsilon_{HC}}{t_{hosp}}I(t) - \frac{1}{t_{crit}}C(t), \\ \frac{dD}{dt} = \frac{\mu}{t_{crit}}C(t) \end{cases} \tag{2.1}$$

with initial conditions

$$S(t_0) = N - E_0 - I_0 - R_0 - C_0 - D_0, E(t_0) = E_0, \tag{2.2}$$

$$I(t_0) = I_0, R(t_0) = R_0, H(t_0) = H_0, C(t_0) = C_0, D(t_0) = D_0 \tag{2.3}$$

The outline of model 2.1 was described above, and the description and values of parameters and initial conditions for Almaty region are given in Table 1 (the initial time is assumed to be April 15, 2020). In the SEIR-HCD model, the asymptomatic population $E(t)$ transitions to the symptomatic population $I(t)$ after t_{inc} days. Infected people recover after t_{inf} days with probability β and are hospitalized $H(t)$ with probability $1-\beta$. The hospitalized person may then recover or may need to be $C(t)$ connected to a ventilator. In the model, only critical cases can die with probability $D(t)$ [4].

3. Formulation of an inverse problem

Assume that additional information about symptomatic cases $(1-b_k)h_k$, critical cases C_k and death g_k at specified time points in the SEIR-HCD model is known at specified time t_k, $k = 1, .., K$:

$$E(t_k; q) = (1-b_k)h_k, C(t_k; q) = C_k, D(t_k; q) = g_k, k = 1, ..., K. \tag{3.1}$$

$$J(q) = \langle A(q) - f, A(q) - f \rangle. \tag{3.2}$$

Here A is a nonlinear inverse calculation operator, $f = \{f_k^i\}_{k=1,K}^{i \in \mathcal{L}}$, $\mathcal{L}$ is a set of measurable states of the system, $q = (\alpha_E(t), \alpha_I(t), \varepsilon_{HC}, \mu, E_0, I_0)$ is a vector of unknown model parameters, h_k is the number of identified cases in Almaty region, b_k is the percentage of asymptomatic cases according to PCR results.

The inverse problem 3.1-3.2 for the SEIR-HCD model is to determine the parameter vector q. For the SEIR-HCD model, the inverse problem is reduced to the objective functional minimization problem.

Table 1. Description and parameter values of the SEIR-HCD model

Parameter	Description	Value
$a(t)$	Index of self-isolation	$(0, 5)$
$\alpha_E(t)$	The infection parameter between asymptomatic and susceptible populations is $\alpha_E >> \alpha_I$	$(0, 1)$
$\alpha_I(t)$	Parameter of infection between infected and susceptible populations	$(0, 1)$
β	Proportion of infected people who survive the disease without complications	$(0, 1)$
ε_{CH}	Percentage of hospitalized cases requiring mechanical ventilation	$(0, 1)$
μ	Proportion of deaths from COVID-19	$(0, 0.5)$
τ	latent period (describes the delay in the onset of an infectious disease)	2 days
t_{inc}	Duration of the incubation period	$2 - 14$ days
t_{inf}	The length of the infectious period	$2.5 - 14$ days
t_{hosp}	Length of hospital stay	$4 - 5$ days
t_{crit}	The duration of use of the mechanical ventilation	$10 - 20$ days
N	Population (people) in Almaty region	2179600
E_0	Initial number of asymptomatic carriers	$(1, 5000)$
I_0	Initial number of infected	$(1, 5000)$
R_0	Initial number of cases treated	$(1, 100)$
H_0	Initial number of hospitalizations	180
C_0	Initial number of critical cases	51
D_0	Initial number of deaths	10

$$J(q) = \sum_{k=1}^{K} \frac{1}{(1-b_k)^2 h_k^2} \Big(\frac{1}{t_{inc}} E(t_{k-1}; q) - (1-b_k)h_k\Big)^2 + \frac{(C(t_k; q) - C_k)^2}{C_k^2} + \frac{(D(t_k; q) - g_k)^2}{g_k^2}. \quad (3.3)$$

Here, the first term $t_{inc}^{-1} E(t_{k-1}; q)$ describes the number of asymptomatic carriers of the symptomatically infected COVID-19 virus on day t_k. The functional $J(q)$ also includes information on the severity $C(t_k; q)$ and death $D(t_k; q)$ of the COVID-19 cases available in open sources.

4. Firefly algorithm

Nature-inspired metaheuristic algorithms, especially algorithms based on swarm intelligence, have attracted much attention in the last years. The Firefly Algorithm (FA) appeared about five years ago in 2008 [5], and its literature has expanded dramatically with various applications. In this chapter, we first introduce the standard brightness algorithm and then briefly review variants with a selection of recent publications. We can now idealize some of the flashing characteristics of flares so that we can develop algorithms to inspire firefly. For simplicity in describing the standard FA, we use the following three idealized rules:

• All sparklers are of the same sex, so a sparkler is attracted to other sparklers regardless of their gender.

• Attraction is proportional to the brightness of the flame. Thus, for any two flashes, the firefly moves to the less bright light. Attraction is proportional to luminosity, both of which decrease with distance. Unless there is a flash of light from a particular fire, it moves at random.

• The landscape of the objective function affects or is determined by the brightness of the firefly.

For a maximization problem, the brightness can be proportional to the value of the objective function. Other forms of brightness can be defined in a similar way to the fitness function in genetic algorithms.

5. Results

Numerical results of solving the inverse problem of restoring parameter values from some additional information are given in Table 2. The Firefly algorithm was implemented in a high-level language and an interactive environment for programming, numerical calculations and visualization of results - MatLab. The inverse problem was considered with data calculated for each equation of the model at 5 time points uniformly distributed in the interval (0, 60).

Table 2. Results of parameter restoration

True parameter values							
$a(t)$	τ	α_I	α_E	k	β	ε_{CH}	μ
2.5	2	0.491	0.994	0.28	0.165	0.0065	0.0011
Average value of the recovered parameters							
3.4347	0.3323	0.9568	1.2114	0.0091	0.4712	0.0011	0.0032
Relative error							
0.37388	0.83385	0.9486	0.218	0.9675	1.855	0.83	1.9

Thus, the parameter values of the SEIR-HCD model were restored with a relative error of no more than 20-40% (α_E, $a(t)$) for the determined parameters, for 4 parameters this indicator exceeds 80% (τ, α_I, ε_{CH}, k), which indicates that the parameter values recovery is also a good enough result. And the relative error value of the remaining parameters exceeds about 100% (β, μ).

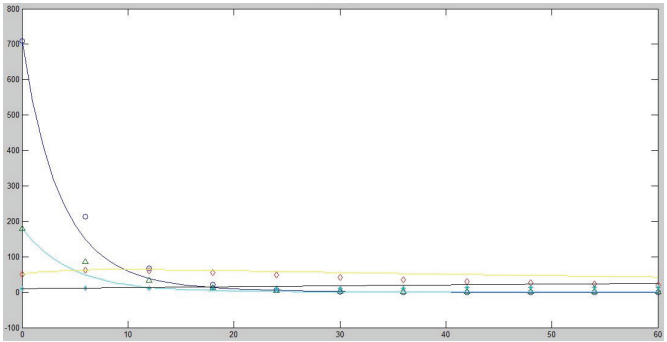


Figure 1. Restored distribution dynamics of I, H, C, D variables.

Observed variables depend differently on different parameters and can be used to rank parameters in order of relative influence on model predictions. Such an effect can be quantified using parametric sensitivity.

6. Conclusion

In the context of the global coronavirus pandemic, mathematical modeling is becoming an important tool for understanding and predicting the dynamics of the spread of the virus, evaluating the effectiveness of control measures, and developing strategies to combat the pandemic.

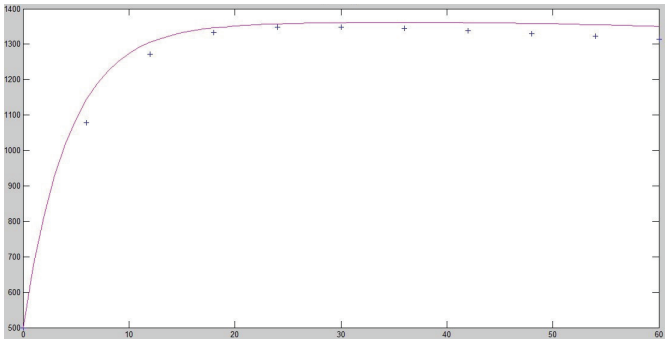


FIGURE 2. Restored distribution dynamics of the R variable.

In this thesis, mathematical modeling of coronavirus infection in Kazakhstan was studied using the SEIR-HCD model and the firefly algorithm in Matlab.

In addition, the results of the study show that the firefly algorithm is an effective and powerful tool for numerically solving the system of differential equations describing the SEIR-HCD model. It has good convergence speed and sufficient accuracy to predict the dynamics of infection spread.

Based on the results of the research, it can be concluded that the SEIR-HCD model, developed by using the algorithm of brightness in the Matlab program, is an effective tool for mathematical modeling of the coronavirus infection in Kazakhstan. It can be used to predict the dynamics of the spread of infection, evaluate the effectiveness of control measures, and make informed decisions in the fight against a pandemic.

Keywords: Inverse problems, optimization algorithms, Firefly algorithm, epidemiology.

AMS Subject Classification: 34A55, 65L09

REFERENCES

[1] Kabanikhin S.I., Bektemesov M.A., Bektemessov J.M., Mathematical model for medium-term COVID-19 forecasts in Kazakhstan, *Journal of Mathematics, Mechanics and Computer Science*, v.111, n.3, 2021, p. 95-106.

[2] Bektemesov Zh.M., Kabanihin S.I., Kasenov S.E., O chislennom modelirovanii obratnoj zadachi epidemiologii. *Vestnik "Fiziko-matematicheskie nauki"*, 75:3, 2021, pp. 7–14. DOI:https://doi.org/10.51889/2021-3.1728-7901.01.

[3] Krivorotko O.I., Kabanikhin S.I., Zyatkov N.Yu., Prikhodko A.Yu., Prokhoshin N.M., Shishlenin M.A., Mathematical modeling and forecasting of COVID-19 in Moscow and Novosibirsk region, *Num.Anal.Appl.*, 13:4, 2020, pp.332–348.

[4] Krivorotko O.I., Kabanikhin S.I., Bektemesov M.A., Sosnovskaya M.I., Neverov A.V., Simulation of COVID-19 Spread Scenarios in the Republic of Kazakhstan Based on Regularization of the Agent-Based Model, *Journal of Applied and Industrial Mathematics*, 17:3, 2023, pp.94-109.

[5] Yang XS., *Nature-inspired metaheuristic algorithms*. Ist ed. Frome, UK: Luniver Press; 2008, 147 p.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

REGULARIZATION OF LINEAR MACHINE LEARNING PROBLEMS

SERGEY KABANIKHIN[1], LIU SHUANG[2]

[1] Siberian Branch of the Russian Academy of Sciences, Novosibirsk, Russian Federation
[2] Novosibirsk State University, Harbin, China
kabanikh@math.nsc.ru

e-mail: liushusng1118@gmail.com

Abstract. In this paper, theoretical knowledge about inverse and ill-posed problems is applied to linear neural networks (LNN) in combination with machine learning methods using the most common regularization methods (including L1, L2, and dropout) to achieve a reduction in the retraining of neural networks. In the L1 regularization method, a penalty is added to the loss function—the absolute value of the model parameters. This leads to the sparsity of the model parameters, which can simplify and speed up their study. For L2 regularization, a penalty is added to the loss function, the square of the L2 norm, which leads to increased noise immunity and, consequently, reduces the likelihood of repeated learning . Dropout means that during the learning process, there is a certain probability that the neurons of the neural network will be disabled, thereby reducing retraining. These methods can be used to select the optimal value of a regularized hyperparameter.

Keywords: Inverse and ill-posed problems; machine learning; linear neural networks.

AMS Subject Classification: 65N20, 68T07.

1. Introduction

With the advent of powerful computers, the field of application of inverse problems and problem theory has expanded to almost all areas of science using mathematical methods [1]. Studies on linear regression appeared at the beginning of the XIX century [2-3], assuming that the relationship between the independent variable x and the dependent variable y is linear, i.e. that y can be expressed as a weighted sum of the elements of x, and usually admitting some noise that observations contain; assuming that any noise is relatively normal, for example, that the noise follows a normal distribution. In 1911, Hermann Weyl discovered and published one of the first examples of solving inverse problems describing the asymptotic behavior of the eigenvalues of the Laplace-Beltrami operator [4]. In 1936, Alan Turing proposed the concept of a Turing machine, which marked the beginning of the era of artificial intelligence [5]. In 1911, Yuri Nesterov made a great contribution to the accelerated version of gradient descent, which converged much faster than the usual gradient descent method [6]. In 1997, Fukumizu Kenji explained linear neural networks [7]. In 2012, John Kalivas described the regularization options L2 and L1, Tikhonov [8]. In 2016, Ian Goodfellow, Joshua Bengio and Aaron Courville in their book "Deep Learning" state: "In the context of deep learning, most regularization strategies are based on regularizing estimates. The regularization of the estimate is achieved by reducing the variance with increasing bias. An effective regularizer can make trades profitable and significantly reduce variance without excessively increasing bias." [9]. In 2021, Haji Saad and Abdulaziz Mohsinz compared various optimization methods based on gradient descent algorithms [10]. In this paper, theoretical knowledge about inverse and ill-posed problems is applied to linear neural networks (LNN) in combination with machine learning methods using the most popular regularization

methods (including L1, L2 and dropout, etc.) to achieve the goal of avoiding overfitting neural networks.

2. Linear neural networks

Let

$$l = \sum_{i=1}^{N} x_i w_i + b,\ i = 1, 2, \cdots N. y = f(l) \tag{1}$$

where Y is the output vector, W is the weight matrix, X is the input vector, b is the displacement term and f is the activation function.

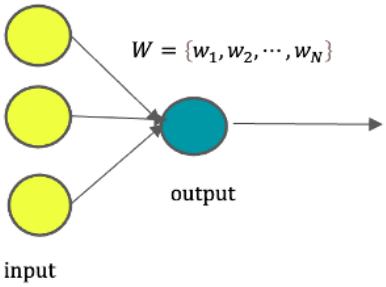


Figure 1. Structure of linear neural networks.

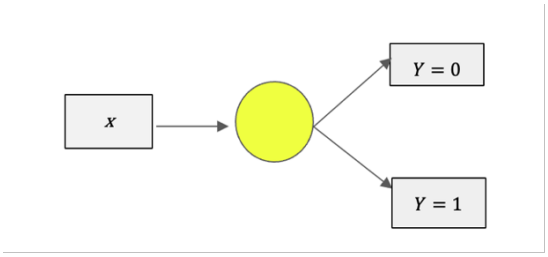


Figure 2. Logistic regression.

The transfer function of a neuron of a linear neural network is a linear function, so the output can be any value, and the output of the perceptron can only be 0 or 1. Linear neural networks (LNN) are a class of neural networks in which each neuron has only one output signal and performs linear operations. In a linear neural network, each neuron receives input data, multiplies it by the appropriate weight, and summarizes the results. Then the result is transmitted to the output of the neuron. Unlike nonlinear neural networks, linear neural networks cannot solve complex tasks such as pattern recognition or image classification. However, they can be useful in regression tasks where values need to be predicted based on input data. Linear neural networks are also used as basic models for more complex architectures, such as convolution neural networks and recurrent neural networks. Training of linear neural networks is carried out by adjusting the weights of neurons to minimize the error between the predicted and actual values in the training data [11]. Optimization methods such as gradient descent are used for this purpose. Before training, it is necessary to define a loss function that measures the difference between the predicted value and the actual value. The standard error or the average absolute error is usually used. After determining the loss function, you can start learning. At each iteration of the gradient descent, the weight of the neuron is updated in the direction opposite to the gradient of the loss function. This minimizes errors and increases the predictive power of the network. Learning linear neural networks can be quite fast and efficient, especially for regression problems with a large number of functions. However, more complex tasks, such as image classification, require a more sophisticated neural network architecture. The main theorists of neural networks are Jeffrey Hinton, Joshua Bengio and Jan Lecun. They are pioneers in the field of deep learning and neural networks, and their work has become the basis for many modern developments in the field of artificial intelligence technologies. Currently, there is not a single major neural network theorist, because the field of artificial intelligence and deep learning is developing very quickly, and many scientists around the world are actively engaged in this field. However, Joshua Bengio, Jeffrey Hinton and Jan Lecun are still the most famous and influential scientists in this field. Jeffrey Hinton was the first to propose an algorithm for "back

propagation of errors" for training neural networks. This algorithm solves the problem associated with the fact that neural networks are difficult to train in multi-layered complex scenarios, and lays the foundation for the subsequent development of deep learning technology. Hinton proposed the famous "dropout" technology, which can effectively prevent the problem of model transformation. At the same time, Hinton also proposed the Deep Belief Network (DBN), which is a deep unsupervised learning network capable of automatically learning functions and used in many fields [8,12,13].

In the literature on neural networks, regularization is understood as a method used to prevent over-fitting of the model. These methods reduce the influence of noise in the data and make the model more resistant to changes in the input data. We will mention only 4 main methods in this regard:

- Regularization of L1 and L2 (a.N.Tikhonov regularization): The penalty term is added to the loss function (for L1-regularization - the absolute value of the parameter, for L2-regularization - the quadratic norm of the parameter). This method can be used to select the best value of a regularized hyperparameter. For example, to assess the quality of the model at different values of the hyperparameter, cross-validation is applied, and the value that gives the best quality is selected [14].
- Dropout: some neurons are randomly switched off during training, thereby reducing retraining [15].
- Early stop: If errors in the test data set begin to increase, stop training [16].
- Batch normalization: Normalize data at each level of the neural network to speed up training and reduce retraining time [17].

The relationship between these methods and classical regularization methods is established.

3. L1 regularization

Regularization L1 is also called lasso regression, Ω - the regularization term is the sum of the absolute values of the weight parameters in the weight matrix.

$$\Omega(W) = \| W \|_1 = \Sigma_i \Sigma_j \mid w_{ij} \mid . \tag{2}$$

We multiply the regularization term by α and add all this to the loss function.

$$\hat{\mathcal{L}}(W) = \alpha \| W \|_1 + \mathcal{L}(W). \tag{3}$$

$$\nabla_W \hat{\mathcal{L}}(W) = \alpha sign(W) + \nabla_W \mathcal{L}(W) \tag{4}$$

The derivative of the new loss function leads to a formula that is the sum of the gradient of the old loss function and the weight value multiplied by α.

4. L2 regularization

L2 regularization is the most common type of all regularization methods and is also commonly referred to as weight reduction or regression. During L2 regularization, the neural network loss function expands by the so-called regularization term, which is here called Ω.

$$\Omega(W) = \| W \|_2^2 = \Sigma_i \Sigma_j w_{ij}^2. \tag{5}$$

$$\hat{\mathcal{L}}(W) = \frac{\alpha}{2} \| W \|_2^2 + \mathcal{L}(W) = \frac{\alpha}{2} \Sigma_i \Sigma_j w_{ij}^2 + \mathcal{L}(W) \tag{6}$$

Calculate the gradient of the new loss function and put the gradient in the weight update rule:

$$\nabla_W \hat{\mathcal{L}}(W) = \alpha W + \nabla_W \mathcal{L}(W). \tag{7}$$

$$W \leftarrow W - \epsilon(\alpha W + \nabla_W \mathcal{L}(W)). \tag{8}$$

Redefinition:

$$W \leftarrow W(1-\epsilon\alpha)W - \epsilon\nabla_W \mathcal{L}(W). \tag{9}$$

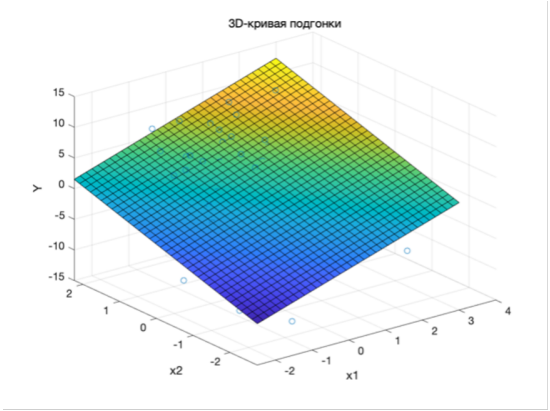


FIGURE 3. Fitting the function $y = 2x + 1$ using the L1 regularization method.

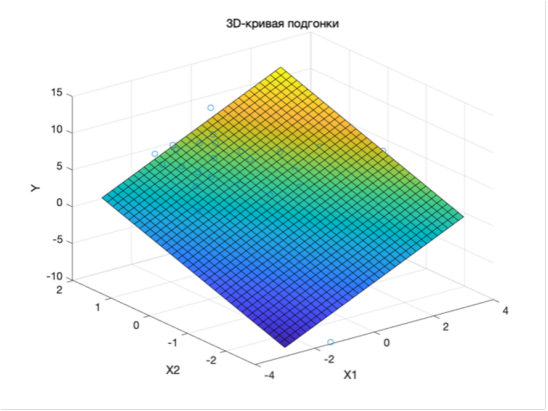


FIGURE 4. Fitting the function $y = 2x + 1$ using the L2 regularization method.

5. DROPOUT

During the learning process, some neurons are randomly switched off, which reduces overtraining [18].

$$f = \sigma\left(\sum_{i=1}^{N} x_i w_i + b\right), \ i = 1, 2, \cdots \ N. \tag{10}$$

Training stage:

$$y = f \bigodot B, B_i \sim Bernoulli\,(p)\,. \tag{11}$$

Training stage:

$$y = (1-p)\, f \bigodot B. \tag{12}$$

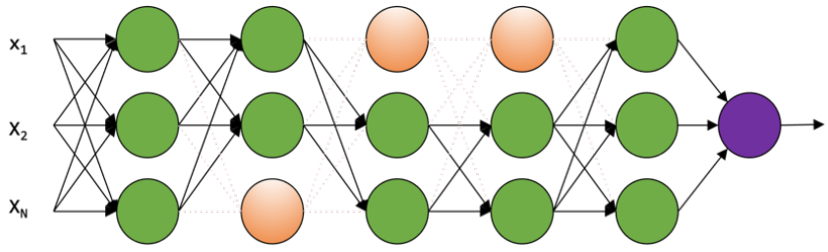


FIGURE 5. Structure of dropout.

Normalization of data on each layer of the neural network to accelerate learning and reduce retraining [19].

$$\mu_B = \frac{1}{m}\sum_{i=1}^{m} x_i, \tag{13}$$

$$\sigma_B^2 = \frac{1}{m}\sum_{i=1}^{m}(x_i - \mu_B)^2, \tag{14}$$

$$\hat{x}_i = \frac{x_i - \mu_B}{\sqrt{\sigma_B^2 + \epsilon}}, \tag{15}$$

$$y_i = \gamma\hat{x}_i + \beta = BN_{\ \gamma\beta}(x_i). \tag{16}$$

Where γ, β are trainable parameters.

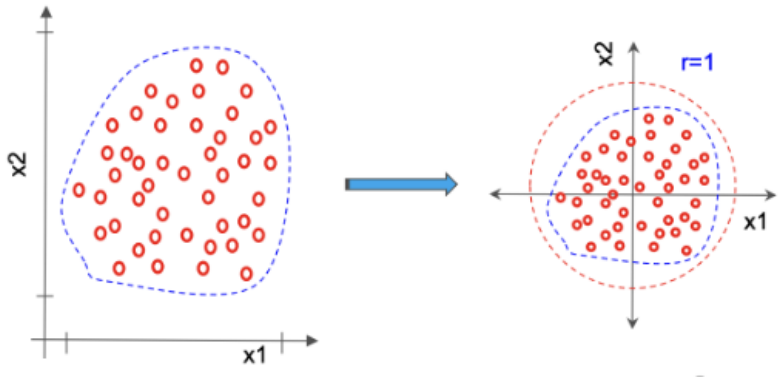


FIGURE 6. The structure of batch normalization [20].

6. Early stopping

An early stop means that we periodically save the trained parameters and monitor the validation error. After the training is stopped, we return the trained parameters to the exact position at which the validation error began to grow, and not to the position of the last parameter. Another way to imagine an early stop is a very efficient hyperparameter selection algorithm that sets the number of epochs as absolutely the best. In fact, it restricts the optimization procedure to a small part of the trained parameter space, close to the initial parameters. It can also be shown that in the case of simple linear models with quadratic error functions and simple gradient descent, early stopping is equivalent to L2 regularization [21].

7. Gradient disappearance/explosion problem

Neural networks rely on back propagation to transmit information back and forth. However, as the number of layers decreases, the gradient often becomes smaller and smaller, which leads to an almost constant weight of the lower layers, and we cannot come to a good solution. This is the gradient vanishing problem. On the other hand, the gradient can get bigger and bigger, which leads to a deviation of the algorithm, which is the problem of the gradient explosion. Xavier and Bendjio pointed out that the sigmoid colon activation function, popular at that time, and the initialization method with an average value of 0 and a standard deviation of 1 had serious problems: the variance of the output data of each layer was much greater than the variance of its input data, and the increase in variance was saturated into the upper layer [22].

The method of initialization by the Sigmoid activation function and the mean value equal to 0 and the standard deviation equal to 1 has a serious problem: the variance of the output data of each layer is much larger than the variance of its input data. An increase in variance leads to saturation of the activation function at the upper level. In fact, this is because the average value of the logical function is 0.5 instead of 0 (the average value of the Tanh function is 0,

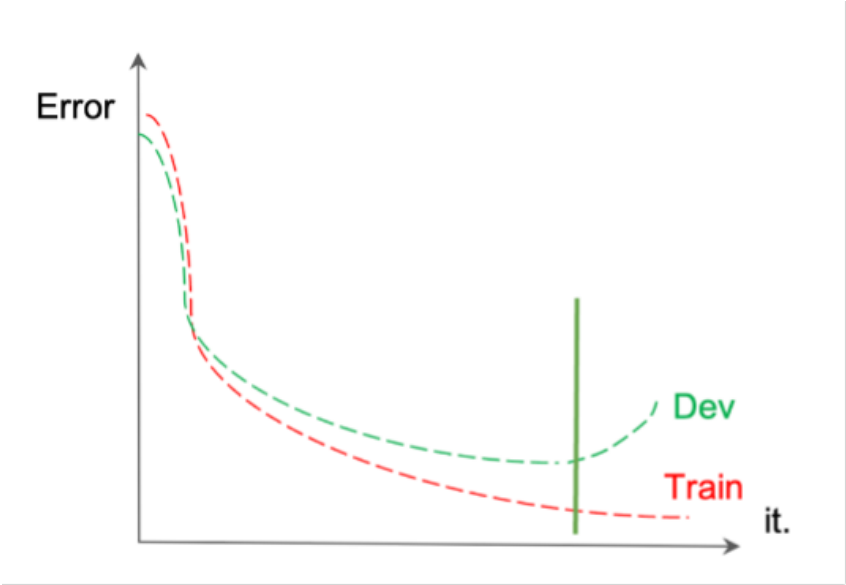


FIGURE 7. Early stop: Stop training if the error on the validation dataset starts to grow.

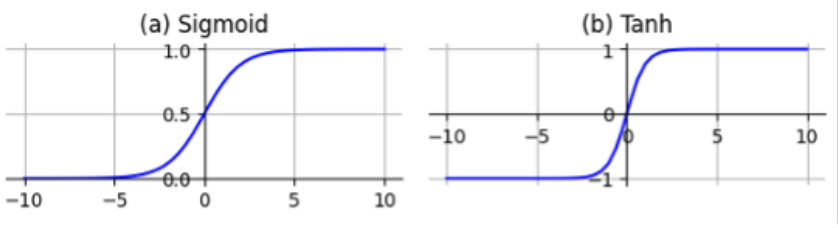


FIGURE 8. Sigmoid activation function (left) and Tanh activation function (right).

which works a little better). As shown in Figure 8, you can see that when the input data of the Sigmoid function becomes larger (negative or positive), the saturation of the function is 0 or 1, and the derivative is very close to 0. Consequently, when back propagation begins, it has almost no gradient for back propagation over the network.

8. Over-fitting and under-fitting

The model studies the relationship between input data (called objects) and output data (called labels) from a training data set. During the training process, the model receives both objects and labels and learns to match the first with the second. For example, suppose we want to build a linear neural network to predict real estate prices. Enter the initial date of data creation, the area of the plot (square feet) and the capacity of the garage, and the result will be the sale price of the house. The trained model is evaluated on a test set. On the test set, we only provide her with characteristics, and she makes predictions. We compare the predicted results with the known labels of the test set to calculate the accuracy. An important part of our data generation is adding random noise to the label. In any real process, whether natural or artificial, the data is not fully suitable for determining the trend. There is always noise or other variables in the relationship that we cannot measure. In the case of housing prices, due to other factors affecting housing prices, prices do not completely match.

- Retraining: excessive dependence on training data.

- Insufficient fit: it is impossible to study the relationship in the training data.
- High variance: the model changes significantly depending on the training data.
- High deviation: assumptions about the model lead to ignoring the training data.
- Excessive and insufficient fitting leads to a deterioration in the ability to generalize in the test set.
- The test kit used to adjust the model can prevent insufficient and excessive fit [23-24].

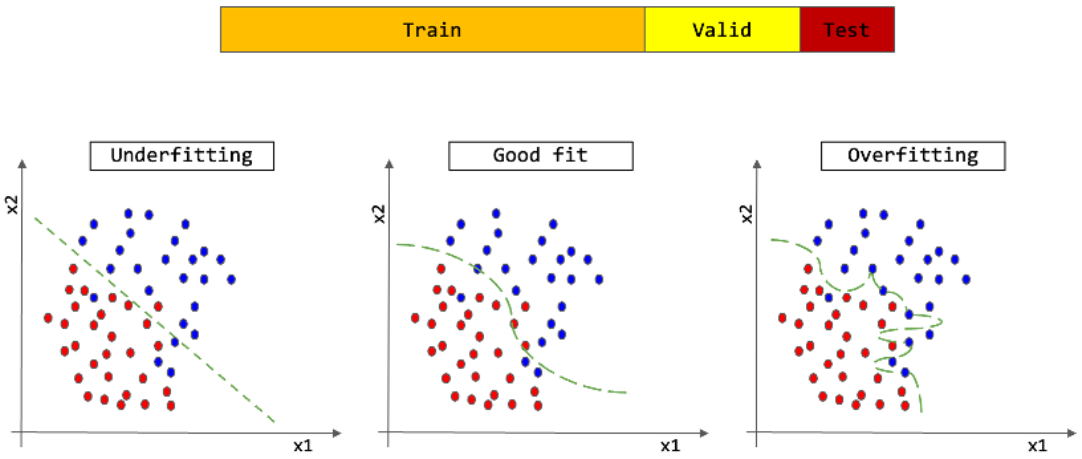


FIGURE 9. Train-Dev-Test in comparison with model fitting.

9. CONCLUSION

Unlike nonlinear neural networks, linear networks cannot solve complex tasks such as pattern recognition or image classification. Linear neural networks are suitable for regression tasks, that is, people want to predict a value based on input data. Linear neural networks can also be used as basic models for more complex architectures, such as convolution neural networks and recurrent neural networks. Regularization is also used in neural networks to solve the problem of excessive network weight and excessive learning. The use of L1-regularization does not have the meaning of "feature selection", as in linear models. Regularization is an integral part of deep learning of neural networks. They either limit the learning parameters or introduce interference somewhere in the learning cycle, depending on the learning data, network architecture, learning parameters or goals. However, regularization does not reduce the number of parameters and does not simplify the network structure. For neural networks, in addition to adding penalties to empirical risks, another method of combating retraining is also actively used — dropout. In this process, the network will be simplified according to the rules - if the error function does not change, the network can be simplified even more. In addition, batch standardization and early stopping can also solve the problem of gradient vanishing/explosion.

REFERENCES

[1] Kabanikhin S. I. *Inverse and Incorrect Problems, Novosibirsk: Siberian Scientific Publishing House.* With And Kabanikhin, 2009.

[2] Gauss C. F. Least squares. 1809.

[3] Legendre A. M. Nouvelles méthodes la determination des orbites des comètes: P. 1-2; *Mit einem Suppl.* Didot, 1805.

[4] Weyl H. Über die asymptotische Verteilung der Eigenwerte. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1911, 1911: 110-117.

[5] Turing A. Turing machine. *Proc London Math Soc*, 1936, 242: 230-265.

[6] Nesterov Y. A method of solving a convex programming problem with convergence rate mathcal O(1/kˆ2)//Sov. Math. Dokl. 27.1983.
[7] Fukumizu K. Special statistical properties of neural network learning//Proceedings of 1997 International Symposium on Nonlinear Theory and Its Applications. 1997, 2: 747-750.
[8] Kalivas J. H. Overview of two-norm (L2) and one-norm (L1) Tikhonov regularization variants for full wavelength or sparse spectral multivariate calibration models or maintenance. Journal of Chemometrics, 2012, 26(6): 218-230.
[9] Goodfellow I., Bengio Y., Courville A. *Deep learning. MIT press*, 2016.
[10] Haji S. H., Abdulazeez A. M. Comparison of optimization techniques based on gradient descent algorithm: A review. *PalArch's Journal of Archaeology of Egypt/Egyptology*, 2021, 18(4): 2715–2743.
[11] Kwon T. M., Zervakis M. E. Design of regularization filters with linear neural networks (image restoration)//[Proceedings] 1992 IEEE International Conference on Systems, Man, and Cybernetics. IEEE, 1992: 416-421.
[12] Rumelhart D. E., Hinton G. E., Williams R J. Learning internal representations by error propagation. California Univ San Diego La Jolla Inst for Cognitive Science, 1985.
[13] Rumelhart D. E., McClelland J. L. Parallel distributed processing, explotation in the microstructure of cognition-Vol. 1: Foundations. Computational Models of Cognition and Perception, 1987.
[14] McClelland J. L., Rumelhart D. E., PDP Research Group. Parallel Distributed Processing, Volume 2: Explorations in the Microstructure of Cognition: Psychological and Biological Models. MIT press, 1987.
[15] Srivastava N., Hinton G., Krizhevsky A., et al. Dropout: a simple way to prevent neural networks from overfitting. The journal of machine learning research, 2014, 15(1): 1929-1958.
[16] Prechelt L. Early stopping-but when? //Neural Networks: Tricks of the trade. *Berlin, Heidelberg: Springer Berlin Heidelberg*, 2002: 55-69.
[17] Ioffe S., Szegedy C. Batch normalization: Accelerating deep network training by reducing internal covariate shift//International conference on machine learning. pmlr, 2015: 448-456.
[18] Oppermann A. Regularization in Deep Learning—L1, L2, and Dropout. 2020.
[19] Luo P., Wang X., Shao W., et al. Towards understanding regularization in batch normalization. arXiv preprint arXiv:1809.00846, 2018.
[20] Ioffe S., Szegedy C. Batch normalization: Accelerating deep network training by reducing internal covariate shift//International conference on machine learning. pmlr, 2015: 448-456.
[21] https://theaisummer.com/regularization/
[22] Glorot X., Bengio Y. Understanding the difficulty of training deep feedforward neural networks//Proceedings of the thirteenth international conference on artificial intelligence and statistics. JMLR Workshop and Conference Proceedings, 2010: 249-256.
[23] Van der Aalst W. M. P., Rubin V., Verbeek H. M. W., et al. Process mining: a two-step approach to balance between underfitting and overfitting. Software and Systems Modeling, 2010, 9: 87-111.
[24] Koehrsen W. Overfitting vs. underfitting: A complete example. Towards Data Science, 2018, 405.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ALGORITHM FOR SOLVING THE INVERSE PROBLEM FOR THE HELMHOLTZ EQUATION ON "DISCRETIZATION-OPTIMIZATION"

SYRYM KASENOV[1], AIGERIM TLEULESOVA[1]

[1]Al-Farabi Kazakh National University, Almaty, Kazakhstan
e-mail: syrym.kasenov@mail.ru

Abstract. This study focuses on the initial boundary value problem associated with the Helmholtz equation. Due to the presence of incorrectness in this problem, it is transformed into an inverse boundary value problem with two unknown boundary conditions. Usually, the numerical solution of this problem is based on the "optimization - discretization" scheme. The original problem is reformulated in the form of an inverse problem presented in operator form. This operator equation is further transformed into the task of minimizing the objective function. By calculating the functional gradient, we develop an algorithm for solving the inverse problem, which is then combined with the discretization of both direct and conjugate problems [1]–[9].

In this paper, we propose to discretize the formulation of the inverse problem and derive the gradient in discrete form, after which we develop an algorithm. This approach to the solution significantly simplifies the process of the inverse problem and contributes to faster convergence of the algorithm.

Keywords: inverse problems, numerical solution, Helmholtz equation, method Landweber, finite difference scheme.

AMS Subject Classification: 65M32, 65N21, 35R30

1. Problem statement

Consider the initial boundary value problem for the Helmholtz equation:

$$u_{xx} + u_{yy} + \left(\frac{\omega^2}{c^2} + a(x,y)\right)u = 0, \qquad (x,y) \in (0,1)\times(0,1), \tag{1}$$

$$u(0,y) = h_1(y), \qquad y \in [0,1], \tag{2}$$

$$u(x,0) = h_2(x), \qquad x \in [0,1], \tag{3}$$

$$u_x(0,y) = f_1(y), \qquad y \in [0,1], \tag{4}$$

$$u_y(x,0) = f_2(x), \qquad x \in [0,1]. \tag{5}$$

The problem (1) — (5) is ill-posed. To solve the problem numerically, we first reduce it to the inverse problem $Aq = f$, with respect to some direct (correct) problem. Next, we reduce the solution of the operator equation $Aq = f$ to the problem of minimizing the objective functional $J(q) = \langle Aq_n - f, Aq_n - f\rangle$.

To minimize the objective functional of $J(q_n)$, we will use the Landweber iteration method.

This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19579325).

2. Reducing the initial problem to the inverse problem.

The solution of the initial problem (1) — (5) is reduced to solving the inverse problem, with respect to some direct (well-posed) problem. As a direct direct, we will consider the following

$$u_{xx} + u_{yy} + \Big(\frac{\omega^2}{c^2} + a(x,y)\Big)u = 0, \qquad (x,y) \in (0,1)\times(0,1), \tag{6}$$

$$u(0,y) = h_1(y), \qquad y \in [0,1], \tag{7}$$

$$u(x,0) = h_2(x), \qquad x \in [0,1], \tag{8}$$

$$u(1,y) = q_1(y), \qquad y \in [0,1], \tag{9}$$

$$u(x,1) = q_2(x), \qquad x \in [0,1]. \tag{10}$$

The inverse problem to the problem (6) – (10) is to define the function $q_1(x), q_2(y)$, according to additional information about the solution of the direct problem.

$$u_x(0,y) = f_1(y), \qquad y \in [0,1], \tag{11}$$

$$u_y(x,0) = f_2(x), \qquad x \in [0,1]. \tag{12}$$

Let's introduce the operator

$$A\colon (q_1,q_2) \mapsto (u_x(0,y), u_y(x,0)). \tag{13}$$

Then we will write the inverse problem in the operator form

$$Aq = f.$$

Let's introduce the objective functional

$$J(q_1,q_2) = \int\limits_0^1 \big[u_x(0,y;q_1,q_2) - f_1(y)\big]^2 dy + \int\limits_0^1 \big[u_y(x,0;q_1,q_2) - f_2(x)\big]^2 dx. \tag{14}$$

The objective functional (14) will be minimized by the Landweber iteration method. Let q^n approximation be known. The following approximation is defined from:

$$q^{n+1} = q^n - \alpha J'(q^n) \tag{15}$$

here $\alpha \in (0, ||A||^{-2})$ – descent parameter [1].

3. Solving the inverse problem for the Helmholtz equation according to the "discretization-optimization" scheme.

Introducing a grid into a given Square Ω in area, we get the steps x and y in the same way.

$$h = \frac{1}{N}, \quad h_x = h_y = h, \quad \omega_h = \big\{(x_i, y_j) = i\cdot h, j\cdot h; i,j = \overline{1, N-1}\big\}$$

We write the problem (6)–(10) in discrete form:

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} + \Big(\frac{\omega^2}{c^2} + a_{i,j}\Big)u_{i,j} = 0 \tag{16}$$

$$u_{0,j} = g_{1,j}, \tag{17}$$

$$u_{i,0} = g_{2,i}, \tag{18}$$

$$u_{N,j} = q_{1,j}, \tag{19}$$

$$u_{i,N} = q_{2,i}. \tag{20}$$

The objective functional is approximated in the form:

$$J(q_1^n, q_2^n) = \sum_{j=0}^{N-1}\left[\frac{u_{1,j} - u_{0,j}}{h} - f_{1,j}\right]^2 \cdot h + \sum_{i=0}^{N-1}\left[\frac{u_{i,1} - u_{i,0}}{h} - f_{2,i}\right]^2 \cdot h. \tag{21}$$

(19)–(20) boundary conditions we give the increment $q_{1,j} + \delta q_{1,j}, q_{2,i} + \delta q_{2,i}$ and enter the following designations

$$
\begin{aligned}
&u_{i,j} \approx u(x_i, y_j; q_{1,j}, q_{2,i}),\\
&\widetilde{u}_{i,j} \approx u(x_i, y_j; q_{1,j} + \delta q_{1,j}, q_{2,i} + \delta q_{2,i}),\\
&\delta u_{i,j} = \widetilde{u}_{i,j} - u_{i,j}.
\end{aligned} \tag{22}
$$

Using the notation (22), we calculate the increment of the objective functional $J(q_1^n, q_2^n)$.

$$
\begin{aligned}
&J(q_{1,j} + \delta q_{1,j}, q_{2,i} + \delta q_{2,i}) - J(q_{1,j}, q_{2,i}) =\\
&\sum_{j=0}^{N-1}\left[\frac{\widetilde{u}_{1,j} - \widetilde{u}_{0,j}}{h} - f_{1,j}\right]^2 \cdot h + \sum_{i=0}^{N-1}\left[\frac{\widetilde{u}_{i,1} - \widetilde{u}_{i,0}}{h} - f_{2,i}\right]^2 \cdot h -\\
&\sum_{j=0}^{N-1}\left[\frac{u_{1,j} - u_{0,j}}{h} - f_{1,j}\right]^2 \cdot h + \sum_{i=0}^{N-1}\left[\frac{u_{i,1} - u_{i,0}}{h} - f_{2,i}\right]^2 \cdot h =\\
&\sum_{j=0}^{N-1}\frac{\left(\delta u_{1,j} - \delta u_{0,j}\right)}{h} \cdot 2\left[\frac{u_{1,j} - u_{0,j}}{h} - f_{1,j}\right] \cdot h +\\
&\sum_{i=0}^{N-1}\frac{\left(\delta u_{i,1} - \delta u_{i,0}\right)}{h} \cdot 2\left[\frac{u_{i,1} - u_{i,0}}{h} - f_{2,i}\right] \cdot h + o(\|\delta u\|) =\\
&= \langle \delta q_{1,j}, J' q_{1,j}\rangle + \langle \delta q_{2,i}, J' \delta q_{2,i}\rangle.
\end{aligned} \tag{23}
$$

Consider the formulation of the perturbed problem to the problem (16) – (20).

$$\frac{\widetilde{u}_{i+1,j} - 2\widetilde{u}_{i,j} + \widetilde{u}_{i-1,j}}{h^2} + \frac{\widetilde{u}_{i,j+1} - 2\widetilde{u}_{i,j} + \widetilde{u}_{i,j-1}}{h^2} + \Big(\frac{\omega^2}{c^2} + a_{i,j}\Big)\widetilde{u}_{i,j} = 0, \tag{24}$$

$$\widetilde{u}_{0,j} = g_{1,j}, \tag{25}$$

$$\widetilde{u}_{i,0} = g_{2,i}, \tag{26}$$

$$\widetilde{u}_{N,j} = q_{1,j} + \delta q_{1,j}, \tag{27}$$

$$\widetilde{u}_{i,N} = q_{2,i} + \delta q_{2,i}. \tag{28}$$

To get the problem for $\delta u_{i,j}$, from the problem (24)—(28) we subtract the problem (16) – (20) and, given (22), we get the following relations:

$$\frac{\delta u_{i+1,j} - 2\delta u_{i,j} + \delta u_{i-1,j}}{h^2} + \frac{\delta u_{i,j+1} - 2\delta u_{i,j} + \delta u_{i,j-1}}{h^2} + \Big(\frac{\omega^2}{c^2} + a_{i,j}\Big)\delta u_{i,j} = 0, \tag{29}$$

$$\delta u_{0,j} = 0, \tag{30}$$

$$\delta u_{i,0} = 0, \tag{31}$$

$$\delta u_{N,j} = \delta q_{1,j}, \tag{32}$$

$$\delta u_{i,N} = \delta q_{2,i}. \tag{33}$$

To calculate the gradient of the objective function set for solving the inverse problem, we use the following partial addition formulas considered in the literature source [10].

$$(\Delta^i v, w^i) = v^N w^{N-1} - v^1 w^0 - (v^i, \Delta^{i-1} w), \tag{34}$$

$$(\Delta^{i-1} v, w^i) = v^{N-1} w^N - v^0 w^1 - (v^i, \Delta^i w). \tag{35}$$

here $\Delta^i v = v^{i+1} - v^i, \qquad (v, w) = \sum_{i=1}^{N-1} v^i w^i h$

Multiplying (29) by an arbitrary function $\psi_{i,j}$, we sum i from 1 to $N-1$ and j from 1 to $N-1$.

$$0 = h^2 \cdot \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\Big[\frac{1}{h^2}(\delta u_{i+1,j} - 2\delta u_{i,j} + \delta u_{i-1,j}) + \frac{1}{h^2}(\delta u_{i,j+1} - 2\delta u_{i,j} + \delta u_{i,j-1}) +$$
$$+ \Big(\frac{\omega^2}{c^2} + a_{i,j}\Big)\delta u_{i,j}\Big]\psi^i_j =$$
$$= \sum_{j=1}^{N-1}\Big[(\Delta^i \delta u_j, \psi^i_j) - (\Delta^{i-1}\delta u_j, \psi^i_j)\Big] + \sum_{i=1}^{N-1}\Big[(\Delta^j \delta u^i, \psi^i_j) - (\Delta^{j-1}\delta u^i, \psi^i_j)\Big] +$$
$$+ h^2 \cdot \sum_{j=1}^{N-1}\sum_{i=1}^{N-1}\Big(\frac{\omega^2}{c^2} + a_{i,j}\Big)\delta u_{i,j}\psi_{i,j} =$$

Using the formulas (34)—(35), we get:

$$= \sum_{j=1}^{N-1}\Big[\delta u_{N,j}\psi_{N-1,j} - \delta u_{1,j}\psi_{0,j} - (\delta u_{i,j}, \Delta^{i-1}\psi_j) - \delta u_{N-1,j}\psi_{N,j} + \delta u_{0,j}\psi_{1,j} + (\delta u_{i,j}, \Delta^i\psi_j)\Big] +$$
$$+ \sum_{i=1}^{N-1}\Big[\delta u_{i,N}\psi_{i,N-1} - \delta u_{i,1}\psi_{i,0} - (\delta u_{i,j}, \Delta^{j-1}\psi_i) - \delta u_{i,N-1}\psi_{i,N} + \delta u_{i,0}\psi_{i,1} + (\delta u_{i,j}, \Delta^j\psi_i)\Big] +$$
$$+ h^2 \cdot \sum_{j=1}^{N-1}\sum_{i=1}^{N-1}\Big(\frac{\omega^2}{c^2} + a_{i,j}\Big)\delta u_{i,j}\psi_{i,j} =$$
$$= \sum_{j=1}^{N-1}\big(\delta u_{i,j}, \Delta^i\psi_j - \Delta^{i-1}\psi_j)\big) + \sum_{i=1}^{N-1}\big(\delta u_{i,j}, \Delta^j\psi_i - \Delta^{j-1}\psi_i)\big) + h^2 \cdot \sum_{j=1}^{N-1}\sum_{i=1}^{N-1}\Big(\frac{\omega^2}{c^2} + a_{i,j}\Big)\delta u_{i,j}\psi_{i,j}$$
$$= \sum_{j=1}^{N-1}\Big[\delta u_{N,j}\psi_{N-1,j} - \delta u_{1,j}\psi_{0,j} - \delta u_{N-1,j}\psi_{N,j} + \delta u_{0,j}\psi_{1,j}\Big] +$$
$$+ \sum_{i=1}^{N-1}\Big[\delta u_{i,N}\psi_{i,N-1} - \delta u_{i,1}\psi_{i,0} - \delta u_{i,N-1}\psi_{i,N} + \delta u_{i,0}\psi_{i,1}\Big].$$

Given the conditions (37)—(40), the sum is of the following type:

$$0 = h^2 \cdot \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\Big[\frac{1}{h^2}(\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}) + \frac{1}{h^2}(\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}) + \Big(\frac{\omega^2}{c^2} + a_{i,j}\Big)\psi_{i,j}\Big]\delta u_{i,j} +$$
$$+ h \cdot \sum_{j=1}^{N-1}\left[\frac{\delta q_{1,j}\psi_{N-1,j} - \delta u_{N_x-1,j}\psi_{N,j} - \psi_{0,j}(\delta u_{1,j} - \delta u_{0,j})}{h}\right] +$$
$$+ h \cdot \sum_{i=1}^{N-1}\left[\frac{\delta q_{2,i}\psi_{i,N-1} - \delta u_{i,N-1}\psi_{i,N} - \psi_{i,0}(\delta u_{i,1} - \delta u_{i,0})}{h}\right].$$

Taking into account the formula (23), we calculate the summary report and the gradient formula as follows:

$$\frac{\psi_{i+1,j}-2\psi_{i,j}+\psi_{i-1,j}}{h^2}+\frac{\psi_{i,j+1}-2\psi_{i,j}+\psi_{i,j-1}}{h^2}+\Big(\frac{\omega^2}{c^2}+a_{i,j}\Big)\delta\psi_{i,j}=0, \tag{36}$$

$$\psi_{0,j}=2\left(\frac{u_{1,j}-u_{0,j}}{h}-f_{1,j}\right), \tag{37}$$

$$\psi_{i,0}=2\left(\frac{u_{i,1}-u_{i,0}}{h}-f_{2,i}\right), \tag{38}$$

$$\psi_{N,j}=0, \tag{39}$$

$$\psi_{i,N}=0. \tag{40}$$

In addition, the gradient formula is in the form:

$$J'q=\begin{pmatrix}J'q_{1,j}\\ J'q_{2,i}\end{pmatrix}=\begin{pmatrix}\frac{\psi_{N-1,j}}{h}\\ \frac{\psi_{i,N-1}}{h}\end{pmatrix} \tag{41}$$

References

[1] Shishlenin M.A., Kasenov S.E., Askerbekova Z.A. Numerical Algorithm for Solving the Inverse Problem for the Helmholtz Equation, *Computational and Information Technologies in Science, Engineering and Education*, **1**:2 (2019), 197-207.

[2] Alimbekova N.B., Berdyshev A.S., Baigereyev D.R. Parallel Implementation of the Algorithm for Solving a Partial Differential Equation with a Fractional Derivative in the Sense of Riemann-Liouville. *2021 IEEE International Conference on Smart Information Systems and Technologies (SIST)*, 2021, P. 1-6. DOI: 10.1109/SIST50301.2021.9465922

[3] Temirbekov N., Malgazhdarov Ye., Tokanova S., Amenova F., Baigereyev D., Turarov A. Information technology for numerical simulation of convective flows of a viscous incompressible fluid in curvilinear multiply connected domains. *Journal of Theoretical and Applied Technology*, 2019. Vol. 97, No. 22. P. 3166-3177

[4] Zhumagulov B.T., Temirbekov N.M., Baigereyev D.R. Efficient difference schemes for the three-phase non-isothermal flow problem. *AIP Conference Proceedings*, 2017. Vol. 1880. P. 060001-1 - 060001-10; DOI: 10.1063/1.5000655.

[5] Wojcik W., Temirbekov N.M., Baigereyev D.R. Fractional flow formulation for three-phase non-isothermal flow in porous media. *Przeglad Elektrotechniczny*, 2016. Vol. 92, № 7. – P. 24-31. - DOI: 10.15199/48.2016.07.04.

[6] Anvar Azimov, Syrym Kasenov, Daniyar Nurseitov, Simon Serovajsky Inverse problemfor the Verhulst equation of limited population growth with discrete experiment data. *AIP Conference Proceedings*. – 2016. – V. 1759, 020037

[7] Syrym Kasenov, Altyn Nurseitova and Daniyar Nurseitov A conditional stability estimate of continuation problem for the Helmholtz equation. *AIP Conference Proceedings*. – 2016. – V. 1759, 020119

[8] Kasenov S., UrmashevB., Temirbekov A., Amantayeva A. Numerical Solution of the Inverse Pharmacokinetic Problem for the Three-Compartment Model.*Journal of Engineering Science and Technology Review*. 2020.–p.122-126.

[9] Kabanikhin, S., Nurseitova, A., Kasenov, S. Stability estimation of the generalized solution to the direct problem for the acoustic equation. *Journal of Physics: Conference Series*, 2021, 2092(1), 012005

[10] Samarskiy A.A. Theory of difference schemes. Moscow, *Nauka* (1989), p.616.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

REGULARISATION OF ILL-POSED ALGEBRAIC SYSTEMS

YANG YANG[1]

[1]Novosibirsk, Russian
e-mail: y.yan1@g.nsu.ru

Abstract. The solution of ill-conditioned system of linear equations has been an active and important area of research, from the point of view of numerical algebra, it has its own value for study. However, from the point of view of inverse problems, it is a necessary display form of an ill-conditioned operator equation after discretization. Thus, many existing methods or techniques for solving ill-conditioned system of linear equations can be used for the numerical solution of inverse problems, for example, the technique of preconditioning can be looked at as a regularization strategy. On the other hand, we can certainly apply the idea of processing inverse problems to study the solution of pathological linear equations, and the combination of the two gives rise to a promising class of effective algorithms in image processing: regularized preconditioned conjugate gradient (regularized PCG) [1]. The methods for finding stable approximate solutions to inverse problems in mathematical physics are often referred to as regularization methods or strategies. These methods were developed separately and independently by Tikhonov and Phillips in the early 1960s.[2-3]

Keywords: Regularization, Ill-conditioned system.

1. Introduction

In the following, we discuss the solution of ill-conditioned system of linear equations from the point of view of dealing with inverse problems.

We discuss:

$$Az = U, z \in Z = R^n, u \in U = R^n \tag{1}$$

Because $A \in R^{n \times n}$, it is a bounded linear operator in a Euclid space and thus it is a compact operator, so the system of equations (1) which corresponds to is the first kind of operator equation. Following the theoretical framework, the system of equations (1) may appear as follows.

When $\det(A) = 0$, its operator does not exist because A is singular, but for any $u \in U$ it has an infinite number of solutions, this is an ill-posed problem.

When $\det(A) \neq 0, A$ has an operator and thus the solution to the system of equations (1) exists and is unique. But depending on the size of its condition number $\text{cond}(A) = \|A\| \left\|A^{-1}\right\|$, there are two further cases:

If it is well-conditioned, i.e., the number of conditions is small, the solution of equation (1) is stable with respect to the perturbation of the right terminal term; thus, the equation (1) is well-posed.

If it is ill-posed, i.e., when the number of conditions is very large, a small change in the right-hand term u will cause a large deviation of the approximate solution from the true solution, and thus equation (1) is ill-posed.

As you know, the best square approximation with polynomials leads to the following Hilbert matrix.

$$\begin{cases} A = H_n = (h_{ij})_{n\times n} \\ h_{ij} = 1/(i+j-1) \end{cases} \quad (i,j = 1,2,\ldots,n)$$

When n is very large, $\mathrm{cond}\,(H_n) \approx e^{3.5n}$ thus being ill-conditioned.

It should be noted that there are several ways to deal with ill-conditioned system of linear equations.

(1) Increasing the word length of the computer (i.e., machine precision). For example, if the PC is used with single precision and the GAUSS principal elimination method or LU decomposition, a stable approximate solution to the system of equations (1) up to H_5 as the coefficient matrix can be found, while the above algorithm fails when it is above order 5; if double precision computing is used, the system of equations of order 20 to 30 can still be solved using the above algorithm to obtain 3 digits above the valid number.[4]

(2) Preconditioning, i.e., choosing a non-singular matrix P such that $\mathrm{cond}(PA) \ll \mathrm{cond}(A)$, can then solve the system of equations

$$PAz = Pu \tag{2}$$

As the solution of Eq. (2) is equivalent to that of Eq. (1), the difficulties posed by ill-conditioning can be avoided, although the implementation of this strategy requires the choice of matrix P, which translates into another difficult problem, although some quite successful methods for constructing P have now been investigated, such as the use of the incomplete LU decomposition and the incomplete Cholesky decomposition.

(3) Next, we will discuss the regularization methods:

Let the operator A be a singular square matrix. First, we give the definition of the solution in the singular case of A in (1), such that:

$$M = \left\{ m \in R^n : \|Am - u\| = \inf_{n\in R^n} \|An - u\| \right\}$$

Then there are infinitely many elements in it.

Definition 1: The vector $z_T \in M$ is said to be a regular fitting solution to the system of equations (1), which satisfies.

$$z_T = \min_{m\in M} \|m\| \tag{3}$$

It can be shown that such a vector z_T exists and is unique and that the regular quasi-solution is equivalent to the solution in the ordinary sense when A^{-1} exists.

In fact, using A^T to multiply both sides of equation (1) have.

$$A^T Az = A^T u \tag{4}$$

and $A^T A$ must be a non-negative definite matrix, so that the only orthogonal matrix P can be found such that.

$$PA^T A = PDP, D = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)\,, \lambda_i \geq 0$$

Let $m = P^T z$ be substituted into the system of equations (4) to obtain:

$$Dm = P^T A^T y \equiv F$$

Since the solution to the above system of equations is:

$$z_i = \begin{cases} F_i/\lambda_i\,(\lambda_i \neq 0) \\ 0\,(\lambda_i = 0) \end{cases} \tag{5}$$

Therefore, we have

$$z_T = Pm$$

But in general, we can only obtain an approximation of A and $y, \tilde{A}$ and $\tilde{y}$.We assume that this perturbation is small:

$$\|A - \tilde{A}\| \leq \delta, \|u - \tilde{u}\| \leq \delta(\delta > 0)$$

At this point, for the system of equations:

$$\tilde{A}\tilde{z} = \tilde{u} \tag{6}$$

Repeat the derivation above and use the perturbation theorem for eigenvalues.

Theorem 1 [5]: The eigenvalues of $\tilde{A}^T\tilde{A}$ are $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n$.For A^TA the size of the eigenvalue perturbation is the same as $\|A - \tilde{A}\|$.

Then those eigenvalues of $\lambda_i = 0$ will be perturbed to non-zero, very small values, thus using.

$$\tilde{m}_i = \tilde{F}_i / \tilde{\lambda}_i$$

When equation (1) is ill-posed, the numerical stability of the approximate solution cannot be guaranteed using any of the ordinary algorithms, but it can be solved with the help of regularization methods.

We consider only the case where the right terminal term u is perturbed and set $\|u - \tilde{u}\| \leq \delta$. To do this, first construct the stable function.

Definition 2: An operator $R(u, \delta)$ is said to be a regular operator of δ the equation $Az = u$ in a neighborhood of $u = u_T$, if it reflects U to F with the following properties.

(1) There exists $\delta_1 > 0$ such that $R(u, \delta)$ is defined for all $\delta : 0 \leq \delta \leq \delta_1$ satisfying the condition $\rho_U(u_\delta, u_T) \leq \delta$ when $u_\delta \in U$;

(2) $\forall \epsilon > 0, \exists 0 \leq \delta_0 := \delta_0(\epsilon, u_T) \leq \delta_1$, such that $\rho_U(u_\delta, u_T) \leq \delta \leq \delta_0$ implies $\rho_U(z_\delta, z_T) \leq \epsilon$, where $z_\delta = R(u_\delta, \delta)$.

Theorem 2: Let A be an operator from F to U and $\tilde{R}(Az, \alpha) : U \to F$ be an operator on u continuous defined for all elements u in U and any $\alpha > 0$. If we have,

$$\lim_{\alpha \to 0} \tilde{R}(Az, \alpha) = z, \forall z \in F \tag{7}$$

then the operator $\tilde{R}(u, \alpha)$ is the regular operator of equation (7).

Proof: Let $z_T \in F, u_T \in U$ be two fixed elements and $Az_T = u_T, \delta$, a fixed positive number. Thus, for everything satisfying the condition:

$$\rho_U(u_\delta, u_T) \leq \delta \tag{8}$$

of $u_\delta \in U$,we have

$$\rho_F\left(\tilde{R}(u_\delta, \alpha), z_T\right) \leq \rho_F\left(\tilde{R}(u_\delta, \alpha), \tilde{R}(u_T, \alpha)\right) + \rho_F\left(\tilde{R}(u_T, \alpha), z_T\right. \tag{9}$$

Since the operator $\tilde{R}(u, \alpha)$ is continuous at $u = u_T$, so for sufficiently small $\delta > 0\,(\delta \leq \delta_1)$ follows from inequality (8),

$$\rho_F\left(\tilde{R}(u_\delta, \alpha), \tilde{R}(u_T, \alpha)\right) \leq \omega(\delta) \tag{10}$$

where, when $\forall \delta \leq \delta_1, \forall \delta \leq \delta_1$, is also due to

$$\lim_{\alpha \to 0} \tilde{R}(Az_T, \alpha) = \lim_{\alpha \to 0} \tilde{R}(u_T, \alpha) = z_T \tag{11}$$

Therefore $\forall \delta > 0$, all exist such that $\alpha = \alpha(\delta, z_T)$ such that when $\alpha \leq \alpha_1$ there is.

$$\rho_F\left(\tilde{R}(u_\delta, \alpha), z_T\right) \leq \omega(\delta) \tag{12}$$

Combining equations (9), (10), (12), it follows that both $\forall \delta \leq \delta_1$ and $\forall \alpha \leq \alpha_1$ hold the inequality.

$$\rho_F\left(\tilde{R}(u_\delta, \alpha), z_T\right) \leq 2\omega(\delta) \tag{13}$$

such that for $\delta \to 0$ with $\omega(\delta) \to 0$, inequalities can be derived from equations (8), (13).

$$\rho_F\left(\tilde{R}(u_\delta, \alpha), z_T\right) \leq \epsilon$$

The theorem yields the proof.

Definition 3: Let F be a metric space and M be a subset of F.Also let the generic function $J : M \to R^1$ exist as a lower exact bound on M. The sequence $\{z_n\} \subset M, M$ is said to be a sequence of minimizations of J. If

$$\lim_{n\to\infty} J(z_n) = d, d = \inf_{z\in M} J(z)$$

In general, the sequence $\{z_n\}$ may not have a limit; even if there exists a limit point z^*,there may not be $z^* \in M$ and

$$\lim_{n\to\infty} J(z_n) = J\left(\lim_{n\to\infty} z_n\right), = J(z^*)$$

Theorem 3[6]: Let A be a continuous operator from the metric space F to the metric space U. Then $\forall\alpha > 0$ and $\forall u \in U, \exists z_\alpha \in F_1$ such that the generic function $M^\alpha[z, u] = \rho_U^2(Az, u) + \alpha\Omega[z], u \in U, z \in F_1 \subset F$ reaches its lower exact bound at z_α,

$$M^\alpha[z_\alpha, u] = \inf_{z\in F_1} M^\alpha[z, u]$$

Theorem 4[6]: For any $u(t) \in L^2[a, b]$ and any $\alpha > 0$, there exists a unique $z_\alpha(s) \in W_2^1[a, b]$ such that the spreading generalized function $M^\alpha[z, u]$ reaches its lower exact bound on $W_2^1[a, b]$,

$$M^\alpha[z_\alpha, u] = \inf_{z\in W_2^1} M^\alpha[z, u]$$

For any given $\alpha > 0$, there is $z_a \in Z_1$ (which is a certain subset of Z) such that.

$$M^\alpha(z_\alpha, u) = \inf_{z\in Z_2} M^\alpha(z, u)$$

Clearly, the solution to the above problem is equivalent to solving the Euler equation.

$$\left(A^T A + \alpha I\right) z_\alpha = A^T u$$

It is the regularized form of equation (4), so that $z_{\alpha^*} = z_{\alpha^*(\widetilde{u},\delta)}$ is a stable approximate solution to the ill-posed equation $Az = \tilde{u}$.

References

[1] Tikhonov A N., On Stability of Inverse Problems, Dokl Acad. Nauk USSR 39(5), 1943, 195-198 p.

[2] Phillips D L., A Technique for the numerical solution of certain integral equations of the first kind, J. Assoc. Comput. Mach., 1962, 9:84 97 p.

[3] Tikhonov A N., On Solving Incorrectly Posed Problems and Method of Regularization, Dokl. Acad. Nauk USSR, 1963, 151 p.

[4] Tikhonov A N., Arsenin V Y. Solutions of Ill-Posed Problems, John Wiley and Sons, New York, 1977.

[5] Kabanikhin S. I., Inverse and ill-posed problems. Textbook, 2009, 97 p.

[6] Wang.,Yu.,Xiao., Numerical solution of inverse problems, Science Publishers, 2003, 179-183 p.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ОБРАТНЫЕ ЗАДАЧИ УРАВНЕНИЯ ПЕРЕНОСА ИЗЛУЧЕНИИ В МНОГОЗОННОЙ ОБЛАСТИ

АМАНГЕЛЬДЫ САРИЕВ[1], АЙНАГУЛЬ ШЫГАНАКОВА[2]

[1,2]Атырауский университет им.Х.Досмухамедова, Атырау, Казахстан
e-mail: ainagul-81-81@mail.ru

В начале по важности вопросов следует отметить, что уравнения переноса описывают ряд практически важных явлений, таких как процессы, происходящие в ядерных реакторах, рассеяние света в атмосфере, прохождение рентгеновского и γ –излучений через вещество. Исследуемые процессы отличаются друг от друга лишь законами столкновения. Первые результаты и подробный вывод уравнений переноса нейтронов можно найти в трудах Аниконов Ю. Е.[1], Владимиров В.С. [4], Г.И.Марчука, В.И.Лебедева [7], К.Кэйз [8]. Разработки численных методов и качественным исследованиям этого уравнения изложено в трудах Аниконов Ю. Е.[1], Г.И.Марчука [7] и Б.Дэвисона [9]. Качественные исследования свойств решений задач переноса получили в работах Т.А.Гермогеновой [2],[3], У.М.Султангазина [5], [6], В.И.Агошкова [10]. Дальнейшие исследования локальных свойств классических решений стационарной задачи переноса внесла Аниконов Ю. Е.[1], Т.А.Гермогенова [2], [3]. Гельдеровость решения нестационарного уравнения переноса в однозонной, выпуклой области по временной и пространственной переменным были доказаны У.М.Султангазиным [5], [6], Г.М.Сыдыков[11], [12], А.Д.Сариевым [11]-[13].

В настоящей работе изучаются локальные свойства классических решений односкоростного нестационарного уравнения переноса, рассматриваемого в многозонной области. Уравнение переноса рассмотрено при следующих предложениях [1]-[6]:

1) все частицы имеют одинаковые по модулю скорости,

2) поток частицы из вакуума на внешнюю границу отсутствует,

3) индикатриса рассеяния $\theta\left(\vec{r},\vec{\omega},\vec{\omega}'\right)$ представлена в виде $\theta\left(\vec{r},\vec{\omega},\vec{\omega}'\right) = (4\pi)^{-1}\delta_S(\vec{r})g(\mu_0)$ где μ_0 -косинус угла между направлениями $\vec{\omega}$ и $\vec{\omega}'$, т.е. $\mu_0 = \left(\vec{\omega},\vec{\omega}'\right)$.

При этих предложениях уравнение переноса имеет вид

$$\frac{\partial u}{\partial t} + Lu = Su + f. \tag{1}$$

Здесь $u = u\left(t,\vec{r},\vec{\omega}\right)$ - функция распределения частиц, $f = f\left(t,\vec{r},\vec{\omega}\right)$ - функция источника, $\delta = \delta\left(\vec{r}\right), \delta_S = \delta_S\left(\vec{r}\right) - (4\pi)^{-1}g(\mu_0)$- индикатриса рассеяния, $\vec{r} = (x,y,z)$ - пространственные координаты, $\vec{\omega} = (\xi,\eta,\zeta)$ - точки единичной сферы Ω со сферическими координатами $\xi = sin\theta cos\varphi$, $\eta = sin\theta sin\varphi$, $\zeta = cos\theta$, $Lu = \left(\vec{\omega}, grad\ u\right) + \delta\left(\vec{r}\right)u$,

$Su = \frac{\delta_S\left(\vec{r}\right)}{4\pi}\int\limits_{\Omega} g(\mu_0)u\left(t,\vec{r},\vec{\omega}'\right)d\vec{\omega}'$.

Будем говорить, что поверхность ∂G области G принадлежит классу $C^p(p \geq 1)$, если в некоторой окрестности каждой точки $\vec{r}_0 \in \partial G$ она представима уравнением $\varphi_{\vec{r}_0}(\vec{r}) = 0$, причем $grad\varphi_{\vec{r}_0}(\vec{r}) \neq 0$, и функция $\varphi_{\vec{r}_0}(\vec{r})$ непрерывна вместе со своими производными до порядка ρ включительно в упомянутой окрестности. Поверхность ∂G называется кусочно-гладкой, если она состоит из конечного числа поверхностей класса C^1 [2], [3]. Будем считать,

что область G в которой происходит процесс переноса, состоит из конечного числа подобластей (зон) G_j ограниченных кусочно-гладкой поверхностью ∂G_j т.е. $G = \bigcup\limits_{j=1}^{J} G_j$, а $\bar{G}$ - выпуклым. Через $\partial\bar{G}$ обозначим внешнюю поверхность области G. Граничная поверхность ∂G области содержит кроме $\partial\bar{G}$ еще поверхности γ_{ij} раздела зон (части поверхности ∂G_j) ∂G_j.

Кроме того полагается, что множество $[0,T]\times\bar{G}$ - удовлетворяет условию «обобщенной выпуклости» см. Гермогеновой [2], [3]

$$\Pi\big(t,\vec{r},\vec{\omega}\big) = \big\{\big(\tau,\vec{r}-\vec{\omega}(t-r)\big),\ 0\le\tau\le t\big\},$$

являющаяся характеристикой дифференциального выражения $\frac{\partial}{\partial t}+L$ и проходящая через любую точку $(t,\vec{r})\in[0,T]\times\bar{G}$ при любом $\vec{\omega}\in\Omega$ имеет конечное число $M\big(t,\vec{r},\vec{\omega}\big)$ точек $\big(t^*_m,\vec{r}^*_m\big)$ $\big(t^*_m = t^*\big(t,\vec{r},\vec{\omega}\big),\ \vec{r}^*_m = \vec{r}-\vec{\omega}(t-t^*_m),\ 0\le t^*_1\le\ldots\le t^*_m<t\big)$ пресечений с граничной поверхностью $(0,T]\times\partial G\cup\{0\}\times G$ и $supM\big(t,\vec{r},\vec{\omega}\big)<\infty$. Здесь $t^*_1\big(t,\vec{r},\vec{\omega}\big)$ есть время пересечения характеристикой $\Pi\big(t,\vec{r},\vec{\omega}\big)$ границы множества $[0,T]\times\bar{G}$.

Для однозначной разрешимости к уравнению (1) необходимо присоединить начальное распределение частиц

$$u\big(0,\vec{r},\vec{\omega}\big) = \Phi\big(\vec{r},\vec{\omega}\big) \tag{2}$$

и режимы на внешней границе и на границе раздела зон

$$u\big(t,\vec{r},\vec{\omega}\big) = 0,\ \vec{r}'\in\partial\bar{G},\ \big(\vec{n}_{\vec{r}'},\vec{\omega}\big)<0 \tag{3}$$

$$\lim_{\tau\to t^*_m+0} u\big(\tau,\vec{r},\vec{\omega}(t-\tau),\vec{\omega}\big) = \lim_{\tau\to t^*_m-0} u\big(\tau,\vec{r},\vec{\omega}(t-\tau),\vec{\omega}\big),\quad m=\overline{2,M}. \tag{4}$$

Пусть $\bar{\Pi}_j = [0,T]\times\bar{G}_j$, $\tilde{\Pi}_j = \bar{\Pi}_j\setminus\Big\{\{0\}\times\partial G_j\Big\},\ j=\overline{1,J}$.

$C\big(\tilde{\Pi}\times\Omega\big)$ - класс функции $\big(t,\vec{r},\vec{\omega}\big)$, непрерывных в каждом множестве $\tilde{\Pi}_j\times\Omega,\ j=\overline{1,J}$ и таких, что $\max\limits_j \sup\limits_{\tilde{\Pi}_j\times\Omega}\big|\big(t,\vec{r},\vec{\omega}\big)\big| = \bar{f}<\infty$.

Заметим, что, если $f\in C\big(\tilde{\Pi}_j\times\Omega\big)$, то при стремлении $\big(t,\vec{r}\big)$ к $\{0\}\times\partial G$ вдоль различных прямых, пределы $\lim f\big(t,\vec{r},\vec{\omega}\big)$ существуют и вообще говоря различны.

Определение 1. Классическим решением задачи ((1)-(4) в области $G=\bigcup\limits_{j=1}^{J}G_j$ назовем функцию $u\big(\tau,\vec{r}-\vec{\omega}(t-\tau),\vec{\omega}\big)$, которая для всех $t\in[0,T],\ \vec{r}\in G,\ \vec{\omega}\in\Omega$

1) непрерывна по τ на отрезках $\big[t^*_K,t^*_{K+1}\big],\ K=\overline{1,M-1},\ \big[t^*_M,t\big]$ и непрерывно дифференцируема по τ в интервалах $\big[t^*_K,t^*_{K+1}\big],\ K=\overline{1,M-1},\ \big(t^*_M,t\big)$;

2) допускает существование интеграла столкновений

$$N\big(t,\vec{r},\vec{\omega}\big) = Su = \frac{\delta_S\big(\vec{r}\big)}{4\pi}\int\limits_{\Omega} g(\mu_0)u\big(t,\vec{r},\vec{\omega}'\big)d\vec{\omega}',$$

принадлежащего $C\big(\tilde{\Pi}\times\Omega\big)$;

3) удовлетворяет уравнению

$$\begin{aligned}\frac{d}{dt}u\big(\tau,\vec{r}-\vec{\omega}(t-\tau),\vec{\omega}\big)+\delta\big(\tau,\vec{r}-\vec{\omega}(t-\tau)\big)u\big(\tau,\vec{r}-\vec{\omega}(t-\tau),\vec{\omega}\big) = \\ = N\big(\tau,\vec{r}-\vec{\omega}(t-\tau),\vec{\omega}\big)+f\big(\tau,\vec{r}-\vec{\omega}(t-\tau),\vec{\omega}\big),\end{aligned} \tag{5}$$

начальному условию (2) и граничным условиям (3) - (4).

Всюду в работе будем полагать, что данные задачи δ,δ_S,Φ,f обладают определенной гладкостью в каждом из зон G_j, а также считать, что выполнены естественные ограничения:

1) $0 \le \delta_S(\vec{r}) \le \delta(\vec{r}) \le \bar{\delta} < \infty$,

2) $g(\mu_0) \ge 0,\ \frac{1}{4\pi}\int\limits_{\Omega} g(\mu_0)d\vec{\omega}' = 1.$

Кроме того, без ограничения общности можно считать, что

3) $\bar{\delta}_S \setminus \underline{\delta} = q < 1$, где $\underline{\delta} = \min\limits_j \inf\limits_{\bar{G}_j} f\left|\delta(\vec{r})\right| > 0$, $\underline{\delta}_S = \max\limits_j \sup\limits_{\bar{G}_j} f\left|\delta_S(\vec{r})\right| > 0.$

Действительно, если соотношение 3) не выполнено, то введя новую неизвестную функцию

$$U = e^{-\lambda t}U,$$

где $\lambda = \underline{\delta}_{S+1}$, приходим к новой задаче (1) - (4) относительно U, для которой соотношения 1)-3) выполнены.

Будем говорить, что задача (1) - (4) удовлетворяет условиям согласования А, если

$$\begin{cases} \Phi(\bar{r}^*, \vec{\omega}) = 0,\ \bar{r}' \in \partial\bar{G},\ (\bar{n}_{\bar{r}'}, \vec{\omega}) < 0, \\ \left[\Phi(\bar{r}, \vec{\omega})\right]_{\partial G} = 0, \end{cases} \tag{6}$$

$$\Phi(h-0, \mu) = \Phi(h+0, \mu), \tag{7}$$

последнее означает, что функция $\Phi(\bar{r}^*, \vec{\omega})$ непрерывна по $\bar{r}$ при переходе через поверхность раздела зон вдоль всех направлений $\vec{\omega} \in \Omega$.

В работе изучены дифференциальные свойства интеграла столкновений, доказана теорема существования и единственности классического решения задачи.

$$N = KN + Kf + B\Phi, \tag{8}$$

где

$$KN(t, \vec{r}, \vec{\omega}) = \frac{\delta_S(\vec{r})}{4\pi} \int\limits_{G \cap Ш(\vec{r}, t)} \left|\vec{r} - \vec{r}'\right|^{-2} g(\mu_0')e^{-a(\vec{r}, \vec{r}')} N(t - \left|\vec{r} - \vec{r}'\right|, \vec{r}', \vec{\omega}')d\vec{r}',$$

$$B\Phi(t, \vec{r}, \vec{\omega}) = \frac{\delta_S(\vec{r})}{4\pi} \int\limits_{\Omega(\vec{r}, t)} g(\mu_0)\Phi(\vec{r} - \vec{\omega}''t, \vec{\omega}'')e^{-\int\limits_0^t \sigma(\vec{r} - \vec{\omega}''\tau)d\tau} d\vec{\omega}'',$$

$$\vec{\omega}' = (\vec{r} - \vec{r}')/\left|\vec{r} - \vec{r}'\right|,\ \mu_0' = (\vec{\omega}, \vec{\omega}'),\ \mu_0 = (\vec{\omega}, \vec{\omega}''),$$

$$a(\vec{r}, \vec{r}') = \left|\vec{r} - \vec{r}'\right| \int\limits_0^1 \sigma(\vec{r}\alpha + \vec{r}'(1-\alpha))d\alpha,$$

$$\Omega(\vec{r}, t) = \left\{\vec{\omega} \in \Omega :\ \vec{r} - \vec{\omega}t \in \bar{G}\right\} \setminus \partial\bar{G}.$$

На основе исследований свойств операторов K и B доказано, что решение уравнения (8) может быть представлено рядом Неймана

$$N = \sum_{n=0}^{\infty} K^n(Kf + B\Phi) \tag{9}$$

Доказаны теоремы о существовании и единственности классического решения задачи, об областях гельдеровости интеграла столкновений.

Пусть $G = \bigcup\limits_{j=1}^{J} G_j$ – многозонная область, в которой происходит процесс переноса, удовлетворяющая условиям, упомянутым во введении. Ω – единичная сфера. Введем обозначения

$\Lambda = [-1;1],\ \bar{\Pi}_j = [0,T]\times\bar{G}_j,\ \tilde{\Pi}_j = \bar{\Pi}_j \setminus \big(0,\partial G_j\big)$.

Если функция $u\big(t,\bar{r},\vec{\omega}\big)$ терпит разрыв при переходе через точку $\bar{r}_\gamma$ границы раздела зон γ_{ij}, тогда обозначим

$$\delta u\big(t,\bar{r}_\gamma,\vec{\omega}\big) = \lim_{\varepsilon\to+0}\Big\lfloor u\big(t,\bar{r}_\gamma+\varepsilon\vec{n}_{r_\gamma},\vec{\omega}\big) - u\big(t,\bar{r}_\gamma-\varepsilon\vec{n}_{r_\gamma},\vec{\omega}\big)\Big\rfloor,$$

где $\vec{n}_{r_\gamma}$ - нормаль к γ_{ij} в точке $\bar{r}_\gamma$.

Интегрируя уравнение (5) по переменной τ от t_1^* до t с учетом начального и граничных условий (2)-(4), имеем

$$u = P\Phi + R(N+f) \tag{10}$$

где операторы P и R определены формулами

$$\Phi\big(t,\bar{r},\vec{\omega}\big) = \begin{cases} 0, & \vec{r}-\vec{\omega}t\notin\bar{G} \\ \Phi\big(\vec{r}-\vec{\omega}t,\vec{\omega}\big)e^{-\int\limits_0^t\sigma\left(\vec{r}-\vec{\omega}(t-\tau)\right)d\tau}, & \vec{r}-\vec{\omega}t\in\bar{G} \end{cases}$$

$$Rf\big(t,\vec{r},\vec{\omega}\big) = \int\limits_{t_1^*\left(t,\vec{r},\vec{\omega}\right)}^{t} f\big(\tau,\vec{r}-\vec{\omega}(t-\tau),\vec{\omega}\big)e^{-\int\limits_0^t\sigma\left(\vec{r}-\vec{\omega}(t-\tau')\right)d\tau'}d\tau.$$

Действуя на уравнение (10) оператором S, для интеграла столкновений N получаем

$$N = KN + Kf + B\Phi. \tag{11}$$

где $K = SR,\ B = SP$.

Рассматривая интеграл как кратный, используя теорему Фубини, а затем замену переменных интегрирования $\vec{r}' = \vec{r} - \vec{\omega}'(t-\tau)$ учитывая, что при $\tau = t_1^*\big(t,\vec{r},\vec{\omega}'\big)$ точка $\vec{r}$ лежит либо на внешней границе области G либо на границе шара $\text{Ш}\big(\vec{r},t\big)$ с центром $\vec{r}$ радиуса t, с помощью известной техники [2], [3] приходим к равенству

$$K\int\big(t,\vec{r},\vec{\omega}\big) = \int\limits_{G\cap\text{Ш}\left(\vec{r},t\right)} K\big(\vec{r},\vec{r}',\mu_0'\big)f\big(t-\big|\vec{r}-\vec{r}'\big|,\vec{r}',\vec{\omega}'\big)d\vec{r}'$$

где

$$K\big(\vec{r},\vec{r}',\mu_0'\big) = \frac{\delta_S\big(\vec{r}\big)g\big(\mu_0'\big)}{4\pi\big|\vec{r}-\vec{r}'\big|}exp\Big\{-\big|\vec{r}-\vec{r}'\big|\int\limits_0^1\delta\big(\alpha\vec{r}+(1-\alpha)\vec{r}'\big)d\alpha\Big\},$$

$$\vec{\omega}' = \frac{\vec{r}-\vec{r}'}{\big|\vec{r}-\vec{r}'\big|},\ \mu_0' = \big(\vec{\omega},\vec{\omega}'\big).$$

Легко видеть, что

$$B\Phi\big(t,\vec{r},\vec{\omega}\big) = \frac{\delta_S\big(\vec{r}\big)}{4\pi}\int\limits_{\Omega\left(\vec{r},t\right)} g\big(\mu_0'\big)\Phi\big(\vec{r}-\vec{\omega}'t,\vec{\omega}'\big)e^{-\int\limits_0^t\delta\left(\vec{r}-\vec{\omega}'\tau\right)d\tau}d\vec{\omega}',$$

где $\mu_0' = \big(\vec{\omega},\vec{\omega}'\big),\ \Omega\big(\vec{r},t\big) = \Big\{\vec{\omega}\in\Omega;\ \vec{r}-\vec{\omega}t\in\bar{G}\backslash\partial\bar{G}\Big\},\ \Omega_j\big(\vec{r},t\big) = \Big\{\vec{\omega}\in\Omega;\ \vec{r}-\vec{\omega}t\in G_j\Big\},\ j=\overline{1,J}$.

Нам нужны следующие леммы:

Лемма 1. Пусть $\big(t,\vec{r}\big)\in\tilde{\Pi}_i,\ i=\overline{1,J}$. Тогда справедливы пределы

$$\lim_{(t_1,\vec{r}_1)\to(t,\vec{r})} mes\big\{\Omega_j\big(\vec{r},t\big)\setminus\Omega_j\big(\vec{r}_1,t_1\big)\big\} = 0, \tag{12}$$

$$\lim_{(t_1,\vec{r}_1)\to(t,\vec{r})} mes\{\Omega(\vec{r},t)\setminus\Omega(\vec{r}_1,t_1)\}=0. \tag{13}$$

Лемма 2. Если $(t_1,\vec{r}_1)\to(0,\vec{r}_\gamma)\in\gamma_j,\ j=\overline{1,J}$, то предел $\lim mes\{\Omega_j(\vec{r}_1,t_1)\}$ зависит от пути приближения $(t_1,\vec{r}_1)$ к $(0,\vec{r}_\gamma)$.

Лемма 3.Если S – площадь единичного сферического пояса, определяемого широтными углами θ_2 и θ_1 $(\theta_2\geq\theta_1)$, то (Рисунок 1)

$$S=2\pi(cos\theta_1-cos\theta_2). \tag{14}$$

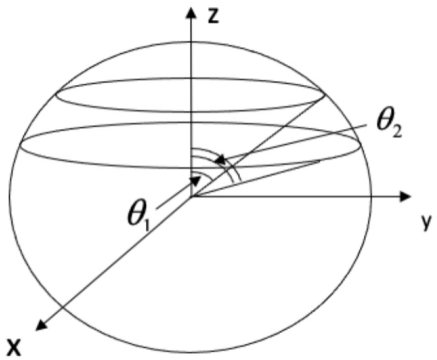


Рисунок 1.

Лемма 4. Пусть даны шары $Ш(\vec{0},R)$ и $Ш(\vec{r},t)$. Тогда для множества

$$\Omega(\vec{r},t)=\left\{\vec{\omega}\in\Omega;\ \vec{r}-\vec{\omega}t\in Ш(\vec{0},R)\right\}$$

верна оценка

$$Z=mes\{\Omega(\vec{r},t_2)\setminus\Omega(\vec{r},t_1)\}\leq\sum_{i=1}^{2}\beta_1(t_i,\vec{r})|t_2-t_1|, \tag{15}$$

где

$$\beta_1(t,\vec{r})=\begin{cases}2\sqrt{2}\pi\left[\dfrac{1}{\sqrt{t^2+(R-|\vec{r}|)^2}}+\dfrac{1}{\sqrt{|\vec{r}|^2+(R-t)^2}}\right],\ \vec{r}\in Ш(\vec{0},R)\\[2ex]\dfrac{2\sqrt{2}\pi}{\sqrt{t^2+(R-|\vec{r}|)^2}}+\dfrac{1}{R},\quad \vec{r}\notin Ш(\vec{0},R).\end{cases}$$

Лемма 5. Пусть шары $Ш(\vec{0},R),Ш(\vec{r}_i,t),i=1,2$. Тогда справедлива оценка

$$Z=mes\{\Omega(\vec{r}_1,t)\setminus\Omega(\vec{r}_2,t)\}\leq\sum_{i=1}^{2}\beta_1(t,\vec{r}_i)|\vec{r}_2-\vec{r}_1|. \tag{16}$$

Лемма 6. Пусть выполнены

$$C_\alpha^{\delta_S},C_\alpha^{\beta},C_\alpha^{g},$$

тогда для задачи С оператор B переводит

$$C^{\alpha,0}(\bar{G}\times\Omega)\text{ в }C^{\alpha}(\tilde{\Pi}\times\Omega)\text{ при }0\leq\alpha<1$$

и в

$$Z\big(\tilde{\Pi}\times\Omega\big) \text{ при } \alpha = 1.$$

Лемма 7. Пусть выполнены условия

$$C_0^{\delta_S}, C_0^{\delta}, C_0^{g},$$

тогда оператор B переводит

$$C\big(\bar{G}\times\Omega\big) \text{ в } C\big(\tilde{\Pi}\times\Omega\big).$$

Лемма 8. Пусть выполнены условия

$$C_\alpha^{\delta_S}, C_\alpha^{\delta}, C_\alpha^{g},$$

и условия согласования A, тогда для задачи (1) – (4) оператор B переводит

$$C^\alpha\big(\bar{G}\times\Omega\big) \text{ в } C^\alpha\big(\tilde{\Pi}\times\Omega\big) \text{ при } 0 \le \alpha < 1$$

и в

$$Z\big(\tilde{\Pi}\times\Omega\big) \text{ при } \alpha = 1.$$

Лемма 9. Пусть выполнены условия

$$C_\alpha^{\delta_S}, C_\alpha^{\delta}, C_0^{g},$$

тогда оператор K переводит

$$C^\alpha\big(\tilde{\Pi}\times\Omega\big) \text{ в } C^\alpha\big(\bar{\Pi}\times\Omega\big)$$

,

$$C^\alpha\big(\bar{\Pi}\times\Omega\big) \text{ в } C^\alpha\big(\bar{\Pi}\times\Omega\big) \text{ при } 0 \le \alpha < 1,$$
$$Z\big(\tilde{\Pi}\times\Omega\big) \text{ и } Z\big(\bar{\Pi}\times\Omega\big) \text{ в } Z\big(\bar{\Pi}\times\Omega\big).$$

Кроме того, справедливо неравенство

$$\|K\|_{C\left(\bar{\Pi}\times\Omega\right)\to C\left(\bar{\Pi}\times\Omega\right)} \le q < 1. \tag{17}$$

Теорема 1. Пусть выполнены условия

$$C_0^{\delta_S}, C_0^{\delta}, C_0^{g},$$

и

$$\Phi \in C\big(\bar{G}\times\Omega\big),\ f \in C\big(\bar{\Pi}\times\Omega\big),$$

тогда существует единственное классическое решение задачи (1) – (4).

Доказательство. Рассмотрим уравнение (2). В силу леммы 7 и леммы 9 существует единственное решение этого уравнение, принадлежащее $C\big(\tilde{\Pi}\times\Omega\big)$, которое представляется рядом Неймана

$$N = \sum_{n=1}^{\infty} K^n(Kf + B\Phi) + Kf + B\Phi. \tag{18}$$

Рассмотрим теперь функцию $u\big(t,\vec{r},\vec{\omega}\big)$, определяемую формулой (1). Действуя на равенство (1) оператором S, видим, что

$$Su = KN + Kf + B\Phi = N.$$

Пусть $t \in [0,T],\ \vec{r} \in G,\ \vec{\omega} \in \Omega$, тогда при

$$u\big(\tau,\vec{r}-\vec{\omega}(t-\tau),\vec{\omega}\big) = P\Phi\big(\tau,\vec{r}-\vec{\omega}(t-\tau),\vec{\omega}\big) + R(N+f)\big(\tau,\vec{r}-\vec{\omega}(t-\tau),\vec{\omega}\big).$$

Из равенств

$$\vec{r}-\vec{\omega}(t-\tau)-\vec{\omega}(\tau-\tau') = \vec{r}-\vec{\omega}(t-\tau'),$$

$$\vec{r}-\vec{\omega}(t-\tau)-\vec{\omega}\tau = \vec{r}-\vec{\omega}t,$$

$$t_1^*\big(t,\vec{r},\vec{\omega}\big) = t_1^*\big(\tau,\vec{r}-\vec{\omega}(t-\tau),\vec{\omega}\big),$$

следует, что

$$P\Phi\big(\tau,\vec{r}-\vec{\omega}(t-\tau),\vec{\omega}\big)=e^{\int\limits_{\tau}^{t}\delta\left(\vec{r}-\vec{\omega}(t-\tau')\right)d\tau'}P\Phi\big(t,\vec{r},\vec{\omega}\big),$$

$$Rf\big(\tau,\vec{r}-\vec{\omega}(t-\tau),\vec{\omega}\big)=\int\limits_{t_1^*\left(t,\vec{r},\vec{\omega}\right)}^{\tau}f\big(\tau',\vec{r}-\vec{\omega}(t-\tau'),\vec{\omega}\big)e^{-\int\limits_{\tau'}^{\tau}\delta\left(\vec{r}-\vec{\omega}(t-\tau'')\right)d\tau''}dr',$$

Из этих формул видим, что функция $u\big(\tau,\vec{r}-\vec{\omega}(t-\tau),\vec{\omega}\big)$ непрерывна по τ на отрезке $[t_1^*\big(t,\vec{r},\vec{\omega}\big),t]$, дифференцируемое по τ в интервалах $(t_k^*,t_{k+1}^*),\ k=\overline{1,M-1},\ (t_M^*,t)$ удовлетворяет уравнению (5). Также нетрудно видеть, что $u\big(t,\vec{r},\vec{\omega}\big)$ удовлетворяет начальному условию (2) и граничному условию (4). Если $\vec{r}_\gamma\in\partial G$ и $(\vec{n}_{r\gamma},\vec{\omega})<0$, то $t_1^*\big(t,\vec{r}_\gamma,\vec{\omega}\big)=t$ и $\vec{r}_\gamma-\omega t\in G$, а потому $R(N+f)\big(t,\vec{r}_\gamma,\vec{\omega}\big)=0,\ P\Phi\big(t,\vec{r}_\gamma,\vec{\omega}\big)=0$, т.е. $u\big(t,\vec{r}_\gamma,\vec{\omega}\big)=0$. Этим доказали существование классического решения задачи (1) – (4).

Докажем единственность. Пусть существуют два решения u_1 и u_2 . Тогда разность $u=u_1-u_2$ является решением уравнения

$$u=RN, \tag{19}$$

где $N=N_1-N_2$ ($N_i=Su_i,\ i=1,2$), который в свою очередь является решением уравнения

$$N=KN. \tag{20}$$

Из (19) и (20) видим, что $u\equiv 0$. Теорема доказана.

Теорема 2. Пусть выполнены условия

$$C_\alpha^{\delta_S},C_\alpha^\beta,C_\alpha^g,\Phi\in C^{\alpha,0}\big(\bar{G}\times\Omega\big),\ f\in C^\alpha\big(\bar{\Pi}\times\Omega\big),$$

тогда для задачи

$$CN\in C^\alpha\big(\tilde{\Pi}\times\Omega\big)\text{ при }0\le\alpha<1$$

и

$$f\in Z\big(\tilde{\Pi}\times\Omega\big)\text{ при }\alpha=1.$$

Теорема 3. Пусть выполнены условия

$$C_\alpha^{\delta_S},C_\alpha^\beta,C_\alpha^g,\Phi\in C^{\alpha,0}\big(\bar{G}\times\Omega\big),\ f\in C^\alpha\big(\bar{\Pi}\times\Omega\big),$$

а также условия согласования А, тогда для задачи (1) – (4)

$$N\in C^\alpha\big(\bar{\Pi}\times\Omega\big)\text{ при }0\le\alpha<1$$

и

$$f\in Z\big(\bar{\Pi}\times\Omega\big)\text{ при }\alpha=1.$$

Ключевые слова: уравнения переноса, многозонная область, граничное условие, обратные задачи.

Предметная классификация AMS: 35

Список литературы

[1] Аниконов Ю. Е., Бубнов Б. А., Обратные задачи теории переноса, *ДАН СССР*, Vol.299, No.5, 1988, сс.1037-1040.

[2] Гермогенова Т. А., *Локальные свойства решения уравнения переноса* , М.: Наука, 1986, 272 с.

[3] Гермогенова Т. А., Некоторые свойства решений первой краевой задачи для уравнения переноса нейтронов, *Вычислительные методы в теории переноса.*, Vol.299, No.5, 1988, сс.1037-1040.*Некоторые свойства решений первой краевой задачи для уравнения переноса нейтронов* , М.: Атомиздат, 1969, сс. 34-49.

[4] Владимиров В.С. , *Уравнения математической физики* , М.: Наука, 1981, 512 с.

[5] Султангазин У.М., *Методы сферических гармоник и дискретных ординат в задачах кинетической теории переноса* , Алма-Ата: Наука, 1979, 269 с.

[6] Султангазин У.М., Садыков Г.М., О дифференциальных свойствах решений нестационарного уравнения переноса, *Качественная теория дифференциальных уравнений* , Иркутск, 1986.

[7] Марчук Г.И., Лебедев В.И., *Численные методы в теории переноса нейтронов* , М.: Атомиздат, 1981, 454 с.

[8] Кейз К., Цвайфель П., *Линейная теория переноса*, М.: Мир,1972, 384 с.

[9] Дэвисон В., *Теория переноса нейтронов* , М.: Атомиздат, 1960, 520 с.

[10] Агошков В.И., *Задачи и уравнения теории переноса частиц* , М.: ОВМ АН СССР, 1985, 100 с.

[11] Сыдыков Г. М. Сариев А. Д., Об обратных задачах для нестационарного уравнения переноса в плоскопараллельной геометрии, *Всесоюзн. Журнал «Дифференциальн. Уравнения» Минск*, No.9, 1991.

[12] Сариев А.Д., Сыдыков Г.М., О корректности «в целом» некоторых обратных задачах для нестационарных уравнений, *АН СССР, ЖВМ и МФ*, No.12, 1991.

[13] Сариев А.Д. и др., О корректности обратных задачи определения пары, в случае простой области для уравнения переноса излучений, *Международный научно-исследовательский журнал, Екатеринбург*, Vol.55, No.1, 2017, сс.95-104.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

О ЛИНЕЙНОЙ ОБРАТНОЙ ЗАДАЧЕ С НЕЛОКАЛЬНОЙ КРАЕВОЙ УСЛОВИИ ПЕРИОДИЧЕСКОГО ТИПА ДЛЯ УРАВНЕНИЯ СМЕШАННОГО ТИПА ВТОРОГО РОДА ВТОРОГО ПОРЯДКА В НЕОГРАНИЧЕННОМ ПАРАЛЛЕЛЕПИПЕДЕ

БИЙБИНАЗ СИПАТДИНОВА[1], БАХТИЁР ХАЛХАДЖАЕВ[2], СИРОЖИДДИН ДЖАМАЛОВ[3]

Институт математики имени В.И.Романовского Академии наук Республики Узбекистан, Ташкент, Узбекистан
e-mail: [1]sbiybinaz@mail.ru, [2] xalxadjaev@yandex.ru, [3]siroj63@mail.ru

Аннотация. В данной статье рассматриваются вопросы корректности одной линейной обратной задачи с нелокальной краевой условии периодического типа для трехмерного уравнения смешанного типа второго рода, второго порядка в неограниченном параллелепипеде. Для этой задачи методами «ε–регуляризации», априорных оценок, последовательностью приближений с применением преобразования Фурье доказаны теоремы существования и единственности обобщенного решения одной линейной обратной задачи с нелокальной краевой условии периодического типа в определенном классе интегрируемых функции.

Ключевые слова: уравнения смешанного типа второго рода второго порядка, линейная обратная задача с нелокальной краевой условии периодического типа, коректность задачи, методы «ε-регуляризации», априорных оценок, последовательность приближения, преобразования Фурье.

Предметная классификация AMS: 35M10

Введение и постановка задачи

В процессе исследования нелокальных задач была выявлена тесная взаимосвязь задач с нелокальными краевыми условиями и обратными задачами. К настоящему времени достаточно хорошо изучены обратные задачи для классических уравнений таких как, параболических, эллиптических и гиперболических типов [1],[2], [14], [21]. Для уравнений смешанного типа в плоскости изучено в работах К.Б. Сабитова и его учеников [19], а для многомерных уравнений смешанного типа как первого, так и второго рода в ограниченных областях, изучены в работах С.З.Джамалова, Р.Р.Ашурова и С.Г.Пяткова [6]-[10] .

Значительно менее изученными являются обратные задачи для уравнений смешанного типа первого рода (в частности, для уравнения Трикоми) в неограниченных областях [11],[12], а для уравнений смешанного типа второго рода такие задачи в неограниченных областях практические не исследовались. Частично восполнить данный пробел мы и попытаемся в рамках этой работы.

В данной работе, для исследования однозначное разрешимости обратных задач для трехмерного уравнения смешанного типа второго рода, второго порядка в неограниченном параллелепипеде предлагается метод, который основан на сведение обратной задачи к прямым полунелокальным краевым задачам для семейство нагруженных интегродифференциальных уравнения смешанного типа второго рода, второго порядка в ограниченной прямоугольной области.

Напомним, что нагруженным уравнением принято называть уравнение с частными производными, содержащее в коэффициентах или в правой части значения тех или иных функционалов от решения уравнения [17],[21].

В области

$$G = (0,1) \times (0,T) \times R = Q \times R = \{(x,t,z); x \in (0,1), 0 < t < T < +\infty, z \in R\}$$

рассмотрим трехмерное уравнение смешанного типа второго рода второго порядка:

$$Lu = k(t)u_{tt} - \Delta u + a(x,t)u_t + c(x,t)u = \psi(x,t,z), \tag{1}$$

где $\Delta u = u_{xx} + u_{zz}$-оператор Лапласа, и пусть $k(0) \leq 0 \leq k(T)$. Здесь $\psi(x,t,z) = g(x,t,z) + h(x,t)f(x,t,z)$, $g(x,t,z)$ и $f(x,t,z)$-заданные функции, а функция $h(x,t)$ подлежит определению.

Уравнение (1) относится к уравнениям смешанного типа второго рода, так как на знак функции $k(t)$ по переменной t внутри области Q не налагается никаких ограничений [3].

Пусть все коэффициенты уравнения (1) достаточно гладкие функции в Q.

В дальнейшем для решения поставленных задач нам необходимо ввести определений несколько функциональных пространств и обозначения.

Обозначим через

$$\hat{u}(x,t,\lambda) = (2\pi)^{-1/2} \int\limits_{-\infty}^{+\infty} u(x,t,z)\, e^{-i\lambda z} dz$$

преобразование Фурье по переменной z, функции $u(x,t,z)$, а через

$$u(x,t,z) = (2\pi)^{-1/2} \int\limits_{-\infty}^{+\infty} \hat{u}(x,t,\lambda)\, e^{i\lambda z} d\lambda$$

обратное преобразование Фурье. Теперь с помощью преобразования Фурье определим пространство $W_2^{l,s}(Q)$ с нормой

$$||u||^2_{W_2^{l,s}(G)} = (2\pi)^{-1/2} \int\limits_{-\infty}^{+\infty} \left(1 + |\lambda|^2\right)^s ||\hat{u}(x,t,\lambda)||^2_{W_2^l(Q)} d\lambda, \tag{A}$$

где s, l-любые конечные положительные целые числа, причем $s \geq 3$.

Через $W_2^l(Q)$ (при $l = 0$, $W_2^0(Q) = L_2(Q)$) определяется пространства Соболева со скалярным произведением $(u, \vartheta)_l$ и нормой

$$\|\vartheta\|_l^2 = \|\vartheta\|^2_{W_2^l(Q)} = \sum_{|\alpha| \leq l} \int\limits_Q |D^\alpha \vartheta|^2 \, dxdt,$$

здесь α мультииндекс, D^α обобщённая производная по переменным x и t.

Очевидно, что пространство $W_2^{l,s}(G)$ с нормой (A) является Гильбертовым пространством [13],[15],[16],[18].

При получении различных априорных оценок мы часто используем неравенство Коши с σ [13]:

$$\forall u,\ \vartheta > 0,\ \forall \sigma > 0,\ u \cdot \vartheta \leq \sigma u^2 + \sigma^{-1}\vartheta^2.$$

Линейная обратная задача.

Найти функции $\{u(x,t,z),\ h(x,t)\}$ удовлетворяющие уравнению (1) в области G, такие что, функция $u(x,t,z)$ удовлетворяет следующим нелокальным краевым условям периодического типа:

$$\gamma u|_{t=0} = u|_{t=T}, \tag{2}$$

$$D_x^p u|_{x=0} = D_x^p u|_{x=1}; p = 0, 1, \tag{3}$$

где γ-некоторое постоянное число, отличное от нуля, величина которого будет уточнена ниже,

$$\begin{array}{l}\text{Далее будем считать, что } u(x,t,z) \text{ и } u_z(x,t,z) \to 0 \text{ при } |z| \to \infty,\\ u(x,t,z) \text{ абсолютно интегрируема по } z \text{ на } R \text{ при любом } (x,t) \text{ в } \overline{Q}.\end{array} \tag{4}$$

Кроме того, решение задачи (1)-(4) удовлетворяет дополнительному условию

$$u(x,t,l_0) = \varphi_0(x,t), \tag{5}$$

где $l_0 \in R$ и с функций $h(x,t)$ принадлежит классу

$$U = \{(u,h) | u \in W_2^{2,3}(G); h \in W_2^2(Q)\}.$$

Определение 1: Обобщённым решением задачи (1)-(5) будем называть функцию $u(x,t,z) \in U$, удовлетворяющую уравнению (1) почти всюду в области G, с условиями (2)-(5).

Пусть все коэффициенты уравнения (1) достаточно гладкие функции в области G и пусть выполнены следующие условия относительно коэффициентов, правой части и заданной функции $\varphi_0(x,t)$;

Условие 1:

Периодичность: $a(x,0) = a(x,T)$, $c(x,0) = c(x,T)$.

Нелокальное условие: $\gamma \cdot g(x,0,z) = g(x,T,z)$, $\gamma \cdot f(x,0,z) = f(x,T,z)$.

Гладкость: $f(x,t,l_0) = f_0(x,t) \in C_{x,t}^{0,1}(Q)$, $|f_0(x,t)| \geq \rho > 0$, $f \in W_2^{3,3}(Q)$, $g \in W_2^{1,3}(Q)$.

Условие 2:

$$\varphi_0(x,t) \in W_2^3(Q);\ \gamma D_t^q \varphi_0|_{t=0} = D_t^q \varphi_0|_{t=T},\ D_x^p \varphi_0|_{x=0} = D_x^p \varphi_0|_{x=1},\ q = 0,1,2; p = 0,1.$$

Однозначное разрешимость задачи (1)-(5) докажем с помощью преобразованием Фурье, т.е. для нахождения решение задачи (1)-(5), применяем преобразование Фурье по переменной z, для задачи (1)-(5).

Для того чтобы сформулировать основной результат, необходимо выполнить некоторые формальности построения.

Рассмотрим следы уравнения (1) при $z = \ell_0$:

$$\begin{array}{c}Lu(x,t,l_0) = k(t)u_{tt}(x,t,l_0) - u_{xx}(x,t,l_0) - u_{zz}(x,t,l_0) +\\ +a(x,t)u_t(x,t,l_0) + c(x,t)u(x,t,l_0) = \psi(x,t,l_0).\end{array}$$

Теперь, учитывая дополнительное условие (5) и то, что $f_0 \neq 0$, определим формально неизвестную функцию $h(x,t)$ в виде интеграла

$$h(x,t) = \frac{1}{f_0(x,t)} \left[\Phi_0 + \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \lambda^2 e^{i\lambda l} \hat{u}(x,t,\lambda) d\lambda \right],$$

где $\Phi_0 = L_0\varphi_0 - g_0$, $L_0\varphi_0 = x\varphi_{0tt} - \varphi_{0xx} + a(x,t)\,\varphi_{0t} + c\,(x,t)\,\varphi_0$, а для определения функций $\hat{u}(x,t,\lambda)$ в области $Q = (0,1) \times (0,T)$ получим нагруженных интегро-дифференциальных уравнений смешанного типа второго рода второго порядка:

$$\begin{array}{c}L\hat{u} = k(t)\hat{u}_{tt} - \hat{u}_{xx} + a(x,t)\hat{u}_t + (c(x,t) + \lambda^2)\hat{u}\\ = \hat{g}(x,t,\lambda) + \frac{\hat{f}(x,t,\lambda)}{f_0(x,t)} \left[\Phi_0 + \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \lambda^2 e^{i\lambda l} \hat{u}(x,t,\lambda) d\lambda \right] \equiv \hat{F}(\hat{u}),\end{array} \tag{6}$$

с нелокальными краевыми условиями периодического типа:

$$\gamma\hat{u}|_{t=0} = \hat{u}|_{t=T}, \tag{7}$$

$$D_x^p\hat{u}|_{x=0} = D_x^p\hat{u}|_{x=1}; p = 0, 1, \tag{8}$$

где, $\lambda \in R = (-\infty, +\infty)$,

$$\hat{f}(x,t,\lambda) = (2\pi)^{-1/2}\int\limits_{-\infty}^{+\infty} f(x,t,z)e^{-i\lambda z}dz,$$

преобразование Фурье по переменной z, функции $f(x,t,z)$.

Основными результатом является

Теорема 1: Пусть выполнены вышеуказанные условия 1 и 2 для коэффициентов уравнения (1), кроме того, пусть $2a - |k_t| + \mu k \geq B_1 > 0$, $\mu c(x,t) - c_t(x,t) \geq b_2 > 0$, для всех $(x,t) \in \overline{Q}$, где $\mu = \frac{2}{T}\ln|\gamma| > 0$, $|\gamma| > 1$ и пусть существует положительное число σ-(коэффициенты неравенство Коши) такое, что для $b_0 = \min\{B_1, \mu, b_2\}$ имеют оценки $b_0 - 11\mu^2\sigma^{-1} = \delta > 0$, $q = M\|f\|^2_{W_2^{3,3}(G)} < 1$, где $M = const\left(\sigma\mu^2 m\delta^{-1}\rho^{-2}\|f_0\|_{C^{0,1}_{x,t}(Q)}\right)$, $m = 10c_1c_2c_3$, $c_1 = \int\limits_{-\infty}^{+\infty}\frac{\lambda^4 d\lambda}{(1+|\lambda|^2)^s} < +\infty$, $c_i (i = 2, 3)$-коэффициенты теоремы вложения Соболева. Тогда следующее функции

$$u(x,t,z) = (2\pi)^{-1/2}\int\limits_{-\infty}^{+\infty}\hat{u}(x,t,\lambda)e^{i\lambda z}d\lambda, \tag{9}$$

$$h(x,t) = \frac{1}{f_0(x,t)}\left[\Phi_0 + \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\lambda^2 e^{i\lambda l}\hat{u}(x,t,\lambda)d\lambda\right] \tag{10}$$

являются единственным решением линейной обратной задачи (1)-(5) из указанного класса U, где функция $\hat{u}(x,t,\lambda)$ подлежит к определению.

Доказательство теоремы 1 проведем по следующей схеме:

1.Покажем, что функция $u(x,t,y)$, определённая по формуле (9) удовлетворяет дополнительному условию (5).

2. Для доказательства однозначной разрешимости задачи (6)-(8) сначала исследуем однозначную разрешимость вспомогательной задачи, то есть исследуем разрешимость семейства нагруженных интегро-дифференциальных уравнений составного типа с малым параметром (14)-(16).

3. Затем с помощью этой вспомогательной задачи (14)-(16), изучим однозначную разрешимость семейства нагруженных уравнений смешанного типа второго рода, второго порядка (6)-(8).

4. Используя однозначную разрешимость задачи (6)-(8), покажем однозначную разрешимость линейной обратной задачи (1)-(5).

Теперь приступим к реализации этой схемы.

Доказательство. Докажем теорему 1 поэтапно. Сначала покажем, что функция $u(x,t,z)$ удовлетворяет дополнительному условию (5), т.е. $u(x,t,\ell_0) = \varphi_0(x,t)$. Положим противное.

Пусть

$$u(x,t,\ell_0) = (2\pi)^{-1/2}\int\limits_{-\infty}^{+\infty}\hat{u}(x,t,\lambda)e^{i\lambda\ell_0}d\lambda = \omega(x,t) \neq \varphi_0(x,t)$$

рассмотрим функции $\vartheta(x,t)=\omega(x,t)-\varphi_0(x,t)$ в области Q. Умножая задачи (6)-(8), на $\frac{e^{i\lambda\ell_0}}{\sqrt{2\pi}}$ и интегрируя по параметру λ от $-\infty$ до ∞ учитывая условие теоремы 1, получим следующую задачу

$$L_0\vartheta=k(t)\vartheta_{tt}-\vartheta_{xx}+a(x,t)\vartheta_t+c(x,t)\vartheta=0 \tag{11}$$

с нелокальными краевыми условиями периодического типа:

$$\gamma\vartheta|_{t=0}=\vartheta|_{t=T}, \tag{12}$$

$$D_x^p\vartheta|_{x=0}=D_x^p\vartheta|_{x=1}; p=0,1. \tag{13}$$

Единственность решения задачи (11)-(13) доказано в работах [4]-[6] отсюда следует $\vartheta(x,t)=0$, т.е. $\omega(x,t)=\varphi_0(x,t)$.

1. Семейство нагруженных интегро-дифференциальных уравнений составного типа с малым параметром

Разрешимость задачи (6)-(8) докажем методами «ε- регуляризации», последовательную приближений и априорных оценок, а именно в области $Q=(0,1)\times(0,T)$ рассмотрим семейство нагруженных интегро-дифференциальных уравнений составного типа с малым парамтром:

$$\begin{gathered}L_\varepsilon\hat{u}_\varepsilon=-\varepsilon\tfrac{\partial\Delta\hat{u}_\varepsilon}{\partial t}+L_0\hat{u}_\varepsilon+\lambda^2\hat{u}_\varepsilon=\\ =\hat{g}(x,t,\lambda)+\tfrac{\hat{f}(x,t,\lambda)}{f_0(x,t)}\left[\Phi_0+\tfrac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\lambda^2e^{i\lambda l_0}\hat{u}_\varepsilon(x,t,\lambda)d\lambda\right]\equiv\hat{F}(\hat{u}_\varepsilon)\end{gathered} \tag{14}$$

с нелокальными краевыми условиями периодического типа:

$$\gamma D_t^q\hat{u}_\varepsilon|_{t=0}=D_t^q\hat{u}_\varepsilon|_{t=T}; q=0,1,2, \tag{15}$$

$$D_x^p\hat{u}_\varepsilon|_{x=0}=D_x^p\hat{u}_\varepsilon|_{x=1}; p=0,1, \tag{16}$$

где ε-малое положительное число.

В дальнейшем при доказательстве корректности задачи (14)-(16), понадобятся следующие обозначения и вспомогательные леммы.

Определим $W_i(Q,\mathbb{R}),(i=0,1,2)$-пространства обобщенных функций в области Q со значениями в $\mathbb{R}$ те

$$W_i(Q,\mathbb{R})=\left\{\hat{\vartheta}\;\middle|\;\hat{\vartheta}\in W_2^i(Q),\ i=0,1,2;\ \left(1+|\lambda|^2\right)^{3/2}\left\|\hat{\vartheta}\right\|_{W_2^i(Q)}\in L_2(\mathbb{R}),\ W_2^0(Q)=L_2(Q)\right\},$$

с нормой

$$\left\langle\hat{\vartheta}\right\rangle_i^2=\int\limits_{-\infty}^{\infty}\left(1+|\lambda|^2\right)^3\left\|\hat{\vartheta}\right\|_{W_2^i(Q)}^2d\lambda. \tag{B}$$

Очевидно, что пространства $W_i(Q,\mathbb{R})$ с заданной нормой являются Гильбертовым [13],[15],[16],[20]. Из определения пространства $W_2^i(Q)$, $i=0,1,2$, следует следующие вложения $W_2(Q,\mathbb{R})\subset W_1(Q,\mathbb{R})\subset W_0(Q,\mathbb{R})$. Через, символом

$$W(Q,\mathbb{R})=\left\{\hat{\vartheta}\;\middle|\;\hat{\vartheta}\in W_2(Q,\mathbb{R}),\ \frac{\partial}{\partial t}\Delta\hat{\vartheta}\in W_0(Q,\mathbb{R})\right\}$$

обозначим класс функций.

Определение 2: Обобщённым решением задачи (14)-(16) будем называть функцию $\hat{\vartheta}(x,t,\lambda)\in W(Q,\mathbb{R})$, удовлетворяющую уравнению (14) почти всюду в области Q, с краевыми условиями (15), (16).

Разрешимость задачи (14)-(16) докажем методами последовательных приближений и априорных оценок [6]-[8],[10]-[12], а именно рассмотрим семейство нагруженных интегро-дифференциальных уравнений составного типа с малым парамтром:

$$L_\varepsilon \hat{u}_\varepsilon^{(\theta)} = -\varepsilon \frac{\partial \Delta \hat{u}_\varepsilon^{(\theta)}}{\partial t} + L_0 \hat{u}_\varepsilon^{(\theta)} + \lambda^2 \hat{u}_\varepsilon^{(\theta)} = \\ = \hat{g} + \frac{\hat{f}(x,t,\lambda)}{f_0(x,t)} \left[\Phi_0 + \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \lambda^2 e^{i\lambda \ell_0}\, \hat{u}_\varepsilon^{(\theta-1)}(x,t,\lambda)\, d\lambda \right] \equiv \hat{F}(\hat{u}_\varepsilon^{(\theta-1)}) \tag{17}$$

и с нелокальными краевыми условиями периодического типа:

$$\gamma D_t^q\, \hat{u}_\varepsilon^{(\theta)}\Big|_{t=0} = D_t^q\, \hat{u}_\varepsilon^{(\theta)}\Big|_{t=T}; q = 0,1,2, \tag{18}$$

$$D_x^p \hat{u}_\varepsilon^{(\theta)}\Big|_{x=0} = D_x^p \hat{u}_\varepsilon^{(\theta)}\Big|_{x=1}; p = 0,1, \tag{19}$$

где $\varepsilon > 0, \quad \theta = 0,1,2,..., \quad \{u_\varepsilon^{(-1)}\} = 0.$

Лемма 1: *Пусть выполнены все условия теоремы 1, тогда для решения задачи* (17)-(19) *справедливы следующие оценки*:

$(I).\ \frac{\varepsilon}{\delta} \left\langle \hat{u}_{\varepsilon tt}^{(\theta)} \right\rangle_0^2 + \left\langle \hat{u}_\varepsilon^{(\theta)} \right\rangle_1^2 \le \mathrm{const}(\widetilde{\theta}, \widetilde{\varepsilon}),$

$(II).\ \frac{\varepsilon}{\delta} \left\langle \frac{\partial \Delta \hat{u}_\varepsilon^{(\theta)}}{\partial t} \right\rangle_0^2 + \left\langle \hat{u}_\varepsilon^{(\theta)} \right\rangle_2^2 \le \mathrm{const}(\widetilde{\theta}, \widetilde{\varepsilon}).$

Символом $\mathrm{const}(\widetilde{\theta}, \widetilde{\varepsilon})$ *здесь и далее обозначим постоянную, независящую от параметров* θ, ε.

Доказательство леммы 1:

Рассмотрим тождество

$$2\left(L\hat{u}_\varepsilon^{(\theta)}, e^{-\mu t} \hat{u}_{\varepsilon t}^{(\theta)} \right)_0 = 2\left(\hat{F}(\hat{u}_\varepsilon^{(\theta-1)}), e^{-\mu t} \hat{u}_{\varepsilon t}^{(\theta)} \right)_0, \tag{20}$$

где постоянную $\mu > 0$, выберем позже.

Учитывая условий теоремы 1, интегрируя по частям тождество (20) и применяя неравенства Коши с σ [13], легко получить слева снизу из (20) следующую неравенству

$$2 \int\limits_Q L\hat{u}_\varepsilon^{(\theta)} e^{-\mu t} \hat{u}_{\varepsilon t}^{(\theta)} dxdt \ \ge\ \varepsilon \left\| \hat{u}_{\varepsilon tt}^{(\theta)} \right\|_0^2 + \int\limits_Q e^{-\mu t} \{ (2a - k_t + \mu k)\, \hat{u}_{\varepsilon t}^{2(\theta)} + \\ + \mu \hat{u}_{\varepsilon x}^{2(\theta)} + \mu\, \lambda^2 \hat{u}_\varepsilon^{2(\theta)} + (\mu\, c - c_t) \hat{u}_\varepsilon^{2(\theta)} \}\, dxdt - \\ - \int\limits_{\partial Q} e^{-\mu t} \{ k(t) \hat{u}_{\varepsilon t}^{2(\theta)} e_t - \hat{u}_{\varepsilon x}^{2(\theta)} e_t - 2\hat{u}_{\varepsilon t}^{(\theta)} \hat{u}_{\varepsilon x}^{(\theta)} e_x + (c + \lambda^2) \hat{u}_\varepsilon^{(\theta)} e_t \} ds, \tag{21}$$

где $\mu - \mathrm{const} > 0$, $\overrightarrow{e} = \left(e_x = (\overrightarrow{e}, x);\ e_t = (\overrightarrow{e}, t) \right)$ единичный вектор внутренней нормали к границе $\partial\, Q$. Условия теоремы 1 обеспечивают не отрицательность интеграла по области Q. Учитывая нелокальные краевые условие периодического типа (18), (19) и условия теоремы 1, с выбором $\gamma^2 = e^{\mu T}$ получим, что граничные интегралы обратятся в нуль. Таким образом, из неравенства (21) получим слева снизу следующее неравенство

$$2 \int\limits_Q L\hat{u}_\varepsilon^{(\theta)} e^{-\mu t} \hat{u}_{\varepsilon t}^{(\theta)} dxdt \ge \varepsilon \left\| \hat{u}_{\varepsilon tt}^{(\theta)} \right\|_0^2 + \int\limits_Q e^{-\mu t} \{ B_1 \hat{u}_{\varepsilon t}^{2(\theta)} + \mu \hat{u}_{\varepsilon x}^{2(\theta)} + \\ + (\mu\lambda^2 + b_2) \hat{u}_\varepsilon^{2(\theta)} \} dxdt \ge \varepsilon \left\| \hat{u}_{\varepsilon tt}^{(\theta)} \right\|_0^2 + b_0 \left\| \hat{u}_\varepsilon^{(\theta)} \right\|_1^2. \tag{22}$$

Применяя неравенство Коши к тождеству (20) с права сверху получим

$$\begin{aligned}
&\left|2\left(\hat{F}(\hat{u}_\varepsilon^{(\theta-1)}), e^{-\mu t}\hat{u}_{\varepsilon t}^{(\theta)}\right)_0\right| \leq \\
&\leq \left|2\left(\hat{g} + \tfrac{\hat{f}(x,t,\lambda)}{f_0(x,t)}\left[\Phi_0 + \tfrac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\lambda^2 e^{i\lambda\ell_0}\hat{u}_\varepsilon^{(\theta-1)}(x,t,\lambda)d\lambda\right], e^{-\mu t}\hat{u}_{\varepsilon t}^{(\theta)}\right)_0\right| \leq \\
&\leq 3\sigma^{-1}\left\|\hat{u}_\varepsilon^{(\theta)}\right\|_1^2 + \sigma\left[\|\hat{g}\|_0^2 + \rho^{-2}\left\|\hat{f}\right\|_{C(Q)}^2\left(T_0\|\varphi\|_{W_2^2(Q)}^2 + \|g_0\|_0^2\right)\right] + \\
&+c_1\sigma\rho^{-2}\left\|\hat{f}\right\|_{C(Q)}\int\limits_{-\infty}^{\infty}\left(1+|\lambda|^2\right)^3\left\|\hat{u}_\varepsilon^{(\theta-1)}\right\|_1^2 d\lambda,
\end{aligned} \tag{23}$$

где $\delta = \min\{B_1, \mu, b_2 + \mu\lambda^2 \geq b_2 > 0\}$; $T_0 = 2\max\{\|k(t)\|_{C[0,T]}, \|a(x,t)\|_{C(Q)}, \|c(x,t)\|_{C(Q)}\}$.

Объединяя неравенства (22) и (23), получим

$$\begin{aligned}
&\varepsilon\left\|\hat{u}_{\varepsilon tt}^{(\theta)}\right\|_0^2 + \left(b_0 - 3\sigma^{-1}\right)\left\|\hat{u}_\varepsilon^{(\theta)}\right\|_1^2 \leq \\
&\leq \sigma\left[\|\hat{g}\|_0^2 + \rho^{-2}\left\|\hat{f}\right\|_{C(Q)}^2\left(T_0\|\varphi\|_{W_2^2(Q)}^2 + \|g_0\|_0^2\right)\right] + \\
&+c_1\sigma\rho^{-2}\left\|\hat{f}\right\|_{C(Q)}\int\limits_{-\infty}^{\infty}\left(1+|\lambda|^2\right)^3\left\|\hat{u}_\varepsilon^{(\theta-1)}\right\|_1^2 d\lambda.
\end{aligned} \tag{24}$$

Применяя теоремы вложения Соболева : $\left\|\hat{f}\right\|_{C(Q)}^2 \leq c_2\left\|\hat{f}\right\|_{W_2^2(Q)}^2$ [13],[18] к неравенству (24), получим

$$\begin{aligned}
&\varepsilon\left\|\hat{u}_{\varepsilon tt}^{(\theta)}\right\|_0^2 + \left(b_0 - 3\sigma^{-1}\right)\left\|\hat{u}_\varepsilon^{(\theta)}\right\|_1^2 \leq \\
&\leq \sigma c_2\left[\|\hat{g}\|_0^2 + \rho^{-2}\left\|\hat{f}\right\|_{W_2^2(Q)}^2\left(T_0\|\varphi\|_{W_2^2(Q)}^2 + \|g_0\|_0^2\right)\right] + \\
&+c_1c_2\sigma\delta^{-1}\rho^{-2}\left\|\hat{f}\right\|_{W_2^2(Q)}^2\int\limits_{-\infty}^{\infty}\left(1+|\lambda|^2\right)^3\left\|\hat{u}_\varepsilon^{(\theta-1)}\right\|_1^2 d\lambda.
\end{aligned} \tag{25}$$

Учитывая условие теоремы $b_0 - 3\sigma^{-1} \geq \delta > 0$, разделяя неравенства (25) на $\delta > 0$, умножая $\left(1+|\lambda|^2\right)^3$ и интегрируя по λ от $-\infty$ и до ∞, получим первую рекуррентную формулу

$$\tfrac{\varepsilon}{\delta}\left\langle\hat{u}_{\varepsilon tt}^{(\theta)}\right\rangle_0^2 + \left\langle\hat{u}_\varepsilon^{(\theta)}\right\rangle_1^2 \leq A + c_1c_2\sigma\delta^{-1}\rho^{-2}\left\langle\hat{f}\right\rangle_2^2\left\langle\hat{u}_\varepsilon^{(\theta-1)}\right\rangle_1^2. \tag{26}$$

Введем обозначение

$$A \equiv \sigma\delta^{-1}c_2\left[\langle\hat{g}\rangle_0^2 + \rho^{-2}\left\langle\hat{f}\right\rangle_2^2\left(T_0\|\varphi\|_{W_2^2(Q)}^2 + \|g_0\|_0^2\right)\right],$$

учитывая условий теоремы 1

$$c_1c_2\sigma\delta^{-1}\rho^{-2}\left\langle\hat{f}\right\rangle_2^2 < q = M\left\langle\hat{f}\right\rangle_3^2 < 1,$$

из рекуррентной формулы (26), получим справедливость (I) оценки. Действительно, для этого в качестве «начального приближения» возьмем функцию $\left\{\hat{u}_\varepsilon^{(-1)}\right\} \equiv \{0\}$. Тогда имеем

$$\tfrac{\varepsilon}{\delta}\left\langle\hat{u}_{\varepsilon tt}^{(0)}\right\rangle_0^2 + \left\langle\hat{u}_\varepsilon^{(0)}\right\rangle_1^2 \leq \sigma c_2\delta^{-1}\left[\langle\hat{g}\rangle_0^2 + \rho^{-2}\left\langle\hat{f}\right\rangle_2^2\left(T_0\|\varphi\|_{W_2^2(Q)}^2 + \|g_0\|_0^2\right)\right] \equiv A.$$

Продолжая этот процесс, методом индукции получим первую априорную оценку для любой функции $\left\{\hat{u}_\varepsilon^{(\theta)}\right\}$, $\forall\theta \geq 1$.

$$\tfrac{\varepsilon}{\delta}\left\langle\hat{u}_{\varepsilon tt}^{(\theta)}\right\rangle_0^2 + \left\langle\hat{u}_\varepsilon^{(\theta)}\right\rangle_1^2 \leq A\sum_{\tau=0}^{\theta}q^{(\tau)} \leq 2A.$$

Теперь докажем справедливость оценки (II). Для этого рассмотрим тождество:

$$-2\int\limits_Q e^{-\mu t}\, L_\varepsilon \hat{u}_\varepsilon^{(\theta)}\, P\hat{u}_\varepsilon^{(\theta)} dxdt = -2\int\limits_Q e^{-\mu t}\hat{F}\left(\hat{u}_\varepsilon^{(\theta-1)}\right) P\hat{u}_\varepsilon^{(\theta)} dxdt, \tag{27}$$

где $P\hat{u}_\varepsilon^{(\theta)} = \left(\frac{\partial}{\partial t}\Delta\hat{u}_\varepsilon^{(\theta)} - \mu\hat{u}_{\varepsilon tt}^{(\theta)} - \mu\hat{u}_{\varepsilon t}^{(\theta)}\right)$.

Аналогичные рассуждения, что и при доказательстве оценки (I), осно-ванные на интегрировании по частям (27), с учетом условий теоремы и не-локальных краевых условий (18), (19) из (27) получим снизу следующую оценку

$$\begin{aligned}&\left|-2\int\limits_Q e^{-\mu t}\hat{F}\left(\hat{u}_\varepsilon^{(\theta-1)}\right) P\hat{u}_\varepsilon^{(\theta)} dxdt\right| \geq \varepsilon\left\|\tfrac{\partial\Delta\hat{u}_\varepsilon^{(\theta)}}{\partial t}\right\|_0^2 + \int\limits_Q e^{-\mu t}\{B_1\hat{u}_{\varepsilon tt}^{2(\theta)} + \\ &+\mu\hat{u}_{\varepsilon xx}^{2(\theta)} + \mu\hat{u}_{\varepsilon tx}^{2(\theta)}\}dxdt + b_0\left\|\hat{u}_\varepsilon^{(\theta)}\right\|_1^2 - 3\mu^2\sigma^{-1}\left\|\hat{u}_\varepsilon^{(\theta)}\right\|_2^2,\end{aligned} \tag{28}$$

где $\sigma, C\left(\sigma^{-1}\right)-$ коэффициенты неравенство Коши $C\left(\sigma^{-1}\right) = 3\mu^2\sigma^{-1}$, учитывая условий теоремы 1, $b_0 - C(\sigma^{-1}) \geq \delta > 0$, where $b_0 = \min\{B_1, \mu, \lambda^2\mu + b_2 \geq b_2 > 0\}$. Тогда из неравенство (28) получим снизу следующую оценку

$$\left|-2\int\limits_Q e^{-\mu t}\hat{F}(\hat{u}_\varepsilon^{(\theta-1)})P\hat{u}_\varepsilon^{(\theta)}dxdt\right| \geq \varepsilon\left\|\frac{\partial\Delta\hat{u}_\varepsilon^{(\theta)}}{\partial t}\right\|_0^2 + \delta\left\|\hat{u}_\varepsilon^{(\theta)}\right\|_2^2. \tag{29}$$

Применяя неравенство Коши с σ к тождеству (27), справа получим неравенство

$$\begin{aligned}&\left|-2\int\limits_Q e^{-\mu t}\hat{F}\left(\hat{u}_\varepsilon^{(\theta-1)}\right) P\hat{u}_\varepsilon^{(\theta)} dxdt\right| \leq \\ &\leq \left|2\left(\hat{g} + \tfrac{\hat{f}(x,t,\lambda)}{f_0(x,t)}\left[\Phi_0 + \tfrac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\lambda^2 e^{i\lambda\ell_0}\hat{u}_\varepsilon^{(\theta-1)}(x,t,\lambda)d\lambda\right]; e^{-\mu t}P\hat{u}_\varepsilon^{(\theta)}\right)_0\right| \leq \\ &\leq 11\mu^2\sigma^{-1}\left\|\hat{u}_\varepsilon^{(\theta)}\right\|_2^2 + 8\sigma\mu^2\left[\|\hat{g}\|_1^2 + \rho^{-2}\left\|\hat{f}\right\|_{C^1(Q)}^2\left(T_0\|\varphi\|_{W_2^3(Q)}^2 + \|g_0\|_1^2\right)\right] + \\ &+10c_1\sigma\mu^2\rho^{-2}\|f_0\|_{C^1(Q)}\left\|\hat{f}\right\|_{C^1(Q)}\int\limits_{-\infty}^{\infty}\left(1+|\lambda|^2\right)^3\left\|\hat{u}_\varepsilon^{(\theta-1)}\right\|_2^2 d\lambda.\end{aligned} \tag{30}$$

Объединяя неравенства (29) и (30), получим

$$\begin{aligned}&\varepsilon\left\|\tfrac{\partial\Delta\hat{u}_\varepsilon^{(\theta)}}{\partial t}\right\|_0^2 + \left(b_0 - c(\sigma^{-1})\right)\left\|\hat{u}_\varepsilon^{(\theta)}\right\|_2^2 \leq \\ &\leq 8\sigma\mu^2\left[\|\hat{g}\|_1^2 + \rho^{-2}\left\|\hat{f}\right\|_{C^1(Q)}^2\left(T_0\|\varphi\|_{W_2^3(Q)}^2 + \|g_0\|_1^2\right)\right] + \\ &+10c_1\sigma\mu^2\rho^{-2}\|f_0\|_{C^1(Q)}\left\|\hat{f}\right\|_{C^1(Q)}\int\limits_{-\infty}^{\infty}\left(1+|\lambda|^2\right)^3\left\|\hat{u}_\varepsilon^{(\theta-1)}\right\|_2^2 d\lambda.\end{aligned} \tag{31}$$

Применяя теоремы вложения Соболева $\left\|\hat{f}\right\|_{C^1(Q)}^2 \leq c_3\left\|\hat{f}\right\|_{W_2^3(Q)}^2$ к неравенству (31), получим

$$\begin{aligned}&\varepsilon\left\|\tfrac{\partial\Delta\hat{u}_\varepsilon^{(\theta)}}{\partial t}\right\|_0^2 + \left(b_0 - c(\sigma^{-1})\right)\left\|\hat{u}_\varepsilon^{(\theta)}\right\|_2^2 \leq \\ &\leq 8\sigma c_3\mu^2\left[\|\hat{g}\|_1^2 + \rho^{-2}\left\|\hat{f}\right\|_{W_2^3(Q)}^2\left(T_0\|\varphi\|_{W_2^3(Q)}^2 + \|g_0\|_1^2\right)\right] + \\ &+10c_1c_3\sigma\mu^2\rho^{-2}\|f_0\|_{C^1(Q)}\left\|\hat{f}\right\|_{W_2^3(Q)}^2\int\limits_{-\infty}^{\infty}\left(1+|\lambda|^2\right)^3\left\|\hat{u}_\varepsilon^{(\theta-1)}\right\|_2^2 d\lambda.\end{aligned} \tag{32}$$

Разделяя неравенства (32) на $b_0 - C(\sigma^{-1}) \geq \delta > 0$ где $C(\sigma^{-1}) = 14\mu^2\sigma^{-1}$, умножая $\left(1+|\lambda|^2\right)^3$ и интегрируя по λ от $-\infty$ и до ∞, получим вторую рекуррентную формулу

$$\frac{\varepsilon}{\delta}\left\langle \frac{\partial \Delta \hat{u}_\varepsilon^{(\theta)}}{\partial t}\right\rangle_0^2 + \left\langle \hat{u}_\varepsilon^{(\theta)}\right\rangle_2^2 \leq A_1 + 10c_1c_3\sigma\mu^2\rho^{-2}\left\|f_0\right\|_{C^1(Q)}\left\langle \hat{f}\right\rangle_3^2\left\langle \hat{u}_\varepsilon^{(\theta-1)}\right\rangle_2^2, \tag{33}$$

Введем обозначение

$$A_1 \equiv 8\mu^2c_3\sigma\delta^{-1}\left[\langle \hat{g}\rangle_1^2 + \rho^{-2}\left\langle \hat{f}\right\rangle_3^2\left(T_0\left\|\varphi\right\|_{W_2^3(Q)}^2 + \left\|g_0\right\|_1^2\right)\right],$$

учитывая условий теоремы 1, что

$$10c_1c_3\sigma\mu^2\rho^{-2}\left\|f_0\right\|_{C^1(Q)}\left\langle \hat{f}\right\rangle_3^2 \leq q = M\left\langle \hat{f}\right\rangle_3^2 < 1,$$

из рекуррентной формулы (33), получим справедливость (II) оценки. Действительно, для этого в качестве «начального приближения» возьмем функцию $\left\{\hat{u}_\varepsilon^{(-1)}\right\} \equiv \{0\}$. Тогда для «нулевого приближения» имеем

$$\frac{\varepsilon}{\delta}\left\langle \frac{\partial \Delta \hat{u}_\varepsilon^{(0)}}{\partial t}\right\rangle_0^2 + \left\langle \hat{u}_\varepsilon^{(0)}\right\rangle_2^2 \leq 8\mu^2c_3\sigma\delta^{-1}\left[\langle \hat{g}\rangle_1^2 + \rho^{-2}\left\langle \hat{f}\right\rangle_3^2\left(T_0\left\|\varphi\right\|_{W_2^3(Q)}^2 + \left\|g_0\right\|_1^2\right)\right] \equiv A_1.$$

Продолжая этот процесс, методом индукции получим вторую апри-орную оценку для любой функции $\left\{\hat{u}_\varepsilon^{(\theta)}\right\}$, $\forall\theta \geq 1$,

$$\frac{\varepsilon}{\delta}\left\langle \frac{\partial \Delta \hat{u}_\varepsilon^{(\theta)}}{\partial t}\right\rangle_0^2 + \left\langle \hat{u}_\varepsilon^{(\theta)}\right\rangle_2^2 \leq A_1\sum_{\tau=0}^{\theta} q^{(\tau)} \leq 2A_1.$$

Отсюда, как доказательстве оценки $I)$, легко получается оценка $II)$.

Лемма-1 доказана.

Введём новую функцию из $W(Q,\mathbb{R})$ по формуле $\hat{\vartheta}_\varepsilon^{(\theta)} = \hat{u}_\varepsilon^{(\theta)} - \hat{u}_\varepsilon^{(\theta-1)};\ \varepsilon > 0:\ \theta = 1,2,3...,$ $\left\{\hat{u}_\varepsilon^{(-1)}\right\} \equiv \{0\}$. Тогда для неё справедлива следующая лемма.

Лемма 2: *Пусть выполнены все условия теоремы 1. Тогда для функции* $\left\{\hat{\vartheta}_\varepsilon^{(\theta)}\right\} \in W(Q,\mathbb{R})$ *справедливы следующие оценки:*

$(III).\ \frac{\varepsilon}{\delta}\left\langle \hat{\vartheta}_{\varepsilon tt}^{(\theta)}\right\rangle_0^2 + \left\langle \hat{\vartheta}_\varepsilon^{(\theta)}\right\rangle_1^2 \leq Aq^{(\theta)},$

$(IV).\ \frac{\varepsilon}{\delta}\left\langle \frac{\partial \Delta \hat{\vartheta}_\varepsilon^{(\theta)}}{\partial t}\right\rangle_0^2 + \left\langle \hat{\vartheta}_\varepsilon^{(\theta)}\right\rangle_2^2 \leq A_1q^{(\theta)}.$

Доказательство леммы 2: Из (17)-(19) для функции $\hat{\vartheta}_\varepsilon^{(\theta)} \in W(Q,\mathbb{R})$ получим следующую задачу.

$$\begin{aligned} L_\varepsilon\hat{\vartheta}_\varepsilon^{(\theta)} &= -\varepsilon\frac{\partial\Delta\hat{\vartheta}_\varepsilon^{(\theta)}}{\partial t} + L_0\hat{\vartheta}_\varepsilon^{(\theta)} + \lambda^2\hat{\vartheta}_\varepsilon^{(\theta)} = \\ &= \frac{\hat{f}(x,t,\lambda)}{\sqrt{2\pi}f_0(x,t)}\int\limits_{-\infty}^{+\infty}\lambda^2e^{i\lambda\ell_0}\,\hat{\vartheta}_\varepsilon^{(\theta-1)}(x,t,\lambda)\,d\lambda \equiv \hat{F}\left(\hat{\vartheta}_\varepsilon^{(\theta-1)}\right) \end{aligned} \tag{34}$$

с нелокальными краевыми условиями периодического типа:

$$\gamma D_t^q\,\hat{\vartheta}_\varepsilon^{(\theta)}\Big|_{t=0} = D_t^q\,\hat{\vartheta}_\varepsilon^{(\theta)}\Big|_{t=T};\ q = 0,1,2, \tag{35}$$

$$D_x^p\hat{\vartheta}_\varepsilon^{(\theta)}\Big|_{x=0} = D_x^p\hat{\vartheta}_\varepsilon^{(\theta)}\Big|_{x=1} = 0;\ p = 0,1, \tag{36}$$

где $\varepsilon > 0, \quad \theta = 1,2,...,\left\{\hat{u}_\varepsilon^{(-1)}\right\} \equiv \{0\}$.

Следовательно, как и в доказательстве леммы 1, для функции $\left\{\hat{\vartheta}_{\varepsilon}^{(\theta)}\right\} = \left\{\hat{u}_{\varepsilon}^{(\theta)}\right\} - \left\{\hat{u}_{\varepsilon}^{(\theta-1)}\right\} \in W(Q,\mathbb{R})$ получим третью рекуррентную формулу

$$\frac{\varepsilon}{\delta}\left\langle \hat{\vartheta}_{\varepsilon tt}^{(\theta)}\right\rangle_0^2 + \left\langle \hat{\vartheta}_{\varepsilon}^{(\theta)}\right\rangle_1^2 \le q\left\langle \hat{\vartheta}_{\varepsilon}^{(\theta)}\right\rangle_1^2 \tag{37}$$

то, повторяя рассуждения леммы 1, из (37) получим III) априорную оценку.

Аналогично доказывается оценка IV).

Лемма 2 доказана.

Теорема 2: Пусть выполнены все условия теоремы 1. Тогда задача (17)-(19) однозначно разрешима в $W(Q,\mathbb{R})$.

Доказательство. Теорему 2 докажем методом сжимающих отображений [9]. Пусть $\hat{L}_{\varepsilon}$ оператор, соответствующий дифференциальному выражению (17) и условиям (18), (19). Обозначим через $\hat{L}_{\varepsilon}^{-1}$ is формальный обратный оператор. Определим в пространстве $W(Q,\mathbb{R})$ оператор.

$$\hat{u}_{\varepsilon}^{(\theta)} = \hat{L}_{\varepsilon}^{-1}F_s\left(\hat{u}_{\varepsilon}^{(\theta-1)}\right) \equiv P\hat{u}_{\varepsilon}^{(\theta-1)},$$

1. Покажем, что оператор P отображает пространства $W(Q,\mathbb{R})$ в себя.

Пусть $\left\{\hat{u}_{\varepsilon}^{(\theta-1)}\right\} \in W(Q,\mathbb{R})$, тогда для решения задачи (17)-(19), справедлива утверждение леммы 1, т.е. справедлива оценка II). Отсюда следует, что для любых $\theta = 1,2,3...$ получим $\left\{\hat{u}_{\varepsilon}^{(\theta)}\right\} \in W(Q,\mathbb{R})$.

Таким образом $P : W(Q,\mathbb{R}) \to W(Q,\mathbb{R})$.

2. Покажем, что P сжимающий оператор. Пусть $\left\{\hat{u}_{\varepsilon}^{(\theta)}\right\}, \left\{\hat{u}_{\varepsilon}^{(\theta-1)}\right\} \in W(Q,\mathbb{R})$. Рассмотрим новую функцию $\left\{\hat{\vartheta}_{\varepsilon}^{(\theta)}\right\} = \left\{\hat{u}_{\varepsilon}^{(\theta)}\right\} - \left\{\hat{u}_{\varepsilon}^{(\theta-1)}\right\}$, для нее справедливо утверждение леммы 2, т.е. справедлива IV) оценка, т.е.

$$\frac{\varepsilon}{\delta}\left\langle \hat{\vartheta}_{\varepsilon ttt}^{(\theta)}\right\rangle_0^2 + \left\langle \hat{\vartheta}_{\varepsilon}^{(\theta)}\right\rangle_2^2 \le q^{(\theta)}\mathrm{const}(\widetilde{\theta}).$$

Таким образом P сжимающий оператор, теперь по известному принципу сжимающих отоб-ражений, задача (17)-(19) имеет единственное решение, принадлежащее пространству $W(Q,\mathbb{R})$ при $\varepsilon > 0$. При этом имеем $\hat{u}_{\varepsilon}^{(\theta)} \to \hat{u}_{\varepsilon}$ при $\theta \to \infty$ [6]-[8],[11],[12].

2. Семейство нагруженных интегро-дифференциальных уравнений сме-шанного типа второго рода, второго порядка.

Теперь докажем однозначную разрешимость задачи (6)-(8). При этом семейство нагруженных интегро-дифференциальных уравнений составного типа (14) с условиями (15),(16) используем в качестве «ε- регуляризиру-ющего» уравнения для уравнения (6) с условиями (7),(8) [3],[4] -[12].

Пусть $\varepsilon > 0$ при фиксированном $\{\hat{u}_{\varepsilon}\} \in W(Q,\mathbb{R})$ есть единственное решение задачи (14)-(16). Тогда при $\varepsilon > 0$ справедливо неравенство IV). По теореме о слабой компактности [13, 22], из ограниченной последовательности $\{\hat{u}_{\varepsilon}\}$ можно извлечь слабо сходящуюся под последовательность функции $\left\{\hat{u}_{\varepsilon_j}\right\}$, такую что $\hat{u}_{\varepsilon_j} \to \hat{u}$ слабо в $W(Q,\mathbb{R})$. Покажем, что предельная функция $\hat{u}(x,t,\lambda)$ удовлетворяет уравнению (6) почти всюду в $W(Q,\mathbb{R})$. Действительно, так как подпоследовательность $\left\{\hat{u}_{\varepsilon_j}\right\}$ слабо сходится в $W(Q,\mathbb{R})$, а оператор L линеен, то имеем

$$\begin{gathered} L\hat{u} - F(\hat{u}) = L\hat{u} - F(\hat{u}_{\varepsilon_j}) - \left[F(\hat{u}) - F(\hat{u}_{\varepsilon_j})\right] = \\ = \varepsilon_j\frac{\partial\Delta\hat{u}_{\varepsilon_j}}{\partial t} + L_0(\hat{u} - \hat{u}_{\varepsilon_j}) + \lambda^2(\hat{u} - \hat{u}_{\varepsilon_j}) - \left[F(\hat{u}) - F(\hat{u}_{\varepsilon_j})\right]. \end{gathered} \tag{38}$$

Переходя к пределу в (38) при $\varepsilon_j \to 0$, получим $L\hat{u} = F(\hat{u})$. Значит функция $\hat{u}(x,t,\lambda)$ будет единственным решением задачи (6)-(8) из $W(Q,\mathbb{R})$.

Тем самым доказано теорема 2. Теперь докажем теорему 1.

Так как выполнены все условие теоремы 1 используя равенства Парсеваля – Стеклова [13],[22] для решения задачи (6)-(8) получим решение задачи (1)-(5) из указанного класса U.

Замечание. Аналогично изучается линейные обратные задачи для многомерного уравнения смешанного типа второго рода, второго порядка с нелокальными кра-евыми условиями периодического типа в призматической неограниченной области.

Список литературы

[1] Аниканов Ю.Е., *К вопросу о разрешимости многомерных обратных за-дач для параболических и гиперболических уравнений*, Новосибирск, Наука, 1978, с.120

[2] Бубнов Б.А., *К вопросу о разрешимости многомерных обратных задач для параболиче-ских и гиперболических уравнений*, Новосибирск, 1987, Препринты №713,714, ВЦ.СО АН СССР, с.43,44

[3] Врагов В.Н., *Краевые задачи для неклассических уравнений математической физики*, Новосибирск: НГУ, 1983.

[4] Джамалов С.З., О корректности одной нелокальной краевой задачи с постоянными коэффициентами для уравнения смешанного типа второго рода второго порядка в пространстве *Мат. заметки СВФУ*, 2017, №4, С.17-28.

[5] Джамалов С.З., О гладкости одной нелокальной краевой задачи для многомерного уравнения смешанного типа второго рода в пространстве. *Журнал Средне Волжского мат общества*, 2019, Т.21, №1, с.24-33.

[6] Джамалов С.З., *Нелокальные краевые и обратные задачи для уравнений смешанного типа*, Монография, Ташкент, 2021, с.176.

[7] Джамалов С.З., Ашуров Р.Р. Об одной линейной обратной задаче для многомерного уравнения смешанного типа второго рода, второго порядка *Дифференциальные уравнения*, 2019, Т.55, № 1, с.34-44.

[8] Джамалов С.З., Ашуров Р.Р. Об одной линейной обратной задаче для многомерного уравнения смешанного типа первого рода, второго порядка, *Известия вузов. Математика*, 2019, №6, с.1-12.

[9] Джамалов С.З., Пятков С.Г. Некоторые классы обратных задач для уравнений смешанного типа второго порядка, *Матем, заметки СВФУ*, 2018, Т.25, №4, с.3-15.

[10] Dzhamalov S.Z., Sipatdinova B.K. On a semi-nonlocal boundary value problem for a three-dimensional mixed-type equation of the second kind in an un-bounded parallelepiped, *Lobachevskii Journal of Mathematics*, 2023, Vol. 44, No. 3, pp. 1145–1152.

[11] Dzhamalov S.Z., Ashurov R.R., Turakulov Kh.Sh. The Linear Inverse Problem for the Three- Dimensional Tricomi Equation in a Prismatic Unbounded Domain, *Lobachevskii Journal of Mathematics*, 2021, 42(15), pp. 3606–3615.

[12] Dzhamalov S.Z., Aliev M.G., Turakulov Kh.Sh. On a linear inverse problem for the three-dimensional Tricomi equation with nonlocal boundary conditions of periodic type in a prismatic unbounded domain *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Math*, 2022, (42)(1), pp.1-12.

[13] Ладыженская О.А. *Краевые задачи математической физики*, М. 1973, с.407.

[14] Лаврентьев М.М., Романов В.Г., Васильев В.Г., *Многомерные обрат-ные задачи для дифференциальных уравнений*, Новосибирск, Наука, 1969, с-67.

[15] Лионс Ж.Л., Мадженес Е., *Неоднородные граничные задачи и их приложения*, М.Мир, 1971.

[16] Хёрмандер Л., *Линейные дифференциальные операторы с частными производными*, Издательство: «Мир», Москва, 1965.

[17] Nakhushev A.M., *Loaded Equations and their Application*, (Nauka, Moscow, 2012) [in Russian].

[18] Соболев С.Л., *Некоторые применения функционального анализа в ма-тематической физике*, Издательство: «Наука», Москва, 1988.

[19] Сабитов К.Б., Мартемьянова Н.В. Нелокальная обратная задача для уравнения смешанного типа *Изв. вузов. Математика*, 2011, №2, с.71-85.

[20] Кабанихин С.И., *Обратные и некорректные задачи* , Новосибирск, Сибирское научное издательство, 2009.

[21] Кожанов А.И., Нелинейные нагруженные уравнения и обратные задачи, *Журн. вычислит. математики и мат. физики*, 2004, Т.44, №4, С.694-716.

[22] Треногин В.А., *Функциональный анализ*, Москва, Наука, с-494.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ЧИСЛЕННЫЕ АЛГОРИТМЫ РЕШЕНИЯ ОБРАТНОЙ ЗАДАЧИ ВОССТАНОВЛЕНИЯ МЛАДШЕГО КОЭФФИЦИЕНТА УРАВНЕНИЯ СУБДИФФУЗИИ

МУРАТ СУЛТАНОВ[1], ВЛАДИМИР МИСИЛОВ[2,3], ЕРКЕБУЛАН НУРЛАНУЛЫ[1]

[1]Международный казахско-турецкий университет им. Х. А. Ясави, Туркестан, Казахстан
[2]Институт математики и механики им. Н. Н. Красовского УрО РАН, Екатеринбург, Россия
[3]Уральский федеральный университет, Екатеринбург, Россия

e-mail: murat.sultanov@ayu.edu.kz

Аннотация. Работа посвящена исследованию дифференциального уравнения с дробной производной по времени. Рассматривается обратная задача, состоящая в нахождении неизвестной функции а также зависящего от времени младшего коэффициента (потенциала). Подход к решению обратной задачи основан на сведении ее к решению нелинейного операторного уравнения, получаемого из дополнительного условия. Для его решения используется итеративный метод Левенберга-Марквардта. На каждой итерации для решения вспомогательных прямых задач используются конечно-разностные схемы и метод прогонки для решения СЛАУ. Разработанный параллельный алгоритм реализован для многоядерных процессоров с использованием технологии OpenMP. Проведены численные эксперименты для исследования производительности разработанного алгоритма.

Ключевые слова: Дробные производные, уравнение субдиффузии, обратные задачи, младший коэффициент, метод Левенберга-Марквардта.

Предметная классификация AMS: 35R30; 65R32.

1. Введение

Дробные дифференциальные уравнения являются математическим аппаратом для описания многих реальных физических процессов, например, аномальной диффузии [1, 2, 3]. Поэтому разработка численных алгоритмов решения прямых и обратных задач для таких уравнений является актуальной задачей.

Классическая начально-краевая задача для дифференциального уравнения состоит в определении решения из уравнения и дополнительных граничных и начальных условий. К обратным задачам относятся задачи определения неизвестных коэффициентов уравнения или неизвестных краевых или начальных условий [4]. При этом для обеспечения единственности решения используется априорная информация. Такие задачи часто являются некорректными. Для достижения корректности используются методы регуляризации.

Существует широкий набор вычислительных методов для численного решения задач для дробных уравнений диффузии [5, 6, 7, 8, 9, 10, 11, 12]. Отметим, что вычислительные затраты при решении задач для дробных дифференциальных уравнений значительно выше,

чем для классических уравнений целого порядка. Поэтому актуальной проблемой является построение эффективных параллельных алгоритмов [13, 14, 15].

Данная работа посвящена построению и реализации параллельного алгоритма решения обратной задачи о восстановлении зависящего от времени младшего коэффициента (потенциала) для уравнения диффузии с дробной производной по времени. После применения конечно-разностных схем прямая (начально-краевая) задача сводится к решению систем линейных алгебраических уравнений (СЛАУ). Для решения систем используется метод параллельной прогонки. Обратная задача сводится к решению нелинейного операторного уравнения. Для его решения используется метод Левенберга-Марквардта. Параллельный вычислительный алгоритм решения обратной задачи реализован на многоядерных процессорах с использованием технологии параллельного программирования OpenMP.

Статья организована следующим образом. В разделе 2 приведены постановки прямой и обратной задач. В разделе 3 описан метод решения начально-краевой задачи, способ дискретизации и аппроксимации уравнения, разностная схема. В разделе 4 описан метод решения обратной задачи. Раздел 5 посвящен параллельной реализации алгоритма на многоядерном процессоре и результатам численных экспериментов.

2. Постановка задачи

Работа посвящена исследованию дифференциального уравнения с дробной производной по времени

$$\begin{aligned} &\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} - \Delta U(x,t) + m(t)n(x,t)U(x,t) = d(x,t), \quad x \in \Omega,\ 0 < t \le T, \\ &U(x,0) = 0, \qquad x \in \Omega, \\ &U(x,t) = 0, \qquad x \in \delta\Omega, 0 < t \le T, \end{aligned} \tag{1}$$

где Ω — ограниченная область в $\mathbb{R}^d$, $d \le 3$ с достаточно гладкой границей $\delta\Omega$, $m(t)$, $n(x,t)$ — коэффициентные функции (потенциал), $d(x,t)$ — правая часть (функция источника).

Производная $\partial^\alpha U(x,t)/\partial t^\alpha$ рассматривается в смысле Капуто:

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int\limits_0^t \frac{f^{(m)}(s)}{(t-s)^{\alpha-m+1}} ds,$$

где $m = \lceil \alpha \rceil$ $(m \le \alpha \le m+1, m \in \mathbb{Z})$, $t > 0$.

Если функции $m(t)$, $n(x,t)$, $d(x,t)$ заданы, то задача нахождения неизвестной функции $U(x,t)$ является классической начально-краевой прямой задачей.

В работе рассматривается обратная задача, состоящая в одновременном нахождении неизвестной функции $U(x,t)$, а также зависящего от времени младшего коэффициента (потенциала) $m(t)$ при известных функциях $n(x,t)$, $d(x,t)$. В качестве дополнительной информации дается интегральное условие

$$R(t) = \int\limits_\Omega U^2(x,t)dx, \quad 0 < t \le T. \tag{2}$$

3. Метод решения прямой задачи

Далее для простоты рассматривается случай $d = 1$ и $\Omega = [0, \gamma]$. Для дискретизации уравнения (1) на сетке пространственный отрезок $[0, \gamma]$ разбивается равномерно M точками с шагом $h = \Delta x = \gamma/M$. Временной отрезок $[0, T]$ разбивается N точками с шагом $\tau = \Delta t = T/N$. Обозначим узловые точки $x_i = ih, \quad i \in \{0, 1, ..., M\}$ и $t_j = j\tau, \quad j \in \{0, 1, ..., N\}$. Значения функций в узловых точках обозначим как $U_{i,j} = U(x_i, t_j)$.

Для аппроксимации дробной производной Капуто воспользуемся формулой первого порядка [8]

$$D_t^\alpha U_{i,p} \cong \sigma_{\alpha,\tau} \sum_{j=1}^{p} w_j^{(\alpha)} (U_{i,p-j+1} - U_{i,p-j}),$$
$$\sigma_{\alpha,\tau} = \frac{1}{\Gamma(1-\alpha)(1-\alpha)\tau^\alpha}, \quad w_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}. \tag{3}$$

Применяя неявную схему к уравнению (1), получим на временном шаге p для внутренних точек $i \in \{1, ..., M-1\}$ разностные уравнения

$$\sigma_{\alpha,\tau} \sum_{j=1}^{p} w_j^{(\alpha)} (U_{i,p-j+1} - U_{i,p-j}) = \frac{U_{i-1,p} - 2U_{i,p} + U_{i+1,p}}{h^2} - m_p n_{i,p} U_{i,p} + d_{i,p} \ . \tag{4}$$

Преобразуя, получаем

$$\sigma_{\alpha,\tau}(U_{i,p} - U_{i,p-1}) + \sigma_{\alpha,\tau} \sum_{j=2}^{p} w_j^{(\alpha)} (U_{i,p-j+1} - U_{i,p-j}) =$$
$$= \frac{1}{h^2} U_{i-1,p} + \left(-m_p n_{i,p} - \frac{2}{h^2} \right) U_{i,p} + \frac{1}{h^2} U_{i+1,p} + d_{i,p} \ ;$$

или

$$-aU_{i-1,p} + c_{i,p} U_{i,p} - bU_{i+1,p} = \sigma_{\alpha,\tau} \left(U_{i,p-1} - \sum_{j=2}^{p} w_j^{(\alpha)} (U_{i,p-j+1} - U_{i,p-j}) \right) + d_{i,p} \ , \tag{5}$$

где

$$a = b = \frac{1}{h^2}, \quad c_{i,p} = \sigma_{\alpha,\tau} + m_p n_{i,p} + \frac{2}{h^2}.$$

Запишем разностные уравнения (5) во внутренних точках $i \in \{1, 2, ..., M-1\}$ в виде системы линейных алгебраических уравнений (СЛАУ)

$$A_p U_p = F_p \ , \tag{6}$$

где

$$A_p = \begin{bmatrix} c_{1,p} & -b & & & \\ -a & c_{2,p} & -b & & \\ & \ddots & \ddots & \ddots & \\ & & -a & c_{M-2,p} & -b \\ & & & -a & c_{M-1,p} \end{bmatrix},$$

вектор неизвестных

$$U_n = [U_{1,p}, \ U_{2,p}, ..., \ U_{M-1,p}]^\intercal \, ,$$

вектор правой части

$$F_n = [f_{1,p} + aU_{0,p},\ f_{2,p}, ..., f_{M-2,p},\ f_{M-1,p} + bU_{M,p}]^\mathsf{T},$$
$$f_{i,p} = \sigma_{\alpha,\tau}\left(U_{i,p-1} - \sum_{j=2}^{p} w_j^{(\alpha)}(U_{i,p-j+1} - U_{i,p-j})\right) + d_{i,p}. \tag{7}$$

Алгоритм численного решения начально-краевой задачи (1) сводится к решению СЛАУ (6) с трехдиагональной матрицей на каждом последовательном временном шаге p. В работе [14] показана безусловная устойчивость применяемой разностной схемы. Для решения системы (6) используется метод прогонки.

4. Метод решения обратной задачи

В работе [7] доказана теорема о единственности решения обратной задачи.

Теорема 1. *Пусть* $n \equiv 1$ *и* $d \in C^1(0,T); H^2(\Omega) \cup H_0^1(\Omega)$. *Возьмем* $m,\ \tilde{m} \in C^1[0,1]$ *такие, что* $m(t),\ \tilde{m}(t) \geq 0,\ t \in [0,T]$ *и* $U_m(x,t), U_{\tilde{m}}(x,t)$ *— решения прямой задачи* (1) *с заданием коэффициентной функции* $m(t)$ *и* $\tilde{m}(t)$ *соответственно.*

Если для $t \in [0,T]$

$$\|U_m(\cdot,t)\|^2_{L^2(\Omega)} = R(t) = \|U_{\tilde{m}}(\cdot,t)\|^2_{L^2(\Omega)}, \tag{8}$$

то

$$U_m = U_{\tilde{m}}, \quad m = \tilde{m}.$$

Для численного решения обратной задачи нахождения пары неизвестных функций $[U(x,t), m(t)]$ из уравнения (1) с дополнительным условием (2) можно сформулировать операторное уравнение

$$Q(m) = R(t), t \in [0,T], \tag{9}$$

где Q — нелинейный оператор

$$Q : m \in H^2(0,T) \mapsto \int_\Omega U^2(x,t;m)dx \in L^2(0,T).$$

Для решения нелинейного уравнения (9) используется итеративный метод Левенберга-Марквардта [7, 16]:

$$m^{k+1} = m^k + \Delta^k, \tag{10}$$

где m^k — приближенное решение на k-й итерации. В точке m^k линеаризуем оператор Q

$$Q(m^{k+1}) \simeq Q(m^k) + Q'(m^k)(m - m^k)$$

и найдем неизвестную поправку Δ^k из предположения, что m^{k+1} минимизирует регуляризованную невязку

$$\Delta^k = \arg\min_{\Delta^k}\left(\left\|Q'(m^k)\Delta^k - [R - Q(m^k)]\right\|^2_{L^2(0,T)} + \mu\left\|\Delta^k\right\|^2_{H^2(0,T)}\right),$$

где μ — параметр регуляризации.

Пусть $\{\xi_l\},\ l = 1, 2, ...$ — множество базисных функций в $H^2(0, T)$. Тогда

$$m^k(t) \approx m_L^k(t) = \sum_{l=1}^{L} z_l^k \xi_l(t),$$

где $L \in \mathbb{N},\ z_l^k,\ l = 1, 2, 3, ..., L$ — коэффициенты разложения. Приближенное решение m^k можно представить как вектор $z^k = (z_1^k, z_2^k, ..., z_L^k)$ размерности L.

Тогда итеративный процесс (10) примет вид

$$z^{k+1} = z^k + \Delta_z^k, \quad k = 1, 2, 3, ...,$$

где Δ_z^k находится из задачи минимизации

$$\Delta_z^k = \arg \min_{\Delta_z^k \in \mathbb{R}^L} \left(\tau \left\| \Delta_z^k (B^k)^* - (R_N - S_N^k) \right\|_2^2 + \mu \Delta_z^k G (\Delta_z^k)^\intercal \right), \tag{11}$$

где

$$B^k = \{b_{pl}\}_{N \times L}, \quad b_{pl} = \frac{S(t_p, (z_1^k, ..., z_l^k + \delta, ..., z_L^k)) - S(t_p, z^k)}{\delta},$$

$$G = \{(g_{lr})\}_{L \times L}, \quad g_{lr} = (\xi_l, \xi_r)_{H^2(0,T)},$$

$$S(t, z^k) = \int_\Omega U^2(x, t; m_L^k) dx, \quad m_L^k(t) = \sum_{l=1}^{L} z_l^k \xi_l(t),$$

$$S_N^k = \left(S(t_1, z^k), S(t_2, z^k), ..., S(t_N, z^k) \right),$$

$$R_N = \left(R(t_1), R(t_2), ..., R(t_N) \right),$$

δ – шаг численного дифференцирования Q.

Задача (11) сводится к системе линейных алгебраических уравнений

$$\left(\frac{\mu}{\tau} G + \left(B^k\right)^* B^k \right) \Delta_z^k = \left(B^k\right)^* \left(R_N - S_N^k\right). \tag{12}$$

Таким образом, алгоритм решения обратной задачи (1) о восстановлении зависимого от времени младшего коэффициента имеет следующий вид

(1) Взять начальное приближение, например $m^0 = 0$ ($z^0 = 0$).
(2) Инициализировать счетчик итераций $k = 0$.
(3) На очередной итерации $k \Leftarrow (k+1)$ решить $(L+1)$ прямых задач (1) с подстановкой:
   - $m(t) = m_L^k \Longleftarrow z^k$,
   - $m(t) = m_L^k \Longleftarrow \left(z_1^k, ..., z_l^k + \delta, ..., z_L^k\right), \quad l = 1, 2, ..., L.$

   При решении каждой прямой задачи нужно на каждом последовательном временном шаге p решить СЛАУ (6) методом прогонки.

(4) Составить и решить СЛАУ (11), найдя Δ_z^k. Для этого используются градиентные методы, например, метод бисопряженных градиентов [17].
(5) Вычислить $z^{k+1} = z^k + \Delta_z^k$.
(6) Проверить критерий останова $\left\|R_N - S_N^k\right\| / \|R_N\| \le \varepsilon, \quad \varepsilon < 1$. Если критерий не выполнен, перейти к следующей итерации (вернуться на шаг 3).

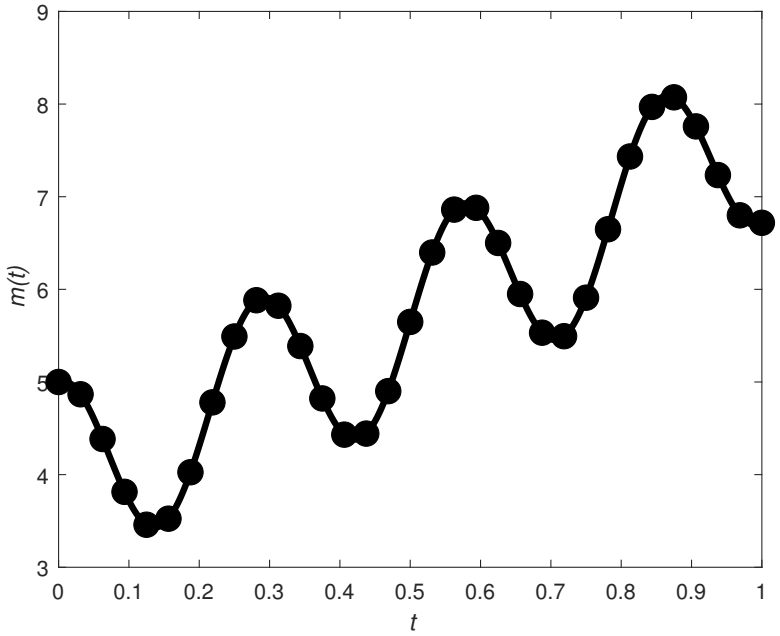


Рис. 1. Точное (сплошная линия) и приближенное (точки) решения $m(t)$ обратной задачи для модельного примера.

5. Параллельная реализация

Для ускорения решения задачи используются параллельные вычисления на многоядерном процессоре с векторной архитектурой использовалась технология OpenMP. В работе [15] предложено использовать алгоритм параллельной прогонки для распределения работы по решению СЛАУ (6) по потокам OpenMP.

В данной работе распараллеливание осуществляется другим способом. Прямые задачи на шаге (3) алгоритма можно решать независимо друг от друга, поэтому эти задачи распределяются по потокам OpenMP и решаются в параллельной секции. СЛАУ (11) имеет малый размер, поэтому может решаться в последовательном режиме.

Рассмотрим применение параллельного алгоритма к численному решению обратной задачи для уравнения диффузии с дробной производной по времени. Приводятся результаты экспериментов, проведенных на процессоре Intel i9-12900K (8 P-ядер).

Рассматривается обратная задача для уравнения

$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} - \frac{\partial^2 U(x,t)}{\partial x^2} + (\cos(7\pi t) + e^t + 3)U(x,t) = \sin(\pi x)\sin(4\pi t)e^{-t}, \tag{13}$$

$$0 < \alpha < 1, \quad 0 \le x \le 1, \quad 0 \le t \le 1,$$

с краевыми условиями

$$U(0,t) = 0, \quad U(1,t) = 0,$$

начальным условием

$$U(x,0) = 0.$$

Эксперименты проводились для порядка производной $\alpha = 0.3$. Использовалась сетка размером $M = 1024$, $N = 256$, шаг численного дифференцирования $\delta = 0.01$.

Задача решена описанным выше алгоритмом. Начальное приближение $m^0 = 0$. Число членов в разложении решения $L = 15$, параллельная программа использовала 16 потоков. Базисные функции

$$\{\xi_l\} = \{1, \sqrt{2}\cos(\pi t), \sqrt{2}\cos(2\pi t), ..., \sqrt{2}\cos((L-1)\pi t)\}, \quad l = 1, ..., L.$$

Условие останова $\varepsilon = 0.01$. Итеративный процесс Левенберга-Марквардта потребовал 15 итераций.

Время счёта однопоточной программы составило 168 с (2.8 мин.) Время счета многопоточной программы составило 40 с. Погрешность решения $\left\|m_L^k - m\right\| / \|m\| = 0.005$.

На рис. 1 показаны точное (сплошная линия) решение $m(t) = \cos(7\pi t) + e^t + 3$ и приближенное (точки) решение $m_L^j(t)$.

6. Заключение

В данной работе построен параллельный алгоритм решения обратной задачи восстановления зависимого от времени младшего коэффициента для уравнения диффузии с дробной производной по времени. Для решения обратной задачи используется итеративный метод Левенберга-Марквардта. На каждой итерации метода необходимо решить набор вспомогательных прямых (начально-краевых) задач. Алгоритм решения начально-краевой задачи основан на использовании неявной конечно-разностной схемы для аппроксимации дифференциального уравнения. После дискретизации и аппроксимации задачи сводятся к решению систем линейных алгебраических уравнений, решаемых методом прогонки. Реализованный параллельный алгоритм обеспечивает ускорение в 4 раза при решении модельной задачи на 8-ядерном процессоре.

Работа поддержана Министерством науки и высшего образования Республики Казахстан (проект №AP09258836).

Список литературы

[1] Kilbas A.A., Srivastava H.M., Trujillo J.J., *Theory and Applications of Fractional Differential Equations*. Amsterdam, The Netherlands: Elsevier, 2006.

[2] Scher H., Montroll E.W., Anomalous transit-time dispersion in amorphous solids, *Phys. Rev. B.* **12**, 1975. https://doi.org/10.1103/PhysRevB.12.2455

[3] Kou S.C., Stochastic modeling in nanoscale biophysics: Subdiffusion within proteins, *Ann. Appl. Stat.*, Vol.2, 2008. pp.501–535. https://doi.org/10.1214/07-AOAS149

[4] Samarskii A. A., Vabishchevich P. N., *Numerical Methods for Solving Inverse Problems of Mathematical Physics*, De Gruyter, 2008.

[5] Jin B., Rundell W., A tutorial on inverse problems for anomalous diffusion processes, *Inverse Probl.*, Vol.31, 2015, Art.No. 035003. https://doi.org/10.1088/0266-5611/31/3/035003

[6] Wei T., Zhang Z.Q. Reconstruction of a time-dependent source term in a time-fractional diffusion equation, *Engineering Analysis with Boundary Elements*, Vol.37, No.1, 2013 pp.23–31. https://doi.org/10.1016/j.enganabound.2012.08.003

[7] Jiang S., Wei T., Recovering a time-dependent potential function in a time fractional diffusion equation by using a nonlinear condition, *Inverse Problems in Science and Engineering*, Vol.29, No.2, 2021, pp.174–195. https://doi.org/10.1080/17415977.2020.1782399

[8] Podlubny, I. *Fractional differential equations*, Academic Press, 2002.

[9] Baleanu D., Diethelm K., Scalas E. *Fractional Calculus: Models and Numerical Methods*, World Scientific, Singapore, 2012.

[10] Li, C., Zeng F., *Numerical Methods for Fractional Calculus*, Chapman and Hall/CRC: London, UK, 2019.

[11] Sultanov M.A., Durdiev D.K., Rahmonov A.A., Construction of an Explicit Solution of a Time-Fractional Multidimensional Differential Equation, *Mathematics*, Vol.9, No.17, 2021, Art.No.2052. https://doi.org/10.3390/math9172052

[12] Omran A.K., Zaky M.A., Hendy A.S., Pimenov V.G. An Efficient Hybrid Numerical Scheme for Nonlinear Multiterm Caputo Time and Riesz Space Fractional-Order Diffusion Equations with Delay, *Journal of Function Spaces*, Vol.2021, 2021. Art. No. 5922853. https://doi.org/10.1155/2021/5922853

[13] Белевцов Н.С., Лукащук С.Ю., Параллельный алгоритм численного решения дробно-дифференциального обобщения уравнения Пуассона, *Параллельные вычислительные технологии (ПаВТ'2019)*, 2019, с.165–174.

[14] Sultanov M.A., Akimova E.N., Misilov, V.E., Nurlanuly, Y., Parallel Direct and Iterative Methods for Solving the Time-Fractional Diffusion Equation on Multicore Processors, *Mathematics*, Vol.10, No.3, 2022, Art.No.323. https://doi.org/10.3390/math10030323

[15] Акимова Е.Н., Султанов М.А., Мисилов В.Е., Нурланулы Е. Алгоритм параллельной прогонки для решения прямой и обратной задач дробного уравнения диффузии, *Вычислительные методы и программирование*, No.23, 2022. с.275-287. https://doi.org/10.26089/NumMet.v23r417

[16] Васин В.В., Еремин И.И., *Операторы и итерационные процессы фейеровского типа. Теория и приложения.*, Екатеринбург: УрО РАН, 2005, 210 с.

[17] Van der Vorst H. A., *Iterative Krylov Methods for Large Linear Systems, Cambridge Monographs on Applied and Computational Mathematics, Vol. 13*, Cambridge University Press, 2003.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

О ЛИНЕЙНОЙ ДВУХТОЧЕЧНОЙ ОБРАТНОЙ ЗАДАЧЕ ДЛЯ МНОГОМЕРНОГО ВОЛНОВОГО УРАВНЕНИЯ С ПОЛУНЛОКАЛЬНЫМИ ГРАНИЧНЫМИ УСЛОВИЯМИ

ШОХРУХ ХУДОЙКУЛОВ[1], КУРБАНИЁЗ МАМБЕТСАПАЕВ[2], СИРОЖИДДИН ДЖАМАЛОВ[3], ШЕРЗОД ЮСУПОВ[4]

Институт математики имени В.И.Романовского Академии наук Республики Узбекистан, Ташкент, Узбекистан
e-mail: [1]xudoykulov111194@mail.ru., [2] mr.kurbaniyaz@gmail.com., [3]siroj63@mail.ru, [4]sherzod.yusupov.2020@inbox.ru

Аннотация. В данной работе исследуется корректность одной линейной двухточечной обратной задачи для многомерного волнового уравнения. Методами априорных оценок, последовательности приближений и сжимающихся отображений доказывается однозначная разрешимость обобщённого решения линейной двухточечной обратной задачи для многомерного волнового уравнения.

Ключевые слова: многомерное волновое уравнение, линейная двухточечная обратная задача, однозначное разрешимости обобщённого решения, методы априорных оценок, последовательности приближение и сжимающихся отображение.

Предметная классификация AMS: 35M10

1. Введение и постановка задачи

В связи с существенно возросшими за последние десятилетия возможностями вычислительной техники в прикладной математике начинают находить применения сложные математические модели, учитывающие значительно большее количество физических факторов [2, 3, 4]. В этой связи следует особо отметить, что процессы вибрации тесно связаны именно с многоточечными обратными задачами для гиперболических уравнений [5, 6]. С этой целью в данной работе с использованием результатов [5, 6] исследуется однозначная разрешимость некоторой линейной двухточечной обратной задаче (ЛДОЗ) для многомерного волнового уравнения.

Пусть Ω-односвязная область в пространстве R^n с достаточно гладкой границей $\partial\Omega$. В области $G = \Omega \times (0,T) \times (0,l) = Q \times (0,l) \subset R^{n+2}$ рассмотрим многомерное волновое уравнение

$$Lu = u_{tt} - \Delta_x u - u_{yy} + c(x,t)u = g(x,t,y) + \sum_{i=1}^{2} h_i(x,t) f_i(x,t,y), \tag{1}$$

где $\Delta_x u = \sum_{m=1}^{n} u_{x_m x_m}$-оператор Лапласа, по переменим x, здесь $c(x,t)$, $g(x,t,y)$ и $f_i(x,t,y)$, $i = 1,2$ – заданные функции, $h_1(x,t)$, $h_2(x,t)$-неизвестные функции.

2. Линейная двухточечная обратная задача (ЛДОЗ).

Найти функции $u(x,t,y)$, $h_1(x,t)$, $h_2(x,t)$,удовлетворяющие уравнению (1) в области G, такие, что функция $u(x,t,y)$ удовлетворяет следующим полунелокальным краевым условиям:

$$\gamma D_t^p u\,|_{t=0} = D_t^p u\,|_{t=T},\ , \tag{2}$$

$$u\,|_{\partial\Omega} = 0, \tag{3}$$

$$u\,|_{y=0} = u\,|_{y=l} = 0, \tag{4}$$

при $p = 0, 1$, где $D_t^p u = \frac{\partial^p u}{\partial t^p}, D_t^0 u = u$, γ–некоторое постоянное число, отличное от нуля, величина которого будет уточнена ниже.

Кроме того, решение задачи (1)-(4) удовлетворяет дополнительным условиям

$$u(x,t,\ell_j) = \varphi_j\,(x,t), \tag{5}$$

где $\ell_j \in (0,\ell)$, $j = 1,2$ такие, что $0 < \ell_1 < \ell_2 < \ell < +\infty$, а функции $u(x,t,y)$ и $h_i(x,t)$, $i = 1,2$ принадлежат классу

$$U = \{(u, h_i,\ i = 1,2);\ u \in W_2^2(G),\ D_y^3(u_{tt}, u_{xx}, u_{xt}) \in L_2(G),\ h_i \in W_2^2(Q)\}.$$

где $D_t^p u = \frac{\partial^p u}{\partial t^p}, D_t^0 u = u$, γ–некоторое постоянное числа, отличное от нуля, величина которое будут уточнено ниже.

Введем следующие обозначения

Пусть $g_j(x,t) = g(x,t,\ell_j)$, $f_{i\,j}(x,t) = f_i(x,t,\ell_j)$, $\forall i,j = 1,2$.

Тогда через $\mathbb{F} = \{f_{i\,j}\}_{i,j=1}^2$ определим квадратную матрицу второго порядка, т.е.

$\mathbb{F} = \begin{pmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{pmatrix}$, а через $H = \det \mathbb{F} = \begin{vmatrix} f_{11}\, f_{21} \\ f_{12}\, f_{22} \end{vmatrix}$ обозначим её детерменант.

Определение 1. Обобщенным решением задачи (1)-(5) будем называть функцию $u(x,t,y) \in U$, удовлетворяющую уравнению (1) почти всюду в области G, с условиями (2)-(5).

Пусть все коэффициенты уравнения (1) достаточно гладкие функции в области Q и пусть выполнены следующие условия относительно коэффициентов и правых частей уравнения (1) и заданной функции $\varphi_j(x,t)$, $j = 1,2$.

Условие 1:

периодичность: $c(x,0) = c(x,T)$, для всех $x \in \bar{\Omega}$

нелокальные условия: $\gamma g(x,0,y) = g(x,T,y)$, $\gamma f_j(x,0,y) = f_j(x,T,y)$, $j = 1,2$;

гладкость: $g_j(x,t) = g(x,t,\ell_j) \in C_{x,t}^{0,1}(Q)$, $f_{ij}(x,t) = f(x,t,\ell_j) \in C_{x,t}^{0,1}(Q)$,

$H = |\det \mathbb{F}| \geq \eta > 0$, $f \in W_2^2(G)$, $g \in W_2^2(G)$.

Условие 2:

$\varphi_j(x,t) \in W_2^2(Q)$, $\gamma D_t^p \varphi_j|_{t=0} = D_t^p \varphi_j|_{t=T}$, $p = 0,1$; $\varphi_j|_{\partial\Omega} = 0$, $j = 1,2$. Здесь $W_2^2(Q)$-пространство Соболева с нормой $\|u\|^2_{W_2^{2,1}(Q)} = \int\limits_Q \left(u_{xx}^2 + u_{tt}^2 + u_x^2 + u_t^2 + u^2\right) dxdt$.

3. Однозначная разрешимость задачи (1)-(5)

Теорема 1.(Основной результат) Пусть выполнены вышеуказанные условия 1 и 2 для коэффициентов и правых частей уравнения (1), кроме того, пусть $\lambda c - c_t \geq \delta_1 > 0$ для всех $(x,t) \in \overline{Q}$, здесь $\lambda = \frac{2}{T}\ln|\gamma| > 0$, $|\gamma| > 1$ и пусть существует малое положительное число σ (коэффициент неравенство Коши) такое, что $\delta_0 - 21\sigma^{-1} \geq \delta > 0$,

$q \equiv M \cdot \sum\limits_{i=1}^{2} \left\|(1 + D_y^3) f_i\right\|_{W_2^1(G)} < 1$, (где $\delta_0 = min\{\lambda, \delta_1, \lambda\left(\frac{\pi}{\ell}\right)^2\}$, M–положительное постоянное число, зависящееся от σ, λ, $mess(G)$, η^{-2}, $\sum\limits_{k=1}^{\infty} \frac{\mu_k^4}{\left(1+\mu_k^2\right)^3}$, $\mu_k = \frac{k\pi}{\ell}$, c_2– постоянное число, определяется из теоремы вложения Соболева). Тогда для любых функций

$(1+D_y^3)g \in W_2^2(G)$; $(1+D_y^3)f_i \in W_2^2(G)$, $\forall i = 1,2$; таких, что существует единственное решение задачи (1)-(5) из указанного класса U.

Для доказательства разрешимости задачи (1)-(5) сначала используем метод Фурье. А именно, решение задачи (1)-(5) ищем в виде

$$u(x,t,y) = \sum_{k=1}^{\infty} u_k(x,t)Y_k(y), \tag{A}$$

где функции $Y_k(y) = \{\sqrt{\frac{2}{\ell}}\sin\mu_k y\}$, $\mu_k = \left(\frac{\pi k}{\ell}\right)^2$, $k = 1,2,3,...$ являются решениями спектральной задачи Штурма- Лиувилля с условиями Дирихле, Известно, что система собственных функций $\{Y_k(y)\}$- полна в пространстве $L_2(0,\ell)$ и в нем образует ортонормированный базис [1, 7, 10].

Для того чтобы определить неизвестные функции необходимо выполнить некоторые формальности построения.

Рассмотрим следы уравнения (1) при $y = \ell_j$, $j = 1,2$

$$\begin{aligned} &Lu(x,t,\ell_j) = u_{tt}(x,t,\ell_j) - \Delta_x u(x,t,\ell_j) - u_{yy}(x,t,\ell_j) + \\ &+ c(x,t)u(x,t,\ell_j) = g(x,t,\ell_j) + h_1(x,t)f_{1j}(x,t) + h_2(x,t)f_{1j}(x,t);\ j = 1,2. \end{aligned} \tag{6}$$

Теперь, учитывая условие (5) и то, что $H = |\det\mathbb{F}| \geq \eta > 0$, определим формально неизвестные функции $h_j(x,t)$, $j = 1,2$ в виде
$h_1(x,t) = \frac{1}{H}[\Phi_1(x,t)f_{22}(x,t) - \Phi_2(x,t)f_{21}(x,t)]$,
$h_2(x,t) = \frac{1}{H}[\Phi_2(x,t)f_{11}(x,t) - \Phi_1(x,t)f_{12}(x,t)]$, здесь

$$\Phi_j(x,t) = \varphi_{jtt}(x,t) - \Delta_x\varphi_j(x,t) + c(x,t)\varphi_j(x,t) - g_j(x,t) + \sum_{k=1}^{\infty}\mu_k^2 u_k(x,t)\sin\mu_k\ell_j =$$

$$= L_0\varphi_j - g_j(x,t) + \sum_{k=1}^{\infty}\mu_k^2 u_k(x,t)\sin\mu_k\ell_j, \quad j = 1,2.$$

где функция $u_k(x,t)$ определяется в области $Q = \Omega\times(0,T)$ как решение следующий системы нагруженных волновых уравнений:

$$\begin{aligned} &Lu_k = u_{ktt} - \Delta_x u_k + (c+\mu_k^2)u_k = g_k + \\ &+ \frac{f_{1k}}{H}[f_{22}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty}\mu_m^2 u_m\sin\mu_m\ell_1) - f_{21}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty}\mu_m^2 u_m\sin\mu_m\ell_2)] + \\ &+ \frac{f_{2k}}{H}[f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty}\mu_m^2 u_m\sin\mu_m\ell_2) - f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty}\mu_m^2 u_m\sin\mu_m\ell_1)] \end{aligned} \tag{7}$$

с полунелокальными краевыми условиями

$$\gamma D_t^p u_k\,|_{t=0} = D_t^p u_k\,|_{t=T},\ p = 0,1; \tag{8}$$

$$u_k\,|_{\partial\Omega} = 0, \tag{9}$$

где $f_1(x,t,y) = \sum\limits_{k=1}^{\infty} f_{1k}(x,t)\sin\mu_k y$, $\quad f_1(x,t,\ell_1) = f_{11}(x,t) = \sum\limits_{k=1}^{\infty} f_{1k}(x,t)\sin\mu_k\ell_1$,
$f_2(x,t,y) = \sum\limits_{k=1}^{\infty} f_{2k}(x,t)\sin\mu_k y$, $\quad f_2(x,t,\ell_1) = f_{21}(x,t) = \sum\limits_{k=1}^{\infty} f_{2k}(x,t)\sin\mu_k\ell_1$,
$f_{ik} = \sqrt{\frac{2}{\ell}}\int\limits_0^{\ell} f_i\sin\mu_k y dy, \forall i = 1,2$; $\quad g_k = \sqrt{\frac{2}{\ell}}\int\limits_0^{\ell} g\sin\mu_k y dy$, $k = 1,2,3,...$

Доказательство. Докажем теорему 1 поэтапно. Сначала покажем, что, функция $u(x,t,y)$ при любом $j = 1,2$ удовлетворяет условию (5), т.е. $u|_{y=\ell_j} = u(x,t,\ell_j) = \varphi_j(x,t)$.

Докажем выполнение этих условие, с помощью обратных предположений. Пусть существует функция $\vartheta_j(x,t)$, $u|_{y=\ell_j} = \vartheta_j(x,t) \neq \varphi_j(x,t)$, то есть

$$u|_{y=\ell_j} = \sum_{k=0}^{\infty} u_k(x,t)\sin\mu_k\ell_j = \vartheta_j(x,t) \neq \varphi_j(x,t),$$

Тогда для функции $z_j(x,t) = \vartheta_j(x,t) - \varphi_j(x,t)$ в области Q, учитывая условия (8)-(9), умножая уравнение (7) на $\sin\mu_k\ell_j$ и суммируя по k от 1 до ∞, получим следующие нагруженные уравнения

$$\begin{aligned}
&\sum_{k=1}^{\infty} Lu_k\sin\mu_k\ell_j = \sum_{k=1}^{\infty} u_{ktt}\sin\mu_k\ell_j - \sum_{k=1}^{\infty}\Delta_x u_k\sin\mu_k\ell_j + \sum_{k=1}^{\infty}(+\mu_k^2)u_k\sin\mu_k\ell_j\\
&= \sum_{k=1}^{\infty} g_k\sin\mu_k\ell_j + \frac{\sum\limits_{k=1}^{\infty} f_{1k}\sin\mu_k\ell_j}{H}[\Phi_1 f_{22} - \Phi_2 f_{21}] + \frac{\sum\limits_{k=1}^{\infty} f_{2k}\sin\mu_k\ell_j}{H}[\Phi_2 f_{11} - \Phi_1 f_{12}]\\
&= \sum_{k=1}^{\infty} g_k\sin\mu_k\ell_j + \frac{\sum\limits_{k=1}^{\infty} f_{1k}\sin\mu_k\ell_j}{H}[f_{22}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty}\mu_m^2 u_m\sin\mu_m\ell_1) - f_{21}(L_0\varphi_2 - g_2\\
&+ \sum_{m=1}^{\infty}\mu_m^2 u_m\sin\mu_m\ell_2)] + \frac{\sum\limits_{k=1}^{\infty} f_{2k}\sin\mu_k\ell_j}{H}[f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty}\mu_m^2 u_m\sin\mu_m\ell_2)\\
&- f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty}\mu_m^2 u_m\sin\mu_m\ell_1)] = g_j + \frac{f_{1j}}{H}[f_{22}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty}\mu_m^2 u_m\sin\mu_m\ell_1)\\
&- f_{21}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty}\mu_m^2 u_m\sin\mu_m\ell_2)] + \frac{f_{2j}}{H}[f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty}\mu_m^2 u_m\sin\mu_m\ell_2)\\
&- f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty}\mu_m^2 u_m\sin\mu_m\ell_1)].
\end{aligned} \tag{10}$$

Чтобы легче было бы разобраться в формуле (10), каждые случай рассмотрим отдельно. Сначало рассмотрим случай $j = 1$. Тогда из формулы (10) получим

$$\begin{aligned}
&\vartheta_{1tt} - \Delta_x\vartheta_1 + \vartheta_1 + \sum_{k=1}^{\infty}\mu_k^2 u_k\sin\mu_k\ell_1 = g_1 +\\
&+ \frac{f_{11}}{H}[f_{22}(L_0\varphi_1 - g_1 + \sum_{k=1}^{\infty}\mu_k^2 u_k\sin\mu_k\ell_1) - f_{21}(L_0\varphi_2 - g_2 + \sum_{k=1}^{\infty}\mu_k^2 u_k\sin\mu_k\ell_2)] +\\
&+ \frac{f_{21}}{H}[f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty}\mu_m^2 u_m\sin\mu_m\ell_2) - f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty}\mu_m^2 u_m\sin\mu_m\ell_1)] =\\
&= g_1 + \frac{L_0\varphi_1 - g_1}{H}[f_{11}f_{22} - f_{12}f_{21}] + \frac{\sum\limits_{k=1}^{\infty}\mu_k^2 u_k\sin\mu_k\ell_1}{H}[f_{11}f_{22} - f_{12}f_{21}] +\\
&\frac{L_0\varphi_2 - g_2}{H}[f_{21}f_{11} - f_{21}f_{11}] + \frac{\sum\limits_{k=1}^{\infty}\mu_k^2 u_k\sin\mu_k\ell_2}{H}[f_{21}f_{11} - f_{21}f_{11}] =\\
&= g_1 + L_0\varphi_1 - g_1 + \sum_{k=1}^{\infty}\mu_k^2 u_k\sin\mu_k\ell_1 = L_0\varphi_1 + \sum_{k=1}^{\infty}\mu_k^2 u_k\sin\mu_k\ell_1.
\end{aligned} \tag{11}$$

Тогда из формул (7)-(11) для функции $z_1(x,t) = \vartheta_1(x,t) - \varphi_1(x,t) \Rightarrow \vartheta_1 = z_1 + \varphi_1$ в области Q получим следующее тождество

$$L_0(z_1+\varphi_1) + \sum_{k=1}^{\infty} \mu_k^2 u_k \sin \mu_k \ell_1 = L_0\varphi_1 + \sum_{k=1}^{\infty} \mu_k^2 u_k \sin \mu_k \ell_1. \tag{12}$$

Отсюда, получим следующую задачу

$$L_0 z_1 = z_{1\,tt} - z_{1\,xx} + z_1 = 0 \tag{13}$$

$$\gamma D_t^p z_1|_{t=0} = D_t^p z_1|_{t=T}, p = 0, 1; \tag{14}$$

$$z_1|_{\partial\Omega} = 0, \tag{15}$$

Теперь докажем единственность решения задачи (13)-(15) методом интегралов энергии [5, 6]. Для этого рассммотрим тождество $2(L_0 z_1, e^{-\lambda t} z_{1\,t}) = 0$ и интегрируя по частям тождество (13), с учетам условий теоремы 1 и краевых условий (14), (15) при $|\gamma| > 1$, получим неравенство $\|z_j\|_{W_2^1(Q)} \le 0$, откуда следует $z_1(x,t) = 0$.

Значит, задача (13)-(15) имеет единственное решение т.е. $\vartheta_1(x,t) \equiv \varphi_1(x,t)$. Отсюда получим, что задача (1)-(4) удовлетворяет условию (5), при $j = 1$, т.е. $u(x,t,\ell_1) = \varphi_1(x,t)$. Аналогично доказывается $u(x,t,\ell_2) = \varphi_2(x,t)$ в случай, когда $j = 2$.

Теперь докажем разрешимость задачи (7)-(9) методами априорных оценок и последовательных приближений [4]-[6], а именно в области Q рассмотрим семейство нагруженных волновых уравнений

$$\begin{aligned} Lu_k^{(l)} &= u_{k\,t\,t}^{(l)} - \Delta_x u_k^{(l)} + (c+\mu_k^2) u_k^{(l)} = g_k + \\ &+ \frac{f_{1k}}{H}[f_{22}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m^{(l-1)} \sin \mu_m \ell_1) - f_{21}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m^{(l-1)} \sin \mu_m \ell_2)] \\ &+ \frac{f_{2k}}{H}[f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m^{(l-1)} \sin \mu_m \ell_2) - f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m^{(l-1)} u_m \sin \mu_m \ell_1)] \\ &= F_k(u_k^{(l-1)}) \end{aligned} \tag{16}$$

$$\gamma D_t^p u_k^{(l)}|_{t=0} = D_t^p u_k^{(l)}|_{t=T},\ p = 0, 1; \tag{17}$$

$$u_k^{(l)}|_{\partial\Omega} = 0, \tag{18}$$

В далнейшем для доказательства однозначной разрешимости задачи (16)-(18) нам необходимы следующие обозначения и леммы. Определим пространство вектор-функций $W_i(Q) = \{\vartheta_k |\, \vartheta_k \in W_2^i(Q), k = 1, 2, ...\}$, где $i = 0, 1, 2$ с конечной нормой

$$\langle \vartheta_k \rangle_i^2 = \sum_{k=1}^{\infty} (1+\mu_k^2)^3 \|\vartheta_k\|_{W_2^i(Q)}^2, \quad i = 0, 1, 2; \quad W_2^0(Q) = L_2(Q), \tag{19}$$

Очевидно, что пространство $W_i(Q)$ с определённой нормой (19) является гильбертовым пространством [5]-[9]. Из определения пространств $W_2^i(Q), \quad i = 0, 1, 2$ следует $W_2(Q) \subset W_1(Q) \subset W_0(Q)$. Через $W(Q)$ ниже будем обозначать класс вектор-функций $\{\vartheta_k(x,t)\}_{k=1}^{\infty}$ таких, что $\{\vartheta_k(x,t)\}_{k=1}^{\infty} \in W_2(Q)$, удовлетворяющих соответствующим условиям (17), (18).

Определение 2. Решением задачи (16)-(18) будем называть вектор-функцию $\{\vartheta_k(x,t)\}_{k=1}^{\infty} \in W(Q)$, удовлетворяющую уравнению (16).

Лемма 1. *Пусть выполнены все условия теоремы, тогда для решения задачи* (16)-(18) *справедливы следующие оценки*:

(I). $\left\langle u_k^{(l)} \right\rangle_1^2 \le const(\hat{k},\, \hat{l}) < +\infty$,

(II). $\left\langle u_k^{(l)} \right\rangle_2^2 \le const(\hat{k},\ \hat{l}) < +\infty$.

Символом $const(\hat{k},\ \hat{l})$*здесь и далее обозначим постоянную, не зависящую от параметров* $k,\ l$.

Доказательство леммы 1:

Рассмотрим тождество

$$2(Lu_k^{(l)}, e^{-\lambda t}u_{kt}^{(l)})_0 = 2(F_k(u_k^{(l-1)}), e^{-\lambda t}u_{kt}^{(l)})_0, \tag{20}$$

где постоянную $\lambda > 0$ выберем позже.

Учитывая условия теоремы 1, интегрируя по частям тождество (20) и применяя неравенства Коши с σ [7, 8], легко получить снизу следующее неравенство

$$\begin{aligned} 2\int\limits_Q Lu_k^{(l)} \cdot e^{-\lambda t} \cdot u_{kt}^{(l)} dxdt \geq \int\limits_Q e^{-\lambda t}\{\lambda \cdot u_{kt}^{2(l)} + \lambda \cdot u_{kx}^{2(l)} + (\lambda c - c_t + \lambda\mu_k^2) \cdot u_k^{2(l)}\}dxdt - \\ - \int\limits_{\partial Q} e^{-\lambda t}\{u_{kt}^{2(l)}e_t - 2u_{kt}^{(l)}u_{kx}^{(l)}e_x + u_{kx}^{2(l)}e_t + (c+\mu_k^2)u_k^{2(l)}e_t\}ds, \end{aligned} \tag{21}$$

где $u_x^2 = \sum\limits_{m=1}^{n} u_{x_m}^2$; $\overrightarrow{e} = (e_x = (\overrightarrow{e}, x);\ e_t = (\overrightarrow{e}, t))$- единичный вектор внутренней нормали к границе ∂Q. Условия теоремы 1 обеспечивают не отрицательность интеграла по области Q. Учитывая полунелокальные краевые условия (17), (18) и условия теоремы 1, с выбором $\gamma^2 = e^{\lambda T}$, получим обращение граничных интегралов в нуль. Таким образом, из неравенства (21) получаем снизу следующее неравенство

$$2\int\limits_Q Lu_k^{(l)} \cdot e^{-\lambda t} \cdot u_{kt}^{(l)} dxdt \geq \int\limits_Q e^{-\lambda t}\{\lambda \cdot u_{kt}^{2(l)} + \lambda \cdot u_{kx}^{2(l)} + (\delta_1 + \lambda\Big(\frac{\pi}{\ell}\Big)^2) \cdot u_k^{2(l)}\}dxdt \geq \delta_0 \left\|u_k^{(l)}\right\|_1^2, \tag{22}$$

где $\delta_0 = \min\{\lambda,\ \delta_1, \lambda\big(\frac{\pi}{\ell}\big)^2\}$, $\lambda c - c_t \geq \delta_1 > 0$.

Применяя неравенство Коши с σ к тождеству (20), сверху получим

$$\begin{aligned} \left|2(F(u_k^{(l-1)}), e^{-\lambda t}u_{kt}^{(l)})_0\right| \leq \left|2\left(g_k + \frac{f_{1k}}{H}[f_{22}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty}\mu_m^2 u_m^{(l-1)}\sin\mu_m\ell_1) - \right.\right. \\ - f_{21}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty}\mu_m^2 u_m^{(l-1)}\sin\mu_m\ell_2)] + \frac{f_{2k}}{H}[f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty}\mu_m^2 u_m^{(l-1)}\sin\mu_m\ell_2) \\ \left.\left. - f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty}\mu_m^2 u_m^{(l-1)}u_m\sin\mu_m\ell_1)], e^{-\lambda t}u_{kt}^{(l)}\right)_0\right| \leq 9\sigma^{-1}\left\|u_k^{(l)}\right\|_1^2 + \\ + \sigma\left[\|g_k\|_0^2 + 2\eta^{-2}\mathfrak{F}\sum_{i,j=1}^{2}\left(T_0\|\varphi_j\|_{W_2^2(Q)}^2 + \|g_j\|_0^2\right)\|f_{ik}\|_{C(Q)}^2\right] + \\ + 2c_1\eta^{-2}\sigma\mathfrak{F}\sum_{i=1}^{2}\|f_{ik}\|_{C(Q)}^2\sum_{m=1}^{\infty}\left(1+\mu_m^2\right)^3\left\|u_m^{(l-1)}\right\|_1^2 \end{aligned} \tag{23}$$

где, $T_0 = \max\{1, \|c\|_{C(Q)}\}$, $\mathfrak{F} = \max\{\|f_{11}\|_{C(Q)}^2, \|f_{12}\|_{C(Q)}^2, \|f_{21}\|_{C(Q)}^2, \|f_{22}\|_{C(Q)}^2\}$.
Объединяя неравенства (22) и (23), получим

$$\begin{aligned} (\delta_0 - 9\sigma^{-1})\left\|u_k^{(l)}\right\|_1^2 \leq \sigma\left[\|g_k\|_0^2 + 2\eta^{-2}\mathfrak{F}\sum_{i,j=1}^{2}\left(T_0\|\varphi_j\|_{W_2^2(Q)}^2 + \|g_j\|_0^2\right)\|f_{ik}\|_{C(Q)}^2\right] + \\ + 2c_1\eta^{-2}\sigma\mathfrak{F}\sum_{i=1}^{2}\|f_{ik}\|_{C(Q)}^2\sum_{m=1}^{\infty}\left(1+\mu_m^2\right)^3\left\|u_m^{(l-1)}\right\|_1^2 \end{aligned} \tag{24}$$

Применяя теорему вложения Соболева $\|f_{ik}\|^2_{C(Q)} \le c_2 \|f_{ik}\|^2_{W_2^2(Q)}$ [7, 9] из неравенства (24), получим

$$(\delta_0 - 9\sigma^{-1}) \left\|u_k^{(l)}\right\|_1^2 \le \sigma \left[\|g_k\|_0^2 + 2c_2\eta^{-2}\mathfrak{F} \sum_{i,j=1}^{2} \left(T_0 \|\varphi_j\|^2_{W_2^2(Q)} + \|g_j\|_0^2\right) \|f_{ik}\|^2_{W_2^2(Q)}\right] + \\ + 2c_1c_2\eta^{-2}\sigma\mathfrak{F} \sum_{i=1}^{2} \|f_{ik}\|^2_{W_2^2(Q)} \sum_{m=1}^{\infty} \left(1+\mu_m^2\right)^3 \left\|u_m^{(l-1)}\right\|_1^2 \tag{25}$$

Учитывая условие теоремы $\delta_0 - 9\sigma^{-1} > \delta_0 - 21\sigma^{-1} \ge \delta > 0$, разделяя неравенства (25) на δ, умножая неравенства (25) на $\left(1+\mu_m^2\right)^3$ и суммируя по k от 1 до ∞, получим первую рекуррентную формулу

$$\left\langle u_k^{(l)} \right\rangle_1^2 \le \sigma\delta^{-1} \left[\langle g_k\rangle_0^2 + 2c_2\eta^{-2}\mathfrak{F} \sum_{i,j=1}^{2} \left(\left(T_0 \|\varphi_j\|^2_{W_2^2(Q)} + \|g_j\|_0^2\right) \langle f_{ik}\rangle_2^2\right)\right] + \\ + 2c_1c_2\eta^{-2}\sigma\delta^{-1}\mathfrak{F} \sum_{i=1}^{2} \langle f_{ik}\rangle_2^2 \left\langle u_m^{(l-1)} \right\rangle_1^2 \tag{26}$$

где $c_1 = \sum\limits_{k=1}^{\infty} \frac{\mu_k^4}{\left(1+\mu_k^2\right)^3}$, c_2- коэфициент вложения Соболева.

Введем обозначение $\sigma\delta^{-1}\left[\langle g_k\rangle_0^2 + 2c_2\eta^{-2}\mathfrak{F} \sum\limits_{i,j=1}^{2} \left(\left(T_0 \|\varphi_j\|^2_{W_2^2(Q)} + \|g_j\|_0^2\right) \langle f_{ik}\rangle_2^2\right)\right] \equiv A$, учитывая условия теоремы 1, что $2c_1c_2\eta^{-2}\sigma\delta^{-1}\mathfrak{F} \sum\limits_{i=1}^{2} \langle f_{ik}\rangle_2^2 < q = M \sum\limits_{i=1}^{2} \langle f_{ik}\rangle_2^2 < 1$, из рекуррентной формулы (26) получим справедливость оценки I), т.е получаем первую оценку. Действительно, для этого в качестве начального приближения возьмем функцию $\{u_k^{(-1)}\} \equiv \{0\}$.

Тогда для нулевого приближения имеем

$$\left\langle u_k^{(0)} \right\rangle_1^2 \le \sigma\delta^{-1} \left[\langle g_k\rangle_0^2 + 2c_2\eta^{-2}\mathfrak{F} \sum_{i,j=1}^{2} \left(\left(T_0 \|\varphi_j\|^2_{W_2^2(Q)} + \|g_j\|_0^2\right) \langle f_{ik}\rangle_2^2\right)\right] \equiv A$$

Продолжая этот процесс, методом индукции получим первую априорную оценку для любой функции $u_k^{(l)}$, $\forall l \ge 1$.

$$\left\langle u_k^{(l)} \right\rangle_1^2 \le A \cdot \sum_{s=0}^{l} q^s \le \frac{A}{1-q}$$

Теперь докажем справедливость второй оценки II). Для этого рассмотрим тождество

$$-2\int\limits_Q e^{-\lambda t} L u_k^{(l)} \cdot P u_k^{(l)} dxdt = -2\int\limits_Q e^{-\lambda t} F_k(u_k^{(l-1)}) \cdot P u_k^{(l)} dxdt. \tag{27}$$

где $Pu_k^{(l)} = (u_{kttt}^{(l)} + \Delta_x u_{kt}^{(l)} - \lambda u_{tt}^{(l)})$.

Рассуждая аналогично доказательству оценки I), основанному на интегрировании по частям (27), с учетом условий теоремы и полунелокальных краевых условий (17), (18), приходим к оценке снизу

$$\begin{aligned}
&\left|-2\int\limits_Q e^{-\lambda t}L(u_k^{(l)})\cdot Pu_k^{(l)}dxdt\right| \geq \int\limits_Q \left(\lambda u_{ktt}^{2(l)}+\lambda\Delta_x u_k^{2(l)}+2\lambda u_{kxt}^{2(l)}\right)dQ-\\
&-3\sigma^{-1}\int\limits_Q \left(u_{ktt}^{2(l)}+\Delta_x u_k^{2(l)}+u_{kxt}^{2(l)}\right)dQ+2\lambda\mu_k^2\int\limits_Q e^{-\lambda t}u_{kt}^{2(l)}dQ+\\
&+2\lambda\mu_k^2\int\limits_Q e^{-\lambda t}u_{kx}^{2(l)}dQ-\sigma\left\|c\right\|_{C^1(Q)}^2\left\|u_k^{(l)}\right\|_1^2-\sigma\lambda^2\left\|c\right\|_{C^1(\overline{Q})}^2\left\|u_k^{(l)}\right\|_1^2+\\
&+2\int\limits_{\partial Q} e^{-\lambda t}[u_{ktt}^{2(l)}e_t+u_{ktt}^{(l)}u_{xt}^{(l)}e_x+u_{kxt}^{2(l)}e_t+\Delta_x u_k^{(l)}u_{ktt}^{(l)}e_t+u_{ktt}^{(l)}u_{kxt}^{(l)}e_x+u_{kxt}^{2(l)}e_t+\\
&+\Delta_x u_k^{2(l)}e_t+cu_k^{(l)}\Delta_x u_k^{(l)}e_t-\mu_k^2u_k^{(l)}u_{kxt}^{(l)}e_t]ds \geq \int\limits_Q \left(\lambda u_{ktt}^{2(l)}+\lambda\Delta_x u_k^{2(l)}+2\lambda u_{kxt}^{2(l)}\right)dQ-\\
&-3\sigma^{-1}\int\limits_Q \left(u_{ktt}^{2(l)}+\Delta_x u_k^{2(l)}+u_{kxt}^{2(l)}\right)dQ+2\lambda\Big(\frac{\pi}{\ell}\Big)^2\int\limits_Q e^{-\lambda t}u_{kt}^{2(l)}dQ+\\
&+2\lambda\Big(\frac{\pi}{\ell}\Big)^2\int\limits_Q e^{-\lambda t}u_{kx}^{2(l)}dQ-\sigma\left\|c\right\|_{C^1(Q)}^2\left\|u_k^{(l)}\right\|_1^2-\sigma\lambda^2\left\|c\right\|_{C^1(\overline{Q})}^2\left\|u_k^{(l)}\right\|_1^2\geq\delta_0\left\|u_k^{(l)}\right\|_2^2-\\
&-3\sigma^{-1}\left\|u_k^{(l)}\right\|_2^2-\sigma\left\|c\right\|_{C^1(Q)}^2\left\|u_k^{(l)}\right\|_1^2-\sigma\lambda^2\left\|c\right\|_{C^1(\overline{Q})}^2\left\|u_k^{(l)}\right\|_1^2
\end{aligned}\tag{28}$$

Условия теоремы 1 обеспечивают не отрицательность интеграла по области Q. Учитывая полунелокальные краевые условия (17), (18) и условия теоремы 1, с выбором $\gamma^2=e^{\lambda T}$, получим обращение граничных интегралов в нуль. Таким образом, из неравенства (28) получаем снизу следующее неравенство

$$\begin{aligned}
&\left|-2\int\limits_Q e^{-\lambda t}F(u_k^{(l-1)})\cdot Pu_k^{(l)}dxdt\right|\geq\delta_0\left\|u_k^{(l)}\right\|_2^2-3\sigma^{-1}\left\|u_k^{(l)}\right\|_2^2-\\
&-\sigma\left\|c\right\|_{C^1(Q)}^2\left\|u_k^{(l)}\right\|_1^2-\sigma\lambda^2\left\|c\right\|_{C^1(\overline{Q})}^2\left\|u_k^{(l)}\right\|_1^2.
\end{aligned}\tag{29}$$

Теперь применяя неравенства Коши с σ к тождеству (27), получим сверху следующее неравенство

$$\begin{aligned}
&\left|-2\int\limits_Q e^{-\lambda t}F(u_k^{(l-1)})\cdot Pu_k^{(l)}dxdt\right|\leq 18\sigma^{-1}\left\|u_k^{(l)}\right\|_2^2+\sigma(1+\lambda^2)\left\|g_k\right\|_1^2+\\
&+\frac{4\sigma(1+\lambda^2)}{\eta^2}\mathfrak{F}\sum_{i,j=1}^{2}\left(T_0\left\|\varphi_j\right\|_{W_2^2(Q)}^2+\left\|g_j\right\|_0^2\right)\left\|f_{ik}\right\|_1^2+\\
&+\frac{4\sigma(1+\lambda^2)c_1}{\eta^2}\mathfrak{F}\sum_{i,j=1}^{2}\left\|f_{ik}\right\|_{C(Q)}^2\sum_{m=1}^{\infty}\left(1+\mu_m^2\right)^3\left\|u_m^{(l-1)}\right\|_2^2
\end{aligned}\tag{30}$$

Объединяя неравенства (29) и (30), получим

$$(\delta_0 - 21\sigma^{-1}) \left\| u_k^{(l)} \right\|_2^2 \le \sigma(1+\lambda^2) \|c\|_{C^1(\overline{Q})}^2 \left\| u_k^{(l)} \right\|_1^2 + \sigma(1+\lambda^2) \|g_k\|_1^2 +$$
$$+ 4\sigma\eta^{-2}(1+\lambda^2)\mathfrak{F} \sum_{i,j=1}^{2} \left(T_0 \|\varphi_j\|_{W_2^2(Q)}^2 + \|g_j\|_0^2\right) \|f_{ik}\|_1^2 +$$
$$+ 4\sigma\eta^{-2}c_1(1+\lambda^2)\mathfrak{F} \sum_{i,j=1}^{2} \|f_{ik}\|_{C(Q)}^2 \sum_{m=1}^{\infty} \left(1+\mu_m^2\right)^3 \left\| u_m^{(l-1)} \right\|_2^2 \tag{31}$$

Учитывая теорему вложения Соболева $\|f\|_{C(Q)}^2 \le c_2 \|f\|_{W_2^2(Q)}^2$ из неравенства (31), получим

$$(\delta_0 - 21\sigma^{-1}) \left\| u_k^{(l)} \right\|_2^2 \le 2\sigma^2\delta^{-1}(1+\lambda^2) \|c\|_{C^1(\overline{Q})}^2 \left\| u_k^{(l)} \right\|_1^2 +$$
$$+ \sigma(1+\lambda^2) \|g_k\|_1^2 + 4\sigma\eta^{-2}(1+\lambda^2)\mathfrak{F} \sum_{i,j=1}^{2} \left(T_0 \|\varphi_j\|_{W_2^2(Q)}^2 + \|g_j\|_0^2\right) \|f_{ik}\|_1^2 +$$
$$+ 4\sigma\eta^{-2}(1+\lambda^2)c_1c_2\mathfrak{F} \sum_{i,j=1}^{2} \|f_{ik}\|_2^2 \sum_{m=1}^{\infty} \left(1+\mu_m^2\right)^3 \left\| u_m^{(l-1)} \right\|_2^2 \tag{32}$$

Разделяя неравенства (32) на $\delta_0 - 21\sigma \ge \delta > 0$, умножая на $\left(1+\mu_m^2\right)^3$ и суммируя по k от 1 до ∞, получим вторую рекуррентную формулу

$$\left\langle u_k^{(l)} \right\rangle_2^2 \le 2\sigma\delta^{-2}(1+\lambda^2) \|c\|_{C^1(\overline{Q})}^2 \left[\langle g_k \rangle_0^2 + 2c_2\eta^{-2}\mathfrak{F} \sum_{i,j=1}^{2} \left(\left(T_0 \|\varphi_j\|_{W_2^2(Q)}^2 + \|g_j\|_0^2\right) \langle f_{ik} \rangle_2^2 \right) \right] +$$

$$+\sigma\delta^{-1}(1+\lambda^2) \left[\langle g_k \rangle_1^2 + 4\eta^{-2}\mathfrak{F} \sum_{i,j=1}^{2} \left(T_0 \|\varphi_j\|_{W_2^2(Q)}^2 + \|g_j\|_0^2\right) \langle f_{ik} \rangle_1^2 \right] + \tag{33}$$

$$+4(1+\lambda^2)c_1c_2\sigma\delta^{-1}\eta^{-2}\mathfrak{F} \sum_{i,j=1}^{2} \langle f_{ik} \rangle_2^2 \left\langle u_k^{(l-1)} \right\rangle_2^2$$

Введем обозначение

$$2\sigma^2\delta^{-2}(1+\lambda^2) \|c\|_{C^1(\overline{Q})}^2 \left[\langle g_k \rangle_0^2 + 2c_2\eta^{-2}\mathfrak{F} \sum_{i,j=1}^{2} \left(\left(T_0 \|\varphi_j\|_{W_2^2(Q)}^2 + \|g_j\|_0^2\right) \langle f_{ik} \rangle_2^2 \right) \right] +$$
$$+ \sigma\delta^{-1}(1+\lambda^2) \left[\langle g_k \rangle_1^2 + 4\eta^{-2}\mathfrak{F} \sum_{i,j=1}^{2} \left(T_0 \|\varphi_j\|_{W_2^2(Q)}^2 + \|g_j\|_0^2\right) \langle f_{ik} \rangle_1^2 \right] \equiv A_1,$$

учитывая условия теорема 1, что $4(1+\lambda^2)c_1c_2\sigma\delta^{-1}\eta^{-2}\mathfrak{F} \sum\limits_{i,j=1}^{2} \langle f_{ik} \rangle_2^2 \left\langle u_k^{(l-1)} \right\rangle_2^2 \le q < 1$, из рекуррентной формулы (33) получим справедливость оценки II), взяв для этого в качестве начального приближения функцию $\{u_k^{(-1)}\} \equiv \{0\}$. В результате, для нулевого приближения

получим

$$\left\langle u_k^{(0)}\right\rangle_2^2 \le 2\sigma^2\delta^{-2}(1+\lambda^2)\left\|c\right\|^2_{C^1(\overline{Q})}\left[\langle g_k\rangle_0^2 + 2c_2\eta^{-2}\mathfrak{F}\sum_{i,j=1}^{2}\left(\left(T_0\left\|\varphi_j\right\|^2_{W_2^2(Q)} + \left\|g_j\right\|_0^2\right)\langle f_{ik}\rangle_2^2\right)\right] +$$

$$+\sigma\delta^{-1}(1+\lambda^2)\left[\langle g_k\rangle_1^2 + 4\eta^{-2}\mathfrak{F}\sum_{i,j=1}^{2}\left(T_0\left\|\varphi_j\right\|^2_{W_2^2(Q)} + \left\|g_j\right\|_0^2\right)\langle f_{ik}\rangle_1^2\right] \equiv A_1$$

Продолжая этот процесс, методом индукции получим вторую априорную оценку для любой функции $u_k^{(l)}$, $\forall l \ge 1$

$$\left\langle u_k^{(l)}\right\rangle_2^2 \le A_1\cdot\sum_{s=1}^{l} q^s \le \frac{A_1}{1-q}$$

Отсюда, как и при доказательстве оценки I), легко получается оценка II). **Лемма 1 доказана.**

Введем теперь новую функцию из $W(Q)$ по формуле $\vartheta_k^{(l)} = u_k^{(l)} - u_k^{(l-1)}$; $\forall l = N\cup\{0\}$, $k = 1,2,...$ $\{u_k^{(-1)}\}\equiv\{0\}$. Тогда для нее справедлива следующая

Лемма 2. Пусть выполнены все условия теоремы 1 и леммы 1. Тогда для функций $\{\vartheta_k^{(l)}\}\in W(Q)$ справедливы следущие оценки:

$(III).\left\langle \vartheta_k^{(l)}\right\rangle_1^2 \le A\cdot q^{(l)}$,

$(IV).\ \left\langle \vartheta_k^{(l)}\right\rangle_2^2 \le A_1\cdot q^{(l)}$

Доказательство леммы 2:

Из (16)-(18) для функции $\{\vartheta_k^{(l)}\}\in W(Q)$ получим следующую задачу,

$$\begin{aligned} &L\vartheta_k^{(l)} = \vartheta_{ktt}^{(l)} - \Delta_x\vartheta_k^{(l)} + (c+\mu_k^2)\vartheta_k^{(l)} = \\ &= \frac{f_{1k}}{H}[f_{22}\sum_{m=1}^{\infty}\mu_m^2\vartheta_m^{(l-1)}\sin\mu_m\ell_1 - f_{21}\sum_{m=1}^{\infty}\mu_m^2\vartheta_m^{(l-1)}\sin\mu_m\ell_2] + \\ &\frac{f_{2k}}{H}[f_{11}\sum_{m=1}^{\infty}\mu_m^2\vartheta_m^{(l-1)}\sin\mu_m\ell_2 - f_{12}\sum_{m=1}^{\infty}\mu_m^2\vartheta_m^{(l-1)}\sin\mu_m\ell_1] = F_k(\vartheta_k^{(l-1)}) \end{aligned} \tag{34}$$

с полунелокальными краевыми условиями

$$\gamma D_t^p\vartheta_k^{(l)}\mid_{t=0} = D_t^p\vartheta_k^{(l)}\mid_{t=T},\ p=0,1; \tag{35}$$

$$\vartheta_k^{(l)}\mid_{\partial\Omega} = 0, \tag{36}$$

где $l = 0,1,2,...$

Следовательно, как и при доказательстве леммы 1, для функции $\{\vartheta_k^{(l)}\} = \{u_k^{(l)}\} - \{u_k^{(l-1)}\}\in W(Q)$ получим третью рекуррентную формулу

$$\left\langle \vartheta_k^{(l)}\right\rangle_1^2 \le q\left\langle \vartheta_k^{(l-1)}\right\rangle_1^2, \tag{37}$$

Повторяя рассуждения, при доказательства леммы 1, из (37) получим априорную оценку III). Аналогично доказывается оценка IV). Лемма 2 доказана.

Теорема 2. Пусть выполнены все условия теоремы 1. Тогда задача (16)-(18) одназначно разрешима в $W(Q)$.

Доказательство. Определим в пространстве $W(Q)$ отображение $u_k^{(l)} = L^{-1}F(u_k^{(l-1)}) = \mathcal{F}u_k^{(l-1)}$.

1. Покажем, что оператор $\mathcal{F}$ отображает пространства $W(Q)$ в себя. Пусть $\left\{ u_k^{(l-1)} \right\} \in W(Q)$, тогда для решения задачи (16)-(18), справедлива утверждение леммы-1, т.е. справедлива оценка II). Отсюда следует, что для любых $l = 1, 2, 3...$ получим $\left\{ u_k^{(l)} \right\} \in W(Q)$.

Таким образом $\mathcal{F} : W(Q\,) \to W(Q)$.

2. Покажем, что $\mathcal{F}$-сжимающий оператор. Пусть $\left\{ u_k^{(l)} \right\}, \left\{ u_k^{(l-1)} \right\} \in W(Q)$. Рассмотрим новую функцию $\left\{ \vartheta_k^{(l)} \right\} = \left\{ u_k^{(l)} \right\} - \left\{ u_k^{(l-1)} \right\}$, для нее справедливо утверждение леммы-2, т.е. справедлива оценка IV):

$$\left\| \left| \vartheta_k^{(l)} \right| \right\|_2^2 = \left\langle \vartheta_k^{(l)} \right\rangle_2^2 \le A_1 \cdot q^{(l)} \tag{38}$$

Теперь установим фундаментальность последовательности $\left\{ u_k^{(l)} \right\} \in W(Q)$.

Из неравенства треугольника и оценок (38), получим

$$\left\| \left| u_k^{(l+p+1)} - u_k^{(l)} \right| \right\|_2^2 \le \left\| \left| u_k^{(l+p+1)} - u_k^{(l+p)} \right| \right\|_2^2 + \left\| \left| u_k^{(l+p)} - u_k^{(l+p-1)} \right| \right\|_2^2 + ... + \left\| \left| u_k^{(l+1)} - u_k^{(l)} \right| \right\|_2^2 \le$$

$$\le A_1(q^{(l+p+1)} + q^{(l+p)} + ... + q^{(l)}) = A_1 q^{(l)}(1 + q + ... + q^{(p+1)}) \le \frac{A_1 q^{(l)}}{1-q}.$$

Отсюда вытекает фундаментальность последовательность $\left\{ u_k^{(l)} \right\}$.

Таким образом $\mathcal{F}$-сжимающийся оператор по известному принципу сжимающих отображений, задача (16)-(18) имеет единственное решение, принадлежащее пространству $W(Q)$, при $\varepsilon > 0$. При этом имеем $u_k^{(l)} \to u_k$ при $l \to \infty$ и $u_k(x,t)$ является единственным решением задачи (7)-(9) при фиксированном k [6, 10].

Из принципа сжимающих отображений заключаем, что задача (7)-(9) имеет единственное решение из $W(Q)$. **теорема 2 доказана**

Применяя к функциям $\{u_k\} \in W(Q)$ равенство Парсеваля – Стеклова получим утверждение теоремы, то есть $u(x,t,y) \in U$ [7, 10]. **теорема 1 доказана**

Замечание. Для уравнения (1) аналогично изучаются Л.Д.О.З. с условием Коши, то есть в этом случае вместо условия (2) предлагается условие Коши $u\,|_{t=0} = u_0(x), u_t\,|_{t=0} = u_1(x)$.

Список литературы

[1] Березинский Ю.М. *Разложение по собственным функциям самосопряженных операторов*, Киев, Наук.думка,1965.

[2] Бицадзе А.В., Самарский А.А. *О некоторых простейших обобщениях линейных эллиптических краевых задач*, ДАН СССР, 1969. Т.185. № 4. с.740.

[3] Бицадзе А.В. *О нелокальных краевых задачах* , ДАН СССР,1989, Т.277. № 1, с.17-19.

[4] Ильин.В.А., Моисеев.Е.И. *Нелокальная краевая задача первого рода для оператора Штурма-Лиувилля в дифференциальной и разностной трактовке.* Дифференциальные уравнения. 1987.Т.23. №7.С.1198-1207

[5] Джамалов С.З., *О корректности некоторых линейных многоточечных задач управления для волнового уравнения и уравнения Пуассона.*, ДАН РУз.1992. № 6-7., с.9-11.

[6] Джамалов С.З., *Нелокальные краевые и обратные задачи для уравнений смешанного типа*, Монография, Ташкент, 2021, с.176.

[7] Ладыженская О.А. *Краевые задачи математической физики*, М.1973.с.407

[8] Ладыженская О. А.,Солонников В. А.,Уральцева Н. Н. *Линейные и квазилинейные уравнения параболического типа.*, М.: Наука, 1967.

[9] Соболев С.Л. *Некоторые применения функционального анализа в математической физике.*, Л. ЛГУ,1950.

[10] Треногин.В.А. *Функционалный анализ.* Москва, 2007, с. 488.

Оңтайлы басқару

Optimal control

Оптимальное управление

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

THE NONLINEAR DIFFERENTIAL GAME IN THE CASE OF THE l–CATCH

ULMASJON SOYIBBOEV[1], MOHISANAM TURGUNBOYEVA[2]

[1,2]Namangan State University, Namangan, Uzbekistan
e-mail: ulmasjonsoyibboev@gmail.com[1], mohisanamturgunboyeva95@gmail.com[2]

Abstract. In the present work, we discuss the l-catch problem in a nonlinear differential game with two objects, a Pursuer and an Evader, whose movements occur in a dynamic flow field with various influences in space $\mathbf{R}^n$. Geometric constraints are imposed on controls of the objects. In order for the problem to be solved, a convergence (approach) strategy, which is constructed depending on Krasovskii-Pontryagin's formalization and Pshenichnyi-Chikrii's method of the resolving functions, is proposed for the Pursuer, and a sufficient conditions for solvability of the l-catch are obtained.

Keywords: Differential game, players, l-catch, convergence (approach) strategy.

AMS Subject Classification: 49N70,49N75,91A23,91A24.

1. Introduction

The history of differential games dates back to the mid-20th century and is closely tied to the fields of mathematics, control theory, and game theory. It emerged as a branch of applied mathematics that deals with the study of competitive interactions involving dynamic systems. This theory was studied systematically by R.Isaacs and published in the form of monographs in 1965 [1], in which numerous examples were examined and theoretical questions were only touched upon. Isaacs' work focused on studying dynamic situations where the actions of multiple players, each controlling a dynamic system described by differential equations, interacted in a competitive setting. The players' objectives were usually antagonistic, and the goal was to find strategies that would optimize each player's position while considering the actions of the others. The foundation of the modern theory of the Differential Game was settled mathematicians R. Isaacs [1], L.D. Bercovitz [8], W.H Fleming [9], A. Friedman [10], Y. Ho, A. Bryson, S. Baron [2], L.S. Pontryagin [3], N.N Krasovskii [4], L.A. Petrosjan [6], B.N. Pshenichnii [11].

According to the fundamental approaches in the theory of differential games developed by L.S. Pontryagin [3] and N.N. Krasovskii [4], a differential game is considered as a control problem from the point of view of either the pursuer or the evader. According to this view, the game reduces to either pursuit (convergence) problem or to evasion (escape) problem.

The problem for the case of l-approach [14] was first studied by Indian mathematician Ramchundra. Analogous effects in the case of geometrical constraint were considered in the works of N.N. Krasovsky [4], B.N.Pshenichnyi, A.A.Chikrii and I.S.Rappoport [12], Chikrii [13] L.A.Petrosjan [6], N.Yu.Satimov [18], A.Azamov and B.T.Samatov [20], B.K.Khaidarov [33], G.I.Ibragimov and B.B.Rikhsiev [21], P.B. Gusyatnikov and E.Z. Mokhonko [28], B.N. Sokolov [29].

This work was partially supported by B.Samatov.

In Satimov's works, l-capture problems for differential games with inertial and noninertial objects under integral and geometric constraints on controls was studied. l-capture and evasion problem for simple motion, The "Boy and crocodile" problem and The "Hunter and crocodile" problem can be such examples [18]. Samatov solved the problem of group pursuit for the case of l-catch in simple motion of the players, which integral constraints are imposed on controls, on the basis of Chikrii's method of resolving functions [25]. In [27], Samatov and et.al. studied the problem of Ramchundra for a problem of l-capture with geometrical constraint. They constructed a strategy depending only the constraint for approaching the ship to the boat. In the work of Petrosyan and Dutkevich [7], the l-capture problem have been considered for the players moving at the limited velocities by the coordinates on the plane and also, a lifeline game was solved by geometrical method. The work [30] generalizes the result of B.N.Pshenichnyi to the case of l-capture.

Some optimal control problem formulations have taken into account the effect of an external flow field. For example, in [5], the authors considered the problem of optimal guidance of a Dubin's vehicle [15] to a specified position under the influence of an external flow. The minimum-time guidance problem for an isotropic rocket in the presence of wind has been studied in [8]. The problem of minimizing the expected time to steer a Dubin's vehicle to a target set in a stochastic wind field has also been discussed in [16]. However, the same level of attention in the literature has not been devoted to pursuit-evasion games with two (or more) competing agents under the influence of external disturbances (e.g. winds or currents). In paper [22]–[24], a multi pursuer and one-evader for pursuit evasion game in an external dynamic flow field is considered. Due to the generality of the external flow, Isaacs's approach is not readily applicable [1]. Instead, in [23], a different approach is used and the optimal trajectories of the players through a reachable set method is found.

In the present work, we discuss the l-catch problem in a nonlinear differential game with two objects, a Pursuer and an Evader, whose movements occur in a dynamic flow field with various influences in space $\mathbf{R}^n$. Geometric constraints are imposed on controls of the objects. In order for the problem to be solved, a convergence (approach) strategy, which is constructed depending on Krasovskii-Pontryagin's formalization and Pshenichnyi-Chikrii's method of the resolving functions, is proposed for the Pursuer, and a sufficient conditions for solvability of the l-catch are obtained.

2. Statement of problems

Suppose that in $\mathbf{R}^n$, a controlled object $\mathbb{P}$ called the Pursuer, chases another object $\mathbb{E}$ called the Evader. Denote the position of the Pursuer by x and denote the position of the Evader by y in $\mathbf{R}^n$. Let the objects move in accordance with the equations

$$\dot{x} = u + f(t, x), \quad x(0) = x_0, \tag{1}$$

$$\dot{y} = v + g(t, y), \quad y(0) = y_0, \tag{2}$$

respectively, where $x, y, u, v \in \mathbf{R}^n$, $n \geq 2$, $t \in \mathbf{R}_+ := [0, +\infty)$.

$f : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}^n$ ($g : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}^n$) is an effective flow field for the Pursuer (for the Evader) and they may describe the exogenous dynamic flows; x_0, y_0 are the initial locations of the players, respectively that is $|x_0 - y_0| > l$, $l > 0$. The functions $f(t, x)$, $g(t, y)$ may describe the exogenous dynamic flows, however, they could express the endogenous drift on account of the nonlinear dynamics of the players [19], [26], [31], [32].

In equation (1), the control parameter u serves as the control of the Pursuer and it is chosen as a measurable function $u(\cdot) : \mathbf{R}_+ \to \mathbf{R}^n$ satisfying the following inequality

$$|u(t)| \leq \rho \ \text{ for almost every } t \geq 0, \tag{3}$$

where ρ is a given nonnegative number.

Similarly, in equation (2), the control parameter v function as the control of the Evader and it is chosen as a measurable function $v(\cdot) : \mathbf{R}_+ \to \mathbf{R}^n$ satisfying the following inequality

$$|v(t)| \leq \sigma \quad \text{for almost every } t \geq 0, \tag{4}$$

where σ is a given nonnegative number.

In the Theory of Differential Games, the inequalities (3), (4) are usually called the geometric constraints (briefly, G-constraints) for controls. A measurable function $u(\cdot) = (u_1(\cdot), ..., u_n(\cdot))$ is said to be an admissible control of $\mathbb{P}$ and we denote a class of all such controls, which satisfy the constraint (3) by U_G. A measurable function $v(\cdot) = (v_1(\cdot), ..., v_n(\cdot))$ is called to be an admissible control of $\mathbb{E}$ and we denote a class of all such controls, which satisfy the constraint (4) by V_G.

Note that the pair $(U_G,\ V_G)$ of the introduced classes defines a differential game.

Assumption 1. (*Caratheodory's conditions*) Let the functions $f(t, x)$ and $g(t, y)$ be defined on the domain $D := \mathbf{R}_+ \times \mathbf{R}^n$ and satisfy the following three conditions respectively: 1) $f(t, x)$ and $g(t, y)$ are continuous in x and y for each fixed t; 2) $f(t, x)$ and $g(t, y)$ are measurable functions in t for each fixed x and y; 3) for each compact set Q of D, there can be found Lebesgue-integrable functions $h_P(\cdot) : \mathbf{R}_+ \to \mathbf{R}_+$ and $h_E(\cdot) : \mathbf{R}_+ \to \mathbf{R}_+$ such that $|f(t, x)| \leq h_P(t)$ and $|g(t, y)| \leq h_E(t)$ for all $(t, x), (t, y) \in Q$.

In equations (1) and (2), the functions $f(t, x)$ and $g(t, y)$ represent the exogenous dynamic flows, but they could also represent the endogenous drift owing to the nonlinear dynamics of the objects. It is reasonable to assume that the magnitude of these flows (e.g. winds or currents) is bounded from above by some functions $h_P(t)$ and $h_E(t)$ in the third condition of Assumption 1.

Assumption 2. (*Lipshitz's condition*) For each compact sets Q_P and Q_E of D, there exist Lebesgue-integrable functions $k_P(\cdot) : \mathbf{R}_+ \to \mathbf{R}_+$ and $k_E(\cdot) : \mathbf{R}_+ \to \mathbf{R}_+$ that

$$|f(t, x_1) - f(t, x_2)| \leq k_P(t)|x_1 - x_2|, \tag{5}$$

$$|g(t, y_1) - g(t, y_2)| \leq k_E(t)|y_1 - y_2| \tag{6}$$

for all $(t, x_1), (t, x_2) \in Q_P$ and $(t, y_1), (t, y_2) \in Q_E$.

For admissible control functions $u(\cdot) \in U_G$, $v(\cdot) \in V_G$ the Caratheodory differential equations

$$\dot{x} = u(t) + f(t, x(t)), \quad x(0) = x_0,$$

$$\dot{y} = v(t) + g(t, y(t)), \quad y(0) = y_0$$

generate unique trajectories $x(t) = x(t; x_0, u(\cdot))$ and $y(t) = y(t; y_0, v(\cdot))$ respectively. Here $x(t)$ is termed the trajectory of the Pursuer, and $y(t)$ is termed that of the Evader.

The main objective of the Pursuer is to approach the Evader at a distance $l > 0$ (l-catch problem [10], [13], [17]), that is, to reach the inequality

$$|x(\tau) - y(\tau)| \leq l$$

for some $\tau > 0$. While the Evader strives to guarantee the relation

$$|x(t) - y(t)| > l$$

for all $t \geq 0$, and if it is impossible, to prolong the moment τ as far as possible.

Obviously, for the object $\mathbb{P}$, control functions depending only on the time $t \geq 0$ are not sufficient to solve the l-catch problem and the acceptable types of controls should be strategies.

Definition 1. A map $\mathbf{u}(t, x, y, v) : \mathbf{R}^n \setminus lS \times V_G \to U_G$ is said to be a strategy of Pursuer $\mathbb{P}$ if:

1^o. $\mathbf{u}(t, x, y, v) \in U_G$ for an arbitrary $v(\cdot) \in V_G$ and for each $t \geq 0$;

2^o. If, for arbitrary $v_1(\cdot)$, $v_2(\cdot) \in V_G$ and for every t, $t \geq 0$, the relation $v_1(\varepsilon) = v_2(\varepsilon)$ is valid almost everywhere on $[0, t]$, then $\mathbf{u}[\varepsilon, x(\varepsilon), y(\varepsilon), v_1(\varepsilon)] = \mathbf{u}[\varepsilon, x(\varepsilon), y(\varepsilon), v_2(\varepsilon)]$ holds almost everywhere on $[0, t]$, where $S = \{s \in \mathbb{R}^n : |s| \leq 1\}$.

3. The main result

In this section, we propose an approach strategy for the Pursuer and give a solution for the l-catch problem.

First of all, let's introduce the denotations

$$z(t) = x(t) - y(t), \;\; z(0) = z_0, \;\; z_0 = x_0 - y_0. \tag{7}$$

and the functions

$$\omega = \omega(t, x, y, v) = v + g(t, y) - f(t, x). \tag{8}$$

For the object $\mathbb{P}$, only alone program strategies, which is admissible controls depending only on time t, are not enough to achieve its objective. Thus, we will define an approach strategy for the Pursuer $\mathbb{P}$ according to Krasovskii-Pontryagin's formalization [3, 4] and Pshenichnyi-Chikrii's method of the resolving functions [11, 13]. It will turn on the form of dependence only on the current positions of the function $\omega(t)$, $t \geq 0$ and the given constants l, z_0.

Definition 2. For $\rho \geq |\omega|$, we call the control function

$$\mathbf{u}_l^*(z_0, \omega) = \omega + \gamma_l^*(z_0, \omega)\big(\xi_l^*(z_0, \omega) - z_0\big) \tag{9}$$

a convergence (approach) strategy of the Pursuer (for brevity, Π_l^*-strategy) in the l-catch problem, where $\gamma_l^*(z_0, \omega) = \left[\langle \omega, z_0\rangle + \rho l + \sqrt{\big(\langle \omega, z_0\rangle + \rho l\big)^2 + \big(|z_0|^2 - l^2\big)\big(\rho^2 - |\omega|^2\big)}\right] / \big(|z_0|^2 - l^2\big)$.

$\xi_l^*(z_0, \omega) = -\frac{\omega - \gamma_l^*(z_0,\omega) z_0}{|\omega - \gamma_l^*(z_0,\omega) z_0|} l$.

$\langle \omega, z_0\rangle$ is the scalar product of the vectors ω and z_0 in $\mathbf{R}^n$, and the function $\gamma_l^*(z_0, \omega)$ is generally termed as the resolving function.

Lemma 1. If $|\omega| \leq \rho$, then the function $\gamma_l^*(z_0, \omega)$ is continuous, nonnegative in $\mathbf{R}^n$ and it is bounded as

$$\frac{\rho - |\omega|}{|z_0| - l} \leq \gamma_l^*(z_0, \omega) \leq \frac{\rho + |\omega|}{|z_0| - l}. \tag{10}$$

Lemma 2. If $|\omega| \leq \rho$, then the function $\mathbf{u}_l^*(z_0, \omega)$ is continuous in $\mathbf{R}^n$ and the equality

$$|\mathbf{u}_l^*(z_0, \omega)| = \rho$$

is satisfied.

Assume that $v(\cdot) \in V_G$ be any control of Evader $\mathbb{E}$ and let Pursuer $\mathbb{P}$ realizes the Π_l^*-strategy (9). Then according to Assumptions 1 and 2 and from (9) it proceeds that the system of the Caratheodory differential equations

$$\begin{cases} \dot{x} = v(t) + g(t, y(t)) + \gamma_l^*(z_0, \omega(t, x(t), y(t), v(t)))\big(\xi_l^*(z_0, \omega(t, x(t), y(t), v(t))) - z_0\big), & x(0) = x_0, \\ \dot{y} = v(t) + g(t, y(t)), & y(0) = y_0, \end{cases}$$

where the above equations generate the unique trajectories $y(t) := y(t; y_0, v(\cdot))$ of Evader $\mathbb{E}$ and $x(t) := x(t; x_0, \mathbf{u}_l^*(\cdot))$ of Pursuer $\mathbb{P}$.

Definition 3. In the l-catch problem, we say that the strategy $\mathbf{u}_l^*(z_0, \omega)$ is winning on the time interval $[0, t^*]$ if for any control $v(\cdot) \in V_G$:
a) the relation $|z(\varepsilon)| \leq l$ yields at some instant $\varepsilon \in [0, t^*]$;
b)$\mathbf{u}_l^*(z_0, \omega(t, x(t), y(t), v(\cdot))) \in U_G$ holds on the interval $[0, \varepsilon]$,
where we call the number t^* a guaranteed time of the l-catch.

Lemma 3. (*The Grönwall-Bellman inequality* [34]) Let $\varphi(t)$ be a real valued, continuous function and $\psi(t) \geq 0$ be an integrable function on $t \geq 0$. Then if the relation

$$|z(t)| \leq \varphi(t) + \int_0^t \psi(s)|z(s)|ds$$

is valid for those functions, then the inequality

$$|z(t)| \le \varphi(t) + \int_0^t \psi(s)\varphi(s) \left(\exp \int_s^t \psi(\tau) d\tau \right) ds$$

holds.

Conjecture 1. Assume that we can find Lebesgue-integrable functions $k(\cdot) : \mathbf{R}_+ \to \mathbf{R}_+$ and $h(\cdot) : \mathbf{R}_+ \to \mathbf{R}$, which $k(t) \le k$ and $h(t) \le h$ fullfilling

$$|f(t,x) - g(t,y)| \le k(t)|x - y| + h(t)$$

for any $x, y \in \mathbf{R}^n$, where k and h are given positive numbers.

Conjecture 2. Let

a) $t^* := \min\{t : \Lambda(t) = l,\ t \ge 0,\ l > 0\}$, where

$$\Lambda_l(t) := |z_0| - \int_0^t \exp \left(- \int_0^s k(\eta) d\eta \right) (\rho - \sigma - h(s)) ds$$

b) the condition

$$\rho > \sigma + k(t)\Omega_l(t) + h(t)$$

is satisfied for all $t \in [0, t^*]$, where

$$\Omega_l(t) = \Lambda_l(t) \exp \int_0^t k(s) ds.$$

Theorem 1. Let Assumptions 1–2 and Conjectures 1–2 be satisfied. Then Π_l^*-strategy (9) of Pursuer $\mathbb{P}$ is winning on the time interval $[0, t^*]$ in the differential game (1)–(4).

Proof. Let $v(\cdot) \in V_G$ be any control of the Evader and let the Pursuer realize the Π_l^*-strategy (9). Then according to Conjecture 1, Conjecture 2 and from the strategy (9) it proceeds that the system of the Caratheodory differential equations

$$\begin{cases} \dot{x} = \omega(t) + \gamma_l^*(z_0, \omega(t))\big(\xi_l^*(z_0, \omega(t)) - z_0\big) + f(t, x(t)), & x(0) = x_0, \\ \dot{y} = v(t) + g(t, y(t)), & y(0) = y_0, \end{cases} \tag{11}$$

where $0 \le t \le t^*$. By virtue of the denotation (7) and function (8), we reduce system (11) to the unique Cauchy problem

$$\dot{z} = \gamma_l^*(z_0, \omega(t))\big(\xi_l^*(z_0, \omega(t)) - z_0\big), \quad z(0) = z_0, \quad 0 \le t \le t^*. \tag{12}$$

Integrate both sides of (12) and obtain the solution

$$z(t) = z_0 + \int_0^t \gamma_l^*(z_0, \omega(s))\big(\xi_l^*(z_0, \omega(s)) - z_0\big) ds \tag{13}$$

or

$$z(t) = z_0 \Lambda_l(t, \omega(\cdot)) + \int_0^t \gamma_l^*(z_0, \omega(s)) \xi_l^*(z_0, \omega(s)) ds \tag{14}$$

where $\Lambda_l(t, \omega(\cdot)) = 1 - \int_0^t \gamma_l^*(z_0, \omega(s)) ds$. Since $\Lambda_l(t, \omega(\cdot)) \ge 0$ on the interval $[0, t^*]$ and accommodating the form of the $\gamma_l^*(z_0, \omega(t))$ in Definition 2, we perform the following estimations for the absolute value of $z(t)$ in (14):

$$|z(t)| \le |z_0| \Lambda_l(t, \omega(\cdot)) + l \int_0^t \gamma_l^*(z_0, \omega(s)) ds = |z_0| - \big(|z_0| - l\big) \int_0^t \gamma_l^*(z_0, \omega(s)) ds =$$

$$= |z_0| - \big(|z_0| - l\big) \int\limits_0^t \frac{\langle \omega(s), z_0\rangle + \rho l + \sqrt{\big(\langle \omega(s), z_0\rangle + \rho l\big)^2 + \big(|z_0|^2 - l^2\big)\big(\rho^2 - |\omega(s)|^2\big)}}{|z_0|^2 - l^2} ds.$$

By virtue of Lemma 1 and the denotation (7), we will write the followings:

$$|z(t)| \le |z_0| - \big(|z_0| - l\big) \int\limits_0^t \frac{\rho - |\omega(s)|}{|z_0| - l} ds = |z_0| - \rho t + \int\limits_0^t |v(s) + g(s, y(s)) - f(s, x(s))| ds.$$

From this and by virtue of (4) and Conjecture 1 we form the inequality

$$|z(t)| \le |z_0| - (\rho - \sigma)t + \int\limits_0^t |f(s, x(s)) - g(s, y(s))| ds \le$$

$$\le |z_0| - (\rho - \sigma)t + \int\limits_0^t \big[k(s)|x(s) - y(s)| + h(s)\big] ds$$

or the Grönwall-Bellman inequality in the form

$$|z(t)| \le |z_0| - (\rho - \sigma)t + \int\limits_0^t h(s) ds + \int\limits_0^t k(s)|z(s)| ds. \tag{15}$$

As a result of applying Lemma 3 to (15), the following is obtained:

$$|z(t)| \le |z_0| - (\rho - \sigma)t + \int\limits_0^t h(s) ds +$$

$$+ \int\limits_0^t \left[|z_0| - (\rho - \sigma)s + \int\limits_0^s h(\tau) d\tau \right] k(s) \left(\exp \int\limits_s^t k(\tau) d\tau \right) ds. \tag{16}$$

By means of the integration method by parts for the right-hand side of (16), it can be easily calculated that

$$|z_0| - (\rho - \sigma)t + \int\limits_0^t h(s) ds + \int\limits_0^t \left[|z_0| - (\rho - \sigma)s + \int\limits_0^s h(\tau) d\tau \right] k(s) \left(\exp \int\limits_s^t k(\tau) d\tau \right) ds =$$

$$= \left[|z_0| - \int\limits_0^t \exp \left(- \int\limits_0^s k(\tau) d\tau \right) \big(\rho - \sigma - h(s)\big) ds \right] \exp \int\limits_0^t k(s) ds =$$

$$= \Lambda_l(t) \exp \int\limits_0^t k(s) ds = \Omega_l(t),$$

and therefor, from (16) we generate the final estimation of $z(t)$ in the form

$$|z(t)| \le \Omega_l(t) \tag{17}$$

In accordance with Conjecture, it is known that $\Omega_l(t^*) = l$. As a consequence, from (16) it follows that $|z(t^*)| \le l$. Due to that, $|x(t^*) - y(t^*)| \le l$ is obtained.

Now let's confirm the admissibility of the strategy (9) for $t \in [0, t^*]$. Here we will present that the inequality $\rho \ge |\omega(t)|$ is true on $[0, t^*]$. Let Conjecture 2 be satisfied. Then by virtue of (4), (17) and Conjecture 1, we get a range of estimations:

$$\rho > \sigma + k(t)\Omega_l(t) + h(t) \ge |v(t)| + k(t)|z(t)| + h(t) \ge$$

$$\geq |v(t)| + |f(t, y(t)) - g(t, x(t))| \geq |v(t) + f(t, y(t)) - g(t, x(t))| = |\omega(t)|.$$

This completes the proof of Theorem.

4. Examples

Example 1. Consider the differential game

$$\dot{x} = u + x + x_0 \cos t, \quad x(0) = x_0, \quad |u(t)| \leq \rho, \tag{18}$$

$$\dot{y} = v + y - y_0 \sin t, \quad y(0) = y_0, \quad |v(t)| \leq \sigma \tag{19}$$

respectively, where $x, y, u, v, x_0, y_0 \in \mathbb{R}^n$, $n \geq 2$, and it is required that $|x_0 - y_0| > l$, $l > 0$ for x_0 and y_0,

For Conjecture 1, we can take as $k(t) = 1$ and $h(t) = \sqrt{|x_0|^2 + |y_0|^2}$. Thus, Conjecture 2 will be as follows:

(a) $|z_0| + \left(\rho - \sigma - \sqrt{|x_0|^2 + |y_0|^2}\right)\left(e^{-t} - 1\right) = l$, where $z_0 = x_0 - y_0$;

(b) $\rho > \sigma + |z_0| - l + \sqrt{|x_0|^2 + |y_0|^2}$.

As a result, we obtain the following assertion.

Theorem 2. If $\rho > \sigma + |z_0| - l + \sqrt{|x_0|^2 + |y_0|^2}$, then Π_l^*-strategy (9) is winning on the time interval $\left[0, T^*\right]$ in the game (18)–(19), where $T^* = \ln\left(\frac{\rho - \sigma - \sqrt{|x_0|^2 + |y_0|^2}}{\rho - \sigma - |z_0| + l - \sqrt{|x_0|^2 + |y_0|^2}}\right)$.

Example 2. Let the dynamics of the game be in the form

$$\dot{x} = u + x + z_0 \cosh^2 t, \quad x(0) = x_0, \quad |u(t)| \leq \rho, \tag{20}$$

$$\dot{y} = v + y + z_0 \sinh^2 t, \quad y(0) = y_0, \quad |v(t)| \leq \sigma \tag{21}$$

respectively, where $x, y, u, v, x_0, y_0 \in \mathbb{R}^n$, $n \geq 2$, $z_0 = x_0 - y_0$, and it is required that $|z_0| > l$, $l > 0$ for z_0.

For Conjecture 1, we find $k(t) = 1$ and $h(t) = |z_0|$. Thus, Conjecture 2 becomes:

(a) $|z_0| + (\rho - \sigma - |z_0|)\left(e^{-t} - 1\right) = l$;

(b) $\rho > \sigma + 2|z_0| - l$.

Consequently, we obtain the following assertion.

Theorem 3. If $\rho > \sigma + 2|z_0| - l$, then Π_l^*-strategy (9) is winning on the time interval $\left[0, T_*\right]$ in the game (20)–(21), where $T_* = \ln\left(\frac{\rho - \sigma - |z_0|}{\rho - \sigma - 2|z_0| + l}\right)$.

Example 3. Let us take the linear differential game

$$\dot{x} = u + Ax + f(t), \quad x(0) = x_0, \quad |u(t)| \leq \rho, \tag{22}$$

$$\dot{y} = v + Ay + g(t), \quad y(0) = y_0, \quad |v(t)| \leq \sigma \tag{23}$$

respectively, where $x, y, u, v \in \mathbb{R}^n$, $n \geq 2$. A is an $n \times n$ real constant matrix; $f(t)$ and $g(t)$ are Lebesgue-integrable functions and it is supposed that there exists some $\delta > 0$ such that $|f(t) - g(t)| \leq \delta$ for all $t \geq 0$. Consequently, in Conjecture 1, it is obtained that $k(t) = ||A||$ and $h(t) = \delta$. Hence Conjecture 2 has the form:

(a) $|z_0| + \frac{(\rho - \sigma - \delta)}{||A||}\left(e^{-||A||t} - 1\right) = l$ if $||A|| \neq 0$ or $|z_0| - (\rho - \sigma - \delta)t = l$ if $||A|| = 0$;

(b) $\rho > \sigma + ||A||\,(|z_0| - l) + \delta$.

In consequence, we get the following statement.

Theorem 4. If $\rho > \sigma + ||A||\,(|z_0| - l) + \delta$, then Π_l^*-strategy (9) is winning on the time interval $\left[0, T_{||A||}\right]$ in the game (22)–(23), where

$T_{||A||} = \frac{1}{||A||} \ln\left(\frac{\rho - \sigma - \delta}{\rho - \sigma - \delta - (|z_0| - l)||A||}\right)$ if $||A|| \neq 0$ or $T = \frac{|z_0| - l}{\rho - \sigma - \delta}$ if $||A|| = 0$.

References

[1] Isaacs R. Differential games, *John Wiley and Sons,* New York, 1965.

[2] Ho Y., Bryson A., Baron S. Differential games and optimal pursuit-evasion strategies. *IEEE Trans Autom Control*, Vol. 10, 1965, pp. 385-389. DOI 10.1109/TAC.1965.1098197

[3] Pontryagin L.S. Selected Works, *MAKS Press,* Moscow, 2014. (in Russian)

[4] Krasovskii N.N. Control of a Dynamical System (in Russian), *Nauka*, Moscow, 1985, pp 520.

[5] Krasovskii N.N., Subbotin A.I. Game-Theoretical Control Problems. *Springer*, New York, 2011.

[6] Petrosjan L.A. Differential games of pursuit. Series on optimization, Vol.2. *World Scientific Poblishing*, Singapore, 1993.

[7] Petrosyan L.A., Dutkevich V.G. Games with "a Survival Zone", Occasion L-catch (in Russian), *Vestnic Leningrad State Univ.,* No.13, Vol.3, 1969 pp.31-38.

[8] Berkovitz L.D. Differential game of generalized pursuit and evasion. *SIAM J. Contr.* Vol.24, No.3, 1986, pp. 361–373. DOI 10.1137/0324021

[9] Fleming W. H. The convergence problem for differential games, II. *Advances in Game Theory (2nd ed.), Annals of Math.*, Vol.52. 1964, pp.195-210. DOI 10.1515/9781400882014

[10] Friedman A. Differential Games (2nd ed.), *Pure and Applied Mathematics,* Vol. 25, Wiley Interscience, New York, 1971. DOI MR 0421700

[11] Pshenichnii B.N. Simple pursuit by several objects. *Cybernetics and System Analysis.* Vol. 12, No. 5, 1976, pp. 484–485. DOI 10.1007/BF01070036

[12] Pshenichnii B.N., Chikrii A.A., and Rappoport J.S. An effective method for solving differential games with many pursuers. *Dokl. Akad. Nauk SSSR ,* Vol. 256, No. 3, 1981, pp. 530-535. DOI Mi dan44191

[13] Chikrii A.A. Conflict-Controlled Processes, *Kluwer,* Dordrecht, 1997. DOI 10.1007/978-94-017-1135-7

[14] Nahin P.J. Chases and Escapes: The Mathematics of Pursuit and Evasion, *Princeton University Press, Princeton,* 2012. pp 272.

[15] Dar'in A. N., Kurzhanskii A. B. Control Under Indeterminacy and Double Constraints. *Differential Equations*, Vol. 39, No. 11, 2003, pp. 1554-1567. DOI 10.1023/B:DIEQ.0000019347.24930.a3

[16] Aubin J.P., Cellina A. Differential Inclusions. Set-Valued Maps and Viability Theory, *Springer-Verlag*, XIII, Berlin-Heidelberg-New York-Tokyo, 1984. DOI 10.1002/zamm.19870670206

[17] Subbotin A.I., Chentsov A.G. Optimization of Guaranteed Result in Control Problems, *Nauka,* Moscow, 1981. (in Russian)

[18] Satimov N.Yu. Methods for Solving the Pursuit Problem in the Theory of Differential Games, *Izd-vo NUUz,* Tashkent, 2003. (in Russian)

[19] *Azamov A.A.* A class of nonlinear differential games, *Mathematical Notes,* Vol. 30, No. 4, 1981, pp. 805–808.

[20] Azamov A.A., Samatov B.T. The Π-Strategy: Analogies and Applications. *The Fourth International Conference Game Theory and Management, St. Petersburg,* Vol.4, 2010, pp. 33–47.

[21] Ibragimov G.I., Rikhsiev B.B. On Some Sufficient Conditions for Optimality of the Pursuit Time in the Differential Game with Multiple Pursuers, *Automation and Remote Control,* Vol. 67, No. 4, 2006, pp. 529-537.

[22] Samatov B.T. On a pursuit-evasion problem under a linear change of the pursuer resource. *Siberian Advances in Mathematics,* Vol.23, No. 10, 2013, pp 294–302. DOI 10.3103/S1055134413040056

[23] Samatov B.T. The pursuit-evasion problem under integral-geometric constraints on pursuer controls. *Automation and Remote Control,* Vol.74 , No. 7, 2013, pp. 1072–1081. DOI 10.1134/S0005117913070023

[24] Samatov B.T. Problems of group pursuit with integral constraints on controls of the players I. *Cybernetics and Systems Analysis,* Vol.49, No. 5, 2013, pp. 756–767. DOI 10.1007/s10559-013-9563-7

[25] Samatov B.T. Problems of group pursuit with integral constraints on controls of the players II, *Cybernetics and Systems Analysis*, Vol.49, No.6, 2013, pp.907-921. DOI 10.1007/s10559-013-9581-5.

[26] Samatov B.T., Sotvoldiyev A.I. Intercept problem in dynamic flow field, *Uzbek Mathematical Journal*, Vol.12, No.2, 2019, pp.103-112.

[27] Samatov B.T., Uralova S.I., Mirzamaxmudov U.A. The problem of Ramchundra for a problem of l-capture. *Scientific Bulletin of Namangan State University,* Vol. 1, No. 2, 2019, pp. 10–14. DOI: https://uzjournals.edu.uz/namdu/vol1/iss2/2

[28] Gusyatnikov P.B., Mokhonko E.Z. On ℓ_∞-Escape in a Linear Many-Person Differential Game with Integral Constraints, *Prikl. Matem. Mekh.,* Vol. 44, No. 4, 1980, pp. 618–624.

[29] Sokolov B. N. Differential pursuit game delayed information in the available of integral constraints, *Differential equation,* Vol. 8, No. 10, 1972, pp. 1797–1804.

[30] Vasilyeva L.G. On one differential game of evasion. *Differential, non-cooperative, cooperative and statistical games,* Publishing house Kalinin university, 1979, pp. 26-33.

[31] Bakolas E. Optimal Guidance of the Isotropic Rocket in the Presence of Wind, *Journal of Optimization Theory and Applications,* Vol. 162, No. 3, 2014, pp. 954–974.

[32] Sun W., Tsiotras P. Pursuit Evasion Game of Two Players under an External Flow Field, *American Control Conference,* Piscataway, 2015, pp. 5617–5622.
[33] Khaidarov B.K. Positional l-catch in the game of one evader and several pursuers, *Prikl. Matem. Mekh,* Vol. 48, No. 4, 1984, pp. 574–579.
[34] Pachpatte B.G. Inequalities for Differential and Integral Equations, Vol. 197 of Mathematics in Science and Engineering *Academic*, London, 1998.

Математикалық және компьютерлік модельдеу

Mathematical and computer modeling

Математическое и компьютерное моделирование

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

INVESTIGATION OF THE THEORY OF OWN AND FORCED OSCILLATORY PROCESSES OF DIFFERENT TYPES OF FLAT ELEMENTS

A. SEITMURATOV, Z. YERGALAUOVA, S. MENLIKHOZHAYEVA, A.IBRAYEVA, G. YENSEBAYEVA

Korkyt Ata Kyzylorda University, Kyzylorda, Kazakhstan
e-mail: angisin_@mail.ru

Abstract. This paper presents the results of the study of natural and forced vibrations of flat elements, taking into account the layering of the element material, rheological viscous properties, environmental influences, deformable base, anisotropy, etc. Various formulations of boundary value problems of vibrations of a rectangular flat element, both taking into account viscosity, are considered. so, taking into account the above factors of a geometric and mechanical nature, which are transcendental frequency equations, which reduces to algebraic ones, and the influence of both boundary conditions along the edges of a rectangular plate and parameters of a geometric and mechanical nature on the natural oscillation frequencies of rectangular plane elements is considered, and the previous results for a rectangular plate, the material of which satisfies the viscoelastic Maxwell model, are generalized [1].

When studying vibration processes in a solid deformable body, it is advisable to take the nuclei of viscoelastic operators as regular, since only such operators describe instantaneous elasticity, and then viscous flow, which is characteristic of deformable solids. Integro-differential equations with regular kernels are known to be equivalent to partial differential equations.Depending on the particular types of the plane element under consideration, the main unknown functions are selected in the general solutions of the three-dimensional problem: displacement or deformation at points in the fixed plane of the plane element, in particular, in the median plane of the plate of constant thickness. Displacements and stresses at an arbitrary point of a plane element are expressed in terms of basic unknown functions, which are determined from boundary conditions on the surfaces of the plane element. The obtained equations for the basic unknown functions are the general equations of oscillation of a plane element containing derivatives of functions in coordinates and time of any arbitrarily large order. General solutions are presented in the form of power series over the thickness of a flat element. The general solution refers to an equation of hyperbolic type, which describes the oscillatory and wave process in a plane element. Limiting the series of the general equation to a finite number of the first terms, we obtain approximate equations of oscillation of a plane element.

The proposed approach allows us to strictly construct approximate theories of vibrations of plane elements of various types.

Keywords: oscillation theories, boundary conditions, plane element, proper oscillation, displacement, deformation.

AMS Subject Classification: 70K05.

1. Introduction

When solving applied problems of oscillation of rectangular flat elements, a wide class of oscillation problems arises related to various boundary value problems: approximate oscillation equations, various boundary conditions at the edges of a flat element and initial conditions. In

the theory of oscillation, an important point is to determine the frequencies of natural oscillations, solve problems about forced oscillations of a plane element and study the propagation of harmonic waves in them.

In problems of determining the frequencies of natural oscillations of plane elements pivotally supported at the edges and based on approximate theories obtained on the basis of hypotheses and assumptions of a mechanical and geometric nature, in particular, on the basis of approximate equations such as the Kirchhoff equation of the parabolic type, poorly describing the wave and oscillatory nature of the behavior of a plane element under non-stationary external influences. In the study of harmonic waves in deformable bodies, the concept of phase velocity is introduced as the rate of change in the state of the medium, while the phase velocity is expressed in terms of the frequencies of natural oscillations, and therefore the study of the propagation of harmonic waves is directly related to the problems of determining the eigenforms and frequencies of vibrations of plane elements limited in plan.

This paper presents the results of the study of natural and forced vibrations of flat elements, taking into account the layering of the element material, rheological viscous properties, environmental influences, deformable base, anisotropy, etc.

The influence of these factors significantly complicates the study of problems about natural and forced vibrations of a plane element, about the propagation of harmonic waves in them.

The paper is devoted to the formulation of various boundary value problems of oscillations of a rectangular flat element, both taking into account viscosity. and taking into account the above factors of a geometric and mechanical nature.

2. Formulation of the problem

We confine ourselves to solving the task on the basis of an approximate equation of transverse oscillations of the fourth order [2]

$$P_0(W)+p\frac{\partial^2 W}{\partial t^2}+\frac{h^2}{6}\Big[p^2(N^{-1}+3M^{-1})\frac{\partial^4 W}{\partial t^4}-4p(3-2MN^{-1})\Delta\frac{\partial^2 W}{\partial t^2} + 8M(1-MN^{-1})\Delta^2 W\Big]=\Phi(\varphi_z, f_{lz}) \tag{1}$$

Since the edges of $(y=0; l_2)$ of the plate are hinged and supported, the solution of equation (1) will be sought in the form

$$W(x,y,t)=\exp\left(i\frac{b}{h}\xi t\right)\sum_{k=1}^{\infty}W_k(x)\sin\left(\frac{k\pi y}{l_2}\right) \tag{2}$$

Substituting (2) into equation (1), for W_k we obtain an ordinary differential equation

$$\frac{d^4W_k}{dx^4}+B_0\frac{d^2W_k}{dx^2}+B_1W_k=0 \tag{3}$$

where the coefficients B_0, B_1 are equal

$$B_0=\left[\frac{A_1}{A_2}\xi^2\Big(\frac{b}{h}\Big)^2-2\Big(\frac{k\pi}{l_2}\Big)^2\right]$$

$$B_1=\left[\left(\frac{k\pi}{l_2}\right)^4+\frac{A_0}{A_2}\xi^4\Big(\frac{b}{h}\Big)^4-\frac{A_1}{A_2}\xi^2\Big(\frac{b}{h}\Big)^2\left(\frac{k\pi}{l_2}\right)^2-\frac{1}{A_2}\xi^2\Big(\frac{b}{h}\Big)^2\right] \tag{4}$$

The general solution of the equation (3) is written as

$$W_k(x) = C_1\left[\frac{\cos(a_0 x)}{a_0^n} + \frac{\cos(a_1 x)}{a_1^n}\right] + C_2\left[\frac{\cos(a_0 x)}{a_0^n} + \frac{\cos(a_1 x)}{a_1^n}\right] + C_3\left[\frac{\sin(a_0 x)}{a_0^m} + \frac{\sin(a_1 x)}{a_1^m}\right] + C_4\left[\frac{\sin(a_0 x)}{a_0^m} + \frac{\sin(a_1 x)}{a_1^m}\right] \tag{5}$$

where C_j are the integration constants, the a_i, a_j roots of the characteristic equation

$$a^4 + B_0 a^2 + B_1 = 0 \tag{6}$$

and equal

$$a_{0,1} = \sqrt{\frac{B_0}{2}} \pm \sqrt{\frac{B_0}{2}^2 - B_1} \tag{7}$$

The integers (n, m) are chosen from the condition of simplifying the solution when the boundary condition on the left edge $x = 0$ is satisfied, and the other boundary conditions on $X = l_1$ lead to a transcendental equation for determining the self-frequencies of the plate [3].

Let us consider some of the formulated tasks.

In this case, at the edges of the plate, we have the boundary conditions

$$W_k = \frac{dW_k}{dx} = 0; \quad (x = 0; l_1) \tag{8}$$

Under boundary conditions (8) in the general solution (5), the numbers $n = 0$; $m = 0$, from the condition on the left end of the constant integration C_1, C_2 are zero, and from the conditions on the right end we get

$$C_2\Big[\cos(a_0 l_1) - \cos(a_1 l_2)\Big] + C_4\left[\frac{\sin(a_0 l_1)}{a_0} - \frac{\sin(a_1 l_1)}{a_1}\right] = 0$$

$$C_2\Big[a_0 \sin(a_0 l_1) - a_1 \sin(a_1 l_1)\Big] - C_4\Big[\cos(a_0 l_1) - \cos(a_1 l_1)\Big] = 0 \tag{9}$$

whence from the condition of non-triviality of the solution we get the transcendental frequency equation

$$2 - \frac{a_0^2 + a_1^2}{a_0 a_1}\sin(a_0 l_1)\sin(a_1 l_1) - 2\cos(a_0 l_1)\cos(a_1 l_1) = 0 \tag{10}$$

Firstly, let us consider the simplest transcendental equation

$$\alpha_0 \cos(\alpha_0 l_1)\sin(\alpha_1 l_1) - \alpha_1 \sin(\alpha_0 l_1)\cos(\alpha_1 l_1) = 0 \tag{11}$$

We introduce the notation

$$l = \frac{l_1}{h};$$

$$\alpha^1_{0,1} = \sqrt{\frac{B^1_0}{2} \pm \sqrt{\left(\frac{B^1_0}{2}\right)^2 - B^1_1}}$$

$$B_0'' = \Big[(2-\nu)\xi^2 - 2\gamma\Big]; \quad \gamma = \left(\frac{\pi k h}{l_2}\right)^2$$

$$B_1^{:} = \left[\gamma^2 + \frac{7-8\nu}{8}\xi^4 - (2-\nu)\gamma\xi^2 - \frac{3}{2}(1-\nu)\xi^2\right] \tag{12}$$

and we will omit the strokes in the future for simplicity.

Since the sines and cosines of any argument are equal

$$\sin z = \sum_{i=0}^{\infty}(-1)^i \frac{z^{2i+1}}{(2i+1)!} \quad \cos z = \sum_{j=0}^{\infty}(-1)^j \frac{z^{2j}}{(2j)!};$$

That equation (11) is equivalent to the following

$$\alpha_0\alpha_1 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j} \frac{\alpha_1^{2i}\alpha_0^{2j} - \alpha_o^{2i}\alpha_1^{2j}}{(2i+1)!(2j)!} l^{2(i+j)} = 0 \tag{13}$$

If we accept the quantity a_0 is determined from the expression (7) with a plus sign under the root, then it follows this root does not vanish for any values of γ, ν, ξ. [4]

Therefore, at the beginning it is possible to $a_1 = 0$ or for ξ we obtain the equation

$$\xi^4 - \frac{8\Big[(2-\nu)\gamma + \frac{3}{2}(1-\nu)\Big]}{(7-8\nu)}\xi^2 + \frac{8\gamma^2}{(7-8\nu)} = 0; \tag{14}$$

whose roots are equal

$$\begin{aligned}\xi_{1,2} =& (7-8\nu)^{-\frac{2}{2}}\sqrt{4\left[(2-\nu)\gamma + \frac{3}{2}(1-\nu)\right]} \\ &\pm \sqrt{8(1+\nu^2)\gamma^2 + 3\gamma(1-\nu)(2-\nu) + \frac{9}{4}(1-\nu)^2}\end{aligned} \tag{15}$$

since the series in the expressions of trigonometric functions are convergent and the series in equation (12), equivalent to equation (11) is also convergent, in the study of the partial equation (13) it can be limited to a finite number of first terms.

Taking the first three terms in the series (3), we write it in the form

$$\alpha_0\alpha_1(\alpha_1^2 - \alpha_0^2)\left\{\frac{1}{3}l^2 - \frac{1}{30}(\alpha_1^2 + \alpha_0^2)l^4 + \left[\frac{1}{840}(\alpha_1^4 + \alpha_0^2\alpha_1^2 + \alpha_0^4) + \frac{1}{360}\alpha_0^2\alpha_1^2\right]l^6 + ...\right\} \tag{16}$$

The roots of the expression $\alpha_1 = 0$ are equal to (15). The quantity of $(a_1^2 - a_0^2)$ is non-zero for any values of γ, ν, ξ.

If in the expression (16) we take only the first two terms, we get

$$(\alpha_1^2 + \alpha_0^2) - 10l^{-2} = 0$$

or

$$B_0 - 10l^{-2} = 0 \tag{17}$$

and frequency equation

$$\xi^2 = \frac{2\gamma + 10l^{-2}}{(2-\nu)}; \tag{18}$$

positive root of which is equal to

$$\xi = \sqrt{\frac{2\gamma + 10l^{-2}}{(2-\nu)}} \tag{19}$$

If we take all the first three terms in the expression, we get

$$\left[(\alpha_1^4 + \alpha_0^4) + \frac{10}{3}\alpha_0^2\alpha_1^2\right] - 28(\alpha_1^2 + \alpha_0^2)l^{-2} + 280l^{-4} = 0 \tag{20}$$

or

$$\left[B_0^2 + \frac{4}{3}B_1\right] - 28B_0l^{-2} + 280l^{-4} = 0$$

and their corresponding frequency equation

$$\left[(2-\nu)^2+\frac{7+8\nu}{6}\right]\xi^4-\left[(2-\nu)\left(\frac{16}{3}\gamma+28l^{-2}\right)+2(1-\nu)\right]\xi^2 + \left[\frac{16}{3}\gamma^2+56\gamma l^{-2}+280l^{-4}\right]=0 \tag{21}$$

which has two positive roots.

Similarly, one can take the first four or more terms in expression (13) and obtain a more accurate frequency equation and corresponding frequencies ξ. [6]

To find the frequency equation from the series of equation (13), it is necessary to clarify the condition of appropriate retention of a finite number of terms.

Let us apply the d'Alembert principle of series convergence to the series in equation (13). We get [7]

$$\left|\frac{\alpha_0^2\alpha_1^2l^2}{(2i+3)(2j+2)}\right|\le q^2<1 \tag{22}$$

where $0<q<1$.

From the inequality (22) implies that

$$\left|a_0^2a_1^2\right|\le q_{i,j}^2=q_{i,j}^2=q^2\frac{(2i+3)(2j+2)}{l^2} \tag{23}$$

The analysis of inequality (23) shows that it is valid when the solving the inequality

$$-\left(\frac{8}{7-8\nu}\right)q_{i,j}^2\le\xi^4-2D\xi^2+E\le\left(\frac{8}{7-8\nu}\right)q_{i,j}^2=C_{i,j}^2$$

where the coefficients D,E are equal

$$D=\frac{4\left[(2-\nu)\gamma+\frac{3}{2}(1-\nu)\right]}{(7-8\nu)};\qquad E=\frac{8\gamma^2}{(7-8\nu)}$$

or inequality

$$D^2-E\le C_{i,j}^2 \tag{24}$$

By the given parameters of a geometric and mechanical character from the inequality (24) one can determine the necessary number of first terms in series (13) for finding the frequency equationof relative frequencies ξ .

We consider the transcendental equation (10). Like transcendental equation (11), equation (10) is equivalent to the following

$$a_0a_1\left\{2\left[1-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\frac{a_0^{2i}a_1^{2j}}{(2i)!(2j)!}J^{2(i+j)}\right] - (a_0^2+a_1^2)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\frac{a_0^{2i}a_1^{2j}}{(2i+1)!(2j+1)!}J^{2(i+j+1)}\right\}=0 \tag{25}$$

From the equation it follows that, at first, $a_1=0$ and we get the frequencies (15). We write equation (25) by writing the first terms

$$\left\{(a_0^2+a_1^2)J^2-\frac{1}{6}(5a_0^4+5a_1^4+a_0^2a_1^2)J^4+\frac{1}{90}\left[a_0^6+a_1^6+7a_0^2a_1^2(a_0^2a_1^2)\right]J^6+...\right\}=0 \tag{26}$$

From (26) it also follows that we can assume $(a_0^2+a_1^2)=0$ and obtain

$$B_0 = 0$$

or

$$\xi^2 - \frac{2\gamma}{(2-\nu)} = 0 \tag{27}$$

whose root is equal to

$$\xi = \sqrt{\frac{2\gamma}{(2-\nu)}} \tag{28}$$

Similarly, we can suppose approximately

$$(a_0^2 + a_1^2) - \frac{J^2}{6}(5a_0^4 + 5a_1^4 + a_0^2 a_1^2) = 0 \tag{29}$$

and get the frequency equation

$$(5B_0^2 - 9B_1) - \frac{6}{J^2}B_0 = 0 \tag{30}$$

having positive roots.

Thus, transcendental frequency equations can be reduced to algebraic and to investigate the influence of both boundary conditions along the edges of a rectangular plate or a rectangular flat element, as well as geometric and mechanical character on the self-oscillation frequencies of rectangular flat elements.

Let us generalize the previous results for a rectangular plate or a flat element, the material of which satisfies the Maxwell viscoelastic model.[8]

Suppose we have a rectangular homogeneousisotropic plate.

In this case, the solution of an approximate fourth order equation will be sought in the form

$$W = \exp\left(\frac{b}{h}\xi t\right)\sum_{k=1}^{\infty} W_k \sin\left(\frac{pky}{J_2}\right), \tag{31}$$

where ξ is the complex frequency, the real part of which determines the law of damping of oscillations, and the imaginary part determines the frequencies of self-oscillations.

For W_k we get an ordinary differential equation

$$\frac{d^4 W_k}{dx^4} - \overline{B_0}\frac{d^2 W_k}{dx^2} + \overline{B_1} W_k = 0 \tag{32}$$

where $\overline{B_0}, \overline{B_1}$ are equal

$$\overline{B_0} = \left[2\gamma + \frac{A_1}{A_2}\frac{b}{h}\left(\frac{b}{h}\xi^2 + \frac{1}{\tau}\xi\right)\right];$$

$$\overline{B_1} = \left[\frac{A_0}{A_2}\left(\frac{b}{h}\right)^4 \xi^4 + 2\frac{A_0}{A_2\tau}\left(\frac{b}{h}\right)^3 \xi^3 + \left(\frac{b}{h}\right)^2\left(\frac{1}{A_2} + \frac{1}{\tau^2}\frac{A_0}{A_2} + 2\gamma\frac{A_1}{A_2}\right)\xi^2 + \frac{1}{A_2\tau}\left(\frac{b}{h}\right)\left(1 + 2\gamma\frac{A_1}{A_2}\right)\xi + \gamma^2\right] \tag{33}$$

The coefficients A_j are given in the previous paragraphs.

The general solution of equation (32) we write in the form

$$W_k = C_1\left[\frac{ch(a_0x)}{a_1^n} + \frac{ch(a_1x)}{a_1^n}\right] + C_2\left[\frac{ch(a_0x)}{a_0^n} + \frac{ch(a_1x)}{a_1^n}\right] + \left[\frac{sh(a_0x)}{a_0^m} + \frac{sh(a_1x)}{a_1^m}\right] + C_4\left[\frac{sh(a_0x)}{a_0^m} + \frac{sh(a_1x)}{a_1^m}\right], \tag{34}$$

that is instead of trigonometric functions, we have hyperbolic.

All the boundary value tasks considered above leading to transcendental equations are solved in a similar way. Transcendental equations are obtained from the previous ones, in which values a_j must be replaced by values i, a_j, where i is an imaginary unit.

For example, the transcendental equation (11) goes into the equation

$$a_0 ch(a_0l_1)sh(a_1l_1) - a_1 sh(a_0l_1)ch(a_1l_1) = 0 \tag{35}$$

which is equivalent to the following

$$a_0a_1\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{a_1^{2i}a_1^{2j}a_0^{2i}}{(2i+1)!(2j)!}l^{2(i+j)} = 0;$$

$$a_{0,1} = \sqrt{\frac{\overline{B_0}}{2}} \pm \sqrt{\frac{\overline{B_0^2}}{4}} - B_1 \tag{36}$$

One of the frequency equations of the formula (36) follows from the condition $a_j = 0$ and we get

$$\xi^4 + \frac{2}{\tau_0}\xi^3 + \frac{8}{(7-8\nu)}\left[(2-\nu)\gamma + \frac{(7-8\nu)}{8\tau_0^2} + \frac{3}{2}(1-\nu)\right]\xi^2 + \frac{12(1-\nu)}{(7-8\nu)\tau_0}[1+2(2-\nu)\gamma]\xi + \frac{8}{(7-8\nu)}\gamma^2 = 0 \tag{37}$$

which coincides with the frequency equation for a rectangular hinged and supported plate on all four sides of the plate, has two complexly conjugate roots.[9]

3. Methods

The theory of oscillation and the method of calculating the vibrations of a flat element, taking into account the above-mentioned factors of a mechanical, rheological and geometric nature, is based on the consideration of a flat element in a three-dimensional formulation of the mechanics of a solid deformed body, under the same boundary and initial conditions. The three-dimensional problem is solved using the methods of integral Fourier and Laplace transformations. In transformations, general solutions of a three-dimensional dynamic problem are constructed.

We present the results obtained by the described method for a different type of flat element. The general equation of plate oscillation with respect to the transverse displacement W of the points of the median plane of the plate $z = 0$ has the form

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{\lambda_1^{(n)}\left[\lambda_2^{(1)} - \Delta\right]^2 DQ_m + \left[\lambda_2^{(1)} + \Delta\right]\lambda_1^{(n+m)} + 4\lambda_2^{(1)}\Delta DQ_n\lambda_1^{(n)}\right\}W\frac{h^{2(n+m+1)}}{(2n+1)!(2m)!} = -\sum_{n=0}^{\infty}M^{-1}\left[(\lambda_2^{(1)} - \Delta)DQ_n + \lambda_1^{(n)}\right]f_z\frac{h^{2n}}{(2n)!} \tag{38}$$

where N and M are viscoelastic operators

$$N(\zeta) = (\lambda + 2\mu)\left[\zeta(t) - \int_0^\infty f_1(t-\xi)\zeta(xi)d\xi\right],$$

$$M(\zeta) = \mu\left[\zeta(t) - \int_0^\infty f_1(t-\xi)\zeta(\xi)d\xi\right], \tag{39}$$

$f_z(x, y, t)$- external unsteady forces;
$\lambda_1^{(1)}, \lambda_2^{(1)}, Q_n$ - operators of the form;

$$\lambda_1^{(1)} = \left[pN^{-1}\left(\frac{\partial^2}{\partial t^2}\right) - \Delta\right]; \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2};$$

$$\lambda_1^{(1)} = \left[pM^{-1}\left(\frac{\partial^2}{\partial t^2}\right) - \Delta\right]; \quad Q_n = \sum_{q=0}^{n=1} \lambda_1^{(n-q-1)}\lambda_2^{2q} \tag{40}$$

The general equation (40) is complex in structure and of little use for solving applied problems. Approximate oscillation equations can be obtained from the general equation. For example, limiting the series of sums of the equation to the first two terms, we obtain an approximate equation

$$\begin{aligned} P_0(W) \equiv p\frac{\partial^2 W}{\partial t^2} + \frac{h^2}{6}\Bigg[&p^2(N^{-1} + 3M^{-1})\frac{\partial^4 W}{\partial t^4} \\ &- 4p(3 - 2MN^{-1})\Delta\frac{\partial^2 W}{\partial t^2} + 8M(1 - MN^{-1})\Delta^2 W\Bigg] = \frac{1}{h}f_z, \end{aligned} \tag{41}$$

$2h$ – where is the thickness of the plate. Equation (41) is a generalization of the equations of Kirchhoff, S.P. Timoshenko and other authors.

If a homogeneous plate lies on a deformable base, then in equation (41) it is necessary to add the law of rebuff, which has the form

$$P_1(W) = \frac{S}{2h}\left\{\frac{\partial W}{\partial t} + \frac{h^2}{2}\frac{\partial}{\partial t}\left[p(M^{-1} + 3N^{-1})\frac{\partial^2 W}{\partial t^2} - 4\Delta W\right]\right\}; \tag{42}$$

$$S = M_1^{-1} M N_1^{1/2}\sqrt{p_1}$$

M_1, N_1, p_1-parameters of the base. As you can see, the law of rebuff is different from Winkler's.

When solving certain problems of oscillation of, say, rectangular plates, it is necessary to formulate boundary and initial conditions. The obtained general solutions and the dependences of displacements and stresses on the desired functions allow us to unambiguously strictly derive the boundary conditions. It is shown that for the hinge and rigid fastening, the boundary conditions coincide with the classical ones, and for the stress-free edge, boundary conditions for a homogeneous isotropic plate of the form ($x = const$) are obtained:

$$(2 + 3D)\frac{\partial^2 W}{\partial x^2} + (1 + D)\frac{\partial^2 W}{\partial y^2} - p(1 + D)M^{-1}\left(\frac{\partial^2 W}{\partial t^2}\right) = 0;$$

$$\frac{\partial^3 W}{\partial x^3} = 0, \tag{43}$$

where one of the conditions contains an inertial component, which corresponds to the Dalembert principle of mechanics. If the flat edge of the plate is in rigid contact with the deformable vertical

plate, then the boundary condition of elastic sealing has the form

$$\frac{h^3}{3}DM\frac{\partial^2 W}{\partial x}+\left[\frac{h^3}{3}DM-2h_1^2D_1M_1\right]\frac{\partial^2 W}{\partial y^2}=\left(\frac{h^2}{6}Dp-h_1^2D_1p_1\right)\frac{\partial^3 W}{\partial x\partial t^2}; \tag{44}$$

$$W=h_1\frac{\partial W}{\partial x};$$

p_1, D_1, M_1- plate parameters.

4. Conclusion

Depending on the particular types of the plane element under consideration, the main unknown functions are selected in the general solutions of the three-dimensional problem: displacement or deformation at points in the fixed plane of the plane element, in particular, in the median plane of the plate of constant thickness. Displacements and stresses at an arbitrary point of a plane element are expressed in terms of basic unknown functions, which are determined from boundary conditions on the surfaces of the plane element.The obtained equations for the basic unknown functions are the general equations of oscillation of a plane element containing derivatives of functions in coordinates and time of any arbitrarily large order. General solutions are presented in the form of power series over the thickness of a flat element. The general solution refers to an equation of hyperbolic type, which describes the oscillatory and wave process in a plane element. Limiting the series of the general equation to a finite number of the first terms, we obtain approximate equations of oscillation of a plane element. Thus, the proposed approach allows us to strictly construct approximate theories of vibrations of plane elements of various types.

References

[1] Almagambetova A., Tileubay S., Taimuratova L., Seitmuratov A.,Kanibaikyzy K., Problem on the distribution of the harmonic type Relay wave, *News of the National Academy of Sciences of the Republic of Kazakhstan, Series of Geology and Technical Sciences*, Vol.1, No.433, 2019, pp.242-247.

[2] Seitmuratov A., Tileubai S.,Toxanova S.,Ibragimova N.,Doszhanov B.,Aitimov M., Axismetric problems of elastic layer oscillation limited by rigid or deformed boundries, *News of the National Academy of Sciences of the Republic of Kazakhstan, Series of Geology and Technical Sciences*, Vol.1, No.427, 2018, pp.127-135.

[3] Seitmuratov A., Tileubay S., Toxanova S., Ibragimova N., Doszhanov B., Aitimov M., The problem of the oscillation of the elastic layer bounded by rigid bouhdaries, *News of the National Academy of Sciences of the Republic of Kazakhstan, Series of Physics and Mathematics*, Vol.5, No.321, 2018, pp.42-48.

[4] Filippov I., Filippov S. Kostin V., *Proceedings of the International Conference on Mechanics and Materials* USA, Los Angeles, 1995, pp. 75–79.

[5] Seitmuratov A., Seylova Z., Kanibaikyzy K., Smakhanova A., Serikbol M., Approximate equation plate oscillation for transverse displacement of points of the median plane, *News of the National Academy of Sciences of the Republic of Kazakhstan, Series of Geology and Technical Sciences*, Vol.3, No.429, 2018, pp.258-265.

[6] Seitmuratov A., Seylova Z., Kanibaikyzy K., Smakhanova A., Serikbol M., Approximate equation plate oscillation for transverse displacement of points of the median plane, *News of the National Academy of Sciences of the Republic of Kazakhstan, Series of Geology and Technical Sciences*, Vol.3, No.429, 2018, pp.258-265.

[7] Ashirbayev, Nurgali K and Banaś, Józef and Dubiel, Agnieszka and others, Solvability of an integral equation of Volterra-Wiener-Hopf type, *Abstract and Applied Analysis*, 2014, 9 pages.

[8] Medeubaev N., Seytmuratov A., Ramazanov M., Solving Problems of Vibrational Processes of Isotropically Homogeneous Elastic Plates, *Lobachevskii Journal of Mathematics*, Vol.41, No.9, 2020, pp.1846-1853.

[9] Seitmuratov A., Medeubaev N. Yeshmurat G., Kudebayeva G., Approximate solution of the an elastic layer vibration task being exposed of moving load, *News of the National Academy of Sciences of the Republic of Kazakhstan, Series of Physics and Mathematics*, Vol. 2, No.318, 2018, pp.54-60.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ON ESTIMATION OF DISTRIBUTION FUNCTION BY PRESMOOTHED RELATIVE-RISK FUNCTION

ABDURAHIM ABDUSHUKUROV[1], SUKHROB BOZOROV[2]

[1]Moscow State University named after M. V. Lomonosov, Tashkent Branch, Tashkent, 100060 Uzbekistan
[2]Gulistan State University, Gulistan,120100 Uzbekistan
e-mail: a_abdushukurov@rambler.ru
e-mail: suxrobbek_8912@mail.ru

Abstract. In paper we present some comparison of distribution function estimators when the lifetime data subjecting to right random censoring. Nonparametric estimators based on conception of presmoothed estimation of relative-risk function. We give some numberical results also.

Keywords: Random, censoring, product-limit, relative risk, presmoothed, proportional hazards, asymptotic representation, strong consistency.

AMS Subject Classification (MSC 2020): 62N02, 62N05.

1. Introduction

Censored data occur in survival analysis, bio-medical trials, industrial experiments. There are several schemas of censoring (from the right, left, both sides, mixed with competing risks and others). However, in statistical literature right random censoring is wide spread, in so far as it was easily described from the methodological point of view. Here we consider also this kind of censorship in order to comparing our results with others.

Let $X_1, X_2, ...$ and $Y_1, Y_2, ...$ be two independent sequences of independent and identically distributed (i.i.d.) random variables (r.v.-s) with common unknown continuous distribution functions (d.f.-s) F and G, respectively. Let the X_j be censored on the right by Y_j, so that the observations available for us at the n – th stage consist of the sample:

$$C^{(n)} = \{(Z_j, \delta_j), 1 \leq j \leq n\},$$

where $Z_j = \min(X_j, Y_j)$ and $\delta_j = I(X_j \leq Y_j)$ with $I(A)$ meaning the indicator of the event A.

1. If $X_j \leq Y_j$, then $Z_j = min(X_j; Y_j) = X_j$ is equal to $\delta_j = 1$ and in this case we can observe X.

2. Otherwise if $Y_j \leq X_j$, then $Z_j = min(X_j; Y_j) = Y_j$is equal to $\delta_j = 0$, and this can be censoring condition.

The main problem consists a nonparametrical estimating of d.f. F with nuisance d.f. G based on censored sample $C^{(n)}$, where the number of observed $X_j - s$, $\nu_n = \delta_1 + ... + \delta_n$ is a random amount with a Binomial distribution.

Authors are supported by Uzbekistan-India Joint Research Project.

Kaplan and Meier [1] were the first to suggest the product-limit (PL) estimator ${F_n}^{PL}$ of F defined as

$$ {F_n}^{PL}(t) = \begin{cases} 1 - \prod\limits_{\{j:Z_{(j)} \le t\}} \left(1 - \frac{\delta_{(j)}}{n-j+1}\right), & t \le Z_{(n)}, \\ 1, \qquad t > Z_{(n)}, \quad \delta_{(n)} = 1, \\ undefined, \quad t > Z_{(n)}, \quad \delta_{(n)} = 0, \end{cases} \tag{1} $$

where $Z_{(1)} \le ... \le Z_{(n)}$ are the order statistics of Z- sample $\{Z_j,\ 1 \le j \le n\}$ and $\left\{\delta_{(j)},\ 1 \le j \le n\right\}$ the sequence of indicators adjunct to the ordered Z - sample. There are a different versions of PL - estimators. However, those do not coincide, if the largest Z_j is a censoring time. There is an enormous set of the works on investigating of several properties of PL - estimators and their application on statistical problems, specially in case of right random censorship.

Now, in order to study the extent to which the above estimator (1) is, we draw an estimation by creating a software that gives an option on this model using computer modeling (Figure 1).

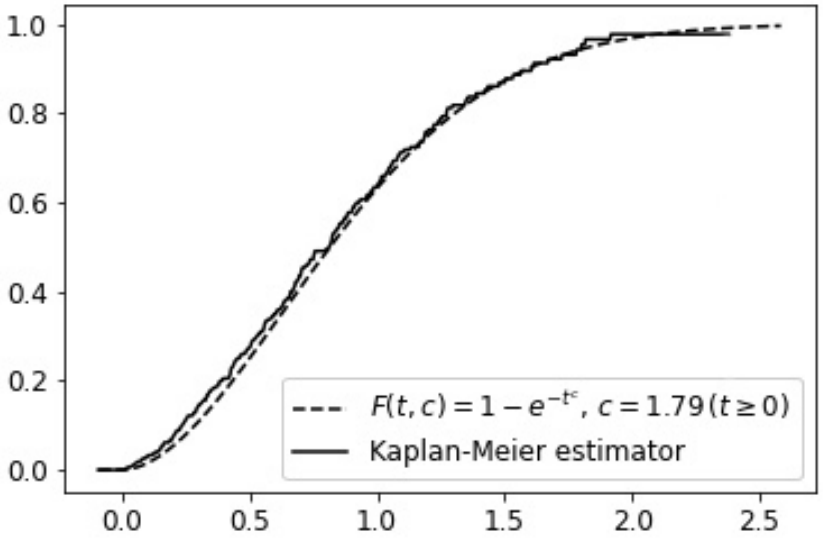


FIGURE 1. F_n^{PL}– Kaplan-Meier estimator, (Weibull distribution)

However ${F_n}^{PL}$is not a unique estimator of d.f. F. Abdushukurov [2, 3] proposed another estimator of F, of relative-risk power type:

$$ {F_n}^{RR}(t) = 1 - (1 - H_n(t))^{R_n(t)} = \begin{cases} 0, & t < Z_{(1)}, \\ 1 - \left(\frac{n-j}{n}\right)^{R_n(t)}, & Z_{(j)} \le t < Z_{(j+1)}, \quad 1 \le j \le n-1, \\ 1, & t \ge Z_{(n)}, \end{cases} \tag{2} $$

where $H_n(t) = \frac{1}{n}\sum_{j=1}^{n} I(Z_j \le t),\ t \in \mathbb{R}^1 \equiv (-\infty; +\infty)$ is an empirical estimator of d.f. $P(Z_j \le t) = 1 - (1 - F(t))(1 - G(t)) \equiv H(t)$ and $R_n(t) = (\Lambda_n(t))^{-1}\Lambda_{1n}(t)$ is an estimator of relative-risk function $R(t) = (\Lambda(t))^{-1}\Lambda_1(t),\ t \in \mathbb{R}^1$. Here cumulative hazard functions (c.h.f.-s) Λ, Λ_0 and Λ_1 corresponding to d.f.-s H, G and F defined as

$$ \Lambda(t) = \int_{-\infty}^{t} \frac{dH(u)}{1 - H(u-)} = \Lambda_0(t) + \Lambda_1(t), $$

$$\Lambda_0(t) = \int_{-\infty}^{t} \frac{dG(u)}{1-G(u-)} = \int_{-\infty}^{t} \frac{dH_0(u)}{1-H(u-)}, \tag{3}$$
$$\Lambda_1(t) = \int_{-\infty}^{t} \frac{dF(u)}{1-F(u-)} = \int_{-\infty}^{t} \frac{dH_1(u)}{1-H(u-)},$$

with subdistribution functions $H_0(t) + H_1(t) = H(t)$, $t \in \mathbb{R}^1$,

$$H_0(t) = P(Z_j \le t,\, \delta_j = 0) = \int_{-\infty}^{t} (1 - F(u-))\, dG(u),$$

$$H_1(t) = P(Z_j \le t,\, \delta_j = 1) = \int_{-\infty}^{t} (1 - G(u-))\, dF(u).$$

The corresponding estimators of c.h.f.-s (3) are

$$\Lambda_n(t) = \int_{-\infty}^{t} \frac{dH_n(u)}{1-H_n(u-)} = \Lambda_{0n}(t) + \Lambda_{1n}(t),$$

where

$$\Lambda_{kn}(t) = \int_{-\infty}^{t} \frac{dH_{kn}(u)}{1-H_n(u-)}, \quad k = 0,1; H_{kn}(t) = \frac{1}{n}\sum_{j=1}^{n} I(Z_j \le t, \quad \delta_j = k)$$

are empirical counterparts of $H_k(t)$, $k = 0,1$ with $H_{0n}(t) + H_{1n}(t) = H_n(t)$, $t \in \mathbb{R}^1$.

Now in order to study what the abovementioned (2) estimator's degree is, we will construct software that an option gives through computer modeling and demonstrate the estimation in drawing (Figure 2).

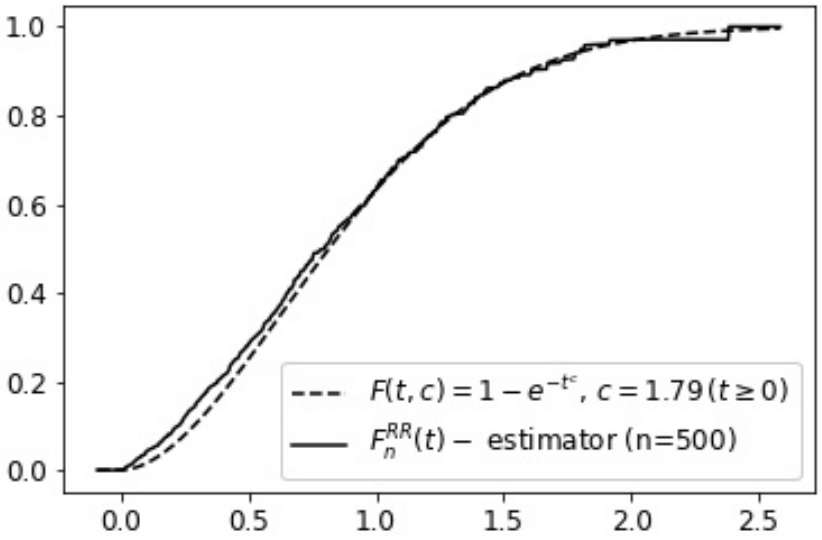


FIGURE 2. F_n^{RRP} – Relative – Risk Power estimator

In [2-4] it was shown that both of estimators (**??**) and (**??**) have a similar asymptotic properties tending to the same limiting Gaussion process. However, the relative-risk power estimator (**??**) has some small-sample advantages with respect to PL - estimator (1). For example, it is not sensitive to censoring in last observed point $Z_{(n)}$, since ${F_n}^{RR}\left(Z_{(n)}\right) = 1$ and it is identifiable with the model: $\left(1 - {F_n}^{RR}(t)\right)\left(1 - {G_n}^{RR}(t)\right) = 1 - H_n(t)$, $n \ge 1$, $t \in \mathbb{R}^1$, where

${G_n}^{RR}(t) = 1 - (1 - H_n(t))^{1-R_n(t)}$ is an corresponding estimator of d.f. $G(t)$. In [4] it was proposed several extended versions of estimator (2) in generalized models of incomplete observations mixed with competing risks. These estimators were also extensively studied in some statistical problems. It is not difficult to observe that estimator (2) is a natural extension of well-know ACL - (Abdushukurov-Cheng-Lin) estimator of F in simple Proportional Hazards Model (PHM):

$$ {F_n}^{ACL}(t) = 1 - (1 - H_n(t))^{p_n}, \quad t \in \mathbb{R}^1, $$

where $p_n = \frac{\nu_n}{n}$ is an estimator of probability $p = P(\delta_j = 1)$, which is value of the constant relative-risk function $R(t) \equiv p$ (so far as in PHM, $\Lambda_1(t) = p\Lambda(t)$, $t \in \mathbb{R}^1$). Note that ${F_n}^{ACL}$ was independently proposed and studied by Abdushukurov [5] and Chen, Lin [6] (for more information, see also Csörgő [7]). This estimator was studied, extended and used by many other authors up present. The main property of PHM is its characterization by independence of subsamples $\{Z_1, ..., Z_n\}$ and $\{\delta_1, ..., \delta_n\}$. This property is equivalent to relation $1 - G(t) = (1 - F(t))^{\beta}$, $t \in \mathbb{R}^1$ for some positive β. In PHM, $p = \frac{1}{1+\beta}$ and therefore β is a censoring parameter. Estimator ${F_n}^{ACL}$ in PHM is asymptotically efficient with respect to ${F_n}^{PL}$. This advantage of estimator is well preserved for plug-in estimators of many functionals (see, [2, 5, 7]).

Now, with the aim of studying to what extent the above ACL estimation is, we draw up a software that gives an option on this model using computer modeling and draw the estimation (Figure 3 and Figure 4).

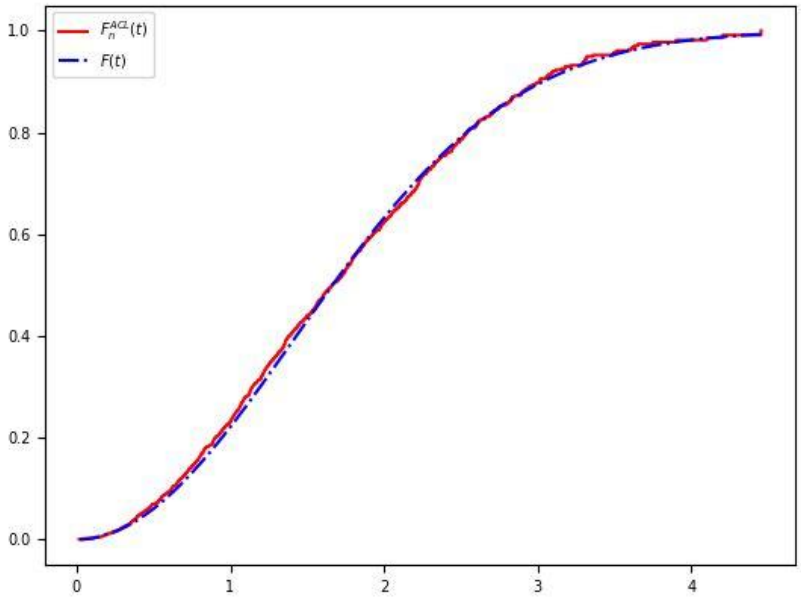


FIGURE 3. F_n^{ACL} – ACL estimator, (Weibull distribution (c=2))

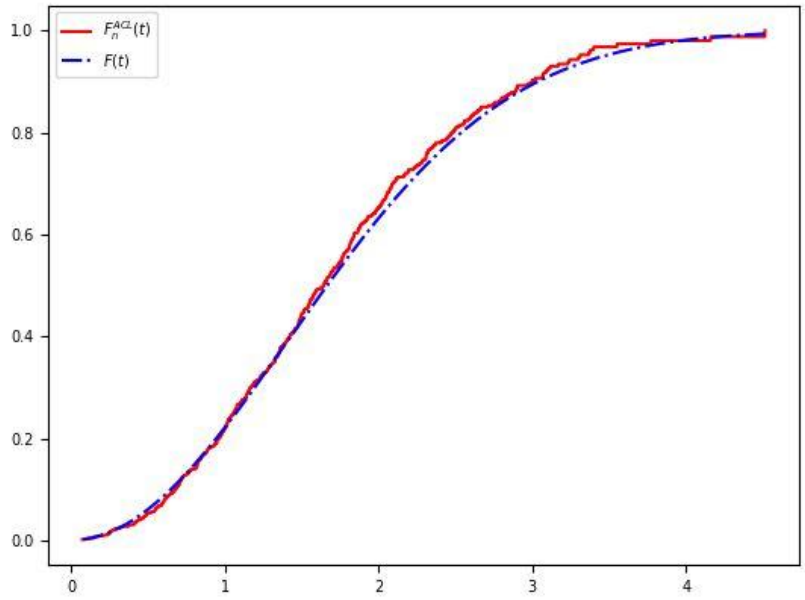


FIGURE 4. $F_n^{ACL}-$ACL estimator's persistency B-robustness, (Weibull distribution (c=2)))

That is why, in this framework the conditional probability that datum is not censored given its observed value

$$p(t) = P(\delta_j = 1/Z_j = t) = \mathrm{E}[\delta_j/Z_j = t], \quad t \in \mathbb{R}^1, \tag{4}$$

is a very important function, which in PHM is constant $p(t) = \frac{1}{1+\beta}$, $t \in \mathbb{R}^1$. Moreover, the key role of probability (4) takes part in expressing c.h.f. Λ_1 via Λ as

$$\Lambda_1(t) = \int_{-\infty}^{t} p(u)\, d\Lambda(u), \quad t \in \mathbb{R}^1,$$

and, therefore, relitive-risk function given as

$$R(t) = (\Lambda(t))^{-1} \int_{-\infty}^{t} p(u)\, d\Lambda(u), \quad t \in \mathbb{R}^1.$$

Probability (4) is a regression of δ_j on Z_j. Hence, it can be estimated by some of regression statistics. We have used following nonparametric regression estimator of Nadaraya [8] and Watson [9]:

$$p_n(t) = \left[\frac{1}{nh(n)}\sum_{j=1}^{n} k\left(\frac{t-Z_j}{h(n)}\right)\right]^{-1} \left[\frac{1}{nh(n)}\sum_{j=1}^{n} \delta_j k\left(\frac{t-Z_j}{h(n)}\right)\right], \tag{5}$$

where the kernel $k(\cdot)$ is a given probability density function and $\{h = h(n), n \geq 1\}$ is a bandwith sequence such that: $h \downarrow 0$, $n \to \infty$. In case of dependence of probability (4) on unknown parameters it may be estimated parametrically (see, Dikta [10] in this context). Cao et. al. [11] proposed following presmoothed PL-estimator of d.f. F by replacing the censoring indicators

$\delta_{(j)}$ in the expression of PL-estimator (1) by the estimator (5) at the observed data points:

$$F_n{}^p(t) = 1 - \prod_{\{j:Z_{(j)}\le t\}} \left(1 - \frac{p_n\left(Z_{(j)}\right)}{n-j+1}\right), \quad t \in \mathbb{R}^1. \tag{6}$$

Now, in order to study the extent to which the above estimator (6) is, we draw an estimation by creating a software that gives an option on this model by computer modeling (Figure 5).

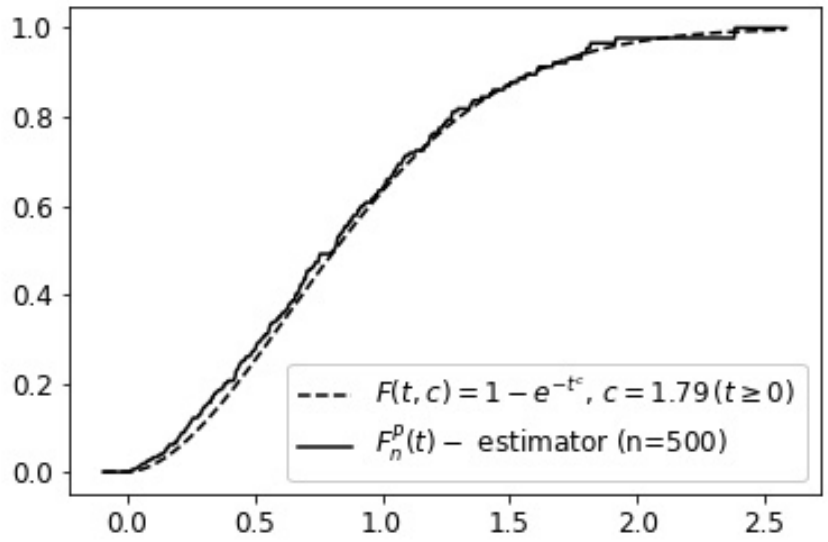


FIGURE 5. F_n^p-estimator

Some asymptotic properties of estimator (6) were investigated in [11, 12]. Taking into account some advantages of estimator (2) with respect to (1), we propose a new presmoothed relative-risk power (PRRP) estimator:

$$F_n{}^{PR}(t) = 1-(1-H_n(t))^{R_n^p(t)} = \begin{cases} 0, & t < Z_{(1)}, \\ 1-\left(\frac{n-j}{n}\right)^{R_n^p(t)}, & Z_{(j)} \le t < Z_{(j+1)},\ 1 \le j \le n-1, \\ 1, & t \ge Z_{(n)}, \end{cases} \tag{7}$$

were

$$R_n^p(t) = (\Lambda_n(t))^{-1}\Lambda_{1n}^p(t) = (\Lambda_n(t))^{-1}\int_{-\infty}^{t} p_n(u)\,d\Lambda_n(u), \ t \in \mathbb{R}^1,$$

is a partially presmoothed analogue of estimator $R_n(t)$. For probability mass function (4) smooth estimator (5) is used in formula for c.h.f. $\Lambda_1(t)$. But the estimator (7) is not smooth. We can see that estimator (7) also well defined in whole line without any conditions on censorship.

Now, with the aim of studying to what extent the above estimator (7) is, we draw up a software that gives an option on this model using computer modeling and draw an estimation (Figure 6).

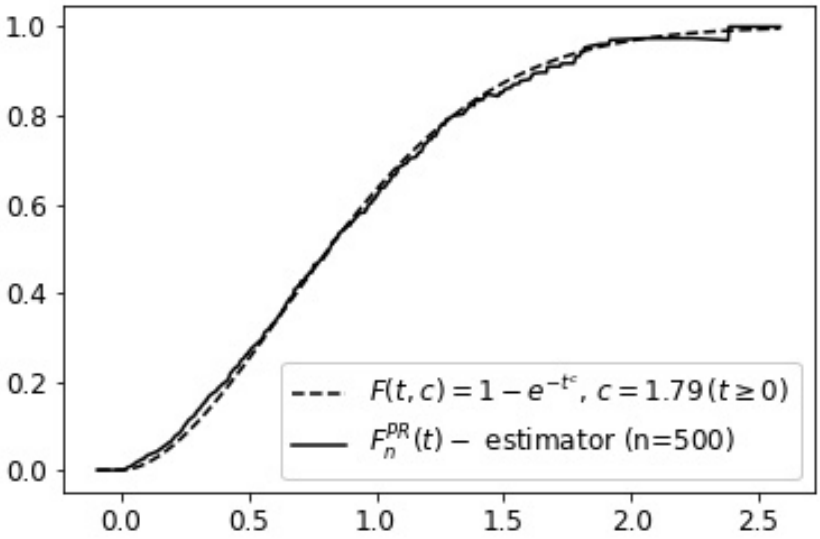


FIGURE 6. $F_n^{PRRP}-$ Presmoothed Relative-Risk Power estimator

2. Asymptotic properties of PRRP estimator

Let's denote $r(n) = h^2(n) + (nh(n))^{-1/2} (\log n)^{1/2}$. In order to investigating the properties of estimator (7) we need the following conditions:
$(C1)$ $(F,G) \in K = \{(F,G) : N_F \bigcap N_G \neq \emptyset,\ P(X_j \leq Y_j) \in (0,1)\}$, where $N_F = \{t : 0 < F(t) < 1\}$ and $N_G = \{t : 0 < G(t) < 1\}$;
$(C2)$ Numbers α, β and γ are such that $\min\{H(\alpha), 1 - H(\beta)\} \geq \gamma\,(0,1)$, $\alpha > \tau_H = \sup\{t : H(t) = 0\}$ and $\beta < T_H = \inf\{t : H(t) = 1\}$, $[\alpha,\beta] \neq \emptyset$;
$(C3)$ For all $n \geq 1$ there takes plase $P(0 < \nu_n < n) = 1$;
$(C4)$ k is a symmetric, twice continuously differentiable and bounded variation density function with compact support;
$(C5)$ Density $q(t) = H'(t)$ exists, is four times continuously differentiable at $t \in [\alpha,\beta]$ and $\sup_{\alpha \leq t \leq \beta} q(t) > 0$;
$(C6) p(t)$ is four times continuously differentiable at $t \in [\alpha,\beta]$;
$(C7) n^{1-\varepsilon} \cdot h(n) \to \infty$ for some $\varepsilon > 0$, $\sum_{n=1}^{\infty} h^{\lambda}(n) < \infty$ for some $\lambda > 0$ and $h^2(n) = o\left((nh(n))^{-1/2} \cdot \left(\log\left(\frac{1}{h(n)}\right)\right)^{1/2}\right)$.
Consider random functions

$$\varphi_1(t;z) = \frac{p(t)}{1 - H(t)} (I(Z \leq t) - H(t)),$$

$$\varphi_2(t;z) = \int_{-\infty}^{t} \frac{I(Z \leq u) - H(u)}{1 - H(u)} p'(u)\, du,$$

$$\varphi_3(t;z,\delta) = \int_{-\infty}^{t} k\left(\frac{u - Z}{h}\right) \frac{(\delta - p(u))}{1 - H(u)} du.$$

In the next theorem, we will show that PRRP estimator can be approximated by summ of i.i.d. random functions on t with the rate for the remainder term tending to zero at $n \to \infty$ almost surely.

Theorem 2.1. [13] *If the conditions* $(C1)-(C7)$ *are fulfilled, then there holds*

$$F_n{}^{PR}(t)-F(t)=(1-F(t))\,\Omega_n(t)+Q_n(t) \tag{8}$$

with $\sup\limits_{\alpha\le t\le\beta}|Q_n(t)|\overset{a.s.}{=}\mathrm{O}\left(\max\left\{(r(n)\log n)^2,\ \frac{\log n}{n}\right\}\right)$, *where*
$\Omega_n(t)=\frac{1}{n}\sum_{i=1}^{n}\left[\varphi_1(t;Z_i)-\varphi_2(t;Z_i)+\varphi_3(t;Z_i,\delta_i)\right]$.

Theorem 2.2. [13] *Let the assumptions of Theorem 1 are fulfilled. Then at* $n\to\infty$ *there holds*

$$\sup_{\alpha\le t\le\beta}\left|F_n^{PR}(t)-F(t)\right|\overset{a.s.}{=}\mathrm{O}\left(\max\left\{r(n)\log n,\left(\frac{\log n}{n}\right)^{1/2}\right\}\right). \tag{9}$$

The approximating sequence of normalized sum of random functions $\Omega_n(t)$ in Theorem 1is the same that for presmoothed PL – estimator (6). Therefore, from theorem 3.7 in [12] follows the asymptotic normality of PRRP estimator, under tacking into account the representation (8).

Theorem 2.3. [13] *Let the assumtions of Theorem 1 are fulfilled and* $(C8)$ $nh^2(n)(\log n)^{-6}\to\infty$,
$nh^8(n)(\log n)^4\to 0$ *and* $h^3(n)(\log n)^5\to 0$ *as* $n\to\infty$*for any* $t\in[\alpha,\beta]$. *Then there hold*

(1) *If* $nh^4(n)\to 0$, *then* $n^{1/2}\left(F_n^{PR}(t)-F(t)\right)\xrightarrow{d}N\left(0,\sigma^2(t)\right)$,

(2) *If* $nh^4(n)\to C^4$, *then* $n^{1/2}\left(F_n^{PR}(t)-F(t)\right)\xrightarrow{d}N\left(b(t),\sigma^2(t)\right)$,

where

$$b(t)=C^2(1-F(t))\,\alpha(t)\,d(k),d(k)=\int_{-\infty}^{\infty}u^2k^2(u)\,du,$$

$$\alpha(t)=\int_{-\infty}^{t}\frac{\left(\frac{1}{2}p''(u)\,q(u)+p'(u)\,q'(u)\right)du}{1-H(u)},\sigma^2(t)=(1-F(t))^2\,\gamma(t),$$

$$\gamma(t)=\int_{-\infty}^{t}\mu(u)\,du,\ \ \mu(t)=\frac{p(t)\,q(t)}{(1-H(t))^2}.$$

3. Numerical study of estimators

In this section we investigate the above estimates using numerical methods. By python programming language we are preparing a high-quality sample. We select $F(t,c)=1-e^{-t^c}$, $c=1.79$ $(t\ge 0)$ and get a sample of volume $n=500$. This sample is censored from the right with r.v.-s having a d.f. $G(t)=1-e^{-t}$ $(t\ge 0)$. The resulting sample have a degree of censorship 47%. We will study the above estimates on the resulting sample.

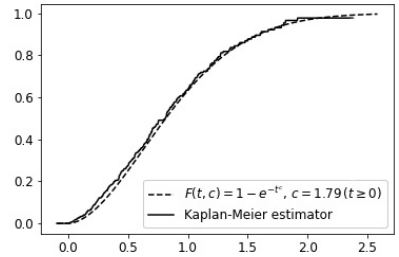


Figure 7. F_n^{PL}−Kaplan-Meier estimator

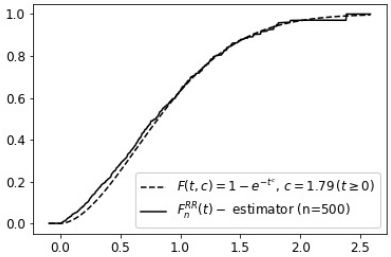


Figure 8. F_n^{RRP}– Relative – Risk Power estimator

The red line in the figure shows the theoretical d.f. $F(t,\, c)$ and the green line shows the Kaplan-Meier estimate (Figure 7). One disadvantage of this estimate is that it may not matter at this endpoint.

Now we draw the evaluation graph (Figure 8) of estimator proposed by Abdushukurov (2). In the figure, the red line shows the theoretical d.f., the blue line shows the Abdushukurov's estimate. It can be seen from the graphs drawn that both estimates are very good. But in practice, it is difficult for us to see on the graph which score is better. Therefore, we study the sum $\sum_{i=1}^{n-1}\left(F_n\left(Z_i\right)-F\left(Z_i\right)\right)^2$. Let's make the appropriate tables for it.

Sum	$\sum_{i=1}^{n-1}\left(F_n^{PL}\left(Z_i\right)-F\left(Z_i\right)\right)^2$	$\sum_{i=1}^{n-1}\left(F_n^{RRP}\left(Z_i\right)-F\left(Z_i\right)\right)^2$	Relative
Value	0.204858	0.123901	0.604814

From the table above, it can be concluded that the estimate (2) proposed by Abdushukurov is closer to the d.f. $F(t,\, c)$.

Now we draw the estimates (6) and (7).

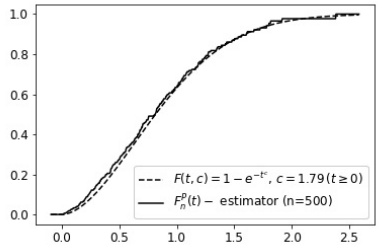


FIGURE 9. F_n^p-estimator

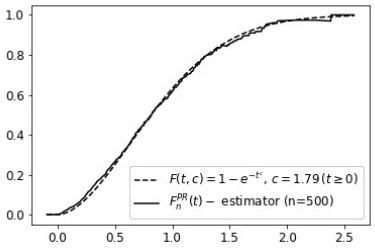


FIGURE 10. $F_n^{PRRP}-$ Presmoothed Relative-Risk Power estimator

As can be seen from the graph, despite the high level of censorship, both estimates are very close to the theoretical d.f.. The table below shows that the price actually depends on the selected bandwith sequence in formula (5).

Bandwith	$\sum_{i=1}^{n}\left(F_n^{p}\left(Z_i\right)-F\left(Z_i\right)\right)^2$	$\sum_{i=1}^{n}\left(F_n^{PRRP}\left(Z_i\right)-F\left(Z_i\right)\right)^2$	Relative
$h(n)=\frac{1}{\sqrt{n}}$	0.294402	0.277757	0.943462
$h(n)=\frac{1}{\sqrt[3]{n}}$	0.23152	0.210734	0.910219
$h(n)=\frac{1}{\sqrt[5]{n}}$	0.181307	0.178650	0.985345

From the above table, we can conclude that the $F_n^{PRRP}-$estimator is better than F_n^p-estimator.

REFERENCES

[1] Kaplan E. L and Meier P. *"Nonparametric estimation from incomplete observation"*, *J. Am. Stat. Assoc.*, **53**, 457-481 (1958).

[2] Abdushukurov A. A. *"Nonparametric estimation of the distribution function based on relative-risk function"*, *Commun. Stat.: Theory Meth.*, **27**, 1991-2012 (1998).

[3] Abdushukurov A. A. *"On nonparametric estimation of reliability indices by censored samples"*, *Theory Probab. Appl.*, **43**, 3-11 (1999).

[4] Abdushukurov A. A. *"Estimation of Unknown Distributions from Incomplete Observations and their Properties"*, LAP Lambert Academic, Saarbrtücken, (2011).

[5] Abdushukurov A.A. *"Nonparametric estimation in proportional hazards model of random censorship"*, VINITI **3448** (B87) (1987).

[6] Cheng P. E. and LinG. D. *"Maximum likelihood estimation of survival function under the Koziol-Green proportional hazards model"*, *Stat. Probab. Lett.*, **5**, 75-80(1987).

[7] CsörgőS. *"Estimation in the proportional hazards model of random censorship"*, *Statistics*, **19**, 437-463 (1988).
[8] NadarayaE. A. *"On estimating regression"*, *Probab. Theor. Relat. Fields*, **61**, 405-415 (1964).
[9] Watson G. S. *"Smooth regression analysis"*, Sankhya, Ser. A **26**, 359-372 (1964).
[10] Dikta J. *"On semiparametric random censorship models"*, *J. Stat. Plann. Infer.*, **66**, 253-279 (1998).
[11] Cao R., Lopez-de-Ullibarri I., Janssen P and Veraverbeke N. *"Presmoothed Kaplan-Meier and Nelson-Aalen estimators"*, *J. Nonparam. Stat.*, **17**, 31-56(2005).
[12] Jacome M. A and Cao R. *"Almost sure asymptotic representation for the presmoothed distribution and density estimators for censored data"*, *Statistics*, **41**, 517-534 (2007).
[13] Abdushukurov A.A., Bozorov S.B., Nurmukhamedova N.S. *"Noparametric Estimation of Distribution Function Under Right Random Censoring Based on Presmoothed Relative - Risk Function"*, ISSN 1995-0802, Lobachevskii journal of mathematics, 2021, Vol. 42, No. 2, 257-268. Pleiades Publishing, Ltd., 2021. https://link.springer.com/content/pdf/10.1134/S1995080221020049.pdf.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

MATHEMATICAL MODEL OF OPTIMAL MANAGEMENT OF THE CUSTOMS CLEARANCE PROCESS

ABDUSOBIR SAIDOV[1], TOHIR ABDURAKHMONOV[2]

[1]Customs Institute of the Customs Committee under the Ministry of Economy and Finance of the Republic of Uzbekistan, Tashkent, Uzbekistan
[2]Department of Information Technology and Cybersecurity of the Customs Committee under the Ministry of Economy and Finance of the Republic of Uzbekistan, Tashkent, Uzbekistan

e-mail: sobirs59@mail.ru

Abstract. The article considers the issue of modeling the multi-step process of customs clearance of goods in foreign trade. A mathematical model of control of the process under consideration has been developed. A brief review of existing methods for solving the linear programming problem with variable coefficients of the target function is given. The essence of customs risks has been studied and a method for identifying customs risks of reliability using threshold matrixes has been proposed. An algorithm for controlling the reliability of the customs value of goods is developed and the results of the implementation of this algorithm are given.

Keywords: customs clearance, mathematical modeling, linear optimization, objective function with a variable coefficient, customs risks, threshold matrix, reliability criteria.

AMS Subject Classification: 68T10

1. Introduction

International trade has long been considered the fundamental form of international economic relations. The pace of globalization in the first quarter of the XXI century confirms the role of international trade as the main driver of socio-economic development of countries. Analysis of the dynamics of international trade in the period from 2000 to 2015 shows that the export of developed countries increased from $4.243212 trillion to $8.613816 trillion (+103.00%), developing countries - from $2.059532 trillion to $7.344534 trillion (+256, 61%), countries with economies in transition - from 149.573 billion dollars to 525.571 billion dollars (+251.38%). On average, the volume of exports of world trade goods increased by 254.2% [1].

At the same time, the customs services of the countries participating in international trade play an important role in the international supply chain. There is a theory in the scientific literature, according to which any customs system successively passes several separate phases of its activity, characterized by the specificity of its relations both with foreign trade participants and with the state. Today, the customs systems of developed countries are in the "customs for foreign trade participants" phase, while for most developing countries the "customs for the government" phase is characteristic, and in a number of disadvantaged and underdeveloped countries, the "customs for themselves" phase is observed [2].

Conversely, the faster a country's customs service approaches the "customs for traders" phase of development, the faster the country approaches the level of developed countries. Today, the development of the activities of the customs service of each country is possible only through the use of modern information and communication technologies. Therefore, such requirements

are imposed to the methods of customs service management as orientation of models on artificial intelligence, possibility of synthesis of adaptive control system and application to complex analysis of multilevel system.

From this point of view, the task of optimal management of the process of organizing customs control and customs clearance of non-trading goods is relevant.

2. The problem of optimal management of the customs clearance process

The purpose of any type of control is to change the state of the control object in accordance with a predetermined task. Control methods should answer the question, "how can we construct an algorithm that can control a given object in a way that achieves a predetermined goal?". To do this, the developer needs to know how the control object will respond to different influences, that is, the control object model is needed.

There are many definitions to the concept of "model". One of them is close to our question: "A model is an object that allows you to study the behavior of another object, which is called the original". The model and the original should be similar so that the conclusions drawn from studying the model can be applied to the original [3].

As noted above, in order to study the control object, it is necessary to know how it reacts to various influences on it. If we denote these influences as "input" signals for the control object, then the changes occurring under the influence of these "input" signals can be regarded as "output" signals. That is, the object interacts with the external environment using "incoming" and "outgoing" signals.

If the documents submitted to the customs authorities in this process are taken as "inputs" for modeling the multi-stage customs clearance process and denote them by $X(t)$, then customs clearance will be carried out in accordance with these documents. In this case, you can take as an "output" signal the information received as a result of customs clearance, and designate it $Y(t)$ (Fig. 1.).

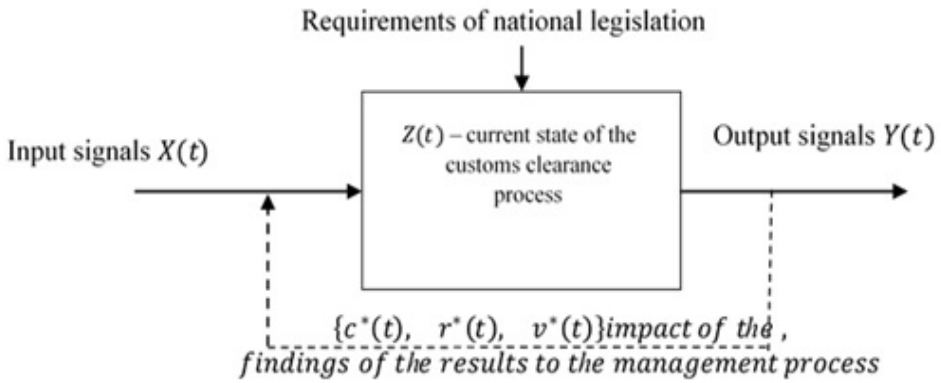


Figure 1. The main factors of the customs clearance process

here:

$c(t),\ r(t),\ v(t)$- set of impacts of customs clearance results to management;

$c(t)$ - objects of organization of customs control (goods, vehicles, persons), documents, additional information flows, etc.;

$r(t)$ – risks that may affect the process (information, human, financial and other resources);

$v(t)$ - obstacles to achieving the goal (refusal, etc.).

This set of influences moves the customs clearance system $Z(t)$ towards a given goal and generates a vector of output results $Y(t)$. From a mathematical point of view, the function $Y(t)$ is the reaction of the control object - the customs clearance process - to external influences.

Optimal management of the multi-stage customs clearance process can be carried out with respect to a number of objectives. In particular, optimization is envisaged in relation to one of the following goals:

- ensuring the completeness of revenues to the state budget, etc. maximization of revenues to the state budget;
- minimize the amount of arrears that may not go to the state budget, that is, possible damage to the budget;
- Reduction of expenses of the entrepreneur - participant of foreign trade, etc. minimization of damage caused to him in the process of customs clearance;
- minimize time for customs clearance processes;
- other purposes.

The purpose of this study is the optimal management of the customs clearance process, as well as minimizing the time spent on this process. Because it will ultimately lead to minimization of time for the customs clearance process, as well as to maximize revenues to the state budget and minimize the costs of the entrepreneur - participant of foreign trade.

It is known that in the general case the question of linear optimization can be expressed as follows [4]:

$$\max(\min) f(x) = \sum_{k=1}^{n} c_k x_k \tag{1}$$

$$\begin{cases} \sum\limits_{k=1}^{n} a_{jk} x_k = b_j, & if \quad j = \overline{1, m_1} \\ \sum\limits_{k=1}^{n} a_{jk} x_k \geq b_j, & if \quad j = \overline{m_1 + 1, m_2} \\ \sum\limits_{k=1}^{n} a_{jk} x_k \leq b_j, & if \quad j = \overline{m_2 + 1, m} \\ x_k \geq 0, \ k = \overline{1, n} \end{cases} \tag{2}$$

In this research paper, in the formation of a mathematical model of optimal control of the process of multi-stage customs clearance in relation to the time consumed, it is proposed to use the stages of customs clearance shown in Table 1.

The presented list of functions is grouped into four main blocks. These functions are performed by customs authorities, foreign trade participants and enterprises providing services in the customs field. The organization of customs clearance is characterized by a large number of operations performed, the complexity of which is determined by a combination of factors: type of goods, country of origin, declared customs, etc. [5]:

From the above table it follows that the process of multi-stage customs clearance covers 18 stages, of which stages 1-7 are performed by a foreign trade participant or a customs broker, stage 8 - by a cargo carrier, stages 9-11 - stages are performed by owners of customs warehouses, and for execution 12 - 18 - the stages are responsible for the employees of the customs service. That is, in formula (1), $n = 18$.

$$f(x) = \sum_{k=1}^{18} r_k x_k \to \min \tag{3}$$

where: $r_k = r_k(X)$ – the level of risk of execution of the k-stage;

$X = X(x_1, x_2, \ldots, x_{58})$ – vector, x_i elements which are determined depending on the value of the graph of the cargo customs declaration. In practice, the level of risk of customs clearance is determined depending on the documents submitted for customs clearance, including the cargo customs declaration.

TABLE 1. Stages of the process of multi-stage customs clearance

Estimated execution time	**Function names**	**Implementer**
t_1	preparation of necessary primary documents for customs clearance	Foreign trade participant or customs broker
t_2	determination of the code of goods according to the harmonized system Commodity nomenclature	
t_3	calculation of the customs value of goods	
t_4	calculation of customs payments	
t_5	preparation of cargo customs declaration	
t_6	ensuring customs payments	
t_7	Submit a preliminary declaration to the customs authorities prior to the arrival of the shipment	
t_8	cargo delivery under customs control	Carrier
t_9	Storage related to customs inspection procedures	Customs warehouse jointly with a foreign trade participant and the customs and the customs service
t_{10}	Delay of goods due to inability to release due to reasons	
t_{11}	Storage initiated by the owner of the goods	
t_{12}	Request additional documents when necessary	Customs Service
t_{13}	direct the cargo to the appropriate (red, yellow or green) customs control lane based on the analysis of submitted documents and risk profiles	
t_{14}	implementation of procedures either according to the principles of red road customs	
t_{15}	implementation of procedures either on the basis of the yellow road customs	
t_{16}	implementation of procedures either according to the principles of green road customs	
t_{17}	control over the completeness of receipt of customs payments	
t_{18}	release of cargo into free circulation or for export in accordance with the established procedure	

At the same time, the estimated time of duration of the customs clearance process for responsible executors is determined by normative and directive documents, in particular, documents approved by the Cabinet of Ministers of the Republic of Uzbekistan. If we denote them as b_1, b_2, b_3 and b_4, respectively, conditions (2) will come to the following form :

$$\begin{cases} \sum\limits_{k=1}^{n} a_{jk}t_k \le b_1; \;\; a_{jk} = 1 \quad if \quad j = \overline{1, m_1} \\ \sum\limits_{k=1}^{n} a_{jk}t_k \le b_2; \;\; a_{jk} = 1 \quad if \quad j = \overline{m_1 + 1, m_2} \\ \sum\limits_{k=1}^{n} a_{jk}t_k \le b_3; \;\; a_{jk} = 1 \quad if \quad j = \overline{m_2 + 1, m3} \\ \sum\limits_{k=1}^{n} a_{jk}t_k \le b_4; \;\; a_{jk} = 1 \quad if \quad j = \overline{m_3 + 1, m_4} \\ t_k \ge 0, \;\; k = \overline{1, n} \end{cases} \tag{4}$$

here: $n = 18,\ m_1 = 7,\ m_2 = 8,\ m_3 = 11,\ m_4 = 18.$

$a_{jk} = 0$ at the values of index j, which are not included in conditions (4)

The above formulas (3) and (4) give a mathematical model of the problem of optimal control of the process of multi-stage customs clearance.

3. Analysis of existing methods for solving the linear programming problem with variable coefficients

The obtained results (3) and (4) show that the mathematical model of the problem of optimal control of the process of multistage customs clearance has the form of a linear programming problem with variable coefficients of the objective function. Currently, there are a number of effective methods available for solving the linear programming problem.

In particular, for constant values of the coefficients (rk) of the objective function (3) under the limiting conditions (4), a number of methods are used in practice to determine its minimum value. These include methods such as the simplex method, the deployment of a function on algebraic polynomials, Fourier series, the use of spline functions, and others. A sufficient number of computer programs for numerical solution of this problem have been implemented.

However, the features of the problem of optimal control of the multi-stage customs clearance process, which is described in (3) - (4) are variable coefficients of the target function. The functions which represent the degree of risk of the - process, is a function of the variables of the cargo customs declaration. This requires a specific approach to solve this problem.

In particular, the 3rd and 4th stages *"calculation of the customs value of goods"* and *"calculation of the amounts of customs payments"* of Table 1. are important stages in the customs clearance process in ensuring the fulfillment of fiscal tasks assigned to the customs authorities.

This fact indicates the relevance of improving mathematical modeling of the customs clearance process and its comprehensive study.

At the same time, the study of scientific papers on the study of similar problems showed that a sufficient number of studies have been carried out and certain methods have been developed for solving the problem of linear programming with variable coefficients.

For example, the work of [6] is one of the relatively early studies in this area. The problem of parametric programming of the following form is considered:

$$\begin{cases} f(X) = \sum\limits_{j=1}^{n} c_j x_j, \quad X = \{x_j\}, \quad c_j \in \Im \\ \sum\limits_{k=1}^{n} a_{ij} x_j = b_i; \qquad i = \overline{1,m} \\ x_j \geq 0,\ j = \overline{1,n} \end{cases} \tag{5}$$

Here c_j - elements of some ordered functional prospace $\Im$, a_{ij}, b_i - known, a x_j - unknown real numbers, X - plan of the problem (non-negative solution of the problem (5)) Plan X^* is optimal if for any plan $X\ f(X) < f(X^*)$. Note that the values of the target function (5) belong to the space $\Im$, in which the usual properties of numerical inequalities are known to hold.

The existence of a solution to problem (5) is proved in this paper by introducing the concept of resolving combinations of problem (5), considered as elements of the space $\Im$, are comparable with the zero of this space, etc., if $x \in R$, then one and only one of three relations holds: $x > 0,\ x < 0$ and $x = 0$. Under these conditions, the following theorem is proved:

Theorem. The problem (5) with a non-empty set of plans and a target function bounded from above, satisfying the condition: all solving combinations of coefficients of the target function are comparable to zero, has a solution.

However, the paper does not provide a methodology for determining the existing solution.

In [7], a parametric programming problem of the following kind is considered:

$$\begin{cases} \min\,(\max)\, z(x) = \sum\limits_{j=1}^{n} c_j(x)x_j \\ \sum\limits_{j=1}^{n} a_{ij}(x)x_j \geq b_j(x), \qquad i = \overline{1,m} \end{cases} \tag{6}$$

Here $a_{ij}(x)$, $b_i(x)$ and $c_j(x)$ – some piecewise constant argument functions $x = (x_1, x_2, \ldots, x_n)$.

The functions $a_{ij}(x)$, $b_i(x)$ and $c_j(x)$ are defined on the same set $G = \{x \in G \subset R_n\}$. There exists a finite partition $G = \bigcup G_k, (k = \overline{1,l})$ such that the functions are constant in each subset G_k and G_{k+1} can intersect only along their boundaries.

By requiring that in problem (1)-(2) the target function $z(x)$ and the constraint functions $f_i(x) = \sum\limits_{j=1}^{n} a_{ij}(x)x_j - b_i(x);\ \ i = \overline{1,m}$ be continuous and convex, by a simple enumeration of a finite number of regions G_k, in each the usual linear programming problem is solved.

The work [8] is devoted to the study of the problem of parametric programming of the following form:

$$\begin{cases} f(X) = c_1(t)x_1 + c_2(t)x_2 + ... + c_n(t)x_n \to \underset{x \in D}{extr} \\ A(t)X \leq b(t) \\ x_j \geq 0,\ \ j = \overline{1,n} \end{cases} \tag{7}$$

Here: $X = (x_1, \ldots, x_n)^T$ – n-dimensional vector of unknown variables, which satisfies the constraints (7), forming the set of admissible solutions of the proble;

$b(t) = (b_1(t), b_2(t), \ldots, b_m(t))^T$ and $C(t) = (c_1(t), \ldots, c_n(t))^T$ – parametric vectors of free terms of constraints and coefficients of the target function, respectively;

$A(t)_{n \times m} = (a_{ij}(t))\,,\ i = 1, \ldots, n;\ \ j = 1, \ldots, m$ is an $n \times m$-dimensional matrix of parametric constraint coefficients. The functional dependence on the parameter t can be either linear or nonlinear.

In the work on the basis of simplex method and differential transformations the methods of solving linear programming problems with parametric coefficients of the target function and right parts of constraints are considered, allowing to organize simple iterative calculations and excluding the solution of systems of inequalities. In this case, it is assumed that the parametric functions $X = (x_1, \ldots, x_n)^T$, $b(t) = (b_1(t), b_2(t), \ldots, b_m(t))^T$ and $C(t) = (c_1(t), \ldots, c_n(t))^T$ are sufficiently smooth, have smooth differentials, and have explicit expressions.

In the studies of the authors [9] and [10] the linear programming problem with variable parameters is considered, in which not only variables included in its composition, but also coefficients, as well as the right part and parameters (coefficients at variables) of the target function can change.

An approach is proposed that allows solving linear programming problems with interdependent variable coefficients using the simplex method. The formulation of the problem is as follows:

$$\begin{cases} \max F = \max \sum\limits_{j=1}^{n} c_j x_j \\ \sum\limits_{j=1}^{n} a_{ij}x_j \geq b_j, \qquad i = \overline{1,m} \\ x_j \geq 0, \quad j = \overline{1,n} \end{cases} \tag{8}$$

Here: m — number of restrictions, n — number of variables, and additional restrictions on variable coefficients:

$$\begin{cases} a_{ij}^- \le f_{ij} a_{ij} \le a_{ij}^+ \\ s_k^- \le \sum\limits_{i=1}^{m} d_{kij} a_{ij} \le s_k^+; \quad k = \overline{1,K} \\ \sum\limits_{i=1}^{m} p_{lij} a_{ij} = r_l; \quad l = \overline{1,L} \\ c_j^- \le f_{ij} c_j \le c_j^+ \end{cases} \tag{9}$$

In the constraint system (8) - (9), all parameters are constants set during the problem formulation process. Conditions (9) are sometimes called interval conditions and problem (8) is an interval linear programming problem with interdependent variable coefficients.

Under these conditions, the following lemma is proved by constructive method:

Lemma. Let in the linear programming problem (8) - (9) there are variable coefficients that depend on a parameter of the form , having a domain of definition on a certain interval and continuously differentiable on it. Then the simplex method applied to solve such a problem converges if the above constraints are satisfied.

The proof is constructive in the sense that it substantiates the execution of all stages of the implementation of the simplex method, taking into account condition (9). The essence of the proposed method is that at each step the number of the column entering the basis at the next iteration of the simplex method algorithm is determined and calculations are performed on the points of minimum of the function and .

In [10], the following parametric programming problem was considered:

$$\begin{cases} \sum\limits_{i=1}^{n} c_i x_i + c_0 \le Z, \\ \sum\limits_{k=1}^{n} a_{ik} x_k \le b_i, \qquad i = \overline{1,m} \\ x_j \ge 0, \quad j = \overline{1,n} \\ Z_{\min} - ?, \quad x_i(Z_{\min}) - ? \end{cases} \tag{10}$$

To solve the linear parametric programming problem, it is assumed that the coefficients change insignificantly with respect to their average values, as, for example, the cost of goods depending on the exchange rate or inflation rate. In such cases, when the relative changes in the coefficients are of the order of 10% or less, it is proposed to use the asymptotic perturbation method. Its essence is reduced to the search for the decomposition of the desired functions into functional series, the rapidity of convergence of which depends on the "smallness parameter" of the relative change of the functions affecting the problem.

The algorithm for solving the problem is as follows. We assume solutions of the problem (10) in the form of series consisting of corrections of the corresponding order:

$$Z = Z^0 + Z^1 + Z^2 + \ldots;$$

$$x_i = x_i^0 + x_i^1 + x_i^2 + \ldots;$$

The values of Z^k and x_i^k are determined in an iterative way, assuming Z^0 and x_i^0 as the first approximation in the iteration process. The values of Z^0 and x_i^0 are the solution of a simple linear programming problem (1), when their average values a_{ij}^0, b_i^0 and c_j^0 are taken instead of the variable coefficients $a_{ij}(x), b_i(x)$ and $c_j(x)$.

For example, if $c_j(x)$ is continuous in the interval [0;1], then we take as its mean value:

$$c_j^0 = \int\limits_0^1 c_j(x) dx \tag{11}$$

The above brief overview shows that the problem of linear programming with variable coefficients is a well-known problem that has sufficient applications and is being studied everywhere. A certain number of methods for solving this problem have been developed, which are successfully applied depending on the problem formulation and application area.

4. Trigger graph model of the process and customs risk criterion

Based on the results of the above analysis, we can say that all methods of solving the linear programming problem with variable coefficients imply certain requirements for the ratio of the coefficients of the target function. First, an explicit form and smoothness of the parametric function are required. Second, when applying some methods of solving this problem, additional conditions such as continuity and differentiability of the given function are required.

However, in the case of the problem of optimal control of the customs clearance process, the parametric coefficients of the target function do not have an explicit expression and the above conditions cannot be required. Consequently, there are certain difficulties in applying the available methods to solve the problem, and it is required to explore new approaches.

Based on the latter findings, to minimize the target function (3) under conditions (4), it is proposed to ensure minimization of the coefficients of the target function $r_k(X)$, which represent the degree of risk of the k – process. Without violating the conditions of the stated problem and generality, we can assume that:

$$r_k(X) \geq 0, \quad k = \overline{1,18}, \quad c_j^0 = \int_0^1 c_j(x)dx \tag{12}$$

The task of minimizing the coefficients of the target function $r_k(X)$ gives rise to the task of investigating the essence of customs risks.

The above studies have led to the fact that customs risks from the mathematical point of view is a function of many variables, is not presented in an explicit form, there are considerable uncertainties in its characteristics in terms of smoothness, continuity, differentiability and other qualities.

Before moving on to research on the specifics of the customs risk, it is important to consider the essence of the risk as a whole, since it is inherent in various fields of activity. There are many approaches to defining the concept of "customs risk". For example, the working group of the World Customs Organization, preparing the document "World Customs Organization Compendium on Customs Risk Management" in 2011, gave the following definition: "Risk: the result of doubts arising in relation to objects" [12]. In scientific literature it is defined as follows: "Risk is the probability of violation of customs legislation associated with evasion of payment of customs duties and taxes due" [13].

In these or other definitions of customs risk, which are observed in previously published scientific papers, the "probability of violation of customs legislation" passes as the main predicate. This means that the mathematical expression of customs risk must necessarily include the "probability of violation of customs legislation" model.

Based on the above, in order to develop a mathematical approach to this definition, it is necessary to study customs legislation both at the national level and at the international level. At the same time, the importance of studying international conventions in the field of customs should be emphasized, as the process of a foreign trade operation is directly related to foreign partners.

Fig. 2. shows a simplified graph model and Fig. 3. an explanatory scheme of the process of foreign trade operations.

An analysis of the customs legislation on the day of the present research shows the following. By the beginning of the second half of 2023, more than 2216 normative legal documents related

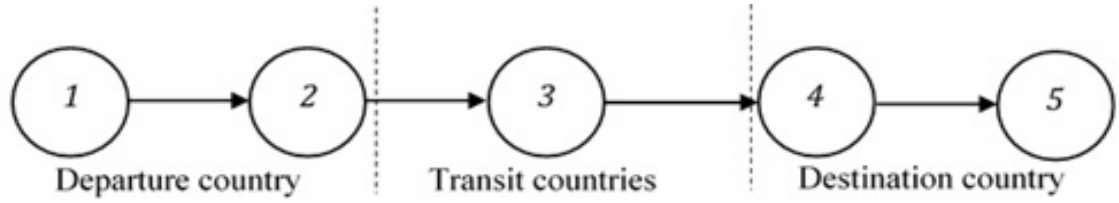


FIGURE 2. Simplified graph model of the process of foreign trade operations



FIGURE 3. Explanatory scheme of the process of foreign trade operations

to the regulation of the activities of customs authorities of the Republic of Uzbekistan were in force (Table 2).

TABLE 2. Information on legal documents related to the regulation of the activities of the customs authorities of the Republic of Uzbekistan

Type of normative document	**Amount**
Laws of the Republic of Uzbekistan	57
Codes of the Republic of Uzbekistan	12
Decrees of the President of the Republic of Uzbekistan	322
Resolutions of the President of the Republic of Uzbekistan	767
Resolutions of the Cabinet of Ministers of the Republic of Uzbekistan	1011
Orders of the Cabinet of Ministers of the Republic of Uzbekistan	11
Legal acts registered with the Ministry of Justice of the Republic of Uzbekistan	36
Total	**2216**

Despite the fact that there is such an extensive customs legal framework, the main task of the customs authorities of the Republic of Uzbekistan is to protect the economic security of the country and almost all the rules of this framework are focused on the following two main tasks [14]:

a) fulfillment of the fiscal task - ensuring the completeness of customs payments collection;

b) prevention, detection and suppression of violations of customs legislation, including smuggling.

The study of customs legislation and practical experiments showed that between the above-mentioned main tasks of managing foreign trade operations, in terms of procedure, there is an irreconcilable contradiction. This contradiction is manifested in the following:

a) the key parameter for increasing customs payments is the time spent on customs clearance of foreign trade goods: the less time spent on customs clearance of a particular foreign trade

cargo, the more cargo will be cleared for a certain period of time, hence, more customs payments will be made to the state budget during this period.

At the same time, the number of violations of customs legislation and the volume of goods of illegal circulation are increasing, since the time for a detailed study of the consignment of goods being processed remains minimal.

b) the key parameter for reducing customs law violations is also the time spent on customs clearance of foreign trade goods: the more time to study a specific consignment of foreign trade cargo, the less chance there is to commit violations of customs legislation.

At the same time, the amount of revenues to the state budget from customs payments decreases, as the foreign trade turnover for a certain period of time decreases.

To minimize irreconcilable contradictions between the main tasks of managing foreign trade operations in procedural terms, in practice, a customs risk management system is used. Trigger graph model of the main tasks of foreign trade operations management with application of risk management system is shown in Fig. 4 and Fig. 5.

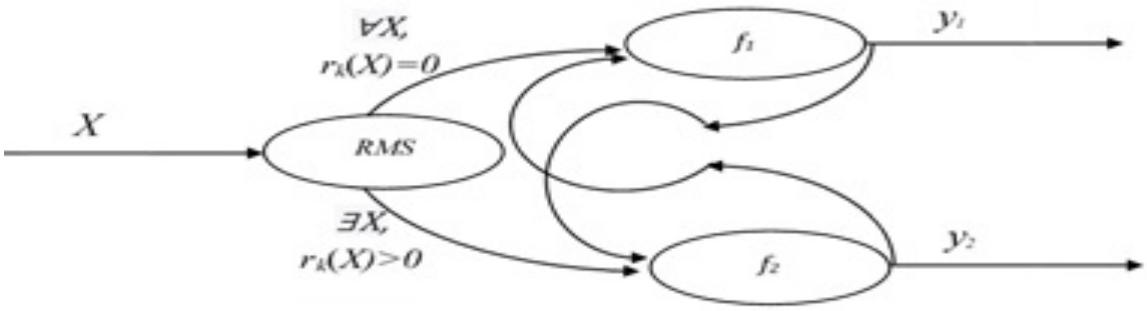


FIGURE 4. Trigger graph model of the main tasks of foreign trade operations management

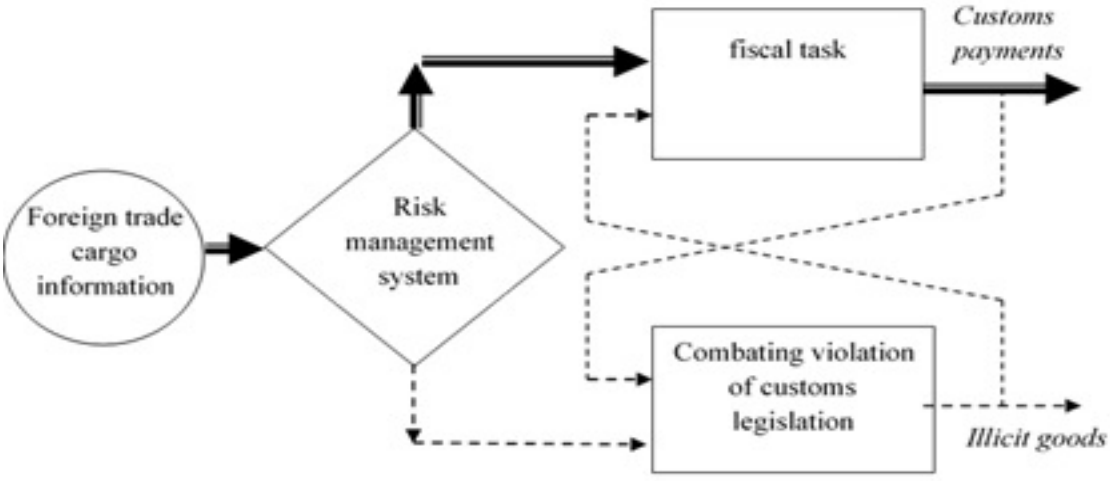


FIGURE 5. Explanatory scheme of the main tasks of foreign trade operations management

It should be noted that both functions - the function y_1, which reflects the volume of receipt of customs payments, and the function y_2, which reflects the volume of illegal goods (customs law violations) are functions of time, etc. $y_i = y_i(t)$. Both functions are inversely proportional to the ratio of customs clearance time, etc.

$$y_i = \varphi_i({}^1/_t), \quad where \;\; \varphi_i - linear\, functions, \quad i = 1, 2 \tag{13}$$

On the other hand, the conducted experiments showed that with an increase in the volume of goods of illegal circulation (offenses of the customs legislation), the receipts of customs payments decrease, etc. functions y_1 and y_2 are inversely proportional to each other:

$$y_1 = \psi_i({}^1/_{y_2}), \quad where \;\; \Psi_i - linear\, function \tag{14}$$

The mathematical contradiction reflected in (13) and (14) gives rise to the need for an optimization problem about the choice of time for customs clearance of foreign trade goods. It is required for each batch of goods to review the duration of customs clearance time and choose it so that there would be maximum receipt of customs payments and minimum volume of goods of illegal turnover (violations of customs legislation).

This task can be solved only if the following conditions are met:

- choose the minimum time for customs clearance, in the absence of customs risk;
- choose the time of customs clearance sufficient to ensure the minimization of the customs risk, if it is detected.

5. Risk assessment of the reliability of customs information

We mentioned above that in the scientific literature there is a theory according to which any customs system successively goes through several separate phases of its activity, characterized by the specifics of its relations both with foreign trade participants and with the state. Three phases of development are noted: the customs systems of developed countries are in the "customs for foreign trade participants" phase, in most developing countries the phase "customs for the government" is characteristic, and in a number of underdeveloped countries there is a "customs for themselves" phase.

Proceeding from the fact that the Republic of Uzbekistan is carrying out large-scale works on transferring the customs service to the phase "customs for foreign trade participants", the authors of this paper study customs risks, categorizing them into three classes:

a) customs risks for business;

b) customs risks of economic security;

c) corruption risks.

a) when it comes to the customs risk for business, it means the submission of an unreliable customs declaration by the business to the customs authorities. Analysis of the database of violations of customs legislation for several years shows that every 4th fact about such violation is the result of false declaration.

Despite the fact that at the present stage of development of foreign trade, favorable conditions are created for a law-abiding participant in foreign trade, the laws react rather harshly towards them if they have submitted an unreliable customs declaration to the customs authorities. The consequence of such phenomena for them can sometimes be undesirable, severe and long-lasting.

Therefore, the primary task of the customs service of the Republic of Uzbekistan today is to minimize customs risks for business.

b) the customs risk of economic security is the probability of violation of customs legislation by a participant in foreign trade, associated with evasion of payment of due customs duties and taxes.

c) corruption risks shall mean abuse of official powers, receiving and giving bribes, bribery, mediation in bribery, commercial bribery or other illegal use by a customs officer of his/her official position contrary to the legitimate interests of the state, in order to obtain benefits for himself/herself or for third parties.

Thus, the first step in solving the problem of optimal management of the customs clearance process (3) - (4) is to minimize the implicit function , which represent the degree of risk of unreliable declaration of foreign trade goods, etc. minimization of customs risks for business. To solve this problem it is necessary to assess the reliability of information about the goods on

all its parameters, etc. it is required to conduct a multivariate analysis of information about the goods. Information about the goods is fully reflected in the cargo customs declaration.

The research of the authors of this paper has shown that the customs cargo declaration is one of the fundamental documents of the customs clearance process. It is formalized in the form of a multidimensional matrix X, which is the source of state customs statistics (Fig.6.) [15].

A brief characterization of the customs cargo declaration is as follows:

- number of columns - 58;
- the level of detail of each column of the cargo customs declaration is determined depending on the complexity of the task, but not more than 40;
- L - the total number of cargo customs declaration per year.

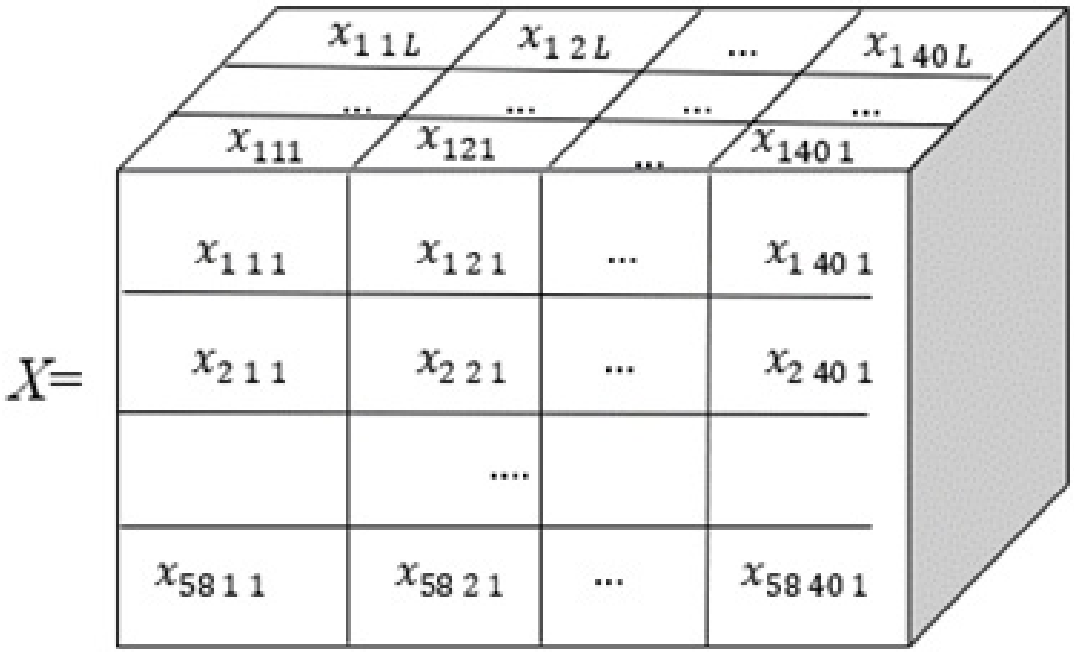


FIGURE 6. Multidimensional information matrix of the cargo customs declaration

It should be noted that each layer of this matrix corresponding to $l = l_0$ reflects a separate cargo customs declaration. It can be labeled as follows:

$$X_0 = \begin{pmatrix} x_{1\,1\,l_0} & x_{1\,2\,l_0} \cdots & x_{1\,40\,l_0} \\ \begin{cases} x_{2\,1\,l_0} \\ \\ \end{cases} \vdots & \begin{cases} x_{2\,2\,l_0} \\ \ddots \end{cases} & \begin{cases} x_{2\,40\,l_0} \\ \vdots \end{cases} \\ x_{58\,1\,l_0} & x_{58\,2\,l_0} \cdots & x_{58\,40\,l_0} \end{pmatrix} \tag{15}$$

Assessing the validity of information about the goods for all its parameters requires controlling and assessing the validity of all elements of the matrix (15). The study of existing methods for solving this problem showed that the problem of identifying unreliable customs declarations is a special case of the general and, as you know, ancient problem of identifying false information, etc. how to distinguish "truth" from "falsehood".

This famous problem is mentioned in many ancient writings, beginning with Aristotle (384 BC), who is the founder of logic as a science [16]. One of the great scientists who devoted his entire conscious life to the study of the task of distinguishing "truth" from "falsehood" is Imam al-Bukhari. His book Al-Jami'as-Sahih has been tested for over 11 centuries and is considered the most authentic book today [17].

The concept of information reliability has different meanings in philosophy, the theory of forensic evidence, epistemology, logic, probability theory, psychology, natural science and other areas. There is no single definition of the term, although many famous philosophers have tried to give their own definition of the term. In logic and philosophy, reliability often acts as a synonym for the concept of "truth" and characterizes indisputable, firmly substantiated and demonstrative knowledge.

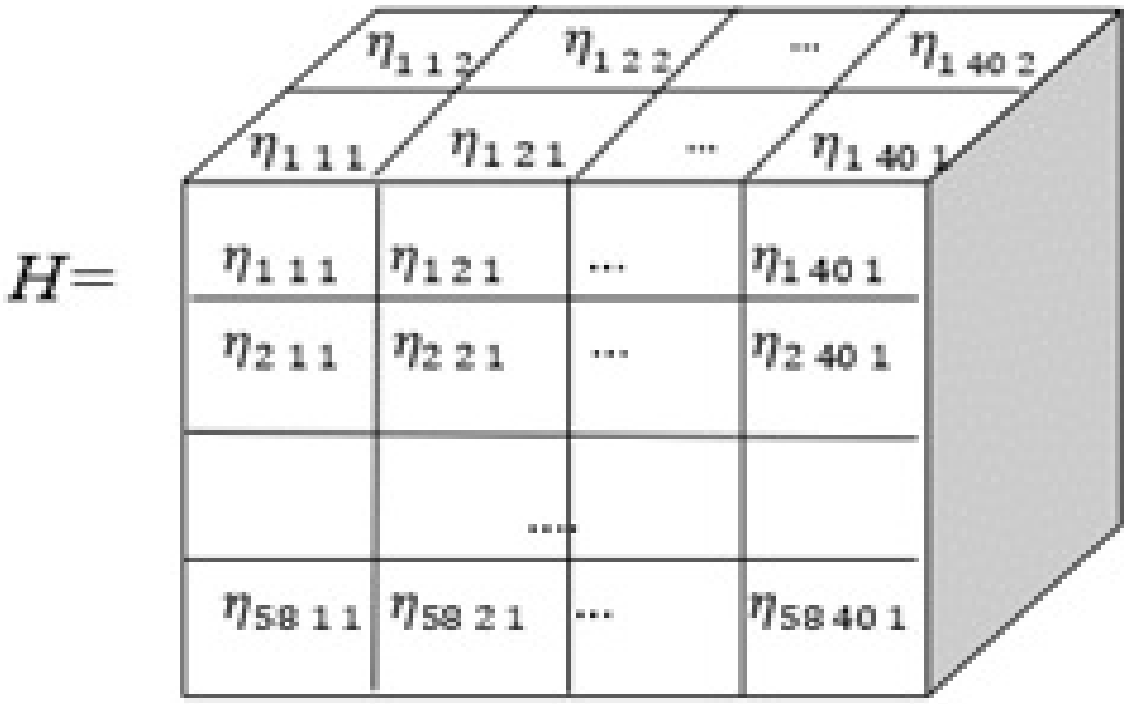


FIGURE 7. "Threshold Matrix" of customs risks

A "threshold matrix" (TM) is proposed to determine the concept of reliability of the elements of the customs cargo declaration (Fig.7. - 15.) (4.3.). here: η_{ij1}, η_{ij2} - some positive real numbers or textual information.

Definition 1. Each element of the matrix is $x_{ijl_0} \in X_0$ called reliable if the following condition is satisfied

$$\eta_{ij1} \leq x_{ijl_0} \leq \eta_{ij2} \tag{16}$$

where: $1 \leq i \leq 58,\ 1 \leq j \leq 40$.

Definition 2. If all elements of the matrix $x_{ijl_0} \in X_0$ are reliable, then the customs cargo declaration is called reliable.

Conditions (16) are called *criteria*, and the elements of the "Threshold Matrix" are called textitindicators of the reliability of the cargo customs declaration.

From Definition 1-2, the following statement is easily proved:

Statement 1. If at least one element of the matrix $x_{ijl_0} \in X_0$ does not satisfy the conditions (16), then the corresponding cargo customs declaration is unreliable. To assess the reliability of the cargo customs declaration, the following function is introduced:

$$\rho_{ij} = \begin{cases} e^{\eta_{ij1} - x_{ijl_0}}, & if \quad \eta_{ij1} \geq x_{ijl_0} \\ 1, & if \quad \eta_{ij1} \leq x_{ijl_0} \leq \eta_{ij2} \\ e^{x_{ijl_0} - \eta_{ij2}}, & if \quad x_{ijl_0} \geq \eta_{ij2} \end{cases} \tag{17}$$

The function can be estimated as follows: when the conditions $\eta_{ij1} \leq x_{ijl_0} \leq \eta_{ij2}$ are satisfied for all :$1 \leq i \leq 58,\ 1 \leq j \leq 40$. function value $\rho_{ij} = 1$; otherwise, -$\rho_{ij} > 1$. In other words,

the function ρ_{ij} reflects the quantitative assessment of the reliability of the element $x_{ijl_0} \in X_0$ of the cargo customs declaration. Then the matrix ρ (18) is the matrix of the reliability of the cargo customs declaration X_0.

$$\rho = \begin{pmatrix} \rho_{1\ 1} & \rho_{1\ 2} \cdots & \rho_{1\ 40} \\ \begin{cases} \rho_{2\ 1} \\ \vdots \end{cases} & \begin{cases} \rho_{2\ 2} \\ \ddots \end{cases} & \begin{cases} \rho_{2\ 40} \\ \vdots \end{cases} \\ \rho_{58\ 1} & \rho_{58\ 2} \cdots & \rho_{58\ 40} \end{pmatrix} \tag{18}$$

With the above notations, the following theorem is proved:

Theorem 1. In order for the cargo customs declaration X_0 to be reliable, it is necessary and sufficient to fulfill the following condition:

$$P = \prod_{i=1}^{58} \prod_{j=1} \rho_{ij} = 1 \tag{19}$$

R is the coefficient of reliability of the customs cargo declaration X_0. It follows from (17) and (19) that the coefficient R takes on the values $R = 1$ only if for all $1 \leq i \leq 58,\ \ 1 \leq j \leq 40$. the conditions $\eta_{ij1} \leq x_{ijl_0} \leq \eta_{ij2}$ are satisfied, otherwise $R > 1$.

6. Algorithm of control of risks of reliability of calculation of customs payments

To verify the above results, let's consider the tasks of controlling the risks of reliability of the calculation of customs payments. The amount of customs payments for the import of goods is determined as follows:

$$S = D + E + V$$

where: D- the amount of customs duty, E- the amount of excise tax, V- the amount of value-added tax on goods. They are determined mainly by the so-called "ad valorem rates". This means that the amount of each of the above types of customs payments is determined depending on the established rate in percentage terms. For example, the rate of value added tax in the Republic of Uzbekistan is set at 12% of the customs value of the goods.

The formulas for calculating them are as follows:

$$D = d \cdot c;$$

$$E = e \cdot c;$$

$$V = 0.12 \cdot (c + D + E)$$

where: c is the customs value of the goods, d is the rate of customs duty, e is the rate of excise tax.

After simple arithmetic transformations, you can get:

$$S = D + E + V = c \cdot (d + e + 0.12\,(1 + d + e))\,.$$

Hence, it can be seen that the amount of customs duties on imports of a particular good depends directly on the customs value of the good c. The lower the customs value of the goods, the lower the receipt of customs payments to the state budget.

Considering this circumstance, the "Threshold Matrices" were formed in the form of "Price Information Bulletins". The table reflects the following data:

η_7 – date of registration of the customs value of goods;

η_{33} – commodity code in accordance with the Harmonised System (HS);

η_{15} -– code of the country of departure of the goods;

η_{34} -– country of origin code;

η_{45} — customs value of goods.

The other elements of the "threshold matrix" for the considered criterion of the reliability of the customs value of goods are of little importance.

Then conditions (16) have the following form:

$$|\eta_7 - x_7| \leq 90$$

$$\eta_{15} = x_{15}$$

$$\eta_{33} = x_{33}$$

$$\eta_{34} = x_{34}$$

$$\eta_{45} \leq x_{45}$$

Taking into account the above designations, an algorithm for controlling the customs value of goods has been developed (Fig. 8).

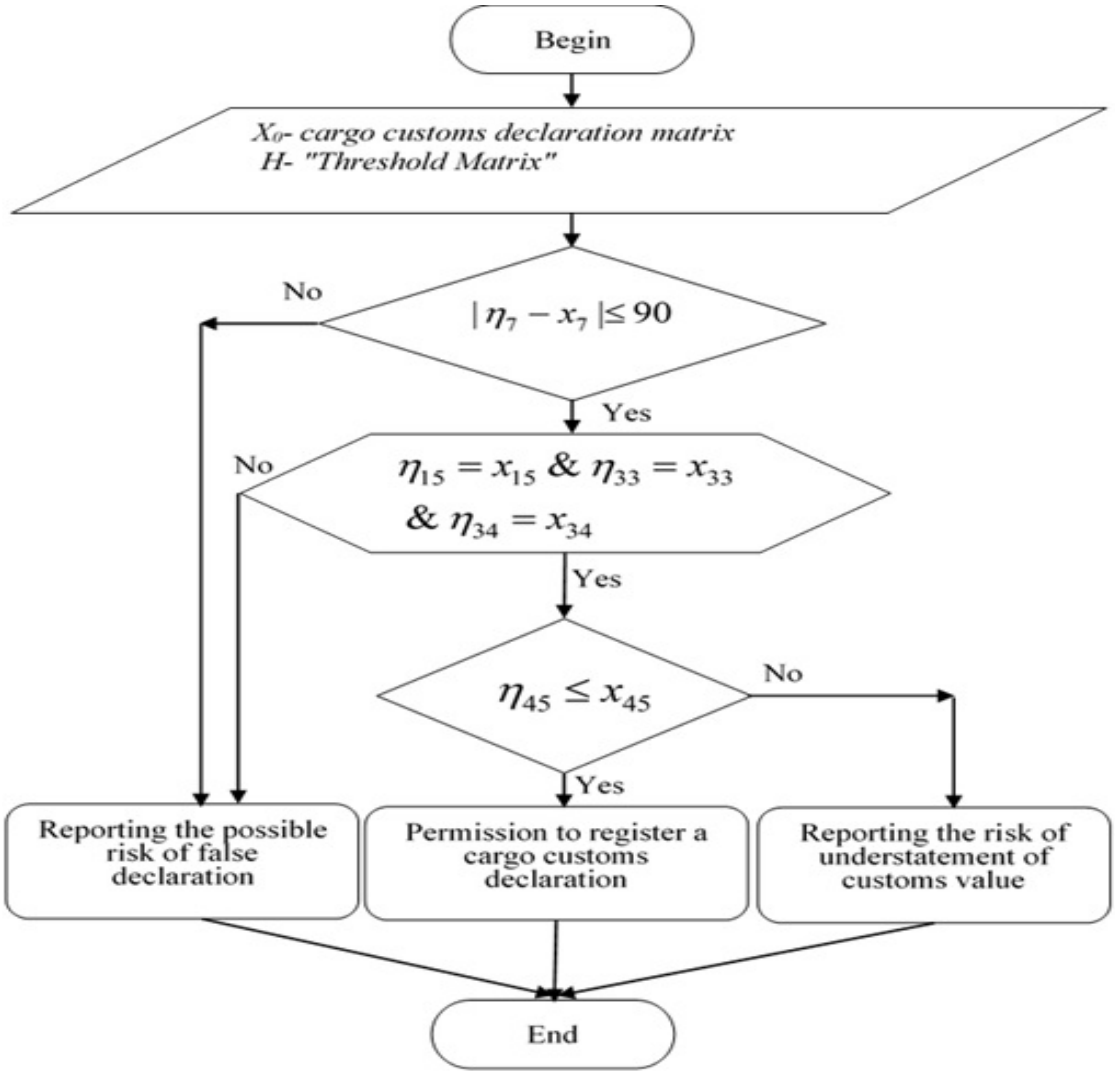


FIGURE 8. Algorithm for controlling the customs value of goods

It should be noted that the "Price Information Bulletin" is a characteristic feature of the national legislation of the Republic of Uzbekistan, and is not observed in the practice of the customs services of other countries.

7. Conclusion

In conclusion, we would like to note that the software that implements the above algorithm determines the errors of the customs cargo declaration and in the automated mode informs the foreign trade participant about it. In relation to the correcting participant in foreign trade, the identified errors are not applied in a timely and voluntary basis.

Currently, 53 logical conditions have been established to control the customs value of goods, which allow localizing customs risks. As a result of the introduction of these risk criteria in 2022, in 88 thousand 897 cases, the risks of *"determination of the customs value of goods"* and arrears to the state budget in the equivalent of more than 9 million 968.8 thousand US dollars were prevented.

References

[1] Rudneva A. O. International trade: specifics and prospects of participation of developed, developing and transition countries // MIR (Modernization. Innovation. Development). 2017. Vol. 8. No. 3. pp. 430-438. DOI: 10.18184/2079–4665.2017.8.3.pp.430–438;

[2] Ershov. A.D. Formation of customs services in foreign economic activity // Scientific notes of the St. Petersburg branch of the Russian Customs Academy. 2005. - № 1(23).pp.174-192;

[3] Zvonarev, S.V. Fundamentals of mathematical modeling: textbook // Yekaterinburg: Ural University Press, 2019. 112 p.;

[4] Ibyatov R.I. Optimization methods in problems of mathematical modeling// methodological guidelines, Kazan: Kazan State Agrarian University Publishing House, 2016. p.32;

[5] Fedorenko R. V. Methodology of management of service complexes in the customs sphere // Dissertation for the degree of Doctor of Economics. Samara. 2015. https://www.sseu.ru/wp-content/uploads/2015/06/Dissertatsiya-Fedorenko-R.V.pdf;

[6] Pinsker, A.G. A linear programming problem with variable coefficients of the purpose function. Sib Math J 20, pp.466–468 (1979). https://doi.org/10.1007/BF00969958. http://www.mathnet.ru/rus/;

[7] M. O. Gavrilova, "On problems of linear programming with piecewise constant coefficients," Scientific journal "Bulletin of the Perm State Technical University. Chemical technology and biotechnology", Perm, 2010, No. 9, pp. 172-179;

[8] Avetisyan A. G., Gyulzadyan L. S. A method for solving problems of parametric linear programming based on differential transformations // Scientific journal "Izvestia of the Tomsk Polytechnic University". 2014. V. 324. No. 2, pp. 25-30;

[9] Salimonenko D.A. A method for solving a linear programming problem with variable coefficients in the form of parametric functions // Scientific journal "Vestnik of the Bashkir University", Ufa, 2015. V. 20. No. 1. pp. 25–29;

[10] Salimonenko D.A., Ziganshin A.M., Mudrov V.A., Salimonenko Yu.D. On interdependent variable coefficients in linear programming problems // Scientific journal "Mathematical Structures and Modeling", Omsk, 2021. No. 2 (58). pp. 96–111. DOI 10.24147/2222-8772.2021.2.96-111;

[11] Kravchuk S.P., Kravchuk I.S., Tatarnikov O.V., Shved E.V. Perturbation method for solving linear programming problems with a parameter // Scientific journal "Fundamental Research", Moscow, 2015, No. 5, pp.299-303;

[12] WCO Customs Risk Management Compendium // Brussels, Belgium. June 2011. //. URL: http://www.wcoomd.org;

[13] Tamrazyan S.E. "Customs Risks: Essence, Management and Evaluation" // Scientific journal "Economics and Management in the XXI century: development trends". 2015. No. 23, pp.168-172;

[14] Saidov A.A., Khakimova F.A., Abdurakhmonov T.T. The concept and model of the "soft component" of the risk management system of customs authorities // Scientific journal "Bulletin of the Russian Customs Academy". Moscow (Russia), 2022, No. 3, pp. 100 - 109. DOI: 10.54048/20727240_2022_03_100;

[15] Saidov A.A. Classical Methods of Controlling the Reliability of Information and Features of Their Application to Customs // Monograph. Tashkent. 2021 498p.;

[16] Aristotle. Metaphysics // Translation from Greek by P. D. Pervov and V. V. Rozanov. Moscow: Institute of Philosophy, Theology and History of St. Thomas. - 2006. - 232p.;

[17] Muhammad ibn Ismail al-Bukhari. Al-Jami' as-sahih. Translation by Vladimir (Abdullah) Nirsha // -M.: Umma Publishing House. - 2017. - 448 p.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ALGEBRAIC METHODS FOR SOLVING RECOGNITION PROBLEMS WITH NON-CROSSING CLASSES

ANVAR KABULOV[1,4,5,6], ALIMDZHAN BABADZHANOV [6], ISLAMBEK SAYMANOV* [1,2,3,5]

[1]National University of Uzbekistan named after Mirzo Ulugbek, Tashkent, Uzbekistan
[2]Central Asian University, Tashkent, Uzbekistan
[3]Tashkent State University of Economics, Tashkent, Uzbekistan
[4]Tashkent State Technical University, Tashkent, Uzbekistan
[5]Gulistan State University, Sirdaryo Region, Uzbekistan
[6]Engineering Federation of Uzbekistan, Tashkent, Uzbekistan

*e-mail: islambeksaymanov@gmail.com

Abstract. In this paper, we propose to consider various models of pattern recognition. At the same time, it is proposed to consider models in the form of two operators: a recognizing operator and a decision rule. Algebraic operations are introduced on recognizing operators, and based on the application of these operators, a family of recognizing algorithms is created. An upper estimate is constructed for the model, which guarantees the completeness of the extension.

Keywords: Zhegalkin polynomial, linear Boolean functions, complex conjunctions, first-order neighborhood, disjunctive normal forms, minimization.

AMS Subject Classification: 68T10

1. Introduction

In recent years, the solution of applied problems of recognition classification and prediction has received great development. In many real cases, the solution scheme remains the same, the set of possible solutions is divided into subsets in such a way that solutions that are close in some metric fall into one subset. In the future, solutions that fall into one subset do not differ, and all objects corresponding to these solutions are assigned to one class. The information described by past experience is presented in the following form: various objects are described in some way and their descriptions are divided into a finite number of non-overlapping classes. When a new object appears, a decision is made to enroll it in one or another class. It is proposed to choose such a generalized algorithm so that it achieves extreme forecasting quality. Consider algorithmic models for solving classification problems. It is possible to distinguish among these models the most frequently encountered in solving problems of an applied nature.

2. Models built on the principle of division

In many problems where descriptions of objects are given by sets of values of numerical features, objects belonging to different classes can be separated by surfaces of a fairly simple form [1, 2]. An example is a hyperplane of the form:

$$\sum_{i=f}^{n} a_i x_i + a_{n+1} = 0$$

Models built according to this principle differ mainly in the assignment of a class of surfaces, among which a surface or a set of surfaces is selected that separates the classified objects of different classes [3, 4, 5].

3. Models built on the principle of potentials

The basis for constructing these models is the following principle. The force of attraction is directly proportional to the product of the masses and inversely proportional to the square of the distance [6]. In other words, when considering objects whose belonging to one of the classes is known, one can set the mass of this set, the distance from the set to the recognized object in various ways, and choose as the function of the object belonging to one or another class a value that is a function that increases monotonically with mass and monotonically decreasing with distance [7, 8].

4. Statistical models

These models are formed on the basis of the apparatus of mathematical statistics and are used for cases where the probabilistic characteristics of classes can be determined. Algorithms belonging to statistical models translate the training information and descriptions of objects into a numerical matrix, and information vectors for the classified objects are formed from the elements of this matrix. The elements of this numerical matrix can be the probabilities of occurrence of objects by class.

5. Models based on the calculation of scores

In works [9] the so-called parametric recognition algorithms were considered, etc. such collections of algorithms in which each algorithm is encoded in a one-to-one way by a set of numerical parameters. In these models, the proximity between parts of previously classified objects and the object to be classified is analyzed. Based on a set of estimates, a general estimate for the object is developed and, according to the introduced decision rule, the belonging of the recognized object to one or another class is determined.

In the dissertation, as the initial model (A), a model is considered that is related to the model for calculating estimates, supplemented by some simple type recognition algorithms: the nearest neighbor algorithm, the average distance algorithm, etc.

A feature of the algorithms of this class is that for calculating estimates that determine the belonging of a recognized object, there are simple analytical formulas that replace complex enumeration procedures that arise when calculating proximity estimates using a system of support sets.

In these models, the division of the algorithm into recognition operators and decision rules is carried out in a natural way [10].

We will consider only algorithms represented in the form $A = B * C$ where B is an arbitrary recognition operator. It turns out that an essential part of the algorithm is the operator - B; $decisionrule-$C can be made standard for all algorithms and programs. Any recognizing vote operator maps task Z to a numeric matrix of votes or scores $B(Z) = \|G_{ij}\|_{q \cdot l}$, $G_{ij} = G_j(S^i)$ moreover, the value G_{ij} has a clear, meaningful interpretation. This value can be considered as the degree of belonging of the examined object S^i to the class expressed by a number K_j. After the introduction of the appropriate normalizations, the quantities G_{ij} can also be considered as the value of the membership function 3 of the elements of the S^i set K_j. In this paper, we will adhere to the term "number of votes" or "assessment" that has been established in the domestic literature, which, in our opinion, is more consistent with the real meaning of the introduced values.

Before describing the evaluation model used in what follows, it should be noted that the complete voting model is not parametric in the previously defined sense.

Indeed, out of the 6-stages in [11]. defining the estimating model recognition operator, one stage is not specified by specifying numerical parameters. This stage is introduced with the help of a variable set-theoretic characteristic. However, in reality, submodels of the class of evaluation operators are usually considered, and the algorithms of these submodels are uniquely encoded by a set of numerical parameters.

Before introducing the submodel used in what follows, let us write how the algebra of recognizing operators is constructed. Let $Z = (J, \tilde{S}^2)$ a fixed recognition problem with classes $K_1, \ldots, K_e$, let also B_1 and B_2 recognize operators, C is a real number

$$B_i(Z) = \left\|G^i_{uv}\right\|_{q\cdot l} \quad i = 1, 2.$$

Then the sum and product of the operators B_1 and B_2 as well as the multiplication of the operator by a real number is defined as follows

$$(B_1 + B_2)(Z) = \left\|G^1_{uv} + G^2_{uv}\right\|_{q\cdot l}$$

$$(B_1 \times B_2)(Z) = \left\|G^1_{uv} \times G^2_{uv}\right\|_{q\cdot l}$$

$$(cB)(Z) = \left\|c \cdot G\right\|_{q\cdot l} \quad i = 1, 2.$$

Obviously, all these operations are commutative and associative, moreover, the operation of addition is distributive with respect to the operation of multiplication by a number. Due to these properties, if (B) is the original set of operators, $a(B)$ is the closure of the family (B) with respect to the introduced operations / algebraic closure / from $a(B)$ can be represented as operator polynomials

$$\sum b_{i_1}, \ldots, b_{i_k}, \quad B_{i_1} \cdot \ldots \cdot B_{i_k}$$

Here $b_{i_k}-$ constants and original operators $B_{i_k}-$ play the role of variables for ordinary polynomials.

Note that if we have one algorithm for applying the original operators B_{iv} to any problem Z, then it is easy to construct an algorithm for applying the operator polynomial to Z.

Example: Let $Z = \{J, \tilde{S}^3\}$ the classes K_1, K_2 be given three initial operators B_1, B_2, B_3 and

$$B_1(Z) = \left\|\begin{matrix} 0 & 1 \\ 0.5 & 1 \\ 1 & 1 \end{matrix}\right\| = M_1$$

$$B_2(Z) = \left\|\begin{matrix} 0.5 & 0.5 \\ 1 & 1 \\ 1 & 0.5 \end{matrix}\right\| = M_2$$

$$B_3(Z) = \left\|\begin{matrix} 0.1 & 0.2 \\ 1 & 1 \\ 0.5 & 0.5 \end{matrix}\right\| = M_3.$$

Consider the operational polynomial

$$2B_2^2 - B_1 \cdot B_3 = B^* \in \{B\}.$$

According to the definition of the operation of addition, multiplication and multiplication by a number $B^*(Z) = 2M_2^2 - M_1 \cdot M_3$.

Note that, unlike the usual matrix multiplication, element-wise, that is, Hadamard multiplication is performed here.

It is easy to see that

$$2M_2^2 = \left\|\begin{matrix} 0.5 & 0.5 \\ 2 & 2 \\ 2 & 0.2 \end{matrix}\right\|,$$

$$M_{1,3} = \begin{Vmatrix} 0 & 0.2 \\ 0.5 & 1 \\ 0.5 & 0.5 \end{Vmatrix},$$

$$B^* = \begin{Vmatrix} 0.5 & 0.3 \\ 1.5 & 1 \\ 1.5 & 0 \end{Vmatrix}$$

For any operator polynomial, the algorithm for applying to the problem is constructed in a similar way: the original operators are applied to the problem, and then the resulting matrices are multiplied by the corresponding scalars and finally added, as in the example just considered. The evaluation matrix obtained by applying the algorithm $a(B)$ from the closure algebra does not allow such a simple interpretation as the matrix of the number of votes in voting algorithms. The new recognizing operators are a formal extension of the original space of meaningful operators. Such formal extensions are often used in mathematics: thus, the field of complex numbers is a formal extension of the field of real numbers, for example, the process of formal Galois extensions is known, etc. In the Galois extension, an algebraic equation of any degree is solved elementarily. However, for a long time no physical interpretation of the Galois expansion was found. Only in recent years has it been established that the elements of the Galois extension are naturally interpreted in connection with problems of error correction in information transmission.

There is currently no meaningful interpretation for the algebraic extension of the space of operators. However, with the help of these extensions, difficult extremal problems are relatively easily solved, including the problem of synthesizing an error-free algorithm for a given recognition problem.

The degree of an operator polynomial is introduced similarly to the degree of an ordinary polynomial in many variables from the terms:

$$b_{i_1}, \ldots, b_{i_n}; \quad B_{i_1}, B_{i_2}, \ldots, B_{i_k}$$

the term with the largest K is chosen equal, for example, and the degree of the polynomial is assumed to be equal to K. So for example, if the polynomial has a degree of – 4.

$$B_1 \cdot B_2 + 2B_2^2 + 3B_1^4$$

Because the term $|M|$ with the largest K is $3B_1^4 = 3B_1 \cdot B_1 \cdot B_1 \cdot B_1$.

Based on the definition of the degree of polynomials, it is easy to distinguish in the extension $a(B)$ the system of nested extensions $a^0\{B\} = \{B\}, a^1\{B\} = \{\sum b_i \{B_i\}\}, \ldots, a^k\{B\} =$ the set of all polynomials of degree no higher than K.

Obviously, $a^0\{B\} \leq a^1\{B\} \leq \ldots \leq a^k\{B\}, \ldots, a\{B\}$ set $a^k\{B\}$ is called the $k-$ th power extension of the original space of operators $\{B\}$. The set is of particular importance $a^1\{B\}$, it consists of all possible linear forms from the original operators $\{B\}$, etc. elements represented as: $\sum b_i B_i, B_i \in \{B\}$.

When forming the elements included in the $a^1\{B\}$ operation, the product of operators is not used. Therefore, this extension is an extension of the original space using the "+" operation, "multiplication by a number", therefore the set $a^1\{B\}$ is also commonly denoted as $L\{B\}$, and is called a linear extension of the set $\{B\}$. And so, let a set of $\{A\}$ algorithms be given $\{A\}$ and each A is represented as $A = B \times C$, then the $\{B\}$ initial model of operators. We take a fixed decision threshold rule $C(d_1, d_2)$. We introduce a family of algorithms $L\{B\} \cdot C(d_1, d_2), \ldots, a^k\{B\} \cdot C(d_1, d_2), \ldots, L\{B\} \cdot C(d_1, d_2), \ldots, a^k\{B\} \cdot C(d_1, d_2), \ldots, a\{B\} \cdot C(d_1, d_2)$ called, respectively, a linear extension $L\{A\}$, an algebraic extension of the $K-$th degree $a^k\{A\}$, an algebraic extension $a\{A\}$ of a family of algorithms $\{A\}$.

We see that the constructed sets of algorithms consist, in the execution of the corresponding operators, and a fixed threshold decision rule c is applied to the result of the operator's action (d_1, d_2).

Let some set of recognition problems be given $\{Z\}$ and the initial set of recognizing operators be chosen in some way $\{B\}$. Let also within the framework $\{B\}$ not for every problem Z there exists an operator B_z and an algorithm $B_z \times C_z$, which gives an error - free solution to the problem Z. Then the scheme for constructing a correct recognition algorithm consist of the following steps:

Stage 1. Some extension is chosen $a^k\{A\}$ in which the existence of an error-free solution is guaranteed for each problem Z from $\{Z\}$. In this case, it is natural to find, if possible, the smallest degree of expansion of K. The implementation of the first stage in the literature is usually called the study of the completeness of the expansion. The theorems proved in this case are usually analous to the existence theorem.

Stage 2. In the chosen extension, $\{a^k\}\{A\}$ for an admissible problem, Z either an error-free /correct algorithm/ is constructed, or, if the former is associated with large computational difficulties, an algorithm for solving with respect to , which is sufficiently acceptable in accuracy Z. The exact calculation of the minimum degree of K is a laborious task, which at present can be solved only for relatively narrow classes of problems Z, and the original family of algorithms A. Therefore, in the studies carried out for the minimum degree of expansion, an upper estimate is constructed that guarantees the completeness of the expansion. So for the voting model, such an estimate is

$$K = \left[\frac{\ln q + \ln l + \ln(d_1 + d_2) + lnd_2 - \ln d_1}{\ln(1 - \frac{1}{q})} \right]$$

built in [12].

Here q, is the number of recognizable objects, l is the number of classes and d_1, d_2 are the parameters of the threshold decision rule.

For most real problems, this estimate is overestimated, and therefore, the algorithm built at the second stage has computational redundancy. Of particular importance is the fact that the above estimate is constructed for problems with intersecting classes, while the majority of real problems are problems with non- intersecting classes [13]. Later we will see that for a wide class of problems it is sufficient to consider the degree of 1, that is, to use only a linear closure.

Algorithms of the class for calculating estimates allow solving recognition problems of all types: assigning an object to one of the given classes, automatic classification, choosing a system of features to describe recognition objects and evaluating their effectiveness.

6. Conclusions

In this paper, we propose to consider various models of pattern recognition. Consideration of models in the form of two operators is studied: a recognizing operator and a decision rule. Algebraic operations are defined on recognizing operators, and based on the use of these operators, a family of recognizing algorithms is created. For the voting model, an upper bound is constructed that guarantees the completeness of the extension.

References

[1] A. Kabulov, I. Yarashov, A. Otakhonov Algorithmic Analysis of the System Based on the Functioning Table and Information Security // 2022 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Toronto, ON, Canada, 2022, pp. 1-5, doi: 10.1109/IEMTRONICS55184.2022.9795746.

[2] A. Kabulov, I. Saymanov, I. Yarashov, A. Karimov Using Algorithmic Modeling to Control User Access Based on Functioning Table // 2022 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Toronto, ON, Canada, 2022, pp. 1-5, doi: 10.1109/IEMTRONICS55184.2022.9795850.

[3] E. Navruzov, A. Kabulov Detection and analysis types of DDoS attack // 2022 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Toronto, ON, Canada, 2022, pp. 1-7, doi: 10.1109/IEMTRONICS55184.2022.9795729.

[4] A. Kabulov, I. Saymanov, I. Yarashov, F. Muxammadiev Algorithmic method of security of the Internet of Things based on steganographic coding // 2021 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Toronto, ON, Canada, 2021, pp. 1-5, doi: 10.1109/IEMTRONICS52119.2021.9422588.
[5] A. Kabulov, I. Normatov, E. Urunbaev, F. Muhammadiev Invariant Continuation of Discrete Multi-Valued Functions and Their Implementation // 2021 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Toronto, ON, Canada, 2021, pp. 1-6, doi: 10.1109/IEMTRONICS52119.2021.9422486.
[6] A. Kabulov, I. Normatov, A. Seytov, A. Kudaybergenov Optimal Management of Water Resources in Large Main Canals with Cascade Pumping Stations // 2020 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Vancouver, BC, Canada, 2020, pp. 1-4, doi: 10.1109/IEMTRONICS51293.2020.9216402.
[7] Kabulov, A. V., Normatov, I. H. (2019). About problems of decoding and searching for the maximum upper zero of discrete monotone functions // Journal of Physics: Conference Series, 1260(10), 102006, 2019. doi:10.1088/1742-6596/1260/10/102006.
[8] Kabulov, A. V., Normatov, I. H., Ashurov A.O. Computational methods of minimization of multiple functions // Journal of Physics: Conference Series, 1260(10), 10200, 2019. doi:10.1088/1742-6596/1260/10/102007.
[9] Anvar Kabulov, Alimdzhan Babadzhanov, Islambek Saymanov Correct models of families of algorithms for calculating estimates // AIP Conference Proceedings 8 June 2023; 2781 (1): 020010. https://doi.org/10.1063/5.0144830.
[10] Anvar Kabulov, Alimdzhan Babadzhanov, Islambek Saymanov Completeness of the linear closure of the voting model // AIP Conference Proceedings 8 June 2023; 2781 (1): 020020. https://doi.org/10.1063/5.0144832.
[11] Anvar Kabulov, Mansur Berdimurodov Gost R 34.12-2015 (Kuznechik) analysis of a cryptographic algorithm // AIP Conference Proceedings 8 June 2023; 2781 (1): 020046. https://doi.org/10.1063/5.0145975.
[12] Valeri Sklyarov, Iouliia Skliarova, Anvar Kabulov Hardware accelerators for solving computationally intensive problems over binary vectors and matrices // AIP Conference Proceedings 8 June 2023; 2781 (1): 020013. https://doi.org/10.1063/5.0145125.
[13] Preface: International Conference on Modern Problems of Applied Mathematics and Information Technology // AIP Conference Proceedings 8 June 2023; 2781 (1): 010001. https://doi.org/10.1063/12.0017180.
[14] Ibrokhimali Normatov Endless individual areas of logic and beginnings of arithmetics // AIP Conference Proceedings 2781, 020008 (2023). https://doi.org/10.1063/5.0144824
[15] I. Normatov, I. Yarashov, A. Otakhonov and B. Ergashev Construction of reliable well distribution functions based on the principle of invariance for convenient user access control // 2022 International Conference on Information Science and Communications Technologies (ICISCT), Tashkent, Uzbekistan, 2022, pp. 1-5, doi: 10.1109/ICISCT55600.2022.10146952.
[16] Normatov I. Principle of Independence of Continuation of Functions Multivalued Logic from Coding // Journal of Physics: Conference Series, 2019, 1210(1), 012107. doi:10.1088/1742-6596/1210/1/012107.
[17] I. Normatov, E. Kamolov Development of an algorithm for optimizing the technological process of kaolin enrichment // 2020 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Vancouver, BC, Canada, 2020, pp. 1-4, doi: 10.1109/IEMTRONICS51293.2020.9216371.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

MODELS, METHODS AND ALGORITHMS FOR MONITORING ENVIRONMENTAL IMPACT ON AGRICULTURAL PRODUCTION

ANVAR KABULOV[1,4,5,6], IBROKHIMALI NORMATOV [1], ISLAMBEK SAYMANOV* [1,2,3,5], ABDUSSATTAR BAIZHUMANOV [7], INOMJON YARASHOV [1]

[1]National University of Uzbekistan named after Mirzo Ulugbek, Tashkent, Uzbekistan
[2]Central Asian University, Tashkent, Uzbekistan
[3]Tashkent State University of Economics, Tashkent, Uzbekistan
[4]Tashkent State Technical University, Tashkent, Uzbekistan
[5]Gulistan State University, Sirdaryo Region, Uzbekistan
[6]Engineering Federation of Uzbekistan, Tashkent, Uzbekistan
[7]South Kazakhstan State Pedagogical University, Shymkent, Kazakhstan

*e-mail: islambeksaymanov@gmail.com

Abstract. The article analyzes the impact of environmental conditions on agricultural production in the Aral Sea region. The development of mathematical models for solving the problems of optimizing agricultural production, the spread of dust and salt particles from above-ground sources of heavy and light aerosols, depending on meteorological conditions, algorithmic and managerial decision-making methods is scientifically substantiated. It is also proposed to develop methods and tools for interactive analysis of water and soil, models, methods and monitoring algorithms to determine the impact of environmental factors on agricultural production.

Keywords: model, method, algorithm, erosion, soil, aerosol, diffusion.

AMS Subject Classification: 93C95

1. Introduction

At present, in order to solve the problems that arise in the development of agriculture and the rational use of natural resources, it is necessary to scientifically substantiate the development of mathematical models for solving problems of optimizing agricultural production, algorithmic and managerial decision-making methods. The principles and important directions of the methodology of a systematic approach to the analysis of the production of agricultural enterprises are presented in Figure 1.

2. Purpose and methods of research

In Central Asia, a striking example is the exposed bottom of the Aral Sea and the salt marshes that have arisen in its place, which are the main source of emission and dispersion of toxic salts, pesticide residues and dust in the atmosphere of the region. To solve the problems of monitoring and predicting the ecological state of regions and making managerial decisions, it is necessary to develop mathematical and software that takes into account soil erosion, weather and climatic conditions, physical and mechanical properties of aerosol particles, and other factors. In order to predict the spread of aerosols in the environment, to find the number of aerosols in the

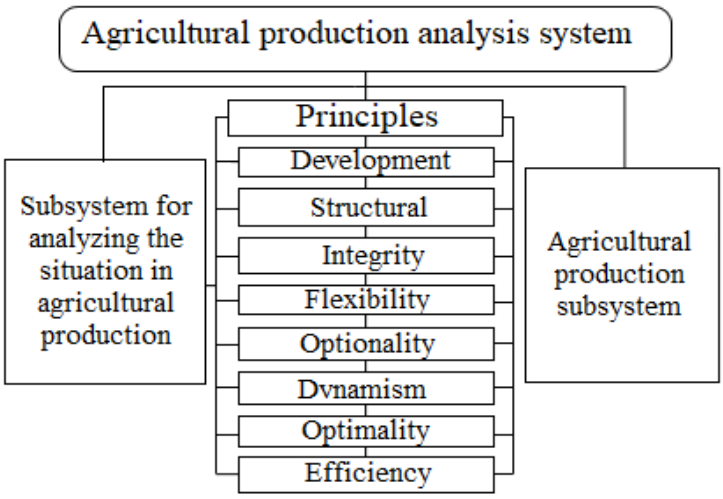


FIGURE 1. Basic principles of system analysis of agricultural production

area under consideration $D = (0 < x < a,\ 0 < y < b,\ 0 < z < H)$ and the amount of aerosols deposited on the underlying surface, we use the mathematical model [1, 2, 3, 4, 5, 6]:

$$\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + (w - w_g)\frac{\partial\theta}{\partial z} + \sigma\theta = \mu\Delta\theta + \frac{\partial}{\partial z}\left(k\frac{\partial\theta}{\partial z}\right) + Q\delta(x,\, y,\, z)\,; \tag{1}$$

$$\theta(x,\, y,\, z,\, 0) = \theta_0(x,\, y,\, z)\,; \tag{2}$$

$$-\mu\frac{\partial\theta}{\partial x}\bigg|_{x=0} = \gamma(\theta - \theta_a)\ ;\ \ \mu\frac{\partial\theta}{\partial x}\bigg|_{x=L_x} = \gamma(\theta - \theta_a)\ ; \tag{3}$$

$$-\mu\frac{\partial\theta}{\partial y}\bigg|_{y=0} = \gamma(\theta - \theta_a)\ ;\ \mu\frac{\partial\theta}{\partial y}\bigg|_{y=L_y} = \gamma(\theta - \theta_a)\,; \tag{4}$$

$$-k\frac{\partial\theta}{\partial z}\bigg|_{z=0} = \gamma(\beta\theta - F_0); \tag{5}$$

$$-k\frac{\partial\theta}{\partial z}\bigg|_{z=H} = \gamma(\theta - \theta_a)\,. \tag{6}$$

At $H = 0$, we have an elevated source at the level $z = H$ $(F_0 = 0)$, and at elevated sources $F_0 \neq 0$ $(Q = 0)$.

Problem (1) - (6) will be considered in the area D when the source is located in the surface layer, where the "crosses" indicate the above-ground sources of atmospheric pollution (Figure 2).

The value of F_0 is a function of x, y, z, t ,and should be determined from experimental data depending on meteorological conditions, properties of the underlying surface, size and density of dust particles. For determine k consider the following models:

(1) $k = const$, u, v, w - $const$;

(2)

$$k = \begin{cases} v + k_1 \cdot \frac{z}{z_1}, & z \le h, \\ v = |v| \cdot z^n, & \\ v + k_1 \cdot \frac{h}{z_1}, & z > h, . \end{cases}$$

(3) $k = k(z)$, $v = v(z)$, $w = w(z)$,

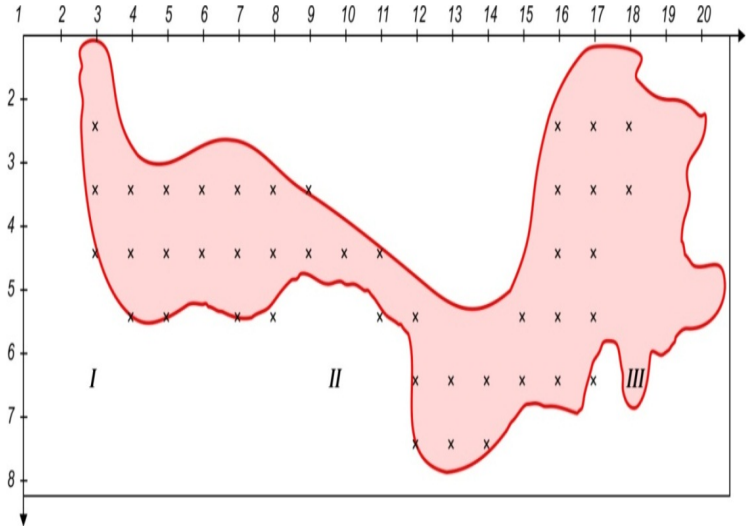


FIGURE 2. Location of sources on the above-ground layer in the drained part of the Aral Sea region.

where h – is the height of the surface layer, v – is the turbulent viscosity. The static processing of data on the influence of various weather and climatic conditions on the removal of particles from the earth's surface shows that the main factor causing soil erosion is the velocity of the oncoming flow, and the main factor is soil moisture. The remaining parameters, although they play a certain role in the development or prevention of erosion, but their influence is ambiguous. Therefore, in mathematical modeling of the process of distribution of harmful particles and changes in their concentration over time, these parameters must be taken into account, and the physicochemical properties of the soil can be considered constants included in the calculation formula.

Wind speed and soil moisture affect the process of distribution of harmful particles in the atmosphere and the development of soil erosion in different ways. If an increase in the speed of the oncoming wind stream enhances the erosion process, then an increase in soil moisture suppresses it. According to [7, 8], we write the general dependence in the form

$$F_0 = f(u, w), \tag{7}$$

where F_0 – is the volumetric flow rate of particles carried away by the atmospheric front m3/s. To determine the type of function (7), we proceed to the analysis of the acting forces that cause the destruction of the soil and resist this destruction. We denote the destructive forces by F. They are always opposed by the resistance forces R, which include moisture and other physical and mechanical properties of the soil [8, 9, 10, 11]. When the force F exceeds the force R , the process of soil erosion and the removal of harmful particles from the earth's surface begins. The destruction force F is mainly determined by the shear stress of the oncoming air flow. At the same time, the more solid particles in the flow, the greater the total shear stress that destroys the soil. To obtain a theoretical dependence, we will consider an equilibrium process [12, 13, 14]. At dynamic equilibrium, the difference between the forces F and R must be equal to zero, etc.

$$F - R = 0. \tag{8}$$

Let's write an expression for these forces. The relationship between the volumetric flow rate F_0 of entrained particles and the flow rate is expressed as

$$F = \frac{\partial F_0}{\partial u} \cdot \chi, \tag{9}$$

where χ is the shear stress, kg/m2. For the resistance force R, by analogy with F, we take the expression

$$R = c_0 \frac{\mu_c}{l} \frac{\partial F_0}{\partial \zeta}, \tag{10}$$

where μ_c – is the viscosity of the mixture $(air + soil), kg * s/m2, l$ is the distance between individual particles, m, c_0 – is the soil constant. Substituting (9) and (10) into (8), we have

$$\frac{\partial F_0}{\partial u} \chi - c_0 \frac{\mu_c}{l} \frac{\partial F_0}{\partial \zeta} = 0 \tag{11}$$

or

$$\frac{\partial F_0}{\partial u} - c_0 \frac{\mu_c}{l\chi} \frac{\partial F_0}{\partial \zeta} = 0. \tag{12}$$

Consider a single expression $c_0 \mu / (l\chi)$ in equation (12). If we approximately assume that the shear stress is determined by the value of the external velocity u_∞, etc. $\chi = u_\infty$, then this expression in equation (12) can be represented as

$$c_0 \mu_c / (l\chi) = c_0 \mu_c / (l u_\infty). \tag{13}$$

In formula (13), the dynamic viscosity of the mixture μ_c is determined mainly by soil moisture, which increases the adhesion force between individual particles. Based on expression (13), we can assume that the soil constant depends on a certain moisture function:

$$c_0 \mu_c / (l u_\infty) \cong c'_0 f(\zeta). \tag{14}$$

In what follows, the function $f(\zeta)$ is replaced by a simple dependence

$$f(\zeta) \cong c_0 c'_0 \zeta. \tag{15}$$

Dependence (14) finally takes the form

$$c_0 \mu_0 / (l u_\infty) \cong c_0 f(\zeta) \cong c_0 c'_0 \zeta = k_p w, \tag{16}$$

where k_p – is the soil constant, s/m. Returning to expression (12), we obtain

$$\frac{\partial F_0}{\partial u} - k_p \zeta \frac{\partial F_0}{\partial \zeta} = 0. \tag{17}$$

3. Research results

As a result, an equation was obtained that makes it possible to calculate the volume of removal of particles from the soil surface depending on its soil moisture and the speed of the incoming air flow. Calculating the value of the volume flow rate of particles F_0, further using the boundary condition (5), we can solve the problem of transport and diffusion of pollutants in the atmosphere. Since problems (1) - (6) are described by multidimensional partial differential equations with appropriate initial and boundary conditions, it is difficult to obtain their solution in an analytical form. To solve them, we similarly used an implicit finite-difference scheme with the second order of accuracy in time [15, 16, 17, 18, 19]. Depending on the accuracy of the technical data, an accurate forecast can be made for the selected crop areas. The data is compared to results every ten days and the data is stored in linked relational databases. As the amount of data grows, it is recommended to use normalization for processing and storing them. For this, a relational model for storing normalized data was created, a datalogical model in the web system provided for the information system for environmental monitoring of the Aral Sea region (Figure 3).

The water level sensor provides information in analog form, where the change in voltage is converted into digital information by determining the mathematical relationship with the environment. There is a converter that converts analog to digital and works by getting power from the source. The power source is a solar panel.

The scheme of information processing processes in environmental monitoring information system (EMIS) is shown in Figure 4.

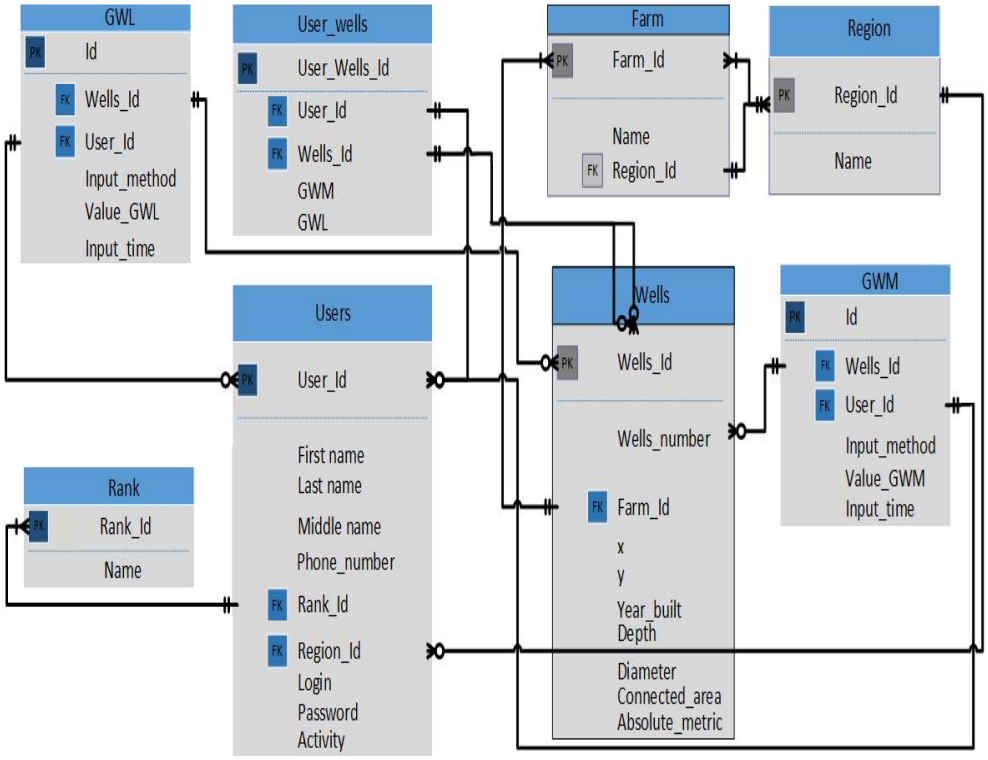


FIGURE 3. Datalogical model of the ecological monitoring information system.

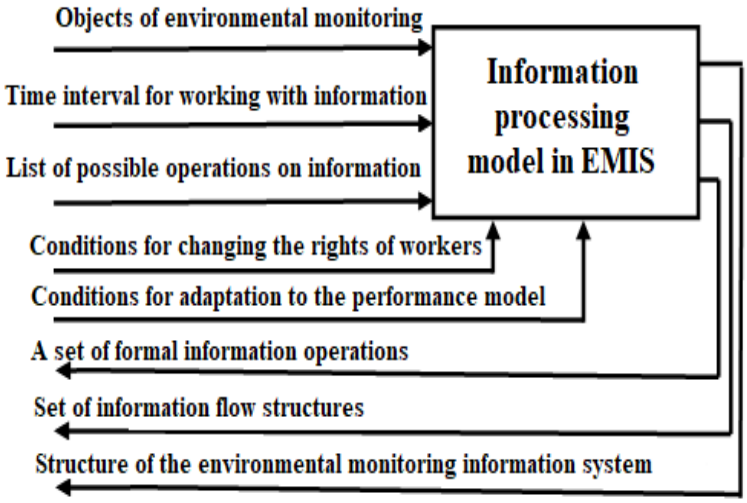


FIGURE 4. Scheme of information processing processes in EMIS.

Let's represent the mathematical model of information processing in EMIS in the form of a tuple:

$$MPI(O, S, P) = F, \tag{18}$$

where F – is the structure of the environmental monitoring system, which is understood as a formalized representation in the form of a tuple model of a set of environmental monitoring

objects O, that change their state as a result of operations P by a set of subjects S.

$$F = O, S, P, T, \tag{19}$$

where $O = \{o_1 | i = \overline{1, n}\}$ – is a set of environmental monitoring objects, $S = \{S_q | q = \overline{1, m}\}$ – is a set of subjects, $P = \{p_i | l = \overline{1, k}\}$ – is a set of operations performed on objects, n, m, k – is the total number of objects, subjects and operations, respectively, T is a set of discrete time points.

Each information corresponds to a set of states , each of which determines the content and attributes of the object in a certain period of its life cycle

$$u_1 \to C_i, \tag{20}$$

where $C_i = \{c_i j | j = \overline{1, l}(l - is\,the\,amount)\}$ is the set of states of the environmental monitoring object $u_i, l(l - is\,the\,amount\,C_i)$ is the number of such states. For designate the state c_{ij}, information u_j used in data processing in specific operations, we will take the designation $u_j(c_{ij})$. Each state is defined as a tuple of a set of object attributes and their values in a given time interval $T_{c_{ij}}$

$$C_{ij} = (\{(a_{in}, d_{ijn}) | a_{in}, d_{ijn} D_{ij}, \in n = \overline{1, N_l}\}, T_{c_{ij}}), \tag{21}$$

where $A_i = \{a_{in} | n = 1, r\,(r - is\,the\,amount\,A_i)\}$ is a set of attributes of object u_j with a corresponding set of attribute values $D_i = \{D_{ij} | j = \overline{1, l}\}$ for each state of c_{ij}.

$$D_{ij} = \{d_{ijn} | n = \overline{1, v}\}, \tag{22}$$

where r – is the number of object attributes; v – is the number of D_{ij}, $T_{c_{ij}} = \{t_m\}$ – T is the set of times.

Digital technologies can make a significant contribution to solving all these problems. Sensors, drones, satellites and robots used in agriculture are examples of digital technologies that have the potential to revolutionize agriculture, even on a small scale. The device model, connection configuration, DBMS, quality controller that controls the level of error of data coming from the sensor combine the capabilities of the device for processing data from the components of the sensor system. (Figure 5).

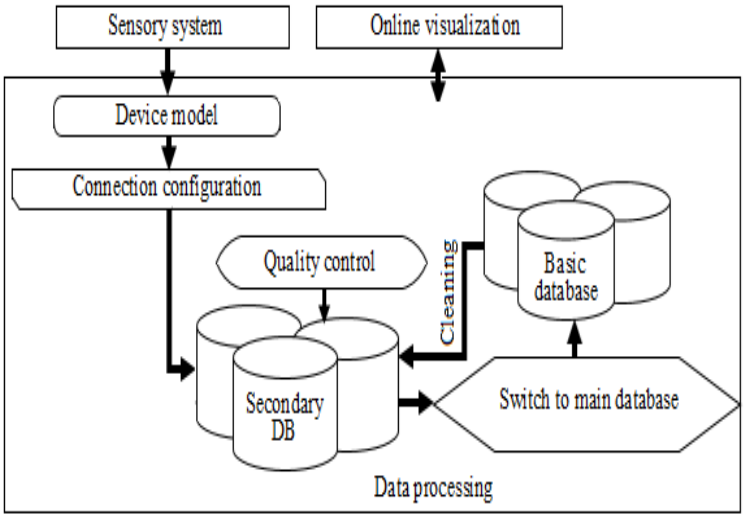


FIGURE 5. Conceptual model of the process of functioning of the environmental monitoring system.

Digital technologies can make a significant contribution to sustainable development. Due to the methods of information protection, it is possible to improve the mechanisms for controlling

social and environmental responsibility by ensuring the security and reliability of DBMS operations. The Internet can perform tasks such as monitoring and simplifying system processes, generating reports, and improving data entry methods.

The environmental monitoring system should perform such activities as the collection, systematization and analysis of information:

- about the state of the observed ecological environment;
- about the causes of observed and possible changes in the situation (sources and factors of influence);
- about possible changes in the situation in the observation well, etc.

Thus, the environmental monitoring system includes monitoring the state of environmental elements and monitoring sources and impact factors.

At present, in many areas of the Aral Sea region, crop yields are directly proportional to the costs allocated to it, however, the purpose of the study is to maintain environmental sustainability in existing crop areas, as well as to monitor changes in environmental parameters to make the necessary changes. In such cases, the problem should be handled carefully. It is necessary to carry out the necessary work, it is necessary to measure the parameters of the environment with the help of workers and control the implementation of actions, to determine the algorithm of seasonal changes depending on the parameters of the environment and regions. Algorithmization of the problem solving process is the process of finding such an algorithm, the execution of which leads to the solution of the problem. To implement this process, it is necessary to know the exact task, structure and functional description of the algorithmic system (automated research system). In a broad sense, the task involves the conscious search for suitable means to achieve a goal that is clearly visible, but not immediately achievable. It is also possible to algorithmize the information system of environmental monitoring, solve problems in it by algorithmic methods.

Here, operations such as registration, authorization, search are performed on the basis of information stored in database management systems (DBMS). When building information systems, the algorithm of the system is always important for the formalization of the process and is relevant in the implementation of such processes as modification, modernization.

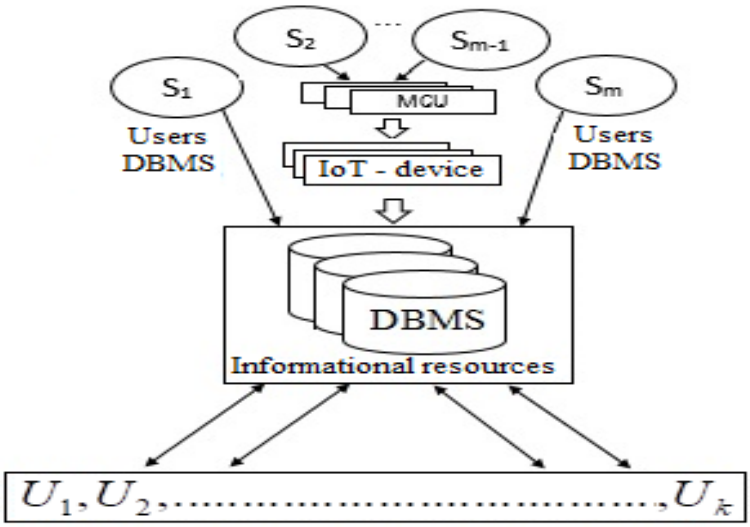


FIGURE 6. Conceptual model for using information from the environmental monitoring system.

A set of IoT information resources is needed from various related data (low level sensor data or aggregated information or functions) (computing functions). IoT data is constantly written to the server (via API). The environmental monitoring system can be used for various tasks of

forecasting and / or tasks of system analysis, process control (Figure 6). It is recommended to use IoT device enrollment to prevent incorrect data entry, learning, deletion, and disconnection through IoT devices that are not part of the IoT system. When IoT data is transmitted over the Internet, etc. URL, HTTP methods, the first step is to use the HTTPS protocol, and to a certain extent, the use of information security.

The Internet of Things provides unlimited opportunities in the field of generation, collection, transmission, analysis and dissemination of information around the world, which ultimately allows humanity not only to survive, but also to gain knowledge and wisdom for its true prosperity for many months, years, decades and centuries. Every year the number of Internet users is growing, applications are being developed in the field of automation and data collection, sensor networks and machine-to-machine communication (M2M). The work done shows that in order for digital technologies to gain popularity among ordinary users, operators and market participants must create applications that significantly improve the quality of life of ordinary citizens, which means that digital technologies must be valuable for each individual.

4. Conclusions

Based on the foregoing, we can conclude that the construction of an environmental monitoring information system based on digital technologies has a significant impact on all aspects of human life, new scientific issues arise in the algorithmic formalization of user access to the environmental monitoring information system. Digital technologies are at the heart of the emergence of new tools in the field of statistics and forecasting, health and medical care. Ultimately, a global telecommunications infrastructure is formed, covering all spheres of human life and all devices that affect his lifestyle.

It should be noted that in the research work, based on the access model, which is an important issue of information security, unauthorized access prevention is carried out, which includes the study of the relationship of subjects present in the information system to objects, their classification and analysis of restrictions on this basis. A functional diagram of user access to the environmental monitoring information system is proposed.

At the same time, the data incoming and outgoing from them when managing sensors operating via the Internet are protected by the https security protocol to increase the security of the IoT infrastructure. The data coming from the sensors, which are one of the main components of this system, were included in the primary and secondary databases. Here, the primary database is a temporary database, and after it is rotated into the main database, the primary database is dropped and recreated.

References

[1] A. Kabulov, I. Yarashov, A. Otakhonov Algorithmic Analysis of the System Based on the Functioning Table and Information Security // 2022 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Toronto, ON, Canada, 2022, pp. 1-5, doi: 10.1109/IEMTRONICS55184.2022.9795746.

[2] A. Kabulov, I. Saymanov, I. Yarashov, A. Karimov Using Algorithmic Modeling to Control User Access Based on Functioning Table // 2022 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Toronto, ON, Canada, 2022, pp. 1-5, doi: 10.1109/IEMTRONICS55184.2022.9795850.

[3] E. Navruzov, A. Kabulov Detection and analysis types of DDoS attack // 2022 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Toronto, ON, Canada, 2022, pp. 1-7, doi: 10.1109/IEMTRONICS55184.2022.9795729.

[4] A. Kabulov, I. Saymanov, I. Yarashov, F. Muxammadiev Algorithmic method of security of the Internet of Things based on steganographic coding // 2021 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Toronto, ON, Canada, 2021, pp. 1-5, doi: 10.1109/IEMTRONICS52119.2021.9422588.

[5] A. Kabulov, I. Normatov, E. Urunbaev, F. Muhammadiev Invariant Continuation of Discrete Multi-Valued Functions and Their Implementation // 2021 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Toronto, ON, Canada, 2021, pp. 1-6, doi: 10.1109/IEMTRONICS52119.2021.9422486.

[6] A. Kabulov, I. Normatov, A. Seytov, A. Kudaybergenov Optimal Management of Water Resources in Large Main Canals with Cascade Pumping Stations // 2020 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Vancouver, BC, Canada, 2020, pp. 1-4, doi: 10.1109/IEMTRONICS51293.2020.9216402.
[7] Kabulov, A. V., Normatov, I. H. (2019). About problems of decoding and searching for the maximum upper zero of discrete monotone functions // Journal of Physics: Conference Series, 1260(10), 102006, 2019. doi:10.1088/1742-6596/1260/10/102006.
[8] Kabulov, A. V., Normatov, I. H., Ashurov A.O. Computational methods of minimization of multiple functions // Journal of Physics: Conference Series, 1260(10), 10200, 2019. doi:10.1088/1742-6596/1260/10/102007.
[9] Anvar Kabulov, Alimdzhan Babadzhanov, Islambek Saymanov Correct models of families of algorithms for calculating estimates // AIP Conference Proceedings 8 June 2023; 2781 (1): 020010. https://doi.org/10.1063/5.0144830.
[10] Anvar Kabulov, Alimdzhan Babadzhanov, Islambek Saymanov Completeness of the linear closure of the voting model // AIP Conference Proceedings 8 June 2023; 2781 (1): 020020. https://doi.org/10.1063/5.0144832.
[11] Anvar Kabulov, Mansur Berdimurodov Gost R 34.12-2015 (Kuznechik) analysis of a cryptographic algorithm // AIP Conference Proceedings 8 June 2023; 2781 (1): 020046. https://doi.org/10.1063/5.0145975.
[12] Valeri Sklyarov, Iouliia Skliarova, Anvar Kabulov Hardware accelerators for solving computationally intensive problems over binary vectors and matrices // AIP Conference Proceedings 8 June 2023; 2781 (1): 020013. https://doi.org/10.1063/5.0145125.
[13] Preface: International Conference on Modern Problems of Applied Mathematics and Information Technology // AIP Conference Proceedings 8 June 2023; 2781 (1): 010001. https://doi.org/10.1063/12.0017180.
[14] Ibrokhimali Normatov Endless individual areas of logic and beginnings of arithmetics // AIP Conference Proceedings 2781, 020008 (2023). https://doi.org/10.1063/5.0144824
[15] I. Normatov, I. Yarashov, A. Otakhonov and B. Ergashev Construction of reliable well distribution functions based on the principle of invariance for convenient user access control // 2022 International Conference on Information Science and Communications Technologies (ICISCT), Tashkent, Uzbekistan, 2022, pp. 1-5, doi: 10.1109/ICISCT55600.2022.10146952.
[16] Balashov, V.A. Direct modeling of moderately rarefied gas microflows in rock samples // Matem. modeling. 2018 pp. 3–20.
[17] Berg C.F., Lopez O., Berland H. Industrial applications of digital rock technology // Journal of Petroleum Science and Engineering. 2017. V. 157. pp. 131–147. https://doi.org/10.1016/j.petrol.2017.06.074.
[18] Normatov I. Principle of Independence of Continuation of Functions Multivalued Logic from Coding // Journal of Physics: Conference Series, 2019, 1210(1), 012107. doi:10.1088/1742-6596/1210/1/012107.
[19] I. Normatov, E. Kamolov Development of an algorithm for optimizing the technological process of kaolin enrichment // 2020 IEEE International IOT, Electronics and Mechatronics Conference (IEMTRONICS), Vancouver, BC, Canada, 2020, pp. 1-4, doi: 10.1109/IEMTRONICS51293.2020.9216371.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

STUDY OF SOLUTIONS TO A CLASS OF VOLTERRA INTEGRAL EQUATIONS WITH A SINGULAR POINT

ASAN BAIZAKOV[1], AIDA BEKTUROVA[2]

[1]Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic, Bishkek, Kyrgyzstan
[2]Institute of Modern Information Technologies in Education, Bishkek, Kyrgyzstan
e-mail: asan_baizakov@mail.ru

Volterra integral equations with a singular point appear in the study of the theory of wave propagation in media with memory and are considered from different points of view in [1-4]. In this paper, we study the analytic structure of vanishing solutions of a class of Volterra integral equations with a regular singular point.

Consider the Volterra integral equations with a singular point

$$tu = \int_0^t [\lambda u(s) + \nu u^{\alpha}(s)]\, ds + \int_0^t (t, s, u(s), u^{\alpha}(s))ds, \tag{1}$$

where $K(t, s, u, w)$ is a holomorphic function in a neighborhood of $t = s = u = w = 0$, and $K(t, s, 0, 0) \equiv 0$, $\frac{\partial(0,0,0,0)}{\partial u} = \frac{\partial(0,0,0,0)}{\partial w} = 0$ λ, ν - complex parameters, $0 < \alpha < 1$.

We will be interested in vanishing solutions of equation (1). The case when $\nu = 0$ and $K(t, s, u, w)$ does not depend on w was studied in [5-7].

Let us show that if equation (1) has a vanishing continuous solution in $[0, \delta], \delta > 0$, then $\nu = 0$ is necessary. Let's assume the opposite.

Equation (1) can be represented as

$\nu \int_0^t u^{\alpha}(s)ds = tu(t) - \lambda \int_0^t u(s)ds - \int_0^t K(t, s, u(s), u^{\alpha}(s))ds.$

Hence, taking into account the properties of the function $K(t, s, u, w)$, we have ,

$$t\,|\nu|\,\|u\|^{\alpha} \le t\,\|u\| + |\lambda|\, t\,\|u\| + t\left|\,\|u\|^{2} g_1(t, \|u\|) + \|u\|^{\alpha} g_2(t, \|u\|, \|u\|^{\alpha})\right|, \tag{2}$$

where $\|\|$ is the norm in $C_{[0,\delta]}$, with $g_1(t, \|u\|) \to const$ and $g_2(t, \|u\|, \|u\|^{\alpha})$ disappearing at $t \to 0, u \to 0$.

I divide both sides of inequality (2) by $t\|u\|^{\alpha}$ and tending t and u to zero, we get a contradiction.

Let

$K(t, s, , w) = \sum b_{ijkl} t^i s^j u^k w^l$

- power series expansion of the function $K(t, s, u, w)$. Let's pretend that

$$(2 - \alpha) b_{0101} \neq 0, \tag{3}$$

$$b_{000l} = 0,\ l \ge 2, \tag{4}$$

$$\lambda - 1 \neq i + \frac{j+1}{1-\alpha} + \frac{k\alpha}{1-\alpha},\ i + j + k \ge 0. \tag{5}$$

We will look for a vanishing solution (1) in the form of a series

$$\lambda - 1 \neq i + \frac{j+1}{1-\alpha} + \frac{k\alpha}{1-\alpha},\ i+j+k \geq 0. \tag{6}$$

Note that when series (6) is substituted into (1), the set of exponents of the right-hand side is contained in the set

$1 + i + \frac{j+1}{1-\alpha} + \frac{k\alpha}{1-\alpha},\ i+j+k \geq 0,$

if on the right side we perform the transformation

$t^{i'+j'} t^{\frac{k'}{1-\alpha}} t^{\frac{\alpha l'}{1-\alpha}} = t^{i'+j'-1} t^{\frac{1+k'}{1-\alpha}} t^{\frac{\alpha(l'-1)}{1-\alpha}},$

for all $i' + j' \geq 1,\ l' \geq 1,\ k' \geq 0.$

We substitute series (6) into equation (1) and compare the coefficients for the same powers of $t,\ t^{\frac{1}{1-\alpha}},\ t^{\frac{\alpha}{1-\alpha}}$. For $i = j = k = 0$ we have

$$P_{000} = \frac{\lambda}{\frac{1}{1-\alpha}+1} P_{000} + b_{1001} \frac{P^{\alpha}{}_{000}}{\frac{\alpha}{1-\alpha}+1} + b_{0101} \frac{P^{\alpha}{}_{000}}{\frac{1}{1-\alpha}+1}. \tag{7}$$

By virtue of (3)-(5), equation (7) has a non-zero solution. Next, we obtain a recursive formula for determining the coefficients P_{ijk}

$$P_{000}P_{ijk} = \frac{\lambda P_{000} P_{ijk}}{1 + i + \frac{j+1}{1-\alpha} + \frac{k\alpha}{1-\alpha}} + \frac{b_{1001}\alpha P^{\alpha}{}_{000} P_{ijk}}{1 + i + \frac{j}{1-\alpha} + \frac{(k+1)\alpha}{1-\alpha}} + \frac{b_{0101} P^{\alpha}{}_{000} P_{ijk}}{1 + i + \frac{j+1}{1-\alpha} + \frac{k\alpha}{1-\alpha}} + B_{ijk}(P^{\alpha}{}_{000}, P_{i'j'k'}), \tag{8}$$

where $i' \leq i,\ j' \leq j,\ k' \leq k,\ i' + j' + k' < i + j + k,\ B_{ijk}$ are polynomials in their arguments.

Suppose again

$$\Lambda \equiv P_{000} - \frac{\lambda P_{000}}{1 + i + \frac{j+1}{1-\alpha} + \frac{k\alpha}{1-\alpha}} + \frac{b_{1001}\alpha P^{\alpha}{}_{000}}{1 + i + \frac{j}{1-\alpha} + \frac{(k+1)\alpha}{1-\alpha}} + \frac{b_{0101} P^{\alpha}{}_{000}}{1 + i + \frac{j+1}{1-\alpha} + \frac{k\alpha}{1-\alpha}} \neq 0 \tag{9}$$

for all $i + j + k \geq 1$. From the recursive formula (8) all the coefficients P_{ijk}, series (6) are successively determined in a unique way.

So, under conditions (3)-(5), (9) there is a formal solution (6) of equation (1). Let us prove its convergence. transformation

$$u = P_{000} t^{\frac{1}{1-\alpha}} (1 + \nu) \tag{10}$$

translates equation (1) into equation

$$\begin{aligned} P_{000} t^{1+\frac{1}{1-\alpha}} \nu = \lambda P_{000} \int_0^t s^{\frac{\alpha}{1-\alpha}} \nu(s) ds + b_{0101}\alpha P^{\alpha}{}_{000} \int_0^t s^{\frac{1}{1-\alpha}} \nu(s) ds + \\ + b_{1001} P^{\alpha}{}_{000} t \int_0^t s^{\frac{\alpha}{1-\alpha}} \nu(s) ds + \int_0^t H(t, s, s^{\frac{1}{1-\alpha}}, s^{\frac{\alpha}{1-\alpha}}, \nu(s)) ds, \end{aligned} \tag{11}$$

where $H(t, s, s^{\frac{1}{1-\alpha}}, s^{\frac{\alpha}{1-\alpha}}, \nu(s))$ is a holomorphic function in its arguments and

$H(t, t, t^{\frac{1}{1-\alpha}}, t^{\frac{\alpha}{1-\alpha}}, \nu) = t^{\frac{1}{1-\alpha}} H_1(t, t, t^{\frac{1}{1-\alpha}}, t^{\frac{\alpha}{1-\alpha}}, \nu),$

$H_1(0,0,0,0,0), \frac{\partial H_1(0,0,0,0,0)}{\partial \nu} = 0.$

Obviously, equation (11) has a formal solution

$$\nu = \sum_{i+j+k \geq 1} P_{ijk} t^i t^{\frac{j}{1-\alpha}} t^{\frac{\alpha k}{1-\alpha}}. \tag{12}$$

Note that there exists $B > 0$ such that $|\Lambda| > B$ for all $i + j + k \geq 1$.

Consider now the equation

$$BV = \bar{H}_1(t, t^{\frac{1}{1-\alpha}}, t^{\frac{\alpha}{1-\alpha}}, V) \tag{13}$$

where $\bar{H}_1(t, t^{\frac{1}{1-\alpha}}, t^{\frac{\alpha}{1-\alpha}}, V)$ is the majorant of the function $H_1(t, t^{\frac{1}{1-\alpha}}, t^{\frac{\alpha}{1-\alpha}}, \nu)$. It is clear that equation (13) has a formal solution

$$V = \sum_{i+j+k\geq 1} \bar{P}_{ijk} t^i t^{\frac{j}{1-\alpha}} t^{\frac{\alpha k}{1-\alpha}}, \tag{14}$$

where $\bar{P}_{ijk}$ are positive constants and

$$|P_{ijk}| \leq \bar{P}_{ijk}. \tag{15}$$

Equation (13) by the implicit function theorem [8] has a unique vanishing solution $t, t^{\frac{1}{1-\alpha}}, t^{\frac{\alpha}{1-\alpha}}$ holomorphic in powers of $V = V(t, t^{\frac{1}{1-\alpha}}, t^{\frac{\alpha}{1-\alpha}})$, and it coincides with formal solution (14). Hence, by virtue of (15), the convergence of series (12) follows. This proves

Theorem 1. Under conditions (3)-(5) and (9), equation (1) has a vanishing solution of the form (6).

Now suppose that $V = V(t, t^{\frac{1}{1-\alpha}}, t^{\frac{\alpha}{1-\alpha}})$ takes place in equation (1).

Let, in addition, condition (3) and $b_{0002} \neq 0$. be satisfied.

Then from equation (1) by transformation (10) we obtain an equation of the form (11), but in this case

$H(t, t, t^{\frac{1}{1-\alpha}}, t^{\frac{\alpha}{1-\alpha}}, \nu) = t^{\frac{1}{1-\alpha}} H_2(t, t^{\frac{2\alpha-1}{1-\alpha}}, t^{\frac{1}{1-\alpha}}, t^{\frac{\alpha}{1-\alpha}}, \nu)$,

$H_2(0,0,0,0,0), \frac{\partial H_2(0,0,0,0,0)}{\partial \nu} = 0.$

The solution of equation (11) will be sought in the form

$$\nu(t) = \sum_{j=1}^{\infty} \Delta_j t^{\delta_j}. \tag{16}$$

Substituting series (16) into equation (11), to determine the exponents δ_j and coefficients Δ_j, an equation of the form

$\delta_j = \gamma_1(\delta_1, ..., \delta_{j-1})$,

$\Delta_j \left(P_{000} - \frac{\lambda P_{000}}{1+\frac{1}{1-\alpha}+\delta_j} - \frac{b_{0101}\alpha P^{\alpha}{}_{000}}{1+\frac{1}{1-\alpha}+\delta_j} - \frac{b_{1001}\alpha P^{\alpha}{}_{000}}{1+\frac{\alpha}{1-\alpha}+\delta_j} \right) = R_j(\Delta_1, ..., \Delta_{j-1})$,

where γ_j are linear functions and R_j are polynomials of their arguments. From these equations we successively find δ_j and Δ_j if

$1 = \frac{\lambda}{1+\frac{1}{1-\alpha}+\delta_j}$,

$\Lambda' = P_{000} - \frac{\lambda P_{000}}{1+\frac{1}{1-\alpha}+\delta_j} - \frac{b_{0101}\alpha P^{\alpha}{}_{000}}{1+\frac{1}{1-\alpha}+\delta_j} - \frac{b_{1001}\alpha P^{\alpha}{}_{000}}{1+\frac{\alpha}{1-\alpha}+\delta_j} \neq 0.$

A formal solution will exist even when Λ' and $R_j(\Delta_1\Delta_{j-1}) = 0$.ss. In this case, it depends on one arbitrary parameter. In the case of the existence of a formal solution (16), let us prove its convergence. To do this, note that there is an $B' > 0$ such that $|\Lambda'| > B'$.

Consider the equation

$$B'V = \bar{H}_2(t, t^{\frac{2\alpha-1}{1-\alpha}}, t^{\frac{1}{1-\alpha}}, t^{\frac{\alpha}{1-\alpha}}, V) \tag{17}$$

where $\bar{H}_2$ is the majorant of the function H_2. Obviously, equation (17) has a formal solution

$$V = \sum_{j=1}^{n} \bar{\Delta}_j t^{\delta_j}, \tag{18}$$

where $\bar{\Delta}_j$ are positive constants and $|\Delta_j| \leq \bar{\Delta}_j$. Equation (17) in the neighborhood of (0, 0, 0, 0, 0) has a unique vanishing solution holomorphic in powers of $t, t^{\frac{2\alpha-1}{1-\alpha}}, t^{\frac{1}{1-\alpha}}, t^{\frac{\alpha}{1-\alpha}}$ [2]

$$V = V(t, t^{\frac{2\alpha-1}{1-\alpha}}, t^{\frac{1}{1-\alpha}}, t^{\frac{\alpha}{1-\alpha}}). \tag{19}$$

Since series (19) converges absolutely and uniformly, it can be re-expanded into series (18), which indicates the convergence of the latter, and consequently of series (16) for sufficiently small t. So fair

Theorem 2. Let in equation (1) $\frac{1}{2} < \alpha < 1$. If for the equation (1) there is a formal solution in the form of a series

$$u(t) = P_{000} t^{\frac{1}{1-\alpha}} (1 + \sum_{j=1}^{\infty} \Delta_j t^{\delta_j})$$

where positive numbers $^{\delta_j}$ have the property $^{\delta_{j+1} > \delta_j, \delta_j \to \infty}$ for $j \to \infty$, then this series converges absolutely and uniformly for sufficiently small t.

Keywords: Volterra integral equations, regular singular point, complex parameters, vanishing solutions of the equation, power series.

45D05

References

[1] Lokshin A.A., Suvorova Yu.V. *Mathematical theory of wave propagation in media with memory*, Moscow: Moscow State University, 1982, 152 p.

[2] Magnitsky N.A., Asymptotics of solutions of the Volterra integral equation of the first kind, *Reports of the Academy of Sciences of the USSR*, Vol.269, No.1, 1983, pp.29-32.

[3] Magnitsky N.A., *Asymptotic methods for the analysis of nonstationary control systems*, Moscow: Nauka, 1992, 160 p.

[4] Anvarov R.A., Larionov G.S., The Volterra model with a weakly linear hereditary characteristic, *Differ. equations*, Vol.14, No.8, 1978, pp.1494-1496.

[5] A. B. Baizakov, T. R. Kydyraliev, Asymptotics of vanishing solutions of Volterra linear integral equations, *Issled. by integro-differential. equations*, Vol.45, 2012, pp.40-45.

[6] Grudo EI, The study of solutions of a class of integro-differential equations, *Differ. equations*, Vol.1, No.4, 1965, pp.535-544.

[7] Bayzakov, A.B., Aitbaev K.A., On solutions of linear Volterra equations with an irregular singularity, *Science, new technologies and innovations*, No.6, 2015, pp.12-14.

[8] Gursa E., *Course of Mathematical Analysis, T1*, Leningrad: ONTI NKTP USSR, 1936, 648 p.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

CONSTRUCTION OF HIGH ORDER MATHEMATICAL MODELS OF THE MAGIC SQUARE AND THEIR COMPUTER SIMULATION

ASAN BAIZAKOV, ZAMIRGUL KAZAKBAEVA

Kyrgyz-Turkish Manas University, Bishkek, Kyrgyz Republic
e-mail:asan.bayzakov@manas.edu.kg, zamira.kazakbaeva@manas.edu.kg

For construction of the magic squares (hereinafter M-matrices) various methods have been proposed [1]. One of the efficient methods of construction of M-matrices is the method of terraces. Earlier authors have implemented algorithmic and programming of construction of M-matrices via this method in electronic form. In particular, the method works for construction of M-matrices of high order 151 × 151 and further. However, there is some disadvantage which lies in the fact that the method works only for constructing M-matrices of odd order: $n = 2k - 1, k \in N$. This paper shows computer modelling of high order M-matrix by a new method (both, odd and even order), that is the method of decomposition [2]. The main feature of the decomposition method is the presence of a database of small M-matrices, of order three to order eight. In addition to this, we can model M-matrices of odd order: 3; 5; 7 by the method of terraces and enter them into the database. Then, on the basis of the selected database we can build higher order M-matrices 3 × 3; 3 × 4; 3 × 5; 3 × 6;...; 4 × 5; 4 × 6; 4 × 7;...

Authors have already modelled M-matrix of 100x100 order constructed by decomposition method $(4 \times 5) \times 5 = 100; (5 \times 5) \times 4 = 100$. The main mathematical tool for constructing the magic squares of high-order is the arithmetic progression. It should be noted that the necessary set of values of square constants can also be computed using a second-order differential equation with corresponding initial conditions:

$$u_{n+2} = 2u_{n+1} - u_n, n \geq 1 \tag{1}$$

According to Eulerian method the general solution of the equation is as follows (1):

$$u(n) = c_1 + c_2 n \tag{2}$$

since the roots of the characteristic equation $\lambda^2 - 2\lambda + 1 = 0; \lambda_1 = \lambda_2 = 1$ correspond to each other. The constants of squares that form an arithmetic progression act as identifiers, or in another word operators. Let us construct M-matrix of the 12^{th} order in order to illustrate the method of decomposition of 3 × 4 type. There is plenty of numbers from 1 to 144 which are classified into 9 groups, with 16 numbers in each in ascending order: I group - 1, 2, 3, ..., 16, II group - 17, 18, ..., 31, 32, etc. ,VIII group - 113,114, ..., 127, 128, IX group - 129,130, ...,143, 144. Then, we will place each group of numbers according to table 3, counting the initial number as shifted number 1, and the last number as shifted number 16. For example, in the II group, the number 17 will be placed instead of 1; 18 is placed instead of 2, etc.., the last number of this group is 32 which is placed instead of 16. The following formula is proposed to calculate the square constants of each group of numbers:

$$S = \frac{n^2 + 1}{2} n = kn \tag{3}$$

$k-$ quantity of shifts: for the second group is equal to $k = 16$, for the third group – $k = 32$, etc. In our case $n = 4$. Consequently, the constants of the squares of the formed groups form an arithmetic progression with the following difference: d $= 16 \times 4 = 64$. Thus, we have the following 9 constituents of the arithmetic progression: $a_1 = 34, a_2 = 98, a_3 = 162, ..., a_9 = 546$ and subsequently, they will be placed in accordance with their numbers as it is shown in Table 1:

$$\begin{pmatrix} a_6 & a_7 & a_2 \\ a_1 & a_5 & a_9 \\ a_8 & a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 354 & 418 & 98 \\ 34 & 290 & 546 \\ 482 & 162 & 226 \end{pmatrix}$$

It should be noted, that the square constant of the last $M-$ matrix is equal to 870.

Furthermore, we will consider these 9 square constants as order-extending operators. $M-$ matrix of the 12^{th} order is considered as a nested 3×4, in particular, as a block matrix. Each component of the $M-$ matrix of the third order is considered as a block and expand it as it is illustrated in Table 1. Accordingly, for example, constant 34 will consist from $M-$ matrix of the 4^{th} order. Each block can be drawn randomly due to square constant. We can apply any of the $M-$ matrix of order 4 from database, for example:

13	2	12	7
16	3	9	6
1	14	8	11
4	15	5	10

For constants as 98 and 162 we can apply$M-$ matrix of the 4th order which were modelled on the basis of the Table 5 in our database:

29	18	28	23
32	19	25	22
17	30	24	27
20	31	21	26

45	34	44	39
48	35	41	38
33	30	40	43
36	47	37	42

We can continuously make these steps regarding each square constant illustrated in Table 3. In consequence, instead of the square constants we will draw $M-$ matrix of our group and get $M-$ matrix of the 12th order. The table below illustrates the fragment of location of $M-$ matrix of order 4 instead of constants 34, 98, 162.

								29	18	28	23
								32	19	25	22
								17	30	24	27
								20	31	21	26
13	2	12	7								
16	3	9	6								
1	14	8	11								
4	15	5	10								
				45	34	44	39				
				48	35	41	38				
				33	30	40	43				
				36	47	37	42				

Finally, we get one type of $M-$ matrix of the 12^{th} order:

93	82	92	87	109	98	108	103	29	18	28	23
93	82	92	87	109	98	108	103	29	18	28	23
93	82	92	87	109	98	108	103	29	18	28	23
93	82	92	87	109	98	108	103	29	18	28	23
93	82	92	87	109	98	108	103	29	18	28	23
93	82	92	87	109	98	108	103	29	18	28	23
93	82	92	87	109	98	108	103	29	18	28	23
93	82	92	87	109	98	108	103	29	18	28	23
93	82	92	87	109	98	108	103	29	18	28	23
93	82	92	87	109	98	108	103	29	18	28	23
93	82	92	87	109	98	108	103	29	18	28	23
93	82	92	87	109	98	108	103	29	18	28	23

It is evident that

$$S = \frac{a_1 + a_{n^2}}{2} n = \frac{34 + 546}{2} 3 = 870. \tag{4}$$

We have modelled a magic square of 12×12 size on the basis of the magic square of 4×4 size built earlier. Then, due to constant square of 4×4 we can get plenty of different magic squares of 12×12 size. If we turn the each 4×4 size block of the magic square of 12×12 size clockwise or back wise to $90^0; 180^0; 270^0$ degrees, then with each turn, we get a new magic square of 12×12 size. This option of getting a new number of manifold matrices creates favorable conditions of application of magic matrices of high order (for instance, 100x100 matrix) in cyber security, information safety and cryptography, in particular. This is the foremost reason of significance of the high order M-matrix modelling. It is noteworthy to build the $M-$ matrix of high order due to its application in information security as well. $M-$ matrix of high order possesses diverse existence of various classical magic squares of size. For example, the total number of classical magic squares of 5×5 size is equal to 68 826 306. The construction of mathematical models of high order magic square and their computer simulation is implemented in C++. Encrypting text with magic squares. The working folder looks like this:

(1) EncryptingUsingMagicSquare.exe - main program
(2) BuilderOfMagicSquares.exe - an additional program that can create magic squares of any size
(3) for encrypt - the folder where the document to be encrypted should be placed
(4) encrypted - the folder where the encrypted documents will be placed
(5) for descrypt - the folder where the document to be decrypted is placed
(6) descrypted - the folder where the decrypted documents will be placed.

Folder names cannot be changed! This encryption method is perfect for both text and tabular documents.

How magic squares are created in the program "Encryption using magic squares"? A magic square is a square of a certain size, consisting of numbers from 1 to n, with the sum of numbers in each column, column, diagonal equal to the so-called magic sum.

It is necessary to take into account that there are three kinds of magic squares, each of which is created in its own way.

(1) Magic squares of odd order (m.k. of size 3, 5, 7, 9 ...).
(2) Magic squares of even - even order (m.k. of size 4, 8, 12, 16 ...).
(3) Magic squares of even - odd order (m.k. of size 6, 10, 14, 18 ...).

The program implements all three algorithms for creating magic squares, so it can create magic squares of any size. All data is recorded in tabular documents and saved in the "Magic Squares" folder.

Keywords: magic square, cryptography, decomposition method, magic square encryption, computer modeling, arithmetic progression.

References

[1] Chebrakov Yu. V., *Theory of Magic Matrices*, St.Petersburg, 2010, 280 p.

[2] Baizakov A. B., Aitbaev K. A., Sharshenbekov M. M., Computer simulation of magic squares of odd order by the terrace method and its application in cryptography, *Izvestiya of the National Academy of Sciences of the Kyrgyz Republic*, No.4, 2020, pp.51-58.

[3] Borubaev A.A., Baizakov A.B., Aitbaev K.A., *Mathematical models for constructing and revealing the properties of high-order magic matrices//Author's certificate No. 3754 of Kyrgyzpatent on copyright. – 11/28/2019*

[4] Baizakov A.B., Mombekov A.D., Aitbaev K.A., On the diversity of square constants in subblocks during matrix decomposition, *Reports of the National Academy of Sciences of the Kyrgyz Republic, Bishkek*, No.2, 2017, pp.19-24.

[5] Baizakov A.B., Mombekov A.D., On some properties of square matrices that preserve symmetry, *Reports of the National Academy of Sciences of the Kyrgyz Republic, Bishkek*,No.3,2018, pp.19-34.

[6] Baizakov A.B., Mombekov A.D., *On the principle of conservation of symmetries in nested M matrices*, II Borubaev Readings, Bishkek, 2018, 21 p.

[7] Baizakov A.B., Sharshenbekov M.M., Aitbaev K.A., Magic square: Terms of arithmetic progression – identifiers, *Bulletin of the Institute of Mathematics NAS KR*, No.1, 2022, pp.38-42.

[8] Baizakov A.B., Aitbaev K.A., Sharshenbekov M.M. *Magic square: Terms of arithmetic progression identifiers//Author's certificate No. 5075 of Kyrgyzpatent on copyright. – 12/21/2022*

[9] Baizakov A.B., Sharshenbekov M.M., Aitbaev K.A., Mombekov A.D.Mathematical modeling of high order magic squares and their applications to information security, *International scientific conference "Actual problems of mathematics and education"*,Osh, 2023

[10] Bayzakov A.B., Aitbaev K.A., Sharshenbekov M.M., Terms of arithmetic progression as identifiers in the decomposition method., *Bulletin of the Institute of Mathematics NAS KR*, No.2, 2022, pp.32-38.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

COMMUNICATING PROCESSES SIMULATION IN AGILE MANAGEMENT FOR STUDENT TEAMS PROJECTS OF SOFTWARE DEVELOPMENT

ELENA DUDYSHEVA[1,2]

[1]Shukshin Altai State University for Humanities and Pedagogy, Biysk, Russia
[2]Altai State Pedagogical University, Barnaul, Russia
e-mail: dudyshevaev@gmail.com

Abstract. The implementation of projects with structural dynamics by student teams, as unstable professional teams, requires agile management of blended interactions in conducting tasks timely by participants. The mechanism of communicating sequential processes (CSP) is one of the ways to model concurrent and distributed processes. The experimental work on student project management in software development showed that CSP model can be applied rather effectively in the management of individual and group project tasks.

Parallelism in this model is manifested in the possibility of simultaneous execution of distributed tasks by different participants. The disadvantage of CSP model is the difficulty to conduct global analysis and monitoring in the practical management of several projects. The disadvantage of Petri nets for team projects, on the contrary, is the impossibility of delimiting the tasks of the personal responsibility of the participants. The possibility of combining two formalisms permits both to control and analyze the interaction of participants in joint project activities. Partially, the CSP model allows synchronization with the Petri net model, which can also be used to manage the work of student teams. Models' matching can be achieved by converting the project process UML activity diagrams into two selected formalisms. In addition, software development learning with process modeling makes it possible to illustrate students' dynamic model application in social system automation.

Keywords: communication processes model, Petri nets, communicating sequential processes, agile project management, distributed processes, software development learning, UML diagrams.

AMS Subject Classification: 90B10, 68Q85, 97P50.

1. Introduction

Management of professional project teams is regulated by both general patterns of business processes and specific technologies within a particular field of activity. Communicating processes simulation is utilized to build formalized and mathematical models for post-factum analysis and subsequent optimization or for a real-time response when team processes deviate from predefined parameters. A wide range of mathematical formalism is employed for modeling different communicating processes depending on research conditions, objectives, and tasks – including classic, stochastic, probabilistic or spatial process calculi, P systems, cellular automata, Petri nets, Grid systems [1].

The common problem lies in constructing models to reliably identify participants' responsibility for delayed joint actions and to determine individual involvement in personal and collaborative project stages without direct participation from instructors or managers, especially when dealing with many unstable teams. Such models can be applied in automation for educational or business process management environments, serving as a basis for timely managerial interventions and final personal rewards or assessments. In practice, process descriptions of the functions of participants in business processes are most often used, such as Business Process Modelling

Notation [2]. They are intuitive for managers who are not experts in process automation. The description of the dynamic features is not semantically accurate and sufficient formalization enough. Therefore, the description of business processes is also used by other methods, such as, for example, Petri nets [3] or algebraic calculus of processes [4].

Existing virtual communication support environments are oriented either towards explicit individual responsibility (e.g., online learning platforms, CRM) or distributing responsibility within stable teams (e.g., virtual whiteboards). The digital trace of participants' actions is mainly intended for verification and possible correction of individual actions in a "manual" or instructional manner, without enabling analysis of the overall project state and progress. On the other hand, reflecting the digital trace of team project results at certain stages (versions of project artifacts) makes it challenging to react to the actions or inactions of individual team members since they remain concealed within informal communications within teams. Therefore, there are contradictions in the construction of formal models for the first or second task of analysis and control.

In the design of communicating business processes, great experience has been accumulated in applying mathematical models of both the first type – process algebras, multi-agents' models such as communicating sequential processes (CSP) [5], and the second type – graph and automaton models such as colored Petri nets [6]. Models of the second type predominate in the management of working projects because of the focus on results. But they practically do not represent cases of possible instability of structures or resources.

Distributed team projects often employ agile management methods due to the inherent instability of such teams. The complexity of managing distributed teams lies in the fact that part of the communications is hidden, so it is not always possible to recognize crises in time. Team structures and participants' responsibilities may change while adhering to project timelines and other requirements. This frequently occurs in resource-constrained environments, small organizations, and startup projects. Situations of instabilities arise in student teams rather often. Communicative activity in blended learning becomes a viable and accessible research subject for dynamic models of agile management in distributed project teams without compromising economic efficiency. An exemplary case for examination is the design of software systems as an area with a wealth of accumulated theoretical and technological tools for both classic and agile design methodologies. A powerful and versatile simulation tool of processes is Unified Modeling Language (UML) which is used in different software engineering methodologies as well for business process modeling [7].

The goal of this paper is to explore the possibility of solving the identified problem of unstable teams' agile management by constructing a dynamic simulation of communicating processes for student teams' projects of software development, allowing solutions for individual formalization tasks and joint participation in design stages, including situations of team structure changes.

2. Materials and Methods

Research question: what mathematical formalisms can be used to describe, agile manage and analyze the individual participation of students along the digital track of project activities and interaction processes in unstable teams?

Literature review. In most theoretical studies, the emphasis of research shifts to the ability to describe the properties of system processes as well as the equivalence of model types describing communicating processes or their extensions or restrictions.

Let's consider comparing alternative ways of describing the system's dynamics with comparing interacting processes simulation [1]. Petri Nets (PN) and Communicating Sequential Processes (CSP) are popular models.

Although the CSP and PN models are not identical in terms of processing power and generated message protocol languages, it is possible to establish a correspondence between their subsets. Olderog directly simulated CSP models with nondetermic machines and shows a better ability

to uncover failure semantics problems [8]. Winskel built the reverse transformation rules, which describe the semantic properties of Petri nets in terms of process algebras [9]. Yang et al. indicate the ability of extended Petri nets to provide a practical estimate of the execution time of processes [10].

Roscoe focuses on the CSP possibility of timed modeling and verification in various systems [11]. Let's note that in works using algebraic models, there is a potential possibility of including and excluding channels. Singer and Teller describe Pi calculus as the ability to send messages through alternative channels [4]. Therefore, we can consider both algebraic models of communicating processes and Petri nets as the appropriate formalisms of the main model components for solving the problem of our study.

Some research is devoted to the formalization and analysis of the properties of business processes and describes ways to transform a formalized description of processes into formal mathematical models.

Trentin gives an example of modeling student communications with Petri nets, but at the level of high abstractness of building knowledge and learning processes [12]. Kucharik and Balogh develop similar ideas, improving the formalized description of the simulation of cognitive processes [13]. But they remain outside the tasks of managing the direct actions of participants in productive communication activities.

The issues of integration of several formalisms in the description of interacting processes for solving automation problems in information systems are encountered in practical works. For example, Katilius et al. [14] propose combining UML and Petri nets, where the use of the network model is used as an additional means of optimizing processes and building a hybrid model for possible changes in the functions of the information system.

Zhang et al. judicial business processes are first modeled with UML diagrams, and then they are transferred to a formal subset of CSP models [15]. Wong and Gibbons build a strict formal process algebra semantics for a subset of Business Process Modeling Notation (BPMN) diagrams based on the formalization of the description of sustainable processes [2]. Mutarra et al. describes Business Process by Petri net formalism [3]. Yang et al. choose UML diagrams to describe the behavior of processes and transform them into extended Petri nets [10].

So, we had not found in the studies an emphasis on formalizing the properties of changing the structure of team members and redistributing responsibility for performing subtasks.

Many researchers point out the advantages and disadvantages of both algebraic and network models ([1], [2], [3], etc.). We assume that it is more effective to consider a multi-model which should consist of two main dynamic components that correspond to the recorded actions of project team members and authentic states of design stages, and include synchronization components for building, transforming, and analyzing the final states and characteristics of the first two components. As two main components, we propose to use protocols of sequential interacting processes and Petri nets states to reflect the progress of project processes.

We believe that the notation of communicating sequential processes formalizing is a possible way to simulate individual activities, and extended Petri nets are suitable tools for describing holistic characteristics with UML activity diagrams as means for synchronization.

The novelty lies in the integration and refinement of the studied and universal mathematical formalisms for cases of a possible violation of the sustainable execution of processes, in our case, due to the situation of unstable work of project team members.

Our research discusses the next issues.

1. Description of the presentation of project tasks for student teams in the form of project processes using UML behavior diagrams.

2. Extension choice of Petri nets to describe the model of common project activity.

3. Allocation of a subset of CSP to describe the model of individual and joint project activity with the solution of the practical problem of automating the collection of digital traces of distant communications.

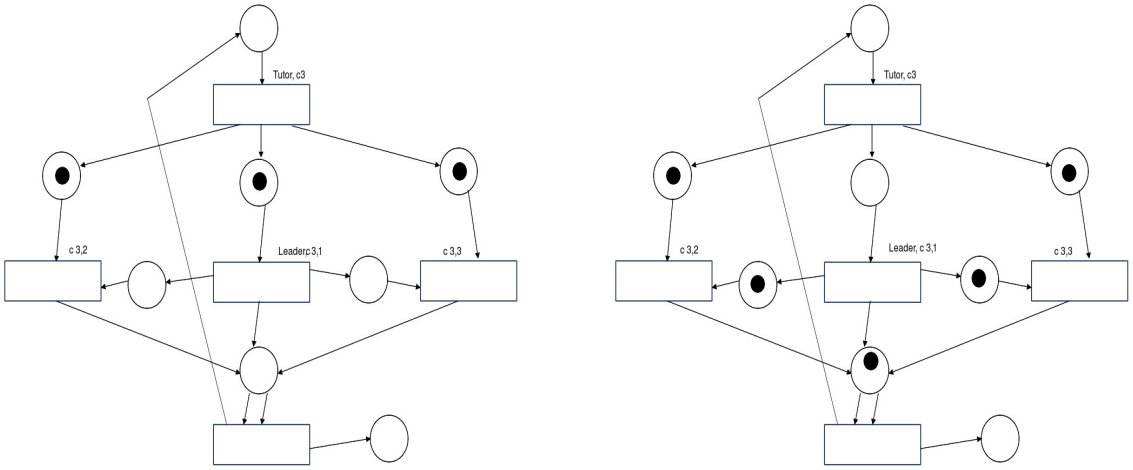
4. Case study and the model discussion on the examples of team and collective projects.

Modeling methods description. We are interested in a more formal description of project activities from the point of view of the logic of processes at an ideal, rather than a specific level of implementation. Therefore, we stuck with a more expressive means - UML behavior diagrams instead of Business Process Modeling Notation. Among all UML diagrams, the most interesting are activity and sequence diagrams, where synchronization and asynchronous communication processes can be accurately described.

In terms of expressiveness and semantics of the elements of the business process model in projects with distributed responsibility, the tracks of the UML activity diagram are more consistent. But since we include a variable number of actors in our consideration, it will not work to build it and use it without changes. Therefore, one can first draw an activity diagram corresponding to the design stages for two participants, and then construct a corresponding UML sequence diagram for each project participant actor that interacts with the organizing actor – leader, teacher, or tutor.

It is for this type of diagram with asynchronous interaction of a variable number of participants that we will consider the transformation in a Petri net. To communicate the model for transitions corresponding to the stages of the project with the joint activity of participants, we set input of multiplicity two to describe the situation of the interaction of at least two students. If the number of students in the team is reduced to one, then an educational meta-strategy will be applied either to terminate the project or to include a teacher and replace him with the role of a participant in the project. All other transitions corresponding to stages that can be performed asynchronously are duplicated for all participants.

Thus, one of the components of our multi-model can be Petri nets synchronized with the project stages and the number of participants. The example of Petri net model shows the asynchronous execution of one project sub-stage with the synchronization of sub-task c_3 by the leader: receiving the task from the tutor (Fig. 1a), execution request from the leader (Fig. 1b).

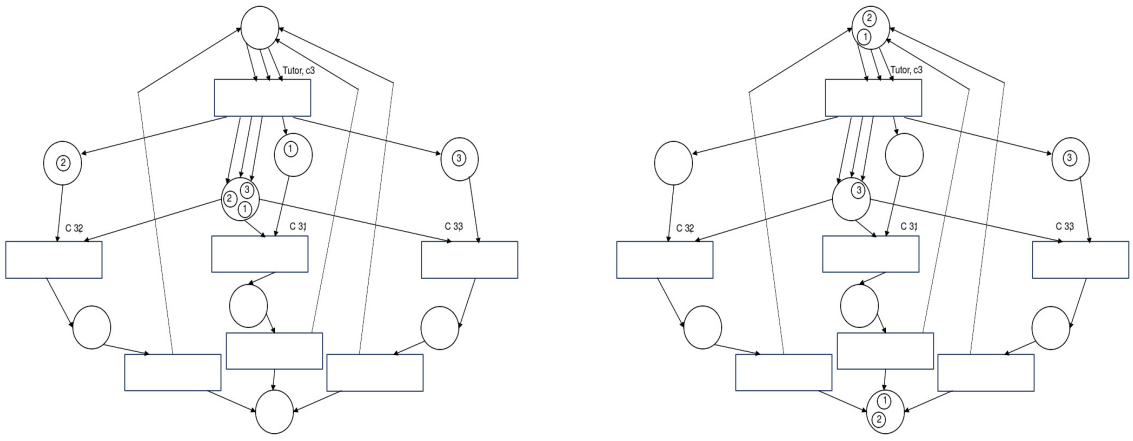


a) b)

FIGURE 1. Petri net for processes synchronization by the leader of the students' team

Colored Petri nets can simulate the asynchronous execution of one project sub-stage with the synchronization of sub-task c_3 by the tutor independently for each participant: receiving the task from the tutor (Fig. 2a), waiting for execution from participant 3 (Fig. 2b).

The task of collecting a digital footprint is critical for the analysis of remote communications. Note that the data of remote communications received in instant messengers and on social networking platforms reflect the facts of communication, but not the effectiveness reflected in the



a) b)

FIGURE 2. Colored Petri net for processes synchronization with the tutor

stages of project implementation. They can serve as additional material for semantic analysis, which goes beyond the boundaries of our study. Cloud services for teamwork should contain a list of tasks for each project participant and points of general or selective coordination of design decisions. Therefore, as a basis for collecting digital data, we have chosen virtual boards of project teams. In particular, the Trello platform (https://trello.com/) is equally accessible for educational and business purposes. Virtual project boards are focused on the phased completion of a list of tasks and can be used to report by stages and by performers. They are directly related to project processes and activities.

Using an asynchronous list of user message history, we can consider the interaction of participants on the whiteboard as CSP message channels. Writing responses to dedicated cards for joint tasks can act as synchronizing events. Participants can complete the rest of the tasks in any order. Therefore, the second component of the multi-model, synchronized with the project stages and the number of participants, can be built based on the CSP formalism.

Next let's consider, for example, a project task P (1) consisting of two successive stages R and Q for the project team:

$$P = R \triangleright Q, \tag{1}$$

where:

R, Q – stages (sub-tasks) of the project P;

$\triangleright$ means sequential execution of processes (in linear notation may be replaced by ;).

Within stages described in the CSP expressions of processes interactive and asynchronous execution, some of the sub-stages c_i (2)-(3) are planned for each team participant:

$$R = (c_1 \to (c_2 \to (c_3))), \tag{2}$$

$$Q = (c_4 \to c_5), \tag{3}$$

where:

$\to$ means performing the event c_i before and then process after event (sometimes replaced by ->).

But there is at least one synchronization event for each stage due to interactivity educational and management principles.

Expression (4) illustrates a process of project example (1)-(3) for the team of three students with sync events c_2 and c_5.

$$P = ((R_1 \|_{(c_2)} R_2)[](R_2 \|_{(c_2)} R_3)[](R_1 \|_{(c_2)} R_(3))) \triangleright ((Q_1 \|_{(c_5)} Q_2)[](Q_2 \|(c_5) Q_3)[](Q_1 \|_{(c_5)} Q_3)) \tag{4}$$

where:
$\|$(c) means sync of processes on event c;
[] means choice one of processes.

In general, project tasks P_j for each participant j (5) are not absolutely identical to planned stages:

$$P_j \neq R_j \triangleright Q_j, \tag{5}$$

where:
R_j, Q_j – stages (sub-tasks) of the project P participant j.

In remote communications, a manager or tutor often distributes responsibility and controls the necessary steps for synchronizing participants, which can be described by the iterative process T (6).

$$T = (((c_1[]Skip) \to T) \| (c_2 \to T) \| ((c_3[]Skip) \to T) \| ((c_4[]Skip) \to T) \| (c_5 \to T))[]Stop \tag{6}$$

where:
Skip means "empty" process;
Stop means that process terminates.

Then, in this example, the description of the project for the first participant will look like (7):

$$P_1 = (((R_1 \|_{(c_2)} R_2)[](R_1 \|_{(c_2)} R_{(}3))) \| T) \triangleright ((((Q_1 \|_{(c_5)} Q_2)[](Q_1 \|_{(c_5)} Q_3)) \| T). \tag{7}$$

The possible instability of the team structure is considered in the combination of events in which the participants communicate in CSP model component when structural instability in the project dynamic can be considered at each stage in Petri net model component.

3. Results and Discussion

We used the case study method and examined the work of student software systems design teams using interactive development methodology and UML diagramming for object-oriented modeling.

As we pointed out in [16], the CSP model can be useful to manage the unstable learning community of student teams. But we did not build mathematical formalism for testing on specific design processes before.

Case study. Student teams can serve as a pilot model for the work of unstable remote teams in the agile development of small projects. Studies on agile software development methodologies suggest that they can be effectively combined with traditional processes, including the object-oriented modeling methodology. A simplified version of the unified object-oriented modeling process was chosen for the work.

The projects included the following initial stages.

1. *Presentation of project themes.*
2. *Announcing common and personal project goals.*
3. *Discussion and preparation of a preliminary report, and reviews in the selected topics.*
4. *Terminological dictionary filling.*
5. *Construction of Use Case diagrams.*
6. *Mastering technology elements.*
7. *Algorithms choice. ...*

Part of the steps must be performed strictly sequentially when ones are in random order, some steps are optional, and part of the process is iterative. We skip illustrating behavior diagrams design since they correspond to simple cases described in various sources ([10], [15]). Instead, let's describe the main components of the multi-model.

In our case, the description of the student project activity P (1) in CSP model can be represented as follows (8)-(9), with tutor activity (10).

$$R = (c_1 \to (c_2 \to (c_3 \to c_4)), \tag{8}$$

$$Q = (c_5 \to ((c_6|Skip) \to ((c_7|Skip) \to Q))), \tag{9}$$

$$\begin{gathered} T = (((c_1[]Skip) \to T)\|((c_2[]Skip) \to T)\|(c_3 \to T)\|((c_4[]Skip) \to T)\| \\ (c_5 \to T))\|((c_6[]Skip)\|((c_7[]Skip). \end{gathered} \tag{10}$$

Parallelism in the model means the ability to simultaneously perform the corresponding actions by different team members. Some of the stages have a special communicative meaning – all students with different roles should take part in them simultaneously, in the sample above c3 and c5 events cannot be located within one of the parallel processes only, they are sync events. Traces of the project events model can be compared with real messages from remote users for individual and collaborative work' analysis. For the illustration let's consider the digital footprint data of the first stage of the project's team of three students obtained from the Trello format.

Tutor commented on card 3. Study the project materials. @student17 Well done!

Student17 commented on card 3. Review project materials. Saved to disk.

Tutor commented on card 3. Study the project materials. @student17 You need to drop it into our project folder.

Student17 commented on card 3. Review project materials. I haven't done the task yet.

Student17 commented on card 3. Review project materials. @student14 I have failed task. Ow, I'll try.

Tutor commented on card 3. Review project materials. Well done! @student14

Student14 commented on card 3. Review project materials. (changed) Hello! I uploaded a model to Google Drive.

Student18 marked the element We are waiting for an answer-Student18 in the checklist of card 2. Write why you are interested in the topic of the project.

Student18 commented on card 2. Write why you are interested in the topic of the project. I am interested in the field of modeling, and video games and work on an unusual project.

Student17 unchecked item Waiting for object-Student17 card checklist 4. Discuss and select special properties.

Student17 marked the item Waiting for the object-Student17 in the checklist of card 4. Discuss and select special properties.

Student17 commented on card 2. Write why you are interested in the topic of the project. It is interesting to get into another program world where everything is different.

Student17 marked the element We are waiting for an answer - Student17 in the checklist of card 2. Write why you are interested in the topic of the project.

*Student17 commented on card 1. Introduce yourself and remember the name of your team project. My name is ****

Student14 marked the element We are waiting for an answer - Student14 in the checklist of card 2. Write why you are interested in the topic of the project.

Student14 commented on card 2. Write why you are interested in the topic of the project. I am interested in game developing, both VR, or others platforms.

*Student14 commented on card 1. Introduce yourself and remember the name of your team project. My name is ****

*Student18 commented on card 1. Introduce yourself and remember the name of your team project. My name is ****

The example involves three students project work represented by channels Student14 ($c_{i,1}$), Student17 (c2), Student18 (c3), Tutor (t). Project tasks are set according to Trello cards. The sequence S = ($c_{1,3}$, $c_{1,1}$, $c_{2,1}$, $c_{2,1}$, $c_{1,2}$, $c_{2,2}$, $c_{2,2}$, $c_{4,2}$, $c_{4,2}$, $c_{2,3}$, $c_{2,3}$, $c_{3,1}$, $t_{3,1}(@c_{3,1})$, $c_{3,2}(@c_{3,1})$, $c_{3,2}$, $t_{3,2}(@c_{3,2})$, $c_{3,2}$, $t_{3,2}(@c_{3,2})$) can be interpreted as trace protocol. So, we can check if S belongs to a set of process protocols traces to make sure that the interaction of the participants goes according to the planned manner. And it can influence agile project management by

calculation different participant's parameters such as *"iteractivity"* - the proportion of returns to the stages already completed, *"systematic"* - the proportion of tasks performed in sequence according to the project task, *"jointness"* - the proportion of participation in coordinating tasks with other participants, and so on.

The undoubted advantage is the flexibility of management and visibility since it is clear which participant is at what stage, and who expects whom. The disadvantage of this model is that it is difficult to conduct a global process performance analysis as well as to carry out general monitoring of the entire community, rather than a separate project. A general assessment of the project state for the entire team can be performed using Petri nets. This case for the synchronous event c_3 corresponds to Figure 2.

The proposed duplication for each participant of copies of the sub nets of the Petri net representing the sub-processes makes the graphical representation unwieldy. But now there are many analysis systems for automated processing of network models [17]. Also, there are effective means of automated analysis of CSP model [18] as the main component of the multi-model, and it is very important in practical applications. Unlike software design methodologies or other specific business processes, we also consider it necessary to further study not only an individual project but a community of student project teams. Firstly, it is close to production situations and shows the real computational complexity of the models; secondly, the community serves as a source for potential performers in unstable teams; thirdly, it demonstrates the variety of project implementation options in different teams and allows you to test the correctness of the model more reliably.

However, the focus on software development projects suggests the use of professionally oriented modeling languages as a practical tool and learning content for students specializing in software engineering. Used in the development of software systems, the UML modeling language is a powerful expressive tool for automating various processes. Therefore, it can be used as a means of organizing self-government in teams of students studying software engineering.

4. Conclusion

The implementation of projects with structural dynamics by student teams, as unstable professional teams, requires agile management of blended interactions in conducting tasks timely by participants. Communicative activity in blended learning is e research subject for the simulation of dynamic communicating processes in activities of this kind. The goal of this paper is to explore the possibility of solving the identified problem of unstable teams' agile management by constructing a dynamic simulation of communicating processes for student team's projects of software development, allowing solutions for individual formalization tasks and joint participation in design stages, including situations of team structure changes.

The mechanism of communicating sequential processes (CSP) is one of the ways to model concurrent and distributed processes. The experimental work on student project management in software development showed that CSP model can be applied rather effectively in the management of individual and group project tasks. Parallelism in this model is manifested in the possibility of simultaneous execution of distributed tasks by different participants. The disadvantage of CSP model is the difficulty to conduct global analysis and monitoring in the practical management of several projects. Petri nets as alternative formalism of process simulation allow dynamic analysis of the state of the project in general. The disadvantage of Petri nets for team projects, on the contrary, is the impossibility of delimiting the tasks of the personal responsibility of the participants.

The combining of two formalisms permits both to control and analyze the interaction of participants in joint project activities. More specifically, the CSP model allows synchronization with the Petri net model, which can also be used to manage the work stages of student teams. Our research discusses of the presentation of project tasks for student teams in the form of project processes using UML behavior diagrams; extension choice of Petri nets to describe the model of

common project activity; allocation of a subset of CSP to describe the model of individual and joint project activity with the solution of the practical problem of automating the collection of digital traces of distant communications; case study and the model discussion on the examples of team and collective projects. We studied mathematical formalisms which can be used to describe, agile manage and analyze the individual participation of students along the digital track of project activities and interaction processes in unstable teams. We showed that the notation of communicating sequential processes formalizing is a possible way to simulate individual activities, and extended Petri nets are suitable tools for describing processes characteristics with UML activity diagrams as means for synchronization. The main models' component matching can be achieved by converting the project process UML activity diagrams into two selected formalisms. In addition, software development learning with process modeling makes it possible to illustrate students' dynamic model application in social system automation. Further studies require detailed construction of transforming rules and model validation of different cases.

References

[1] Toro M., A general overview of formal languages for individual-based modelling of ecosystems, *Journal of Logical and Algebraic Methods in Programming*, Vol. 104, 2019, pp. 117-126.

[2] Wong P.Y., Gibbons J.,A process semantics for BPMN, *Proceedings of the international conference on formal engineering methods, London, 2–4 June 2008*, Berlin; Heidelberg: Springer, 2008, pp. 355–374.

[3] Mutarraf U., Barkaoui K., Li Z., Wu N., Qu T., Transformation of Business Process Model and Notation models onto Petri nets and their analysis *Advances in Mechanical Engineering*, Vol. 10 (12), 2018, pp. 1-21.

[4] Singer R., Teller M., Process algebra and the subject-oriented business process management approach, *Communications in Computer and Information Science*, Vol. 284, 2012, pp. 135–150.

[5] Hoare C.A.R., Communicating sequential processes, *Theories of Programming: The Life and Works of Tony Hoare*, NY: Association for Computing Machinery, 2021, pp.157–186.

[6] Peterson J. L., *Petri net theory and the modelling of systems*, Englewood Cliffs, N.J: Prentice-Hall, Inc, 1981.

[7] Russell N., Aalst W, Ter A., Wohed P., On the suitability of UML 2.0 Activity diagrams for Business Process Modelling, *Conceptual Modelling 2006: Proceedings of APCCM2006*, Australia, 2006, pp. 95-104.

[8] Olderog E. R., TCSP: Theory of communicating sequential processes, *Lecture Notes in Computer Science*, Vol. 255, Berlin, Heidelberg: Springer, 1987.

[9] Winskel G., Petri nets, morphisms and compositionality, *Lecture Notes in Computer Science*, Vol. 222, Berlin, Heidelberg: Springer, 1986.

[10] Yang N., Yu H., Sun H., Qian Z., Modeling UML sequence diagrams using extended Petri nets, *Telecommunication Systems*, Vol. 51(2-3), 2011, pp. 147–158.

[11] Roscoe A. W., *Understanding Concurrent Systems*, Springer, 2010.

[12] Trentin G., Graphic knowledge representation as a tool for fostering knowledge flow in informal learning processes, *Technology and Knowledge Flow: the power of networks*, Cambridge, 2011, pp. 133-156.

[13] Kuchárik M., Balogh Z., Student learning simulation process with Petri nets, *Proceedings of ICCD 2017, Recent Developments in Intelligent Computing, Communication and Devices*, 2019, pp.1115–1124.

[14] Katilius I., Skučas I., Melninkaitė V., Synergetic effect combining UML and Petri nets in designing modern information systems, *Informacinės technologijos ir valdymas*, Vol. 32 (3), 2004, pp. 46-54.

[15] Zhang T., Zeng X., Liu Z., Modeling workflow for judicial business processes: a use case driven method, *2021 7th International Conference on Information Management (ICIM)*, London, 2021, pp. 45-56.

[16] Dudysheva E., Solnyshkova O., Network modeling of blended communications in the community of project teams of students, *Lecture Notes in Networks and Systems*, Vol.181, 2021, pp 347–364.

[17] Carrasquel J. C., Morales A., Villapol M. E., Prosega/CPN: An extension of CPN tools for automata-based analysis and system verification, *Proceedings of ISP RAS*, Vol. 30(4), 2018, pp. 107-128.

[18] Gibson-Robinson T., Armstrong P., Boulgakov A., Roscoe A., FDR3 — A modern refinement checker for CSP, *International Conference on Tools and Algorithms for the Construction and Analysis of Systems*, Vol. 8413, 2014, pp. 187-201.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

COMPLEX RAYS AND THICK LENSES

ELMAN HASANOGLU

Isik University, Department of Mathematics, Istanbul, Türkiye
e-mail: elman.hasanoglu@isikun.edu.tr

Complex rays are one of the mysterious subjects in modern optics and diffraction theory. In some cases, they allow for a more precise estimation of the behavior of diffracted waves compared to real geometric optics methods. As waves (acoustic, optics, etc.) propagate in a medium, diffraction effects cause the rays to broaden and diverge. It turns out that within the framework of geometric optics, it is possible to estimate this divergence without calculating diffraction integrals. This is achieved through the existence of complex solutions to the eikonal equation, the main equation of geometric optics. However, the main drawback of this method is that it requires techniques involving high-dimensional complex spaces ([2,3,4]). Unlike real solutions of the eikonal equation, complex solutions cannot be visually represented in physical space, which causes a significant loss of clarity in wave propagation. In [1], it has been shown that considering the eikonal equation in a non-Euclidean space, instead of the usual Euclidean space, might significantly simplify the geometric picture of propagation. This paper proposes another application of the complex rays. It is shown that the ray picture of the optical system involving thick lenses can be adequately described by using complex rays.

Cosider the standard eikonal equation of geometric optics

$$\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2=1 \tag{1}$$

where $S(x,y,z)$ is the phase of the propgation. Equation (1) represents the mathematical expression of two fundamental principles of geometric optics: Fermat's Principle and Huygen's Principle. The equation is a nonlinear partial differential equation of the first order and can be solved, at least locally, using the well-known characteristic method. In this context, the characteristics correspond to rays of geometric optics, and the surfaces defined by the equation $S(x,y,z)=const$ represent the wave fronts and at each point rays are perpendicular to wave fronts. The solution of (1) subject to the initial value $\varphi(x,y)=S(x,y,0)$, $\varphi(x,y)$ being real valued, is given as

$$\xi=x+\sigma\varphi_x(x,y), \quad \eta=y+\sigma\varphi_y(x,y), \quad S=\varphi(x,y)+\sigma, \quad \sigma=\frac{z}{\sqrt{1-\varphi_x^2-\varphi_y^2}} \tag{2}$$

where σ is a parameter alohg the rays. If $1-\varphi_x^2-\varphi_y^2>0$ then rays are real and if $1-\varphi_x^2-\varphi_y^2<0$ then rays become complex. The case $1-\varphi_x^2-\varphi_y^2=0$ is exeptional case and should be considered separately (so called, Stokes' lines). Complex rays are typically introduced through the analytical continuation of the phase function with respect to all its spatial variables. The connection

This work is supported by Project No 22A205 "Complex Rays and Applications" by Isik University (22.09. 2022).

between complex rays and Gaussian beams has been extensively studied. As demonstrated in [2,3], Gaussian beams can be regarded as emanating from a source located at the complex domain. An alternative interpretation of complex rays is presented in [1], demonstrating that these rays are perpendicular to the initial surface in the sence of Minkowski geometry and in this geometry Gaussian beams correspond to the radii of the unit sphere with an imaginary radius. Exluding σ from (2) we get a mapping between (x,y) and (ξ,η) planes for each fixed z :

$$\xi = x + \frac{z\varphi_x}{\sqrt{1-\varphi_x^2-\varphi_y^2}}, \qquad \eta = y + \frac{z\varphi_y}{\sqrt{1-\varphi_x^2-\varphi_y^2}} \tag{3}$$

We will consider the propagation of rays along the $z-$axis, with the parameter z representing the distance along the rays. Thus, any initial phase function $\varphi(x,y)$ generates a mapping between planes at $z=0$ and $z=const$:

$$\xi = \xi_z(x,y)\,,\ \eta = \eta_z(x,y) \tag{4}$$

For complex rays (3) reads as

$$\xi = x + \frac{iz\varphi_x}{\sqrt{1-\varphi_x^2-\varphi_y^2}}, \qquad \eta = y + \frac{iz\varphi_y}{\sqrt{1-\varphi_x^2-\varphi_y^2}} \tag{5}$$

where again $1-\varphi_x^2-\varphi_y^2>0$. Now solving (5) for φ_x and φ_y we get

$$\varphi_x(x,y) = \frac{x-\xi}{\sqrt{(iz)^2+(x-\xi)^2+(y-\eta)^2}}, \qquad \varphi_y(x,y) = \frac{y-\eta}{\sqrt{(iz)^2+(x-\xi)^2+(y-\eta)^2}} \tag{6}$$

Since $\varphi_{xy}(x,y) = \varphi_{yx}(x,y)$ for twice continuously differentiable function, we get

$$\frac{\partial}{\partial x}\left(\frac{y-\eta}{\sqrt{(iz)^2+(x-\xi)^2+(y-\eta)^2}}\right) = \frac{\partial}{\partial y}\left(\frac{x-\xi}{\sqrt{(iz)^2+(x-\xi)^2+(y-\eta)^2}}\right)$$

or

$$\frac{\partial\xi}{\partial x}(x-\xi)(y-\eta) - \frac{\partial\eta}{\partial x}\left[p^2+(x-\xi)^2\right] + \frac{\partial\xi}{\partial y}\left[p^2+(y-\eta)^2\right] - \frac{\partial\eta}{\partial y}(x-\xi)(y-\eta) = 0 \tag{7}$$

where $p=zi$. Thus complex rays obey equation (7). Now consider an optical system involving a single thick lens ([6]). Plane wave coinsiding with (x,y) plane in the object space after passing the thick lenz converts to plane wave in the image plane coinsiding with (η,ξ) plane.

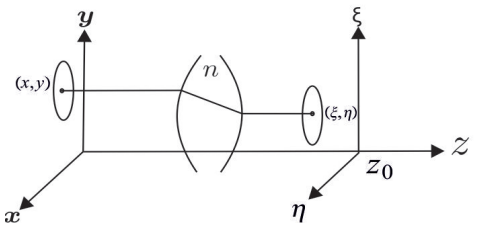


Figure 1

A ray which leaves the incident plane wave at the point (x,y) reaches the refracted plane wave at the point

$$\eta = \eta(x,y)\,, \quad \xi = \xi(x,y)\,.$$

Thus a thick lens generates a mapping between (x,y) and (η,ξ) planes. Let us derive an equation describing this correspondence. From the Fig.1, calculating the optical length through the optical

system, for the Hamilton's characteristic function we get

$$H = (z_1 - z_2)^2 \left(n^2 - 1\right) + 2\left(m - n^2 z_0\right)(z_1 - z_2) + n^2\left[(\xi - x)^2 + (\eta - y)^2\right] - m^2 + n^2 z_0^2 = 0 \quad (8)$$

The Fermat's Principle (or the same Snell's law on the mirrors) can be written as

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial z_1}\frac{\partial z_1}{\partial x} = 0, \quad \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z_1}\frac{\partial z_1}{\partial y} = 0 \quad (9)$$

After calculating the corresponding partial derivatives of H and incorperation them into (9), we find

$$\frac{\partial z_1}{\partial x} = \frac{n^2(\xi - x)}{(z_1 - z_2)(n^2 - 1) + m^2 - n^2 z_0^2}, \qquad \frac{\partial z_1}{\partial x} = \frac{n^2(\eta - y)}{(z_1 - z_2)(n^2 - 1) + m^2 - n^2 z_0^2} \quad (10)$$

Now solving (8) with respect to $z_1 - z_2$ and substituting in (10), we get

$$\frac{\partial z_1}{\partial x} = \frac{n(\xi - x)}{A(x, y, \xi, \eta)}, \qquad \frac{\partial z_1}{\partial y} = \frac{n(\eta - y)}{A(x, y, \xi, \eta)} \quad (11)$$

where

$$A(x, y, \xi, \eta) = \pm\sqrt{n^2 - 1}\sqrt{p^2 - \left[(\xi - x)^2 + (\eta - y)^2\right]} \quad (12)$$

with

$$p = \frac{m - z_0}{\sqrt{n^2 - 1}}.$$

The two signs in (12) corresponds two different optical systems: Cassegrain and Gregory. Since

$$\frac{\partial^2 z_1}{\partial x \partial y} = \frac{\partial^2 z_1}{\partial y \partial x},$$

we have

$$\frac{\partial \xi}{\partial y}A + (\xi - x)\left(\frac{\partial A}{\partial y} + \frac{\partial A}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial A}{\partial \eta}\frac{\partial \eta}{\partial y}\right) = \frac{\partial \eta}{\partial x}A + (y - \eta)\left(\frac{\partial A}{\partial x} + \frac{\partial A}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial A}{\partial \eta}\frac{\partial \eta}{\partial x}\right)$$

After calculating partial derivatives and simplifying finally we have:

$$\frac{\partial \xi}{\partial x}(x - \xi)(y - \eta) - \frac{\partial \eta}{\partial x}\left[p^2 + (x - \xi)^2\right] + \frac{\partial \xi}{\partial y}\left[p^2 + (y - \eta)^2\right] - \frac{\partial \eta}{\partial y}(x - \xi)(y - \eta) = 0$$

which is no other than equation for complex rays (7). In the last equation the parameter p servis as a specific parameter for thick lenses representing the optical thickness of the optical system.

Keywords: eikonal equation complex rays, real rays, geometric optics, Minkowski space, thick lens,

AMS Subject Classification: 78A45, 34E10

References

[1] Hasanov, E.E. (Hasanoglu E.E.) A New Theory of Complex Rays, *IMA Jo urnal of Applied Mathematics* 69, 2004, pp.521-537.

[2] Keller, J.B., Streifer, J., J. Complex Rays with an Application to Gaussian beams. *Opt. Soc. Am.* 61, pp.40-43, 1971.

[3] Chapman, S. J., Lawry, J. M. H., Ockendon, J. R. and Tew, R. H. On the theory of complex rays. *SIAM Rev.* 41 (3), 1999, pp.417-509.

[4] Kravtsov, Yu. A. et all. Theory and Applications of Complex Rays, *Progress in Optics* 39, (Elsevier, Amsterdam), 1998.

[5] Luneburg, R.K., The Mathematical Theory of Optics, Brown University, 1964

[6] Elman Hasanov (Hasanoglu) and Burak Polat, On the realization of Optical Mappings and Transformation of Amplitudes by Means of an Aspherical "Thick" Lens, AEÃœ, International Journal of Electronics and Communications 54, No.2, 2000, pp. 109-113

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

NUMERICAL SOLUTION OF ANOMALOUS FILTRATION PROBLEM IN A POROUS MEDIUM

JAMOL MAKHMUDOV [1], AZIZBEK USMONOV [1], JAKHONGIR KULJANOV [2]

[1] Samarkand State University, Samarkand, Uzbekistan
[2] Samarkand Institute of Economics and Service, Samarkand, Uzbekistan
e-mail:j.makhmudov@inbox.ru; a.usmonov.91@mail.ru; j.kuljanov86@gmail.com

Abstract. The problem of anomalous filtration of disperse systems in a two-dimensional porous medium consisting of two zones with a mobile and slow-mobility fluids is studied taking into account the phenomena of diffusion, convection, colmatation, and suffusion. Consequently, the anomalous filtration and solute transport is described by differential equations with fractional derivatives. The corresponding problem has been solved and the fields of concentration of suspended particles in the fluid, porosity, pressure, etc. have been determined.

Keywords: anamal Darcy's law, fractional derivative, solute transport, anamal filtration, porous medium.

AMS Subject Classification: 76-10

1. Introduction

Multidimensional problems, in which hydrodynamic dissipation in different directions of space proceeds with different intensity, have been little studied. In addition, the issues of multidimensional colmotation-suffusion anomalous filtration, where the colmotation-suffusion phenomena are considered along with longitudinal and transversal diffusion, have not yet been studied . As shown in [1, 2], the presence of zones of poor permeability in macroscopically inhomogeneous media has a noticeable effect on the processes of fluid movement and the solute transport in them. As a rule, in these zones the liquid is inactive or inactive, which allows them to be considered as stagnant.

In the process of fluid filtration, the equilibrium relationship between the filtration rate and the pressure gradient can be broken. When filtering inhomogeneous fluids, the substance (or particles) contained in the fluid can linger in the pore space, which leads to a change in the porosity and porosity characteristics of the porous medium, primarily its permeability and porosity [3, 4].

In [5, 6, 7] the problem of anomalous filtration and solute transport in a two-dimensional formulation is posed and numerically solved. It is considered that the medium has a fractal structure. The fractional order piezoconductivity equation based on anomalous Darcy's law, porosity and fluid density equations of state and continuity equation are proposed. The initial and boundary value problem for the system of equations consisting of the balance equation expressed in relation to the concentration of solids in suspension, fractional order piezoconductivity equation and Darcy's anomalous law is solved by finite difference method.

A number of works are devoted to the problem of modeling and anomalous diffusion in media with a fractal structure, as well as to the numerical analysis of equations with fractional derivatives [8, 9, 10, 11, 12, 13, 14].

2. Statement and numerical solution of the problem

Thus, the study area consists of two zones, one of which is high permeable. (zone $R_1\{0 \leq x < \infty, 0 \leq y \leq l\}$), and the other one is low-permeable ($R_2\{0 \leq x < \infty, l \leq y \leq \infty\}$)(Fig.1). Initially the area R_1 and R_2 filled with liquid without substance.

As R_1 the suspended particles move deeper into the region, they are deposited (colmatation), they are partially separated from the trapped (precipitated) state and further transferred to other pores (suffusion). Convective-diffusion, colmotation-suffusion solute transport also occurs in the region . R_2 Considering the low permeability of this area, we assume that $v_{2x} < v_{1x}, v_{2y} < v_{1y}, D_{2x} < D_{1x}, D_{2y} < D_{1y}$, where $v_{rx}, v_{ry}(r = 1, 2)-$ filtration rate components in R_1 and R_2, D_{rx}, D_{ry} are the longitudinal and transverse diffusion coefficients in R_1 and R_2 (here and below $r = 1$ corresponds to $R_1, r = 2 - R_2$). In view of the above, the process of anomalous solute transport into R_1 and R_2 can be described by the equations [15]

$$\varepsilon_{0r}\frac{\partial c_r}{\partial t} = \varepsilon_{0r}\left\{D_{rx}\frac{\partial^{\beta_{rx}} c_r}{\partial x^{\beta_{rx}}} + D_{ry}\frac{\partial^{\beta_{ry}} c_r}{\partial y^{\beta_{ry}}}\right\} - \frac{\partial\left(v_{rx}c_r\right)}{\partial x} - \frac{\partial\left(v_{ry}c_r\right)}{\partial y} + \frac{\partial \varepsilon_r}{\partial t},\ r = 1, 2 \tag{1}$$

$$\frac{\partial \varepsilon_r}{\partial t} = \omega_1\left(\varepsilon_{0r} - \varepsilon_r\right)\left|\nabla p_r\right| - \omega_2 \varepsilon_r c_r,\ \ r = 1, 2 \tag{2}$$

where C_r are the volumetric concentrations of solid particles in the liquid, $\varepsilon_{0r}, \varepsilon_r$ are the initial and current porosities, ω_1 ,ω_2 are the coefficients characterizing the intensity of suffusion and colmotation of pores, $|\nabla p_r|$ are the modulus of the pressure gradient $p_r, \beta_{rx}, \beta_{ry}$ and are the indices of the derivative.

Filtration velocity components in R_1 and R_2 are defined as [16, 17]

$$v_{rx} = -\frac{k_{rx}\left(\varepsilon_r\right)}{\mu}\frac{\partial^{\gamma_1} p_r}{\partial x^{\gamma_1}}, \quad v_{ry} = -\frac{k_{ry}\left(\varepsilon_r\right)}{\mu}\frac{\partial^{\gamma_2} p_r}{\partial y^{\gamma_2}}, \quad r = 1, 2 \tag{3}$$

where μ is liquid viscosity, $k_r\left(\varepsilon_r\right)$ are the permeability coefficients of the areas R_1 and, which are functions of R_2, γ_{rx} due to colmotation-suffosion effects, $\varepsilon_r, \gamma_{ry}$ are the indicators of the derivative.

The pressure gradient modulus is determined from the following relations

$$\left|\nabla p_r\right| = \sqrt{\left(\frac{\partial p_r}{\partial x}\right)^2 + \left(\frac{\partial p_r}{\partial y}\right)^2}, \quad r = 1, 2 \tag{4}$$

The continuity equation in R_1 is taken in the form

$$\frac{\partial\left(\rho v_{1x}\right)}{\partial x} + \frac{\partial\left(\rho v_{1y}\right)}{\partial y} = \frac{\partial\left(\rho \varepsilon_1\right)}{\partial t} \tag{5}$$

where ρ is the density of the liquid.

We use the equations of state of an elastic fluid and an elastic porous medium [17]

$$\rho = \rho_0\left(1 + \beta_l\left(p_1 - p_0\right)\right), \tag{6}$$

$$\varepsilon_1 = \varepsilon_{01} + \beta_{m1}\left(p_1 - p_0\right), \tag{7}$$

where β_l is the coefficient of volumetric compression of the liquid, β_{m1} is the elasticity coefficient of the area R_1, ρ_0 is the initial density of the liquid, p_0 is the initial pressure.

From (6) and (7) we have

$$\rho\varepsilon_1 = \rho_0\varepsilon_{01} + \left(\rho_0\varepsilon_{01}\beta_l + \rho_0\beta_{m1}\right)\left(p_1 - p_0\right) + \rho_0\beta_l\beta_{m1}\left(p_1 - p_0\right)^2.$$

The last term on the right side of this relation can be neglected due to its smallness in comparison with the others. Then it turns out

$$\rho\varepsilon_1 = \rho_0\left(\varepsilon_{01} + \beta_1^*\left(p_1 - p_0\right)\right), \tag{8}$$

where $\beta_1^* = \varepsilon_{01}\beta_l + \beta_{m1}$ is the coefficient of elasticity of the medium.

For the right side of (5), using the generally accepted hypotheses of the elastic filtration regime, we obtain [18]

$$\frac{\partial(\rho\varepsilon_1)}{\partial t} = \rho_0\beta_1^*\frac{\partial p_1}{\partial t}. \tag{9}$$

Using relations (3), (6) and (7) for the simplest case $k_r(\varepsilon_r) = k_{0r}\varepsilon_r, (k_{0r} = \text{const})$ and, taking into account (8), the first term on the left side of (5) is transformed as follows

$$\begin{gathered}\frac{\partial(\rho v_{1x})}{\partial x} = -\frac{k_{1x}}{\mu}\frac{\partial}{\partial x}\left[\rho\varepsilon_1\frac{\partial^{\gamma_1}p_1}{\partial x^{\gamma_1}}\right] = -\frac{k_{1x}}{\mu}\left(\rho\varepsilon_1\frac{\partial^{\gamma_1+1}p_1}{\partial x^{\gamma_1+1}} + \frac{\partial(\rho\varepsilon_1)}{\partial x}\frac{\partial^{\gamma_1}p_1}{\partial x^{\gamma_1}}\right) = \\ = -\frac{k_{1x}}{\mu}\rho_0\left(\left(\varepsilon_{01} + \beta_1^*(p_1 - p_0)\right)\frac{\partial^{\gamma_1+1}p_1}{\partial x^{\gamma_1+1}} + \beta_1^*\left(\frac{\partial^{\gamma_1}p_1}{\partial x^{\gamma_1}}\right)^2\right)\end{gathered} \tag{10}$$

The expression for the second term on the left side of (5) is similarly obtained.

Substituting (9), (10) into (5), we obtain the piezoconductivity equations in the form

$$\begin{gathered}\frac{\partial p_1}{\partial t} = \chi_{1x}\left(\varepsilon_{01} + \beta_1^*(p_1 - p_0)\right)\frac{\partial^{\gamma_1+1}p_1}{\partial x^{\gamma_1+1}} + \chi_{1y}\left(\varepsilon_{01} + \beta_1^*(p_1 - p_0)\right)\frac{\partial^{\gamma_2+1}p_1}{\partial y^{\gamma_2+1}} + \\ +\beta_1^*\left(\left(\frac{\partial^{\gamma_1}p_1}{\partial x^{\gamma_1}}\right)^2 + \left(\frac{\partial^{\gamma_2}p_1}{\partial y^{\gamma_2}}\right)^2\right),\end{gathered}$$

where $\chi_{1x} = \frac{k_{1x}}{\mu\beta_1^*}$, $\chi_{1y} = \frac{k_{1y}}{\mu\beta_1^*}$, is coefficients of piezoconductivity, β_1^* is elasticity coefficient of the medium.

As in the theory of elastic regime [17], in the last equation, quadratic terms can be discarded as the second order of smallness, which gives

$$\begin{gathered}\frac{\partial p_1}{\partial t} = \chi_{1x}^*(p_1)\frac{\partial^{\gamma_1+1}p_1}{\partial x^{\gamma_1+1}} + \chi_{1y}^*(p_1)\frac{\partial^{\gamma_2+1}p_1}{\partial y^{\gamma_2+1}}, \chi_{1x}^*(p_1) = \chi_{1x}\left(\varepsilon_{01} + \beta_1^*(p_1 - p_0)\right), \\ \chi_{1y}^*(p_1) = \chi_{1y}\left(\varepsilon_{01} + \beta_1^*(p_1 - p_0)\right).\end{gathered}$$

To determine the pressure in the region, R_2 the piezoconductivity equation is obtained in a similar way.

Thus, we obtain a system of equations for the colmatation-suffusion solute transport in the zones R_1 and R_2, consisting of balance equations (1), kinetics equations (2), Darcy laws (3) and equations piezoconductivity

$$\begin{gathered}\varepsilon_{0r}\frac{\partial c_r}{\partial t} = \varepsilon_{0r}\left\{D_{rx}\frac{\partial^{\beta_{rx}}c_r}{\partial x^{\beta_{rx}}} + D_{ry}\frac{\partial^{\beta_{ry}}c_r}{\partial y^{\beta_{ry}}}\right\} - \frac{\partial(v_{rx}c_r)}{\partial x} - \frac{\partial(v_{ry}c_r)}{\partial y} + \frac{\partial\varepsilon_r}{\partial t}, \\ \frac{\partial\varepsilon_r}{\partial t} = \omega_1(\varepsilon_{0r} - \varepsilon_r)|\nabla p_r| - \omega_2\varepsilon_r c_r, \\ v_{rx} = -\frac{k_{rx}(\varepsilon_r)}{\mu}\frac{\partial^{\gamma_{rx}}p_r}{\partial x^{\gamma_{rx}}}, \quad v_{ry} = -\frac{k_{ry}(\varepsilon_r)}{\mu}\frac{\partial^{\gamma_{ry}}}{\partial y^{\gamma_{ry}}}, \\ \frac{\partial p_r}{\partial t} = \chi_{rx}^*(p_r)\frac{\partial^{\gamma_{rx}+1}p_r}{\partial x^{\gamma_{rx}+1}} + \chi_{ry}^*(p_r)\frac{\partial^{\gamma_{ry}+1}p_r}{\partial y^{\gamma_{ry}+1}}, \chi_{rx}^*(p_r) = \chi_{rx}\left(\varepsilon_{0r} + \beta_r^*(p_r - p_0)\right), \\ \chi_{ry}^*(p_r) = \chi_{ry}\left(\varepsilon_{0r} + \beta_r^*(p_r - p_0)\right), r = 1, 2.\end{gathered} \tag{11}$$

It is through the common boundary of the regions R_1 that R_2 the mass of solute (particles) is transferred. At R_1 when $x \to \infty$ and at R_2 when $y \to \infty$ we consider that there are no substance flows. This essentially means that such an initial stage of particle propagation in the reservoir is considered, when the concentration profiles do not yet reach the boundary $x \to \infty$ at R_1 and $y \to \infty$ in R_2. On the common border of the regions R_1 and R_2, i.e. on $y = 0$, we require the continuity of changes in the concentration and flux of the substance. Under the formulated conditions, the initial and boundary conditions of the problem have the form

$$c_1(0, x, y) = 0, c_2(0, x, y) = 0, \tag{12}$$

$$c_1(t, 0, y) = c_0, c_0 = \text{ const}, y = l/2, \tag{13}$$

$$\frac{\partial c_1}{\partial y}(t,x,0)=0,\quad 0\le x<\infty, \tag{14}$$

$$\frac{\partial c_1}{\partial x}(t,\infty,y)=0,\quad 0\le y<\infty, \tag{15}$$

$$\frac{\partial c_2}{\partial x}(t,0,y)=0,-\infty<y<0, \tag{16}$$

$$\frac{\partial c_2}{\partial x}(t,x,\infty)=0,\quad 0\le x<\infty, \tag{17}$$

$$\frac{\partial c_2}{\partial x}(t,\infty,y)=0,\quad -\infty<y<0, \tag{18}$$

$$c_1(t,x,0)=c_2(t,x,0),\quad 0\le x<\infty, \tag{19}$$

$$\left(c_1v_{1y}+D_{1y}\varepsilon_{01}\frac{\partial^{\beta_{1y}-1}c_1}{\partial y^{\beta_{1y}-1}}\right)\Big|_{y=l-0}=\left(c_2v_{2y}+D_{2y}\varepsilon_{02}\frac{\partial^{\beta_{2y}-1}c_2}{\partial y^{\beta_{2y}-1}}\right)_{y=l+0},0\le x<\infty, \tag{20}$$

$$\varepsilon_1(0,x,y)=\varepsilon_{01},\quad \varepsilon_2(0,x,y)=\varepsilon_{02}, \tag{21}$$

$$p_1(0,x,y)=p_0,\quad p_2(0,x,y)=p_0,\quad p_0=\text{ const }, \tag{22}$$

$$p_1(t,0,y)=p_c,p_c>p_0,\quad p_c=\text{ const },y=l/2, \tag{23}$$

$$\frac{\partial p_1}{\partial y}(t,x,h)=0,\quad 0\le x<\infty, \tag{24}$$

$$\frac{\partial p_1}{\partial y}(t,\infty,y)=0,\quad 0<y\le h\quad , \tag{25}$$

$$\frac{\partial p_2}{\partial x}(t,0,y)=0,-\infty<y<0, \tag{26}$$

$$\frac{\partial p_2}{\partial x}(t,x,\infty)=0,\quad 0\le x<\infty, \tag{27}$$

$$\frac{\partial p_2}{\partial y}(t,\infty,y)=0,\quad -\infty<y<0, \tag{28}$$

$$k_{1y}\frac{\partial p_1}{\partial y}(t,x,0)=k_{2y}\frac{\partial p_2}{\partial y}(t,x,0),\quad 0\le x<\infty, \tag{29}$$

$$p_1(t,x,0)=p_2(t,x,0),\quad 0\le x<\infty. \tag{30}$$

Problem (11)-(30) is solved by the finite difference method. To accomplish this, we construct a grid in the $R_1\cup R_2$ area $\omega_{h_1h_2\tau}=\omega^1_{h_1h_2\tau}\cup\omega^2_{h_1h_2\tau}$

where $\omega^1_{h_1h_2\tau}=$
$\left\{(t_k,x_i,y_j),t_k=\tau k,x_i=ih_1,y_j=jh_2,k=\overline{0,K},i=0,1,\ldots,j=0,1,\ldots,J,\tau=T/K,\right\}$
$\omega^2_{h_1h_2\tau}=$
$\left\{(t_k,x_i,y_j),t_k=\tau k,x_i=ih_1,y_j=h_2j,k=\overline{0,K},i=0,1,\ldots,j=J,J+1,\ldots,\tau=T/K\right\}$.

In this grid: h_1 - grid step in direction X, h_2 - grid step in direction y in R_1 and R_2, τ - grid step in time, T maximum time during which the process is being studied, K - number of grid intervals in t, J - number of grid intervals in y in R_1.

Instead of the functions $c(t,x,y)$, $v(t,x,y)$ and $p(t,x,y)$, we will consider network functions whose values at the nodes (t_k,x_i,y_j), respectively, will be denoted by c^k_{ij}, v^k_{ij} and p^k_{ij}.

Consider the $0<\gamma_1,\gamma_2\le 1$ case $1<\beta_1,\beta_2,\beta_3\le 2$.

On the grid, $\omega_{h_1h_2\tau}$ we approximate the first equation of system (1)-(2) as follows [12]

$$\frac{c^{k+1}_{(1)ij}-c^{k}_{(1)ij}}{\tau} =$$
$$= \frac{D_{1x}}{\Gamma(3-\beta_{1x})h_1^{\beta_{1x}}}\sum_{l=0}^{i-1}\left(c^k_{(1)i-(l-1),j}-2c^k_{(1)i-l,j}+c^k_{(1)i-(l+1),j}\right)\left((l+1)^{2-\beta_{1x}}-l^{2-\beta_{1x}}\right)+$$
$$+\frac{D_{1y}}{\Gamma(3-\beta_{1y})h_2^{\beta_{1y}}}\sum_{l=0}^{j-1}\left(c^k_{(1)i,j-(l-1)}-2c^k_{(1)i,j-l}+c^k_{(1)i,j-(l+1)}\right)\left((l+1)^{2-\beta_{1y}}-l^{2-\beta_{1y}}\right) \quad (31)$$
$$-\frac{1}{\varepsilon_{01}}\frac{(v_{1x})^k_{ij}c^k_{(1)ij}-(v_{1x})^k_{i-1,j}c^k_{(1)i-1,j}}{h_1}-\frac{1}{\varepsilon_{01}}\frac{(v_{1y})^k_{ij}c^k_{(1)ij}-(v_{1y})^k_{i,j-1}c^k_{(1)i,j-1}}{h_2}+$$
$$+\frac{1}{\varepsilon_{01}}\frac{\varepsilon^{k+1}_{(1)ij}-\varepsilon^k_{(1)ij}}{\tau}$$
$$i=\overline{1,I-1}, j=\overline{1,J-1}, k=\overline{0,K-1},$$

$$\frac{C^{k+1}_{(2)ij}-C^{k}_{(2)ij}}{\tau} =$$
$$= \frac{D_{2x}}{\Gamma(3-\beta_{2x})h_1^{\beta_{2x}}}\sum_{l=0}^{i-1}\left(c^k_{(2)i-(l-1),j}-2c^k_{(2)i-l,j}+c^k_{(2)i-(l+1),j}\right)\left((l+1)^{2-\beta_{2x}}-l^{2-\beta_{2x}}\right)+$$
$$+\frac{D_{2y}}{\Gamma(3-\beta_{2y})h_2^{\beta_{2y}}}\sum_{l=0}^{j-1}\left(c^k_{(2)i,j-(l-1)}-2c^k_{(2)i,j-l}+c^k_{(2)i,j-(l+1)}\right)\left((l+1)^{2-\beta_{2y}}-l^{2-\beta_{2y}}\right)- \quad (32)$$
$$-\frac{1}{\varepsilon_{02}}\frac{(v_{2x})^k_{ij}c^k_{(2)ij}-(v_{2x})^k_{i-1,j}c^k_{(2)i-1,j}}{h_1}-\frac{1}{\varepsilon_{02}}\frac{(v_{2y})^k_{ij}c^k_{(2)ij}-(v_{2y})^k_{i,j-1}c^k_{(2)i,j-1}}{h_2}+$$
$$+\frac{1}{\varepsilon_{02}}\frac{\varepsilon^{k+1}_{(2)ij}-\varepsilon^k_{(2)ij}}{\tau}$$
$$i=\overline{0,I}, j=\overline{J,J^1-1}, k=\overline{0,K-1},,$$

$$\frac{\varepsilon^{k+1}_{(1)ij}-\varepsilon^k_{(1)ij}}{\tau}=\omega_1\left(\varepsilon_{01}-\varepsilon^k_{(1)ij}\right)\nabla p\Big|^k_{(1)ij}-\omega_2\varepsilon^k_{(1)ij}c^k_{(1)ij}, \quad (33)$$
$$i=\overline{1,I-1}, j=\overline{1,J-1}, k=\overline{0,K-1},$$

$$\frac{\varepsilon^{k+1}_{(2)ij}-\varepsilon^k_{(2)ij}}{\tau}=\omega_1\left(\varepsilon_{02}-\varepsilon^k_{(2)ij}\right)\nabla p|^k_{(2)ij}-\omega_2\varepsilon^k_{(2)ij}c^k_{(2)ij}, \quad (34)$$
$$i=\overline{0,I}, \quad j=\overline{J,J^1-1}, k=\overline{0,K-1},$$

For the filtration velocity component, we use the following schemes

$$(v_{1x})^{k+1}_{ij}=-\frac{k}{\mu}\frac{p^{k+1}_{(1)i+1j}-\gamma_{1x}p^{k+1}_{(1)i,j}}{\Gamma(2-\gamma_{1x})h_1^{\gamma_{1x}}}, i=\overline{0,I-1}, j=\overline{0,J}, k=\overline{0,K-1}, \quad (35)$$

$$(v_{1y})^{k+1}_{ij}=-\frac{k}{\mu}\frac{p^{k+1}_{(1)ij+1}-\gamma_{1y}p^{k+1}_{(1)i,j}}{\Gamma(2-\gamma_{1y})h_2^{\gamma_{1y}}}, i=\overline{0,I}, j=\overline{0,J-1}, k=\overline{0,K-1}, \quad (36)$$

$$(v_{2x})^{k+1}_{ij}=-\frac{k}{\mu}\frac{p^{k+1}_{(2)i+1j}-\gamma_{2x}p^{k+1}_{(2)i,j}}{\Gamma(2-\gamma_{2x})h_1^{\gamma_{2x}}}, i=\overline{0,I}, j=\overline{J,J^1-1}, k=\overline{0,K-1}, \quad (37)$$

$$(v_{2y})^{k+1}_{ij}=-\frac{k}{\mu}\frac{p^{k+1}_{(2)ij+1}-\gamma_{2y}p^{k+1}_{(2)i,j}}{\Gamma(2-\gamma_{2y})h_2^{\gamma_{2y}}}, i=\overline{0,I}, j=\overline{J,J^1-1}, k=\overline{0,K-1}. \quad (38)$$

Equation (4) is approximated as

$$\frac{p^{k+1}_{(1)ij}-p^{k}_{(1)ij}}{\tau} =$$
$$= \frac{\chi^*_{1x}\left(p_{(1)}\right)}{\Gamma(3-\gamma_{1x})h_1^{\gamma_{1x}}}\sum_{l=0}^{i-1}\left(p^k_{(1)i-(l-1),j}-2p^k_{(1)i-l,j}+p^k_{(1)i-(l+1),j}\right)\left((l+1)^{2-\gamma_{1x}}-l^{2-\gamma_{1x}}\right)+ \quad (39)$$
$$+\frac{\chi^*_{1y}\left(p_{(1)}\right)}{\Gamma(3-\gamma_{1y})h_2^{\gamma_{1y}}}\sum_{l=0}^{j-1}\left(p^k_{(1)i,j-(l-1)}-2p^k_{(1)i,j-l}+p^k_{(1)i,j-(l+1)}\right)\left((l+1)^{2-\gamma_{1y}}-l^{2-\gamma_{1y}}\right),$$

$$i=\overline{1,I-1},\quad j=\overline{1,J-1},\quad k=\overline{0,K-1},$$

$$\begin{array}{l}\frac{p^{k+1}_{(2)ij}-p^{k}_{(2)ij}}{\tau}=\\ =\frac{\chi^{*}_{2x}\left(p_{(2)}\right)}{\Gamma(3-\gamma_{2x})h_1^{\gamma_{2x}}}\sum_{l=0}^{i-1}\left(p^{k}_{(2)i-(l-1),j}-2p^{k}_{(2)i-l,j}+p^{k}_{(2)i-(l+1),j}\right)\left((l+1)^{2-\gamma_{2x}}-l^{2-\gamma_{2x}}\right)+\\ +\frac{\chi^{*}_{2y}\left(p_{(2)}\right)}{\Gamma(3-\gamma_{2y})h_2^{\gamma_{2y}}}\sum_{l=0}^{j-1}\left(p^{k}_{(2)i,j-(l-1)}-2p^{k}_{(2)i,j-l}+p^{k}_{(2)i,j-(l+1)}\right)\left((l+1)^{2-\gamma_{2y}}-l^{2-\gamma_{2y}}\right),\end{array}\tag{40}$$

$$i=\overline{0,I},j=\overline{J,J^1-1},k=\overline{0,K-1},$$

The initial and boundary conditions are approximated as

$$c^{k}_{(1)i,j}=0,i=\overline{0,I},\quad j=\overline{0,J},\quad k=0,,\tag{41}$$

$$c^{k}_{(1)i,j}=0,i=\overline{0,I},\quad j=\bar{J}^1,\quad k=0,,\tag{42}$$

$$c^{k}_{(1)i,j}=c_0,\quad i=0,j=J/2,k=\overline{0,K},\tag{43}$$

$$\frac{c^{k}_{(1)i,j+1}-c^{k}_{(1)i,j}}{h_2}=0,i=\overline{0,I},j=0,k=\overline{0,K},\tag{44}$$

$$\frac{c^{k}_{(1)i+1,j}-c^{k}_{(1)i,j}}{h_1}=0,i=0,j\neq J/2,\quad j=\overline{0,J},k=\overline{0,K},\tag{45}$$

$$fracc^{k}_{(2)i,j}-c^{k}_{(2)i-1,j}h_1=0,i=I,\quad j=\overline{0,J^1},k=\overline{0,K},\tag{46}$$

$$\frac{c^{k}_{(2)i,j}-c^{k}_{(2)i,j-1}}{h_2}=0,i=\overline{0,I},j=J^1,k=\overline{0,K},\tag{47}$$

$$c^{k}_{(1)i,J+0}=c^{k}_{(2)i,J-0},\quad i=0,1,\ldots,\quad k=\overline{0,K},\tag{48}$$

$$\begin{array}{l}c^{k}_{(1)iJ}v^{k}_{(1y)iJ}+D_{1y}\frac{c^{k}_{i,J}-(\beta_{1y}-1)c^{k}_{i,J-1}}{\Gamma(2-(\beta_{1y}-1))h_2^{\beta_{1y}-1}}=c^{k}_{(2)iJ}v^{k}_{(2y)iJ}+D_{2y}\frac{c^{k}_{i,J+1}-(\beta_{2y}-1)c^{k}_{i,J}}{\Gamma(2-(\beta_{2y}-1))h_2^{\beta_{2y}-1}},\\ i=0,1,\ldots,\quad k=\overline{0,K},\end{array}\tag{49}$$

$$\varepsilon^{k}_{(1)i,j}=\varepsilon_{01},\varepsilon^{k}_{(2)i,j}=\varepsilon_{02},i=0,j=J/2,k=\overline{0,K},,\tag{50}$$

$$p^{k}_{(1)i,j}=p_0=\text{const},p^{k}_{(2)i,j}=p_0=\text{const}i=\overline{0,I},j=\overline{0,J},k=0,\tag{51}$$

$$p^{k}_{(1)i,j}=p_c,\quad i=0,j=J/2,k=\overline{0,K},\tag{52}$$

$$\frac{p^{k}_{(1)i,j+1}-p^{k}_{(1)i,j}}{h_2}=0,i=\overline{0,I},j=0,k=\overline{0,K},\tag{53}$$

$$\frac{p^{k}_{(1)i,j}-p^{k}_{(1)i,j-1}}{h_2}=0,i=\overline{0,I},j=J,k=\overline{0,K},,\tag{54}$$

$$\frac{p^{k}_{(2)i+1,j}-p^{k}_{(2)i,j}}{h_1}=0,i=0,j\neq J/2,j=\overline{0,J},k=\overline{0,K},\tag{55}$$

$$\frac{p^{k}_{(2)i,j}-p^{k}_{(2)i-1,j}}{h_1}=0,i=I,\quad j=\overline{0,J},k=\overline{0,K},,\tag{56}$$

$$k_{1y}\frac{p^{k}_{(1)i,j}-p^{k}_{(1)i,j-1}}{h_2}=k_{2y}\frac{p^{k}_{(2)i,j}-p^{k}_{(2)i,j-1}}{h_2},i=\overline{0,I},j=0,k=\overline{0,K},,\tag{57}$$

$$p^{k}_{(1)i,J+0}=p^{k}_{(2)i,J-0},\quad i=0,1,\ldots,\quad k=\overline{0,K},\tag{58}$$

where I,J^1 is a sufficiently large number for which the equation is approximately satisfied $c^{k}_{IJ^1}=0$.

The sequence of calculations is as follows: first, p are determined from the difference scheme (39), (40) on the ($k+1$) - layer, then from (35)-(38) the components of the filtration rate are

calculated, after that $\boldsymbol{C}$ is determined on the ($k+1$) - layer from the difference equations (31), (32).

The following values of the initial parameters were used in the calculations: $k_{1x} = 3 \cdot 10^{-13}$ m$^{1+\gamma}$, $k_{1y} = 2 \cdot 10^{-13}$ m$^{1+\gamma}$, $k_{2x} = 1.5 \cdot 10^{-13}$ m$^{1+\gamma}$, $k_{2y} = 10^{-13}$ m$^{1+\gamma}$, $\mu = 5 \cdot 10^{-3}$ Pa $\cdot$ s, $\beta_1^* = 3 \cdot 10^{-8}$ Pa^{-1}, $\beta_2^* = 10^{-8}$ Pa^{-1}, $p_c = 5 \cdot 10^5$ Pa, $p_0 = 10^5$ Pa, $\varepsilon_{01} = 0,2$, $\varepsilon_{02} = 0,1$, $c_0 = 0,01$, $\omega_1 = 10^{-6}$ m/(Pa $\cdot$ s), $\omega_2 = 0,1$ s^{-1}, and various values of $D_{1x} = 5 \cdot 10^{-5}$ m$^{1+\beta}$/s, $D_{1y} = 3 \cdot 10^{-5}$ m$^{1+\beta}$/s, $D_{2x} = 2 \cdot 10^{-5}$ m$^{1+\beta}$/s, $D_{2y} = 10^{-5}$ m$^{1+\beta}$/s.

Based on numerical results, profiles of concentration, pressure gradient, porosity and filtration rate were determined. On fig. 1-3. Pressure profiles and filtration velocities were determined. The

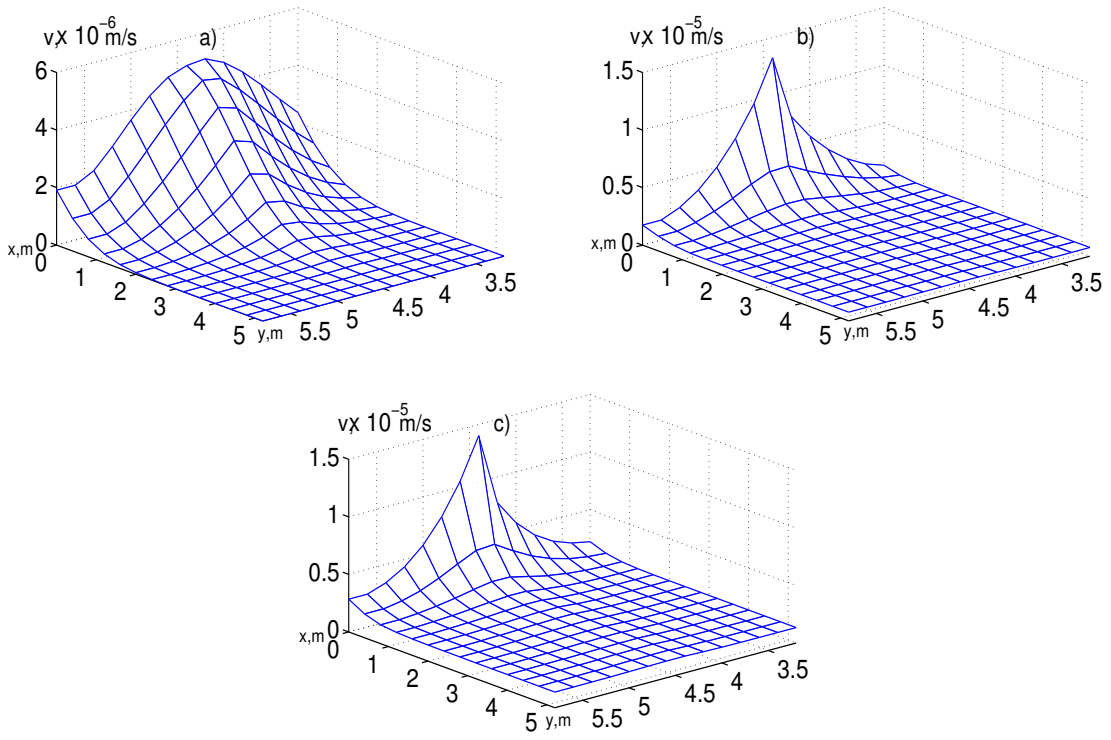


Fig.1. Surfaces of v (1-zone) at: $t = 2000$ s,$\beta_{1x} = \beta_{1y} = \beta_{2x} = \beta_{2y} = 1$;
$\gamma_{1x} = \gamma_{1y} = \gamma_{2x} = \gamma_{2y} = 1(a), \gamma_{1x} = \gamma_{1y} = \gamma_{2x} = \gamma_{2y} = 0,9(b), \gamma_{1x} = \gamma_{1y} = \gamma_{2x} = \gamma_{2y} = 0,8(c)$.

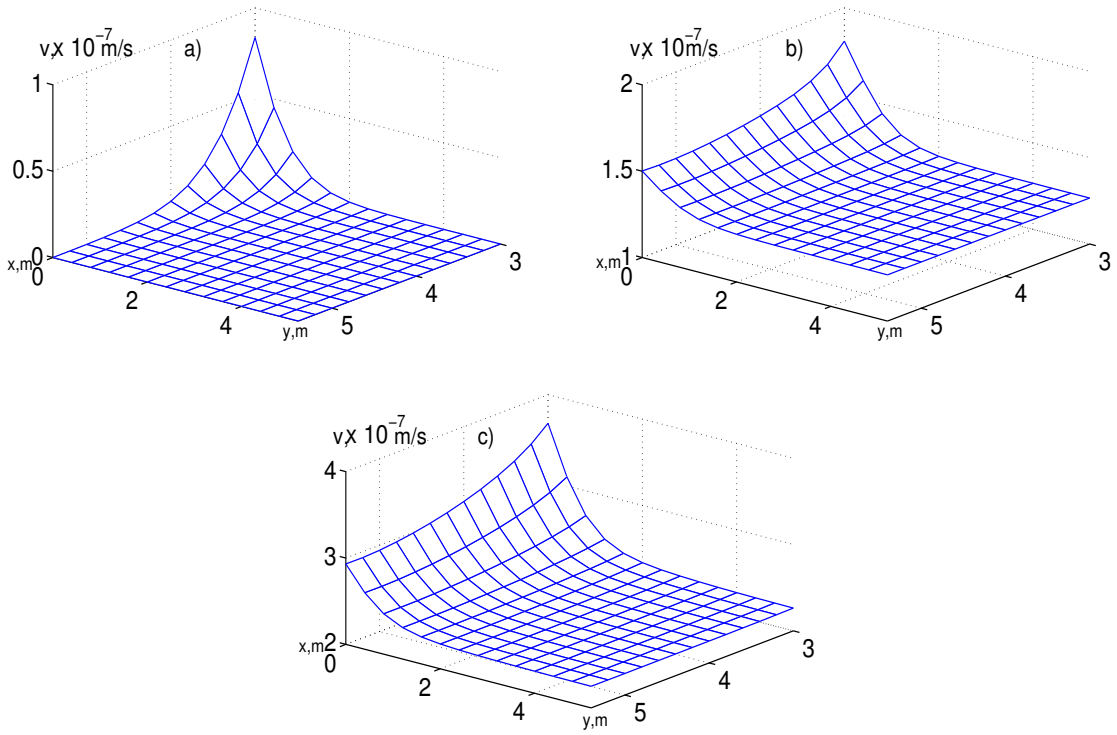


Fig.2. Surfaces of v (2-zone) at: $t = 2000\ (s)$,$\beta_{1x} = \beta_{1y} = \beta_{2x} = \beta_{2y} = 1$;
$\gamma_{1x} = \gamma_{1y} = \gamma_{2x} = \gamma_{2y} = 1(a), \gamma_{1x} = \gamma_{1y} = \gamma_{2x} = \gamma_{2y} = 0,9(b), \gamma_{1x} = \gamma_{1y} = \gamma_{2x} = \gamma_{2y} = 0,8(c)$.

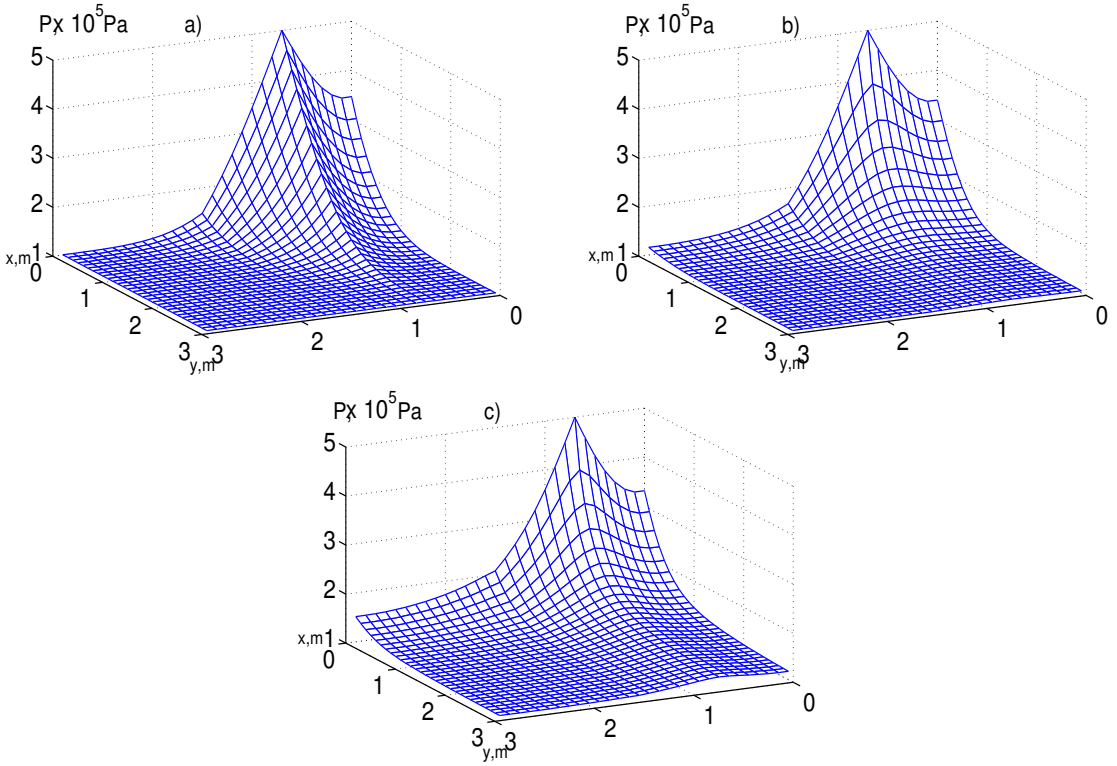


Fig.3. Surfaces of p at: $t = 2000\ (s)$,$\beta_{1x} = \beta_{1y} = \beta_{2x} = \beta_{2y} = 1$;
$\gamma_{1x} = \gamma_{1y} = \gamma_{2x} = \gamma_{2y} = 1(a), \gamma_{1x} = \gamma_{1y} = \gamma_{2x} = \gamma_{2y} = 0,9(b), \gamma_{1x} = \gamma_{1y} = \gamma_{2x} = \gamma_{2y} = 0,8(c)$.

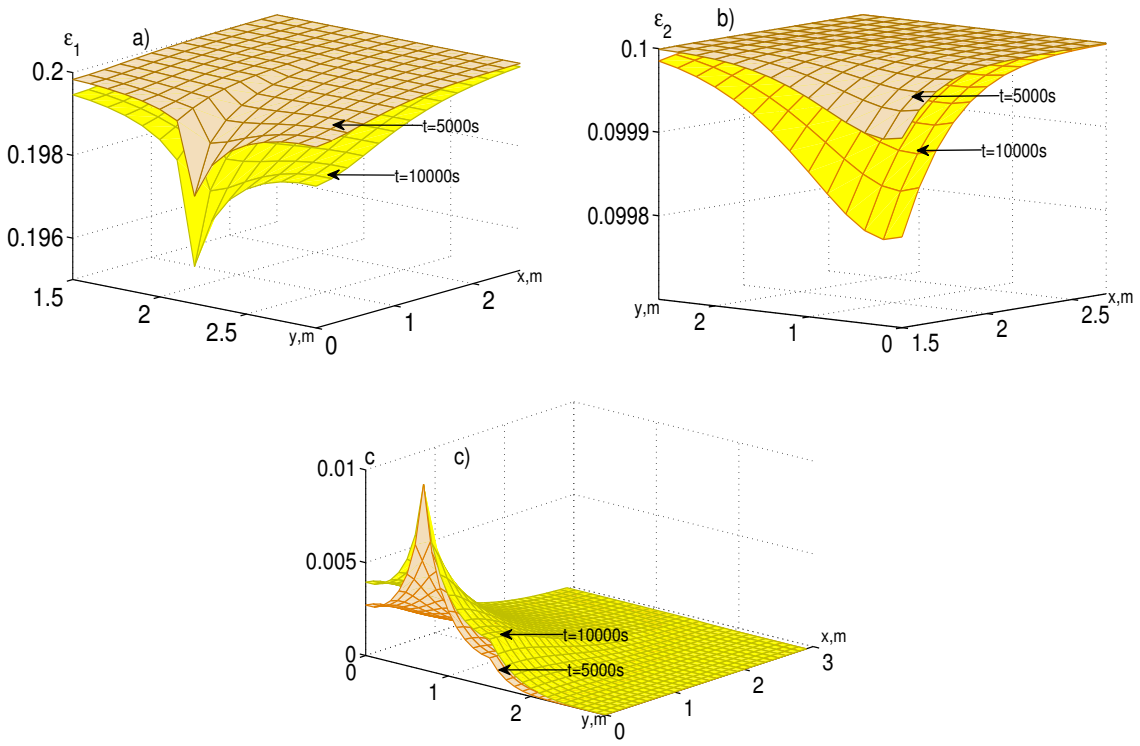


Fig.4. Surfaces of $\varepsilon_1(a), \varepsilon_2(b)$ and $c(c)$ at $\beta_{1x} = \beta_{1y} = \beta_{2x} = \beta_{2y} = 2; \gamma_{1x} = \gamma_{1y} = \gamma_{2x} = \gamma_{2y} = 1$

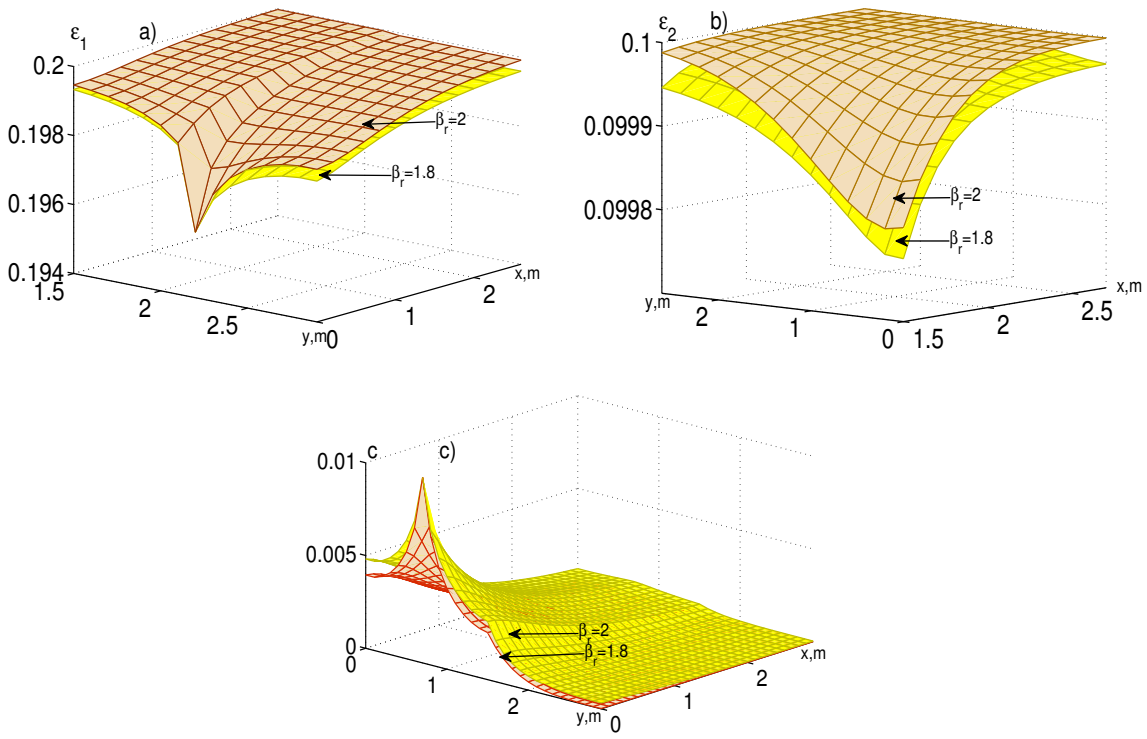


Fig 5. Surfaces of $\varepsilon_1(a), \varepsilon_2(b)$ and $c(c)$ at: $t = 10000s, \gamma_{1x} = \gamma_{1y} = \gamma_{2x} = \gamma_{2y} = 1$

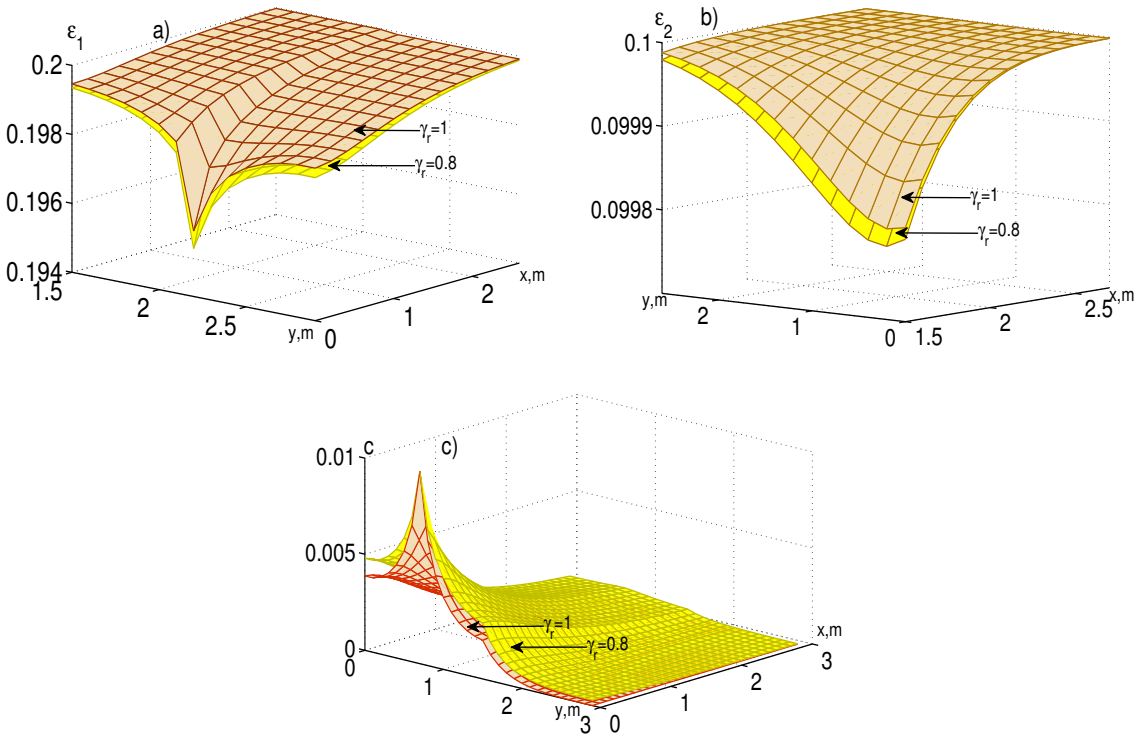


Fig 6. Surfaces of $\varepsilon_1(a), \varepsilon_2(b)$ and $c(c)$ at: $t = 10000$ s, $\beta_{1x} = \beta_{1y} = \beta_{2x} = \beta_{2y} = 2$

obtained results show that decreasing the value of γ from 1 leads to an increase in the pressure influence of the filtration rate in both zones.On fig. 4 shows the porosity and concentration fields at different times. On fig. 5 shows concentration field and porosity at various values of β. As can be seen, if the reductions β in the order of the derivative from 2, the solid particles spread over a wider distance in both zones. On fig. 6 shows the concentration and porosity fields for various values of γ. When decreasing γ from unity, a wider distribution of the concentration field in both zones is observed, which, in turn, leads to "fast diffusion".

3. Conclusion

The problem of colmatation-suffosion filtration in a porous medium consisting of zones with mobile and slow-mobility (or immobile) liquid has been posed and numerically solved. It has been found that convective-diffusion solute transport occurs at the boundary of zones with mobile and slow-mobility liquid, which strongly affects the profiles of substance concentration, porosity, and pressure gradient in the zone with mobile liquid.Based on the numerical results, the concentration of suspended particles, the porosity of the medium, the filtration rate and pressure fields are determined. The influence of orders of the fractional derivative and model parameters on the filtration characteristics of the medium is analyzed.

References

[1] Piquemal J. On the modeling of miscible displacements in porous media with stagnant fluid. *Transport Porous Media* 8. No. 3, 243-262 (1992).

[2] Khuzhayorov B.Kh., Auriault J.L., Royer P. Macroscopic modeling of solute transport in porous media saturated by two immiscible liquids: *Proc. 2nd Int. Biot Conf. on Poromechanics.* 2002, Grenoble, France, 26-28.08.02, pp. 473-479.

[3] Rao P.S.C., Rolston D.E., Jessup R.E., Davidson J.M. Solute transport in aggregated porous media: Theoretical and experimental evaluation.*Soil Sci. soc. Am. J.*, Vol. 44. 1980b ._ P.p._ 1139 - 1146

[4] Makhmudov J.M., Usmonov A.I., Kaytarov Z.D., Sultonov B. Numerical solution of the problem of anomalous solute transport in a two-dimensional nonhomogeneous porous media. *AIP Conference Proceedings* 2637, 040017 (2022);
[5] Makhmudov J.M., Usmonov A.I., Kuljonov J.B. Solution of the Anomalous Filtration Problem in Two-Dimensional Porous Media.*Lecture Notes in Networks and Systemsthis link is disabled*, 2023, 702 LNNS, Pp 68-80
[6] Makhmudov J.M., Usmonov A.I., Kuljanov J.B. The Problem of Filtration and Solute Transport in a Two-Zone Porous Medium. *AIP Conference Proceedingsthis link is disabled*, 2022, 2637, 040020;
[7] Khuzhayorov B.Kh, Dzhiyanov T.O, Mamatov Sh.S, Shukurov V.S. Mathematical Model of Substance Transport in Two-Zone Porous Media. *AIP Conference Proceedings*. 2637, 040011 (2022);
[8] BazzaevA.K. Local-one-dimensional scheme for the diffusion equation of fractional order with a fractional derivative in lower terms with boundary conditions of the first kind. *Vladikavkaz. math. magazine* 2014. V. 16. No. 2. S. 3-13.
[9] Bazzaev A.K., TsopanovI.D. Locally One-Dimensional Difference Schemes for a Fractional-Order Diffusion Equation with a Fractional Derivative in Lower Terms.*Sibirsk. electron. math.* Izv. 2015. Volume 12. P. 80-91.
[10] Bazzaev A.K., Shkhanukov-Lafishev M.Kh. Local-one-dimensional scheme for the fractional-order diffusion equation with boundary conditions of the third kind. *Zh. Vychisl. math. and mat. physical* 2010. V. 50. No. 7. P. 1200-1208.
[11] Beibalaev V.D. Mathematical model of heat transfer in media with a fractal structure. *Matem. modeling.* 2009. V. 21. No. 5. S. 55-62.
[12] Beibalaev V.D., Yakubov A.Z. Analysis of the Difference Scheme of an Analogue of the Wave Equation with a Fractional Differentiation Operator. *Vestn. Sam.state. tech. university Ser. Phys.-Math.* Sciences. 2014. Issue 1(34). pp. 125-133.
[13] Beibalaev V.D., Shabanova M. R. "Numerical method for solving an initial-boundary problem for a two-dimensional heat equation with fractional derivatives." *Vestn. Myself. state tech. university Ser. Phys.-Math.* Sciences. 2010. Issue 5(21). pp. 244-251.
[14] J. Makhmudov, A. Usmonov, Z.D.Kaytarov, B.Sultonov. Numerical solution of the problem of anomalous solute transport in a two-dimensional nonhomogeneous porous medium, *AIP Conference Proceedingsthis link is disabled*, 2022, 2637, 040017;
[15] Khuzhayorov B. Kh., Makhmudov Zh.M. Colmation-suffosion filtration in a porous medium with mobile and immobile fluids. *Engineering and Physical Journal*, 2007, 80(1), 46-53
[16] Belevtsov N.S. On one fractional-differential modification of the non-volatile oil model. *Mathematics and mathematical modeling.* 2020. No. 06. P. 13 - 27. DOI:10.24108/ mathm.0620.0000228 .
[17] Makhmudov J.M., Usmonov A.I., Kuljanov J.B. Problem of anomalous filtration in nonhomogeneous porous medium. *International Journal of Applied Mathematics.* Volume 36 No. 2 2023, 189-203. ISSN: 1311-1728.
[18] Barenblatt G.I., Yentov V.M., Ryzhik V.M. Theory of non-stationary filtration of liquid and gas. -M.: Nedra, 1972. 288p.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

THE METHOD OF SUMMING DIVERGENT SERIES BY REDUCING TO A REPEATED SERIES

MADINA TESHAEVA

Uzbek-Finnish Pedagogical Institute, Samarkand, Uzbekistan
e-mail: madinamath@mail.ru

Abstract. In this thesis, the problem of summing some divergent series by means of special repeated series is considered. If the sum of the series is determined differently from the traditional method, it is accepted to call it the generalized sum of the series.

Keywords: divergent series, generalized sum, repeated series, alternating series.

AMS Subject Classification: 40A30, 40B05.

There are three approaches to the definition of the concept of the sum of a series of numbers [1]: in the first case, the concept of the sum of series is equal to an infinite number, then it is considered meaningless. Supporters of the second approach believed that any sum can be equal to an infinite number, its property and the sum can be determined using arithmetic operations. In the third approach, only those that can be defined the concept of the sum of numerical series are studied, and the rest of them are not. This approach, based on the theory of limits, has proven to be very effective. This approach is still being developed today as a result of the expansion of the set class.

Consider the problem of summing divergent series by reducing to a repeated series. If the sum of the series is determined otherwise than the traditional method, it is usually called the generalized sum of the series ([3], [4]).

Definition. A series of the following form is called a double series:

$$\sum_{n,k=1}^{\infty} a_{nk},$$

here a_{nk} is consists of a sequence of double real numbers.

In the application of mathematical analysis (1), a double series is often considered as a repeated series in the following form:

$$\sum_{n=1}^{\infty}\left\{\sum_{k=1}^{\infty} a_{nk}\right\} \text{ or } \sum_{k=1}^{\infty}\left\{\sum_{n=1}^{\infty} a_{nk}\right\}.$$

In particular, it can be seen that the following repeated series with a special form are effective in summing divergent series:

$$\sum_{n=1}^{\infty}\left\{\sum_{k=n}^{\infty} a_{k}\right\} \text{ or } \sum_{n=1}^{\infty}\left\{\sum_{k=n}^{n} a_{k}\right\}.$$

It is known that divergent alternating series can be conditionally divided into the following three types:

1. Divergent series with a constant source: $\sum_{n=1}^{\infty} l\cdot(-1)^{n+1}$, $(l = const)$.

In this case the generalized sum of divergent series is equal to $\frac{l}{2}$;

2. Divergent series with a bounded source: $\sum_{n=1}^{\infty} a_n \cdot (-1)^{n+1}$, $\lim_{n\to\infty} a_n = l$, $l = const$;

3. Divergent series with an unbounded source: $\sum_{n=1}^{\infty} a_n \cdot (-1)^{n+1}$, $\lim_{n\to\infty} a_n = \infty$.

We consider the summation formulas for divergent series using special repeating series:

$$\sum_{n=1}^{\infty}(an^2+bn)a_n = a\sum_{n=1}^{\infty}\left\{(2n-1)\left(\sum_{k=n}^{\infty}a_k\right)\right\} + b\sum_{n=1}^{\infty}\left\{\sum_{k=n}^{\infty}a_k\right\}. \tag{1}$$

We solve the following examples using the formula (1):

Example 1. Find the sum of the following series:

$$\sum_{n=0}^{\infty}\frac{2n^2+3n}{\alpha^n}.$$

Solution. To find the sum of this series, we first find the sum of the following series using the formula (1):

$$\sum_{n=0}^{\infty}\frac{2n-1}{\alpha^n} = 2\sum_{n=0}^{\infty}\left\{\sum_{k=n}^{\infty}\frac{1}{\alpha^k}\right\} + \sum_{n=0}^{\infty}\frac{1}{\alpha^n} = \frac{2\alpha}{(\alpha-1)^2} + \frac{\alpha}{\alpha-1} = \frac{\alpha^2+\alpha}{(\alpha)^2}.$$

According to the formula (1), we have:

$$\sum_{n=0}^{\infty}\frac{2n^2+3n}{\alpha^n} = \frac{2}{\alpha-1}\cdot\frac{\alpha^2+\alpha}{(\alpha-1)^2} + \frac{3\alpha}{(\alpha-1)^2} = \frac{\alpha(5\alpha-1)}{(\alpha-1)^3}.$$

Theorem 1. The following equality holds:

$$\sum_{n=1}^{\infty}a_{n+1}b_n = \sum_{n=1}^{\infty}\left\{\Delta(a_n)\sum_{k=n}^{\infty}b_k\right\} + a_1\sum_{n=1}^{\infty}b_n, \tag{2}$$

here Δ is a difference operator: $\Delta(a_n) = a_{n+1} - a_n$ and $\sum_{n=1}^{\infty} b_n$ is a converging series.

Theorem 1 is proved in detail in [2].

Theorem 2. The following equality holds:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\cdot a_{n+1} = \frac{a_1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\Delta(a_n) \tag{3}$$

here $\Delta(a_n) = a_{n+1} - a_n$.

Proof. To prove this theorem, it is enough to take $b_n = (-1)^{n+1}$ in the formula (2).

Using the Theorem 2, the sum of the following series can be easily find.

Example 2. Find the sum of the following series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}(an^2+bn+c).$$

Solution. To find the sum of this series, we use the formula (3):

$$\sum_{n=1}^{\infty}(-1)^{n+1}(an^2+bn+c) = a\sum_{n=1}^{\infty}(-1)^{n+1}n^2 + b\sum_{n=1}^{\infty}(-1)^{n+1}n + c\sum_{n=1}^{\infty} = \frac{a}{2}\sum_{n=1}^{\infty}(-1)^{n+1}(2n-1)+$$

$$+b\sum_{n=1}^{\infty}(-1)^{n+1}n + \frac{c}{2} = \frac{a+b}{2}\sum_{n=1}^{\infty}(-1)^{n+1} - \frac{a}{2}\sum_{n=1}^{\infty}(-1)^{n+1} + \frac{c}{2} = \frac{a+b}{4} - \frac{a}{4} + \frac{c}{2} = \frac{b+2c}{4}.$$

Thus, the sum of the series does not depend on the parameter a.

Similarly, we can find the sum of the following lines:

Example 3.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\alpha^n=\frac{\alpha}{\alpha+1},\ \ \alpha\geq 0.$$

Example 4.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\cdot n\cdot\alpha^n=\frac{\alpha}{(\alpha+1)^2},\ \ \alpha\geq 0.$$

References

[1] Alimov Sh.O., Ashurov R.R., *Mathematik tahlil*, Kamalak, Tashkent, 2012, pp.51-56.

[2] Ponomarenko A.N. *Metod summirovaniya rasxodyashixsya ryadov putem svedeniya k povtornomu ryadu.* Molodoy uchyoniy. 2013, vol. 12.

[3] Demidovich B.P.,*Sbornik zadach i uprajneniy po matematicheskomu analizu*, Moskovskogo Universiteta, Moscow, 1997, pp.12-26.

[4] Fixtengolts G.M. *Kurs differensialnogo i integralnogo ischisleniya*, Nauka, Moscow, 1966.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

MATHEMATICAL MODEL OF THE PROBLEM OF OPTIMIZING PRODUCTION CONSIDERING RENTAL ACREAGE BY THE CRITERION OF MAXIMUM INCOME

MAYRAMKAN ASANKULOVA[1], GULSUNAI ISKANDAROVA[2]

[1] Department of Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic
[2] Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

e-mail: may_as@mail.ru, iskandarrova@gmail.com

Abstract. The article develops a mathematical model and algorithm for solving the problem of optimizing the production of a household, considering the leased acreage and the breed of animals kept on the household without limiting the volume of production according to the criterion of maximum net income. The developed model and the method of their solution is verified by a test example.

Keywords: mathematical model, rental acreage, production, net income, household, animal husbandry, consumption, solution algorithm

1. Introduction

In a market economy, each business entity works independently. The goal of each business entity is to obtain maximum income from its activities. The process of production activity of economic entities in market conditions requires the development of new models and methods for solving problems that consider the patterns of market relations. Specialized households have the greatest opportunities in agriculture, among other economic entities. A household's income is determined by the number of productive breeds, as well as from the optimal use of the sown and rental areas on which the fodder crop is grown. At the same time, the annual demand for one animal in each type of agricultural crop is supposed to obtain a specific type of product. In the papers [1]-[6], [9]-[15], economic and mathematical models have been developed for various cases of tasks describing production processes and considering the relationship of economic entities in market conditions. In paper [7], a mathematical model of the problem of choosing a productive breed of animals according to the criterion of minimum costs was developed in case of the execution of the indicative plan. In this paper, a mathematical model has been developed, where the optimal size of the sown and leased area of various categories is determined, and productive animal breeds and their quantity are selected according to the criterion of maximum net income. The developed mathematical model leads to an increase in the efficiency of the specialized households of the agricultural sector of the economy.

2. Problem statement

Let a specialized household that has acreage of various categories (irrigated, rain-fed, etc.) in the amount of $s_k, k \in K$ and sufficient financial means planned to get the maximum income from the production and sale of products from productive breeds of animals (cattle), the expansion of acreage by renting space. The expansion of acreage is planned based on its financial capacity.

It is assumed that for each type of productive animal breed, its productivity, the yield of feed

crops for each category of acreage and the fee ε_k per unit per unit of $k-$ th leased area $k \in K$ are known.

It is required to determine the optimal size of sown and leased areas of various categories, based on the financial capacity of the household and the corresponding number of productive animal breeds, which allows to ensure maximum net income to the household from the production and sale of livestock products. For the mathematical formulation of the problem, we introduce the following descriptions:

$j-$ is index of the type of agricultural crop production used in the daily diet of animal feeding, $j \in J_0$;

J_0- is the set of types of crop production aimed at animal feed, $J_0 = 1, 2, ..., n$;

$k-$ is index of the type of category of acreage in the household, $k \in K$;

$K-$ is the set of types of acreage categories, $K = (1, 2, ..., p)$;

$h-$ is index of the type of livestock products produced on the household, $h \in H$;

$H-$ is the set of types of livestock products, $H = (1, 2, ..., H^-)$;

$l-$ is index of the type of animal breed in the household, $l \in L$;

$L-$ is the set of types of animal breeds, $L = (1, 2, ..., L^-)$;

Known parameters:

s_k- is the size of the sown area of $k-$ th categories in the household, $k \in K$;

$a_{kj}-$ is yield of $j-$ th of the type of crop in the $k-$ th category of the sown area of the household, $k \in K, j \in J_0$;

α_{jl}^h- is the annual demand for the $j-$ th type of crop production per animal of the $l-$ th breed in the production of the $h-$ th type of products, where

$$\alpha_{jl}^h = \beta_{jl}^h \gamma_{jl}^h, j \in J_0, l \in L, h \in H; \ (*)$$

β_{jl}^h- is the share of the $j-$ th crop production in the daily diet per animal of the $l-$ th breed in the household for the production of the $h-$ th type of products $j \in J_0, l \in L, h \in H$;

γ_{jl}^h- is the number of days in the diet of feeding plant products of the $j-$ th type for the $l-$ th breed of animal in the production of the $h-$th type of products, $j \in J_0, l \in L, h \in H$;

θ_l^h- is the volume of products of the $h-$ th type received by the farm from one animal of the $l-$ th breed, $l \in L, h \in H$;

ε_k- is payment per unit of the $k-$ th leased acreage, $k \in K$;

$D-$ is the financial ability of the household to pay for rented space;

α^h- is the selling price of the h-th type of livestock products on the household, $h \in H$;

$c_{kj}-$ are costs per unit size of the $k-$ th category of acreage for the $j-$ th type of culture, $j \in J_0, k \in K$;

c_l^h- is annual consumption per animal of the $l-$ th breed in the production of the $h-$ th type of livestock products, $h \in H, l \in L$;

c^h- are the costs of a household for the sale of a unit of volume of the $h-$ th type of livestock products $h \in H$, where $c^h = r^h + b^h, h \in H$;

r^h- are expenses related to the shipment of a unit of volume of the $h-$ th type of livestock products, $h \in H$;

b^h- is payment to workers for services for organizational work for selling a unit of volume of the $h-$ th type of livestock products, $h \in H$.

3. Required variables

$x_{kj}-$ is the size of the $k-$ th category of the sown area allocated for the $j-$ th type of culture, $j \in J_0, k \in K$;

y_l^h- is the number of animals of the $l-$ th breed in the farm for the production of the $h-$ th type of products, $h \in H, l \in L$;

ν_k- is the size of the $k-$ th category of leased acreage, $k \in K$;

z^h- is the volume of livestock products sold of the $h-$ th type, $h \in H$.

In accordance with the accepted notation, the mathematical model of the above problem will be written in the following form.

To find the maximum;

$$L(x,y,z) = \sum_{h\in H}(a^h - c^h) - \left(\sum_{k\in K}\sum_{j\in J_0} c_{kj}x_{kj} + \sum_{h\in H}\sum_{l\in L} c_l^h y_l^h\right) - \sum_{k\in K}\varepsilon_k\nu_k \tag{1}$$

under the conditions

$$s_k + v_k = \sum_{j\in J_0} x_{kj}, k \in K \tag{2}$$

$$\sum_{h\in H}\sum_{l\in L}\alpha_{jl}^h y_l^h = \sum_{k\in K} a_{kj}x_{kj}, j \in J_0, \tag{3}$$

$$z^h = \sum_{l\in L}\theta_l^h y_l^h, h \in H, \tag{4}$$

$$D \geq \sum_{k\in K^-}\varepsilon_k\nu_k, \tag{5}$$

$$x_{kj} \geq 0, k \in K, j \in J_0, z^h \geq 0, h \in H, y_l^h \geq 0, h \in H, l \in L - \text{is integer} \tag{6}$$

where

$$z = \left[z^h : h \in H\right\}, x = \{x_{kj} : k \in K, j \in J_0\}, y = \left\{y_l^h : h \in H, l \in L - \text{is integer}\right\}, \nu = \{\nu_k : k \in K\}.$$

Target function (1) defines maximum net farm income from production;

Limitation (2) indicates that the sown and leased areas should be fully cleared;

Limitation (3) requires that the volume of the j -th kind of feed crop grown in all categories;

Limitation (4) requires that each type of animal production;

Inequality (5) shows that the involvment of leased cultivated land is limited by the financial capacity; Limitation (6) requires no negative variables;

4. An algorithm for solving the problem

The algorithm for solving the problem (1)-(6) differs from the algorithm described in [7] with minor changes. As in [7], calculations begin with the calculation of the parameter values $\alpha_{jl}^h, j \in J_0, l \in L, h \in H$ according to the formula (*). By using values of parameters $a_{kj}, c_{kj}, s_k, \varepsilon_k, k \in K, j \in J_0, \theta_l^h, c_l^h, a^h, c^h, h \in H, l \in L$, a numeric model of the problem according to (1) to (6) is formulated.

From the solution of the problem the quantitative composition of the cattle in the household $y = \left\{y_l^h :, h \in H, l \in L\right\}$ and the appropriate size of the forage area $x = \{x_{kj} : k \in K, j \in J_0\}$ and the size of leased cultivated land $\nu = \{\nu_k : k \in K\}$, as well as sales of livestock products $z = \left\{z^h : h \in H\right\}$, maximizing net income for the household. The solution algorithm is completed.

Let's test the performance of the mathematical model and algorithm of problem solution on a numerical example.

5. Example

Let a household engaged in the production of livestock products (milk and meat) has irrigated acreage in the amount of $s_1 = 400$ hectares, rainfed $s_2 = 100$ hectares and planned to get maximum income from production by updating its household with more productive animal breeds and increasing the acreage by renting acreage. In order to increase the area planted, a financial contribution of 1,500,000 soms is available. The fee per $k-$ th renting acreage is known, $k = 1, 2, \varepsilon_k = \{\varepsilon_1, \varepsilon_2\} = 2000, 1500$.

The household has the opportunity to choose a productive animal breed for the production of milk and meat. This article uses data on daily diets of dairy and meat feeding, as well as the number of productive breeds of animals published in [7].

Known: - the selling price of a unit of milk volume is $a^1 = 85$ soms, of meat is $a^2 = 440$ soms, as well as transport and service costs for the trade organization's employees $c^1 = 15$ som, $c^2 = 100$ soms;

-Crop yields of irrigated (I) and rain-fed (II) crops included in the feeding ration $a_{kj}, k = 1, 2$, $j = 1, 2, ..., 6$, where

$$|a_{kj}|_{2,6} = \begin{vmatrix} 2070 & 1962 & 2380 & 6281 & 5730 & 12340 \\ 1500 & 0 & 1700 & 0 & 0 & 0 \end{vmatrix}$$

- expenses for cultivating agricultural crops per unit size of (I) and (II) fields, $|c_{kj}|_{2,6}$ where

$$|c_{kj}|_{2,6} = \begin{vmatrix} 2279 & 1096 & 2618 & 5071 & 9225 & 13574 \\ 2000 & 1000 & 2000 & 5000 & 9000 & 13000 \end{vmatrix}$$

The cost of keeping one animal of each breed in the production of milk and meat, respectively, is also known $c_{l=1}^{h=1} = 50040.0$ soms, $c_{l=2}^{h=1} = 55080.0$ soms, $c_{l=1}^{h=2} = 18030.0$ soms, $c_{l=2}^{h=2} = 25020.0$ soms.

It is required to determine the optimal size of the acreage of various categories, the size of the leased acreage, based on the financial capacity of the household, and the appropriate number of productive animal breeds, allowing the maximum net income to the farm from the production and sale of livestock products.

For mathematical formalization of the task, let's determine the annual feed demand using $\alpha_{jl}^h, j \in J_0, l \in L, h \in H$ in [7]

Let's formulate a numerical model of the problem.

To find the maximum

$$L(x, y, z, \nu) = 70z^1 + 340z^2 - (2279.0x_{11} + 1096.0x_{12} + 2618.0x_{13} + 5071.0x_{14} + 9225.0x_{15} +$$
$$+13574.0x_{16} + 2000.0x_{21} + 1000.0x_{22} + 2000.0x_{23} + 5000.0x_{24} + 9000.0x_{25} + 13000.0x_{26} +$$
$$+500040.0y_1^1 + 55080.0y_2^1 + 18030.0y_1^2 + 25020.0y_2^2) - (2000v_1 + 15000v_2) \quad (7)$$

Under the conditions

$$400 + \nu_1 = \sum_{j=1}^{6} x_{1j}, 100 + \nu_2 = \sum_{j=1}^{6} x_{2j}, \quad (8)$$

$$2070, 0x_{11} + 1500, 0x_{21} = 289, 5y_1^1 + 362, 5y_2^1 + 272, 5y_1^2 + 365, 0y_2^2,$$
$$1962, 0x_{12} + 0x_{22} = 907, 5y_1^1 + 1090, 0y_2^1 + 452, 5y_1^2 + 730, 0y_2^2,$$
$$2380, 0x_{13} + 1700, 0x_{23} = 720, 0y_1^1 + 1800, 0y_2^1 + 540, 0y_1^2 + 1825, 0y_2^2,$$
$$6281, 0x_{14} + 0x_{24} = 1080, 0y_1^1 + 1440, 0y_2^1 + 1080, 0y_1^2 + 1800, 0y_2^2,$$

$$5730,0x_{15}+0x_{25}=7200,0y_1^1+9000,0y_2^1+5400,0y_1^2+5400,0y_2^2,$$
$$12340,0x_{16}+0x_{26}=1800,0y_1^1+2160,0y_2^1+900,0y_1^2+1800,0y_2^2, \quad (9)$$
$$3600y_1^1+4500y_2^1=z^1, 300y_1^2+450y_2^2=z^2, \quad (10)$$
$$2000v_1+1500v_2\leq 1500000, \quad (11)$$
$$x_{kj}\geq 0, k=1,2, j=1,2,...,6, \quad (12)$$
$$z^h\geq 0, h=1,2, \quad (13)$$
$$\nu_k\geq 0, k=1,2, \quad (14)$$
$$y_l^h\geq 0, h=1,2, l=1,2 \quad (15)$$

is integer.

Having solved the problem (7) - (15), we will have an optimal plan for the distribution of sown areas under the stern

$$x=\{x_{11}=0; x_{12}=234,5; x_{13}=137,5; x_{14}=87,2; x_{15}=637,0; x_{16}=73,9\};$$
$$\{x_{21}=97,8; x_{22}=0; x_{23}=22,2; x_{24}=0, x_{25}=0, x_{26}=0\},$$

the composition of the content of cattle breeds in the dairy and meat sector
$y=\left\{y_1^1=506,9; y_2^1=0; y_1^2=0; y_2^2=0\right\}$, the volume of products sold by the household
$z=\left\{z^1=1824,8\right\}$ and the volume of leased acreage $v=\{v_1=750, v_2=0\}$, as well as the size of the net income of the farm $L(x,y,z,v)=92693865,0$ soms.

Summary: The use of the developed model in the production activities of the household will allow you to choose the best option from among the possible ones. The implementation of the result of the developed model into economic practice will contribute to improving the efficiency of innovative activities of the economy and obtaining maximum profit.

References

[1] Zhusupbayev A., Barganalieva Zh., Asankulova M. The problem of optimizing the company's beverage production and their distribution among consumers // Izvestiya vuzov Kyrgyzstan. – Bishkek: - 2017, No.5, Part 1. - pp.61-63. (RSCI KR)

[2] Zhusupbayev A., Asankulova M., Zholborsova A. Mathematical model of the problem of choosing a residential building construction object // Izvestia of universities of Kyrgyzstan. –Bishkek: - 2017, No. 5, part 2. - pp.118-120. (RSCI KR)

[3] Zhusupbayev A., Asankulova M., Iskandarov S. The problem of determining the optimal volume of coal purchase and its transportation// Science, new technologies and innovations of Kyrgyzstan. –Bishkek: - 2017, No. 5. –pp.163-165. (RSCI KR)

[4] Asankulova S.A., Nurlanbekov A., Maatov K., Asankulova M., Zhusupbayev A. The problem of determining the optimal size of the sown area for agricultural crops depending on the weather conditions of the season // Materials of the XII Issyk-Kul International school–conference "Radiation solid state physics" SCORPH-2017, August 3-9, 2017, No. 1. –Pp.84-87.

[5] Asankulova S.A., Nurlanbekov A., Maatov K., Asankulova M., Zhusupbayev A. A mathematical model for determining the maximum income of an enterprise considering the seasonal nature of the weather // Materials of the XII Issyk-Kul International School–Conference "Radiation solid state physics" SCORPH-2017, August 3-9, 2017, No. 1. –p.87-90.

[6] Zhusupbayev A., Asankulova M. The problem of forecasting the volume of import and export of products // System engineering, Izhevsk State Technical University named after M.T. Kalashnikov. – Izhevsk, series "Economic Sciences", №1-1(5) , 2017. – Pp.63-67. (RSCI RF)

[7] Zhusupbaev, A. Mathematical model and metod for calculating the optimization problem of livestock production / A. Zhusupbaev, M. Asankulova, G.A. Zhusupbaeva, G.S. Iskandarova // Herald of Institute Mathematics of the National Academy of Sciences of the Kyrgyz Republic. – 2020. – No 1. – P. 115-126.

[8] Borubaev, A.A. Mathematical model for determining the effective option for the development of an agricultural company in the region / A.A. Borubaev, A. Zhusupbayev, K.J. Dzhumabaev, M. Asankulova. –Bishkek, 2019. – 88 p.

[9] Fedoseev, VV Economic-mathematical methods and applied models / V. V. Fedoseev, A. N. Garmash, I. V. Orlova, V. A. Polovnikov. - Moscow: Banks and stock exchanges, UNITI, 1999. - 392 p. – EDN SCWJGR.

[10] Asankulova, M. Determination of the maximum income of an enterprise with a limited amount of finance / M. Asankulova, A. Zhusupbaev, G.A. Zhusupbaeva // Actual directions of scientific research of the XXI century: theory and practice. - 2015. - T. 3. - No. 7-1 (18-1). - S. 101-105. – DOI 10.12737/14811. – EDN TFTRNK.

[11] Zhusupbaev, A. Determination of the volume of production of raw materials and its distribution among consumers under the contract / A. Zhusupbaev, M. Asankulova // Problems of optimization and economic applications: Proceedings of the VI International Conference, Omsk, June 28 - 04, 2015. - Omsk: Omsk State University. F.M. Dostoevsky, 2015. - P. 171. - EDN UGKXBJ.

[12] Asankulova, M., Sharshembieva F.K., Zhusupbaeva G.A. Solution of some problem of optimization of production of a peasant economy // Proceedings of the National Academy of Sciences of the Kyrgyz Republic. - 2017. - No. 4. - S. 24-28. – EDN YUODKH.

[13] Asankulova, M. Determination of the optimal area and project of construction of a housing building / M. Asankulova, A. J. Jolborsova, G. S. Iskandarova , Herald of Institute Mathematics of the National Academy of Sciences of the Kyrgyz Republic. – 2022. – No 1. – P. 152-156. – DOI 10.52448/16948173_2022_1_152. – EDN IZHJTH.

[14] A. Jusupbaev, M. Asankulova, G. S. Iskandarova, N. K. Suynalieva, Determination of the optimal volume of livestock production by the
criterion of maximum income, Herald of Institute Mathematics of the National Academy of Sciences of the Kyrgyz Republic. – 2020. – No 2. – P.
105-118. – DOI 10.52448/16948173_2020_2_105. – EDN GWDPOK.

[15] Karachevskaya, E. V. Modeling and evaluation of the economic efficiency of the functioning of the agro-pharmaceutical cluster of the Republic of Belarus / E. V. Karachevskaya, A. F. Rogachev // Modern Economy Success. - 2016. - No. 1. - P. 87-110. – EDN WWYIUD.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

NUMERICAL STUDY OF THE TEMPERATURE VALUE AT NODAL POINTS OF A COMPLEX-SHAPED BODY

NURLAN TEMIRBEKOV[1], BEKET KENZHEGULOV[2], C. KENZHEGULOVA[3], A. YAKUPOVA[2]

[1]KazNU named after Al-Farabi, Almaty, Kazakhstan
[2]University named after Kh.Dosmukhamedov; Atyrau, Kazakhstan
[3]University of International Business named after K. Sagadiyev, Almaty, Kazakhstan
e-mail: kenzegulov_bz@mail.ru, ntemirbekv314@gmail.com

Abstract. In this article, the values of the temperature at the nodal points of the body of a complex shape are determined by the method of minimization based on the law of conservation of energy. In the process of determining their values, a calculation algorithm is created using a new mathematical model, and numerical studies are conducted based on temperature distribution. One part of this complex body is isolated from heat, one part receives heat flow, and another part has heat exchange with the external environment. The body is affected by several heat sources at the same time. The sources of heat here are heat flow and heat exchange (convective), temperature and heat insulation (insulation), heat conduction. Due to the size and configuration of the main structural elements, the distribution of temperature fields in its different parts is not uniform. Therefore, it is important to determine the value of the temperature in each part of the structural element (nodal points). This is because the body may be physically unusable (destroyed) due to the temperature in some part of the structural element. In order to understand the consequences of this and to save the main structural elements of the installations from damage, it is necessary to perform preliminary mathematical calculations and calculate the value of the temperature at each node point. In the article, the patterns of temperature distribution at the node points of the channel-like body (beam) taken as a structural element are determined and it is determined that the temperature distribution is symmetrical depending on the shape of the body.

Keywords: complex shaped body (beam), nodal point, temperature distribution laws, numerical study, heat flow, heat exchange (convective).

1. Introduction

Experimental: This article discusses a technique for constructing a mathematical model and the corresponding computational algorithm that allow solving problems of studying the patterns of the temperature distribution field in a structural element of a complex shape with the simultaneous presence of heat flow, heat transfer and partial thermal insulation on their local surfaces.

Literature review and problem statement: At present, in our country and abroad, there are many works devoted to the problem of the influence of a thermomechanical process on a change in the structure and composition of the material of any technical installation or design. This article takes into account the simultaneous influence of the heat flow on the body, partial thermal insulation and local heat transfer. A computational algorithm is presented for solving a problem obtained by discretizing bodies of complex shape made of heat-resistant alloys using quadratic finite elements [2, 5].

The aim and objectives of the study: The purpose of this article is to show the regularity of the distribution of the temperature field using a numerical study of heat transfer in the presence of

heat flow, thermal insulation and heat transfer. The objectives of the study are to determine the temperature value at each nodal point of a multidimensional body to develop a computational algorithm based on minimizing the total thermal energy functional.

In order to produce products ready for export, production facilities work with modern aggregate devices for processing raw materials. Most of the main elements of the devices work under the influence of temperature, heat flow, isolated heat, heat exchange, temperature changes of internal nodes. Aggregate devices are made of high-quality materials, i.e., heat-resistant castings, for high-quality, heat-resistant, long-term operation. Temperature distribution along such heat-resistant mixtures in multidimensional bodies, heat insulation and heat exchange properties, thermal conductivity, etc. In addition to thermal physics for the study of the laws, creating special mathematical models, functions, equations, universal digital algorithms and methods for studying the thermomechanical state is one of the most important issues. To solve the problem under consideration, we consider a problem solved by the method of minimization based on the law of conservation of energy for a two-dimensional quadratic finite element. The numerical method and the corresponding calculation algorithm are presented in the report. The structural element considered here is thermally insulated along the entire length of the upper and inner parts, and heat exchange with the external environment occurs along the side surfaces. The heat flow falls on both ends of the lower surface.

The problem presents a channel-like body of infinite length. The structural element under consideration is heat insulated along the entire length of the top $y = h$, and along the internal parts $l \le x \le l+r$, $-\infty \le z \le \infty$, and heat exchange with the external environment takes place on the side surfaces . $q-$ heat flux of constant intensity $y = 0$, $[(0 \le x \le l)\&(r+l) \le x \le (r+2l)]$, $-\infty \le z \le \infty$ falls on the lower surface along its entire length. In this case, the heat exchange coefficient is he and the ambient temperature is T_e. It is necessary to determine the distribution field of constant temperature in the volume of the considered structural element. For this purpose, we discretize the cross section with rectangular finite elements 1.

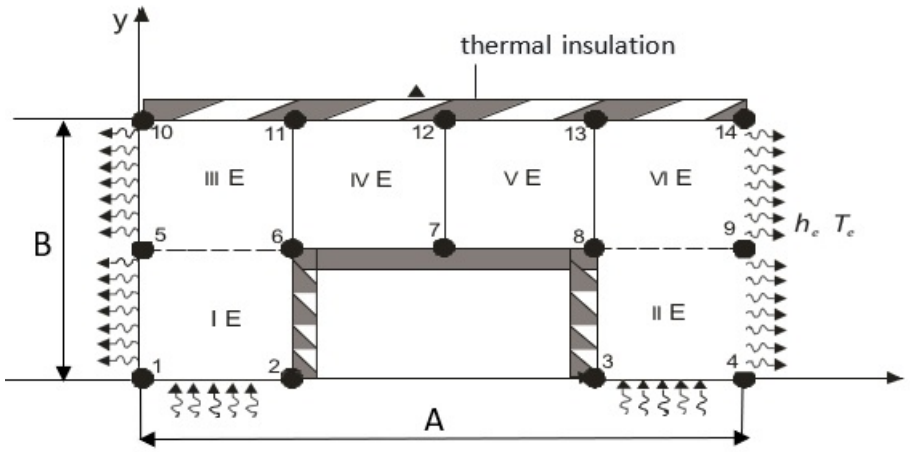


FIGURE 1. Discretization of the computational domain in the context of a structural element

The coordinates of the node points: $1(-a;-b)$; $2(a;-b)$; $3(a;b)$; $4(-a;b)$.

Within each finite element, we approximate the temperature distribution field with the following quadratic polynomial [1, 2]:

$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy = \varphi_1(x,y)\cdot T_1 + \varphi_2(x,y)\cdot T_2 + \varphi_3(x,y)\cdot T_3 + \varphi_4(x,y)\cdot T_4 \quad (1)$$

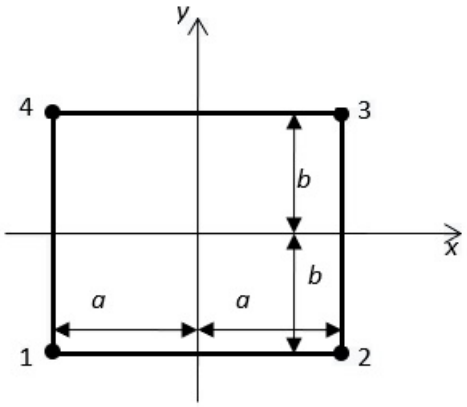


FIGURE 2. Cross section of a rectangular finite element

Here $\varphi_i(x, y)$ is the shape function of the finite element of a quadrilateral with four nodes [2,3]:

$$\varphi_1(x,y) = \frac{(a-x)(b-y)}{4ab}; \varphi_2(x,y) = \frac{(a+x)(b-y)}{4ab}; \varphi_3(x,y) = \frac{(a+x)(b+y)}{4ab}; \varphi_4(x,y) = \frac{(a-x)(b+y)}{4ab}. \tag{2}$$

The size of the limit element in the direction of the coordinate axes Ox and Ox is $(2a \times 2b)$. We discretize the section of the given body into 6 elements. Numbering of elements and nodes is shown (Figure 1). In order to quantitatively study the temperature distribution at the nodes of the body of complex shape under study, taking into account all the given quantities, i.e. thermal insulation, heat sources, heat exchange with the external environment, we write a functional (J) describing its total thermal energy for all finite elements [1–3]:

$$J = \int_V \frac{1}{2}[K_{xx}(\frac{\partial T}{\partial x})^2 + K_{yy}(\frac{\partial T}{\partial y})^2]\, dV + \int_{S(y=0)} qT\, dS + \int_{S(x=0,B)} \frac{h}{2}(T - T_E)^2\, dS \tag{3}$$

where V is the volume of the timber in question; $S(x = 0)$ - the surface area of the beam $(x = 0)$, where the heat flow $q(\frac{W}{cm^2})$; $S(x = A)$ - the surface area $(x = A)$ of the beam through which heat is exchanged with the environment $h(\frac{W}{cm^{2\cdot\circ}C}); K_{xx}; K_{yy}, \frac{W}{cm^{\cdot\circ}C}$ - the coefficient of thermal conductivity of the timber under consideration, respectively, in the directions of the coordinate axes Ox and Oy.

The temperature distribution field in each finite element is determined according to the formula (1) as follows:

For the I element:

$$T_{Ie}(x,y) = \varphi_1(x,y) \cdot T_1 + \varphi_2(x,y) \cdot T_2 + \varphi_3(x,y) \cdot T_6 + \varphi_4(x,y) \cdot T_5.$$

For the II element:

$$T_{IIe}(x,y) = \varphi_1(x,y) \cdot T_3 + \varphi_2(x,y) \cdot T_4 + \varphi_3(x,y) \cdot T_9 + \varphi_4(x,y) \cdot T_8.$$

For the III element:

$$T_{IIIe}(x,y) = \varphi_1(x,y) \cdot T_5 + \varphi_2(x,y) \cdot T_6 + \varphi_3(x,y) \cdot T_{11} + \varphi_4(x,y) \cdot T_{10}.$$

For the IV element:

$$T_{IVe}(x,y) = \varphi_1(x,y) \cdot T_6 + \varphi_2(x,y) \cdot T_7 + \varphi_3(x,y) \cdot T_{12} + \varphi_4(x,y) \cdot T_{11}.$$

For the V element:

$$T_{Ve}(x,y) = \varphi_1(x,y)\cdot T_7 + \varphi_2(x,y)\cdot T_8 + \varphi_3(x,y)\cdot T_{13} + \varphi_4(x,y)\cdot T_{12}.$$

For the VI element:

$$T_{VIe}(x,y) = \varphi_1(x,y)\cdot T_8 + \varphi_2(x,y)\cdot T_9 + \varphi_3(x,y)\cdot T_{14} + \varphi_4(x,y)\cdot T_{13}.$$

For all discrete elements, the integrated form of the function 3 is as follows [1–5]:

$$\begin{aligned}
J = {} & (\frac{alK_{xx}}{6b})_{IE}({T_1}^2 - 2T_1T_2 - T_1T_6 + T_1T_5 + {T_2}^2 + T_2T_6 - T_2T_5 + {T_6}^2 - 2T_6T_5 + {T_5}^2) + \\
& + (\frac{blK_{yy}}{6a})_{IE}({T_1}^2 + T_1T_2 - T_1T_6 - 2T_1T_5 + {T_2}^2 - 2T_2T_6 - T_2T_5 + {T_6}^2 + T_6T_5 + {T_5}^2) + (alq)_{IE}(T_1 + T_2) \\
& + (\frac{alh}{3})_{IE}({T_1}^2 + T_1T_5 - 3T_1T_E + {T_5}^2 - 3T_5T_E + 3{T_E}^2) + \\
& + (\frac{alK_{xx}}{6b})_{IIE}({T_3}^2 - 2T_3T_4 - T_3T_9 + T_3T_8 + {T_4}^2 + T_4T_9 - T_4T_8 + {T_9}^2 - 2T_9T_8 + {T_8}^2) + \\
& + (\frac{blK_{yy}}{6a})_{IIE}({T_3}^2 + T_3T_4 - T_3T_9 - 2T_3T_8 + {T_4}^2 - 2T_4T_9 - T_4T_8 + {T_9}^2 + T_9T_8 + {T_8}^2) + \\
& + (alq)_{IIE}(T_3 + T_4) + (\frac{alh}{3})_{IIE}({T_4}^2 + T_4T_9 - 3T_4T_E + {T_9}^2 - 3T_9T_E + 3{T_E}^2) + \\
& + (\frac{alK_{xx}}{6b})_{IIIE}({T_5}^2 - 2T_5T_6 - T_5T_{11} + T_5T_{10} + {T_6}^2 + T_6T_{11} - T_6T_{10} + {T_{11}}^2 - 2T_{11}T_{10} + {T_{10}}^2) + \\
& + (\frac{blK_{yy}}{6a})_{IIIE}({T_5}^2 + T_5T_6 - T_5T_{11} - 2T_5T_{10} + {T_6}^2 - 2T_6T_{11} - T_6T_{10} + {T_{11}}^2 + T_{11}T_{10} + {T_{10}}^2) + \\
& + (\frac{alh}{3})_{IIIE}({T_5}^2 + T_5T_10 - 3T_5T_E + {T_{10}}^2 - 3T_{10}T_E + 3{T_E}^2) + \\
& + (\frac{alK_{xx}}{6b})_{IVE}({T_6}^2 - 2T_6T_7 - T_6T_{12} + T_6T_{11} + {T_7}^2 + T_7T_{12} - T_7T_{11} + {T_{12}}^2 - 2T_{12}T_{11} + {T_{11}}^2) + \\
& + (\frac{blK_{yy}}{6a})_{IVE}({T_6}^2 + T_6T_7 - T_6T_{12} - 2T_6T_{11} + {T_7}^2 - 2T_7T_{12} - T_7T_{11} + {T_{12}}^2 + T_{12}T_{11} + {T_{11}}^2) + \\
& + (\frac{alK_{xx}}{6b})_{VE}({T_7}^2 - 2T_7T_8 - T_7T_{13} + T_7T_{12} + {T_8}^2 + T_8T_{13} - T_8T_{12} + {T_{13}}^2 - 2T_{13}T_{12} + {T_{12}}^2) + \\
& + (\frac{blK_{yy}}{6a})_{VE}({T_7}^2 + T_7T_8 - T_7T_{13} - 2T_7T_{12} + {T_8}^2 - 2T_8T_{13} - T_8T_{12} + {T_{13}}^2 + T_{13}T_{12} + {T_{12}}^2) + \\
& + (\frac{alK_{xx}}{6b})_{VIE}({T_8}^2 - 2T_8T_9 - T_8T_{14} + T_8T_{13} + {T_9}^2 + T_9T_{14} - T_9T_{13} + {T_{14}}^2 - 2T_{14}T_{13} + {T_{13}}^2) + \\
& + (\frac{blK_{yy}}{6a})_{VIE}({T_8}^2 + T_8T_9 - T_8T_{14} - 2T_8T_{13} + {T_9}^2 - 2T_9T_{14} - T_9T_{13} + {T_{14}}^2 + T_{14}T_{13} + {T_{13}}^2) + \\
& + (\frac{alh}{3})_{VIE}({T_9}^2 + T_9T_{14} - 3T_9T_E + {T_{14}}^2 - 3T_{14}T_E + 3{T_E}^2)
\end{aligned} \tag{4}$$

We minimize the functional (4) through the nodes. It will be a system of linear algebraic equations as follows [2–8]:

$$\frac{\partial J}{\partial T_i} = 0, (i = 1 \div 14) \tag{5}$$

If we open the (5) system and write:

$$\frac{\partial J}{\partial T_1}=0\Rightarrow\frac{aLK_{xx}}{6b}(2T_1-2T_2+T_5-T_6)+\frac{bLK_{yy}}{6a}(2T_1+T_2-2T_5-T_6)+alq+\frac{aLh}{3}(2T_1+T_5-3T_E)=0$$
$$\frac{\partial J}{\partial T_2}=0\Rightarrow\frac{aLK_{xx}}{6b}(-2T_1+2T_2-T_5+T_6)+\frac{bLK_{yy}}{6a}(T_1+2T_2-T_5-2T_6)+alq=0$$
$$\frac{\partial J}{\partial T_3}=0\Rightarrow\frac{aLK_{xx}}{6b}(2T_3-2T_4+T_8-T_9)+\frac{bLK_{yy}}{6a}(2T_3+T_4-2T_8-T_9)+alq=0$$
$$\frac{\partial J}{\partial T_4}=0\Rightarrow\frac{aLK_{xx}}{6b}(-2T_3+2T_4-T_8+T_9)+\frac{bLK_{yy}}{6a}(T_3+2T_4-T_8-2T_9)+alq+\frac{aLh}{3}(2T_4+T_9-3T_E)=0$$
$$\frac{\partial J}{\partial T_5}=0\Rightarrow\frac{aLK_{xx}}{6b}(T_1-T_2+4T_5-4T_6+T_{10}-T_{11})+\frac{bLK_{yy}}{6a}(-2T_1-T_2+4T_5+2T_6-2T_{10}-T_{11})+$$
$$+\frac{aLh}{3}(T_1+4T_5+T_{10}-6T_E)=0$$
$$\frac{\partial J}{\partial T_6}=0\Rightarrow\frac{aLK_{xx}}{6b}(-T_1+T_2-4T_5++6T_6-2T_7-T_{10}+2T_{11}-T_{12})+\frac{bLK_{yy}}{6a}(-T_1-2T_2+2T_5+6T_6+T_7-$$
$$-4T_{11}-T_{12})=0$$
$$\frac{\partial J}{\partial T_7}=0\Rightarrow\frac{aLK_{xx}}{6b}(-2T_6+4T_7-2T_8-T_{11}+2T_{12}-T_{13})+\frac{bLK_{yy}}{6a}(T_6+4T_7+T_8-T_{11}-4T_{12}-T_{13})=0$$
$$\frac{\partial J}{\partial T_8}=0\Rightarrow\frac{aLK_{xx}}{6b}(T_3-T_4-2T_7+6T_8-4T_9-T_{12}+2T_{13}-T_{14})+\frac{bLK_{yy}}{6a}(-2T_3-T_4+T_7+6T_8+2T_9-T_{12}$$
$$-4T_{13}-T_{14})=0$$
$$\frac{\partial J}{\partial T_9}=0\Rightarrow\frac{aLK_{xx}}{6b}(-T_3+T_4-4T_8+4T_9-T_{13}+T_{14})+\frac{bLK_{yy}}{6a}(-T_3-2T_4+2T_8+4T_9-T_{13}-2T_{14})+$$
$$+\frac{aLh}{3}(T_4+4T_9+T_{14}-6T_E)=0$$
$$\frac{\partial J}{\partial T_{10}}=0\Rightarrow\frac{aLK_{xx}}{6b}(T_5-T_6-2T_{10}-2T_{11})+\frac{bLK_{yy}}{6a}(-2T_5-T_6+2T_{10}+T_{11})+$$
$$+\frac{aLh}{3}(T_5+2T_{10}-3T_E)=0$$
$$\frac{\partial J}{\partial T_{11}}=0\Rightarrow\frac{aLK_{xx}}{6b}(-T_5+2T_6-T_7-2T_{10}+4T_{11}-2T_{12})+\frac{bLK_{yy}}{6a}(-T_5-4T_6-T_7+T_{10}+4T_{11}+T_{12})=0$$
$$\frac{\partial J}{\partial T_{12}}=0\Rightarrow\frac{aLK_{xx}}{6b}(-T_6+2T_7-T_8-2T_{11}+4T_{12}-2T_{13})+\frac{bLK_{yy}}{6a}(-T_6-4T_7-T_8+T_{11}+4T_{12}+T_{13})=0$$
$$\frac{\partial J}{\partial T_{13}}=0\Rightarrow\frac{aLK_{xx}}{6b}(-T_7+2T_8-T_9-2T_{12}+4T_{13}-2T_{14})+\frac{bLK_{yy}}{6a}(-T_7-4T_8-T_9+T_{12}+4T_{13}+T_{14})=0$$
$$\frac{\partial J}{\partial T_{14}}=0\Rightarrow\frac{aLK_{xx}}{6b}(-T_8+T_9-2T_{13}+2T_{14})+\frac{bLK_{yy}}{6a}(-T_8-2T_9+T_{13}+2T_{14})+\frac{aLh}{3}(T_9+2T_{14}-3T_E)=0 \quad (6)$$

Solving the system (6) by the Gaussian method, we can determine the values of temperatures at the node points and calculate the temperature value of each limit element according to the formula (1). If we take the following numerical values for the EI-617 heat-resistant ingot and make a calculation [1, 2]:

$$K_{xx}=K_{yy}=72[\frac{Bm}{cm\cdot^{\circ}C}];a=b=1cm;q=-100[\frac{Bm}{cm^2}];h_E=6[\frac{Bm}{cm^2\cdot^{\circ}C}];T_E=40^{\circ}C;r=2cm;l=1cm,$$

we get the following temperature values:

$$T_1 = T_4 = 49,9^\circ C; T_2 = T_3 = 50,71^\circ C; T_5 = T_9 = 48,01^\circ C; T_6 = T_8 = 48,59^\circ C;$$
$$T_7 = 48,37^\circ C; T_{10} = T_{14} = 47,4^\circ C; T_{11} = T_{13} = 48,22^\circ C; T_{12} = 48,44^\circ C.$$

2. Conclusion

What we learned from this numerical experiment is that for any multi-dimensional structural elements of different configurations, the laws of the heat distribution field can be solved using the minimization method based on the energy conservation law for a two-dimensional quadratic finite element. In the report, the value of the temperature at the node points of the cross-section of the body given by the processes of heat flow and heat insulation and heat exchange with the external environment was determined. It is also ensured that the temperature distribution field of numerical values of all discrete limit elements is symmetrical. Therefore, the given method solves the thermal resistance of structural elements operating under the influence of various heat sources and external mechanical forces. Determining the temperature value at each node ensures that the heat-resistant mixture does not exceed the strength limit and the structural element does not collapse.

References

[1] Kenzhegulov B., *Numerical modeling of multidimensional temperature and one-dimensional nonlinear thermomechanical processes in heat-resistant alloys*, ASU Press publishing House, 2021, 310 p.

[2] Segerlind L., *Application of the finite element method*, M.:Mir, 1979, 392 p.

[3] Kenzhegulov B., Shazhdekeyeva N., Myrzhasheva A.N., Kabylhamitov G.T., Tuleuova R.U., Necessary Optimality Conditions for Determining of The Position of The Boundary of Oil Deposit, *International Journal of Engineering Research and Technology*, Vol.13, 2020, pp.1204-1209.

[4] Kenzhegulov B., Kultan J., Alibiyev D.B., Kazhikenova A.Sh., Numerical Modelling of Thermomechanical Processes in Heat-Resistant Alloys, *Bulletin of the Karaganda University. Physics Series*, Vol.2, 2020, pp.101-108.

[5] Kenzhegulov B., Shazhdekeyeva N., Myrzhasheva A.N., Tuleuova R.U., A Numerical Method for Determining the Dependence of The Thermally Stressed State of a Rod on Ambient Temperature with The Simultaneous Presence of Thermal Processes, *Periodico Tche Quimica*, Vol.17, 2020, pp.765-780.

[6] Kenzhegulov B., Tuleuova R., Myrzhasheva A.N., Shazhdekeyeva N., Kabylkhamitov G., Mathematical modelling and development of a computational algorithm for the study of thermo-stressed state of a heat-resistant alloy, *Periodicals of Engineering and Natural Sciences*, Vol.9, 2021.

[7] Kenzhegulov B., Shazhdekeyeva N., Myrzhasheva A., Kabylkhamitov G., Tuleuova R., Numerical methods for solving improper problems of filtration theory, *Istra?ivanjaiprojektovanja za privredu Journal of Applied Engineering Science*, Vol.191, 2021, pp.98-108.

[8] Kenzhegulov B.Z., Kenzhegulova S.B., Alibiyev D.B., Kazhikenova A.Sh., Finite element modeling of heat propagation of a complete rod of constant cross-section, *Bulletin of the Karaganda University. Physics Series*, Vol.4, 2022, pp.94-105.

[9] Wang J., Kinetic and Strength Calculation of Age-Hardening Phases in Heat-Resistant Aluminum Alloys with Silver, *Materials Science Forum*, Vol.993, 2020, pp.1051-1056.

[10] Adaskin A.M., Kirillov A.K., Kutin A.A., Improving the Cutting of Heat-Resistant Chromium Alloy, *Russian Engineering Research*, Vol.40, 2020, pp.748-750.

[11] Min P.G., Sidorov V.V., Vadeev V.E., Kramer V.V., Development of Corrosion and HeatResistant Nickel Alloys and their Production Technology with the Aim of Import Substitution, *Power Technology and Engineering*, Vol.54, 2020, pp.225-231.

[12] Facco A., Couvrat M., Magn? D., Roussel M., Guillet A., Pareige C., Microstructure Influence on Creep Properties of Heat-Resistant Austenitic Alloys with High Aluminum Content, *Materials Science and Engineering*, Vol.783, 2020.

[13] Kvasnytska Y.H., Ivaskevych L.M., Balytskyi O.I., Maksyuta I.I., Myalnitsa H.P., HighTemperature Salt Corrosion of a Heat-Resistant Nickel Alloy, *Materials Science*, Vol.56, 2020, pp.432-440.

[14] Patrin P.V., Karpov B.V., Aleshchenko A.S., Galkin S.P., Capability Process Assessment of Radial-Displacement Rolling of Heat-Resistant Alloy HN73MBTYU, *Steel in Translation*, Vol.50, 2020, pp.42-45.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

A MATHEMATICAL MODEL OF TWO-PHASE FILTRATION IN A POROUS MEDIUM TAKING INTO ACCOUNT ITS DEFORMATION

VLADIMIR BURNASHEV[1], ZOKHIDJON KAYTAROV[2]

[1,2]Samarkand State University, Samarkand, Uzbekistan
e-mail: vladimir.burnash@mail.ru, z.qaytarov@gmail.com

Abstract. The paper considers the process of filtration of water and oil phases, taking into account the deformation of the medium. A suitable mathematical model of the process is proposed. Using the proposed model, the influence of medium deformation on hydrodynamic parameters was analyzed. The problem was solved numerically by the large particle method. Based on the obtained results, the influence of environmental deformation on reservoir pressure, porosity, permeability, oil saturation and oil production was studied.

Keywords: phase, deformation, mass conservation, filtration, saturation.

AMS Subject Classification: 76-10

1. Introduction

The simulation of the fluid flow taking into account the deformation of the medium is actual. In particular, the determination of the deformation state of fluid filtration into wells in an inhomogeneous medium [1]. Joint consideration of fluid flow and deformation of porous media is very important for biomedicine [2], the oil industry [3], the oil refining industry [4], geotechnology [5], water treatment [2, 4], and new energy technologies [2] that depend on large-scale geological and biological processes. In addition, there are environmental problems, including pollution of groundwater as a result of the development of the oil industry. This contamination is very difficult to detect and can take a long time. To determine it, it is necessary to simulate the filtration of multiphase fluids in porous media together with the deformation of the medium [3].

Despite intensive attempts at multiphase modeling taking into account the deformation of the medium, many fundamental physical, mathematical, and computational problems still remain unresolved [6]. Problems of this type are modeled using a system of differential equations, including the force balance equation and mass conservation laws. Due to the complexity of these systems of equations, there are no analytical solutions; therefore, in most cases, numerical methods are used [7, 8, 9].

The process of three-phase flow of two compressible fluids in a deformable porous medium is modeled using a symmetric hyperbolic thermodynamically compatible (HTCS) system [10]. The model of the two-phase flow process was considered in hybrid systems containing regions free from the solid phase and a deformable porous matrix [11].

In [12], a nonlinear model of the flow of a mixture of compressible fluids in porous media with finite deformations and thermal effects was proposed. In [1], the process of fluid filtration in a transversely isotropic deformable medium is modeled by the finite element method. At the same time, the influence of reservoir pressure on the change in the permeability of the medium was analyzed. In [13], a new method for estimating the change in the permeability of the pore medium as a result of changes in reservoir pressure was proposed using analytical and numerical modeling. In the work, the analytical method was used to study the effect of pressure changes on

the permeability of the medium, then numerical methods were used to determine the diagnostic signs of changes in permeability as a result of changes in the pressure of the formation zone and to verify the estimates obtained as a result of solving the inverse problem.

Inverse problems for the equations of elastoplastic filtration of fluids in the modes of pressure reduction and recovery in a porous medium were solved by the method of identifying parameters and deterministic moments [14]. In [15], the process of multiphase flow was modeled taking into account the deformation of the medium.

In this work, the process of filtration of the oil and water phases in a deformable porous medium was modeled, taking into account the oil debit. For the solution, the method of large particles was used. The dependence of the properties of the pore medium on reservoir pressure, the effect of medium deformation on the process of two-phase filtration, the effect of medium deformation and changes in reservoir pressure on oil production are studied.

Consider the filtration of oil and water in a homogeneous deformable oil reservoir penetrated by a system of producing wells. The wells are grouped into rows. We assume that oil and water are incompressible. A quasi-one-dimensional mathematical model of the filtration process in this system can be represented as follows.

Mass conservation law for phases

2. Formulation of the problem

Consider the filtration of oil and water in a homogeneous deformable oil reservoir penetrated by a system of production wells. The wells are grouped into rows. We assume that oil and water are incompressible. A quasi-one-dimensional mathematical model of the filtration process in this system can be represented as follows.

$$A\frac{\partial}{\partial t}\left(ms_o\right)+\frac{\partial\left(Au_o\right)}{\partial x}=A\sum_{k=1}^{n}Q_{ok}\delta_k, \tag{1}$$

$$A\frac{\partial}{\partial t}\left(ms_w\right)+\frac{\partial\left(Au_w\right)}{\partial x}=A\sum_{k=1}^{n}Q_{wk}\delta_k \tag{2}$$

where s_o is the saturation of the pore space with oil, m is the porosity, u_o is the velocity of the oil phase, s_w is saturation of the pore space with water, u_w is water phase velocity, $A=b(x)\cdot H(x)$ is element cross-section, $b(x)$ is element width, $H(x)$ is reservoir thickness, $\delta_k=\delta\left(x-x_k\right)$ - Dirac delta function, Q_{ok} and Q_{wk} oil and water debits reduced to standard conditions at row $k, k=1,2$ is well row number, n is the number of rows.

The filtration rate is determined according to the Darcy law [16].

$$u_o=-\frac{Kk_o}{\mu_o}\frac{\partial p}{\partial x}, \tag{3}$$

$$u_w=-\frac{Kk_w}{\mu_w}\frac{\partial p}{\partial x}. \tag{4}$$

Porosity changes according to the following expression [17]

$$m=m_0+\beta_m\left(p-p_0\right), \tag{5}$$

-absolute permeability [18]

$$K=K_0\left(\frac{m}{m_0}\right)^d, \tag{6}$$

-relative permeability for oil and water phases [15]

$$k_o=\begin{cases}\left(\frac{1-s_w}{1-s_{wE}}\right), & s_w>s_{wE},\\ 1 & s_w\le s_{wE}\end{cases} \tag{7}$$

$$k_w = \begin{cases} \left(\frac{s_w - s_{wE}}{1 - s_{wE}}\right), & s_w > s_{wE}, \\ 0 & s_w \le s_{wE} \end{cases} \tag{8}$$

-obvious equality for saturations [15]

$$s_o + s_w = 1 \tag{9}$$

where p is reservoir pressure,K is absolute permeability, k_o, k_w are phase relative permeability, μ_o, μ_w are oil and water viscosities, m_0 is initial porosity at $p = p_0, \beta_m$ is reservoir elasticity coefficient, p_0 is fixed pressure, d is a constant, K_0 is initial absolute permeability, s_{wE} is residual water saturation .

Phase debits are determined according to the following formulas

$$Q_{ok} = q_k \varphi_{ok}, \tag{10}$$

$$Q_{wk} = q_k \varphi_{wk} \tag{11}$$

where q_k is oil well debet of the kth row

$$q_k = \frac{2\pi \Delta z_k}{\ln \frac{r_{c\kappa}}{r_{k\kappa}}} \left[\bar{k}_w \int_{p_{\kappa\kappa}}^{p_{c\kappa}} \frac{K_k(p)}{\mu_w} dp + \bar{k}_o \int_{p_{\kappa\kappa}}^{p_{c\kappa}} \frac{K_k(p)}{\mu_o} dp \right], \tag{12}$$

$$\varphi_{ok} = \frac{k_o}{\mu_o} \left[\frac{k_o}{\mu_o} + \frac{k_w}{\mu_w} \right]^{-1}, \tag{13}$$

$$\varphi_{wk} = 1 - \varphi_{ok}$$

are the shares of phases, $K_k, \Delta z_k$ are permeability and thickness of the inner region of the k th row. Functions $\bar{k}_o$ and $\bar{k}_w$ are averaged relative permeabilities, p_{kk} is pressure on the contour, p_{ck} is pressure on the well, r_{ck} and r_{kk} are oil well and boundary radiuses.

In order to solve (1)-(13) problem we introduce initial and boundary conditions

-inital conditions

$$p(x,0) = \bar{p}(x), \quad s_o(x,0) = \bar{s}_o, \quad s_w(x,0) = \bar{s}_w, \tag{14}$$

-boundary conditions at production wells

$$p\left(x_k, t\right) = \bar{p}_k(t) \tag{15}$$

-boundary conditions at the boundary

$$\frac{\partial p(0,t)}{\partial x} = 0, \frac{\partial p(L,t)}{\partial x} = 0 \tag{16}$$

Calculate q_k from the expression (12)

$$\bar{k}_\alpha \int_{p_{kk}}^{p_{ck}} \frac{K_k(p)}{\mu_\alpha} dp = \frac{\bar{k}_\alpha}{\mu_\alpha} \int_{p_{kk}}^{p_{ck}} (K_k)_0 \left(\frac{m}{m_0}\right)^d dp = \frac{(K_k)_0 \bar{k}_\alpha}{\mu_\alpha (m_0)^d} \int_{p_{kk}}^{p_{ck}} \left(m_0 + \beta_m \left(p - p_0\right)\right)^d dp =$$

$$= -\frac{(K_k)_0 \bar{k}_\alpha}{\mu_\alpha (m_0)^d} \int_{p_{ck}}^{p_{kk}} \left(m_0 + \beta_m \left(p - p_0\right)\right)^d dp =$$

$$= -\frac{(K_k)_0 \bar{k}_\alpha}{(d+1) \cdot \beta_m \mu_\alpha (m_0)^d} \left[\left(m_0 + \beta_m \left(p - p_0\right)\right)^{d+1} \Big|_{p=p_{kk}}^{p=p_{ck}} = \right.$$

$$\left. -\frac{(K_k)_0 \bar{k}_\alpha}{(d+1) \cdot \beta_m \mu_\alpha (m_0)^d} \left[\left(m_0 + \beta_m \left(p_{kk} - p_0\right)\right)^{d+1} - \left(m_0 + \beta_m \left(p_{ck} - p_0\right)\right)^{d+1} \right] \right.$$

from here

$$q_k = -\frac{2\pi \Delta z_k}{\ln \frac{r_{c\kappa}}{r_{k\kappa}}} \left[\frac{\bar{k}_w (K_k)_0}{(d+1) \cdot \beta_m \mu_w (m_0)^d} \left[\left(m_0 + \beta_m \left(p_{kk} - p_0\right)\right)^{d+1} - \left(m_0 + \beta_m \left(p_{ck} - p_0\right)\right)^{d+1} \right] + \right.$$

$$\left. + \frac{\bar{k}_o (K_k)_0}{(d+1) \cdot \beta_m \mu_o (m_0)^d} \left[\left(m_0 + \beta_m \left(p_{kk} - p_0\right)\right)^{d+1} - \left(m_0 + \beta_m \left(p_{ck} - p_0\right)\right)^{d+1} \right] \right] \tag{17}$$

Adding equations (1) and (2), we obtain the following

$$A\frac{\partial}{\partial t}\left(ms_o + ms_w\right) + \frac{\partial}{\partial x}\left[A\left(u_o + u_w\right)\right] = A\sum_{k=1}^{n} Q_k\delta_k \tag{18}$$

where $Q_k = Q_{ok} + Q_{wk}$

In accordance with equality (9), we write (18) in the following form

$$A\frac{\partial m}{\partial t} + \frac{\partial}{\partial x}\left(u_o + u_w\right) = A\sum_{k=1}^{n}\left(Q_k\right)\delta_k \tag{19}$$

According to (4) and (5), we write (19) in the following form

$$A\beta_M\frac{\partial p}{\partial t} - \frac{\partial}{\partial x}\left[AK\left(\frac{k_o}{\mu_o} + \frac{k_w}{\mu_w}\right)\frac{\partial p}{\partial x}\right] = A\sum_{k=1}^{n}\left(Q_k\right)\delta_k \tag{20}$$

We consider $A = \text{const}$ and write (20) in the following form

$$\beta_M\frac{\partial p}{\partial t} - \frac{\partial}{\partial x}\left[K\left(\frac{k_o}{\mu_o} + \frac{k_w}{\mu_w}\right)\frac{\partial p}{\partial x}\right] = \sum_{k=1}^{n}\left(Q_k\right)\delta_k \tag{21}$$

To solve equation (21), we write the initial conditions (14) and boundary conditions (15)-(16) in the following form.

$$p(x,0) = p_0, \quad s_o(x,0) = s_o^*, \quad s_w(x,0) = s_w^*, \tag{22}$$

$$\frac{\partial p(0,t)}{\partial x} = 0, \quad \frac{\partial p(L,t)}{\partial x} = 0, \tag{23}$$

$$p\left(x_k,t\right) = p_k^*(t) \tag{24}$$

3. Solution method

To solve the problem (1)-(16), we use the method of large particles [19]. To accomplish this, we first introduce the Euler grid.

$$\Omega_{\tau h} = \left\{t_{j+1} = t_j + \tau, \quad j = 0,\overline{T-1}; \quad x_{i+1} = x_i + h, \quad i = 0,\overline{N+1}\right\}.$$

where h is the grid step in space, respectively, τ is the grid step in time, N is number of nodes in spaces, K is number of nodes in time, X is distance in space, T is maximum considered time.

The algorithm of the method is divided into three stages.

Stage 1. We approximate equation (21).

$$\begin{gathered}\beta_m\frac{\tilde{p_i}-p_i^j}{\tau} = \frac{1}{h^2}\left\{\left[K\left(\frac{k_0}{\mu_0}+\frac{k_w}{\mu_w}\right)\right]_{i+\frac{1}{2}}^{j}\cdot\tilde{p_{i+1}} - \right.\\ -\left\{\left[K\left(\frac{k_0}{\mu_0}+\frac{k_w}{\mu_w}\right)\right]_{i+\frac{1}{2}} + \left[K\left(\frac{k_0}{\mu_0}+\frac{k_w}{\mu_w}\right)\right]_{i-\frac{1}{2}}\right\}^{j}\cdot\tilde{p_i} + \\ \left.+\left[K\left(\frac{k_0}{\mu_0}+\frac{k_w}{\mu_w}\right)\right]_{i-\frac{1}{2}}^{j}\cdot\tilde{p_{i-1}}\right\} + \textstyle\sum_{k=1}^{n}\left(Q_k\right)^j\delta_k\end{gathered} \tag{25}$$

Let us write the system of linear algebraic equations (25) in the following form.

$$B_i\cdot\tilde{p_{i-1}} - C_i\cdot\tilde{p_i} + D_i\cdot\tilde{p_{i+1}} = -F_i \tag{26}$$

where

$$B_i = \frac{\tau}{\beta_m h^2}\left[K\left(\frac{k_0}{\mu_0}+\frac{k_w}{\mu_w}\right)\right]_{i-\frac{1}{2}}^{j}$$

$$C_i = 1+\frac{\tau}{\beta_m h^2}\left\{\left[K\left(\frac{k_0}{\mu_0}+\frac{k_w}{\mu_w}\right)\right]_{i-\frac{1}{2}}^{j}+\left[K\left(\frac{k_0}{\mu_0}+\frac{k_w}{\mu_w}\right)\right]_{i+\frac{1}{2}}^{j}\right\}$$

$$D_i = \frac{\tau}{\beta_m h^2}\left[K\left(\frac{k_0}{\mu_0}+\frac{k_w}{\mu_w}\right)\right]_{i+\frac{1}{2}}^{k}$$

$$F_i = p_i^j - \left(\sum_{k=1}^{n}(Q_k)\,\delta_k\right)_i^j$$

We solve the system of tridiagonal linear equations (26) by the Thomas' algorithm [20]. We calculate the coefficients of Thomas' algorithm according to the following formula

$$\begin{cases}\alpha_{i+1} = \frac{C_i}{B_i - A_i\alpha_i}, & i=\overline{1,N}\\ \beta_{i+1} = \frac{F_i + A_i\cdot\beta_i}{B_i - A_i\alpha_i}, & i=\overline{1,N}\end{cases} \tag{27}$$

We calculate the pressure using the found coefficients as follows

$$\tilde{p_i} = \alpha_{i+1}\cdot\tilde{p_{i+1}} + \beta_{i+1}. \tag{28}$$

From the boundary conditions we find

$$\alpha_1 = 1, \quad \beta_1 = 0,$$
$$\widetilde{p_N} = \frac{\beta_N}{1-\alpha_N},$$
$$p_{L_k}^j = p_k^*$$

Approximation of expressions for filtration velocities

$$(u_o)_{i+\frac{1}{2}}^{\sim} = -\left(\frac{Kk_o}{\mu_o}\right)_i^{\sim}\frac{\tilde{p_i}-\widetilde{p_{i+1}}}{h}, \quad (u_w)_{i+\frac{1}{2}}^{\sim} = -\left(\frac{Kk_w}{\mu_w}\right)_i^{\sim}\frac{\widetilde{p_i}-\widetilde{p_{i+1}}}{h} \tag{29}$$

Stage 2. Find the values of porosity and phase saturations on the current time layer

$$m_i^{j+1} = m_0 + \beta_m\left(\tilde{p_i} - p_0\right) \tag{30}$$

$$(s_o)_i^{j+1} = \frac{1}{m_i^{j+1}}\left[m_i^{j+1}(s_o)_i^j + \frac{(u_o)_{i-1/2}^{\sim} - (u_o)_{i+1/2}^{\sim}}{h}\tau\right] \tag{31}$$

$$(s_w)_i^{j+1} = 1 - (s_o)_i^{j+1} \tag{32}$$

$$(k_o)_i^{j+1} = \begin{cases}\left(\frac{1-(s_w)_i^{j+1}}{1-s_{wE}}\right), & s_w > s_{wE}\\ 1 & s_w \le s_{wE}\end{cases} \tag{33}$$

$$(k_w)_i^{j+1} = \begin{cases}\left(\frac{(s_w)_i^{j+1}-s_{wE}}{1-s_{wE}}\right), & s_w > s_{wE}\\ 0 & s_w \le s_{wE}\end{cases} \tag{34}$$

Stage 3. At this stage, we calculate the reservoir permeability and debits
-permeability

$$K_i^{j+1} = K_0\left(\frac{m_i^{j+1}}{m_0}\right)^d \tag{35}$$

We approximate relations (10)-(13).
-oil phase debit

$$(Q_o)_k^j = q_k^j\,(\varphi_o)_k^j$$

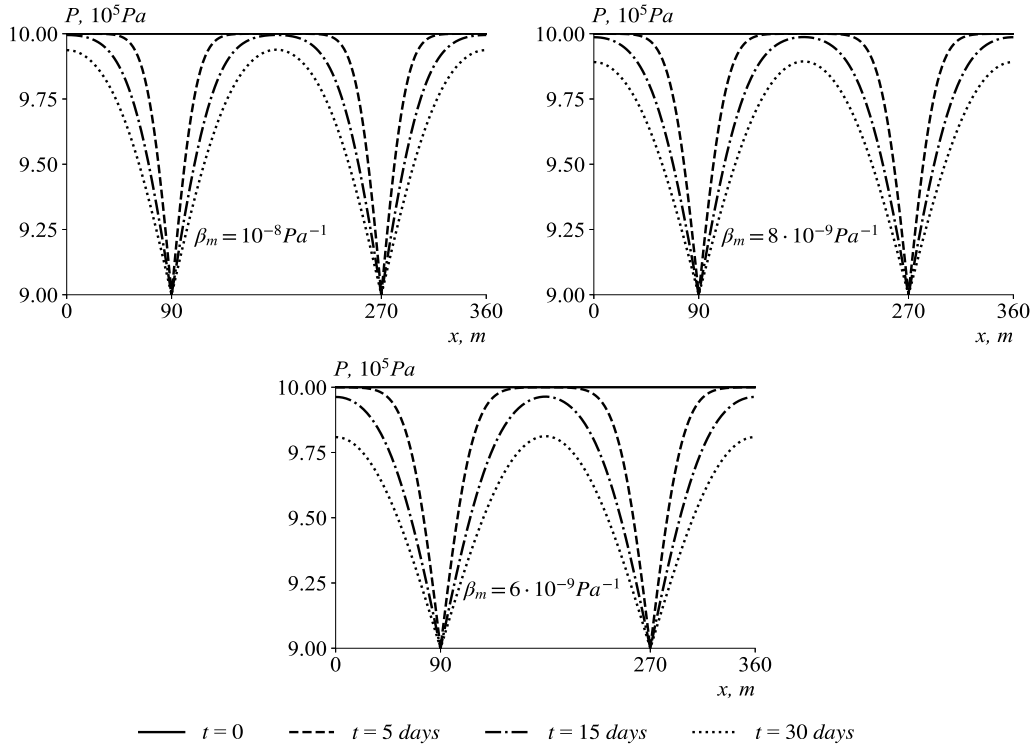


FIGURE 1. Influence of medium compressibility on pressure distribution

-water phase debet

$$(Q_w)_k^j = q_k^j \, (\varphi_w)_k^j$$

-oil well debet of the kth row

$$q_k^j = -\frac{2\pi\Delta z_k}{\ln \frac{r_{ck}}{r_{k\kappa}}} \left[\frac{\bar{k}_w K_0}{(d+1)\cdot \beta_m \mu_w \, (m_0)^d} \left[\left(m_0 + \beta_m \left(p_{kk}^j - p_0 \right) \right)^{d+1} - \left(m_0 + \beta_m \left(p_{ck}^j - p_0 \right) \right)^{d+1} \right] + \right.$$

$$\left. + \frac{\bar{k}_o K_0}{(d+1)\cdot \beta_m \mu_o \, (m_0)^d} \left[\left(m_0 + \beta_m \left(p_{kk}^j - p_0 \right) \right)^{d+1} - \left(m_0 + \beta_m \left(p_{ck}^j - p_0 \right) \right)^{d+1} \right] \right] \quad k = 1, 2$$

-share of oil phase

$$(\varphi_o)_k^j = \frac{(k_o)_k^j}{\mu_o} \left[\frac{(k_o)_k^j}{\mu_o} + \frac{(k_w)_k^j}{\mu_w} \right]^{-1}$$

-share of water phase

$$(\varphi_w)_k^j = 1 - (\varphi_o)_k^j$$

After obtaining the values of the required parameters, we move on to the next time layer and so on.

Results and Discussion. To study the stress-strain state of a saturated porous medium, computational experiments were carried out for the following values of parameters: $m_0 = 0.3$, $K_0 = 4.8 \cdot 10^{-15} m^2$, $(s_o)_0 = 0.6$, $(s_w)_0 = 0.4$, $p_0 = 10^6 Pa$, $p^0 = 10^6 Pa$, $p_k^* = 9 \cdot 10^5 Pa$, $\mu_o = 1.3 \cdot 10^{-3} Pa \cdot c$, $\mu_w = 1.768 \cdot 10^{-3} Pa \cdot c$, $\beta_m = 10^{-8} Pa^{-1}$, , $x_k = 90(2k-1)$, $\Delta z_k = 10m$, $r_{ck} = 0.1m$, $r_{kk} = 90m$, $n = 2$, $k = 1, 2$

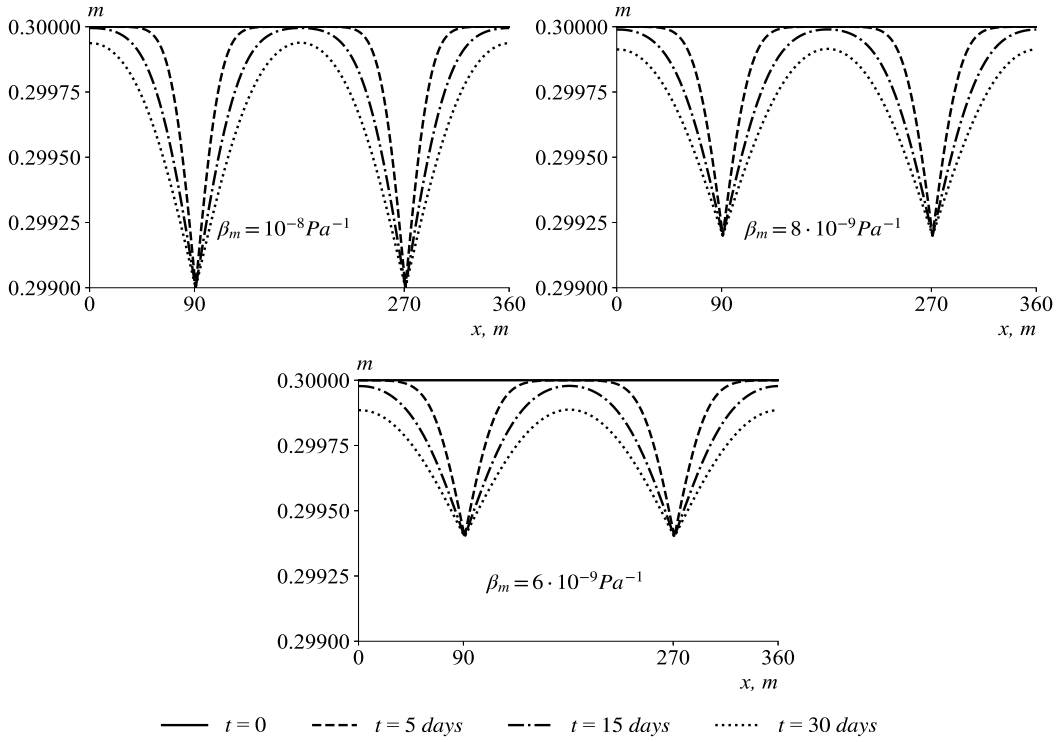


FIGURE 2. Influence of the compressibility coefficient on the porosity of the medium

On Fig 1 the pressure distribution profiles are presented for various values of the medium compressibility coefficient. It can be seen from the graphs that with a decrease in the value of β_m , the reservoir pressure decreases faster.

On Fig 2 the influence of changes in the compressibility coefficient of the medium on the porosity of the reservoir are presented. According to the results, it was noticed that the increase in the values of β_m leads to a sharp decrease in porosity in the bottomhole zone.

From Fig 3, it can be seen that a change in the compressibility coefficient of the medium has a significant effect on the absolute permeability of the formation, and in the bottomhole zone, the permeability decreases sharply at greater values of β_m.

Fig. 4 shows oil saturation for various values of the medium compressibility coefficient. It was observed that an increase in the value of the compressibility coefficient leads to a significant increase in oil saturation in the bottomhole zone.

On Fig. 5, the dynamics of the oil well debit, oil and water debits, as well as the share of oil phase are presented at various values of β_m. From the obtained results, it can be concluded that with a decrease in the compressibility coefficient, it was observed that the oil well debit, water and oil debit first increased and then decreased. As the value of β_m increases the share of oil in the medium increases.

On Fig 6, in contrast to Fig.5, the dynamics of the oil well debit, oil and water debit, the share of oil phase in the environment is presented at different values of β_m. As a result, the oil well debit will increase slightly during the 30 days under consideration. Since the share of oil first increased sharply, and then this growth slowed down, we can observe the same process for the oil debit. On the contrary, the water debit first sharply decreases, and then this decrease slows down. Unlike the results on Fig. 5, the amount of produced fluid increases as the value of β_m of at decreases the share of oil.

On Figs 7-9 show profiles of porosity, permeability and oil saturation at various bottom hole pressures. With a decrease in bottomhole pressure, a significant decrease in porosity and

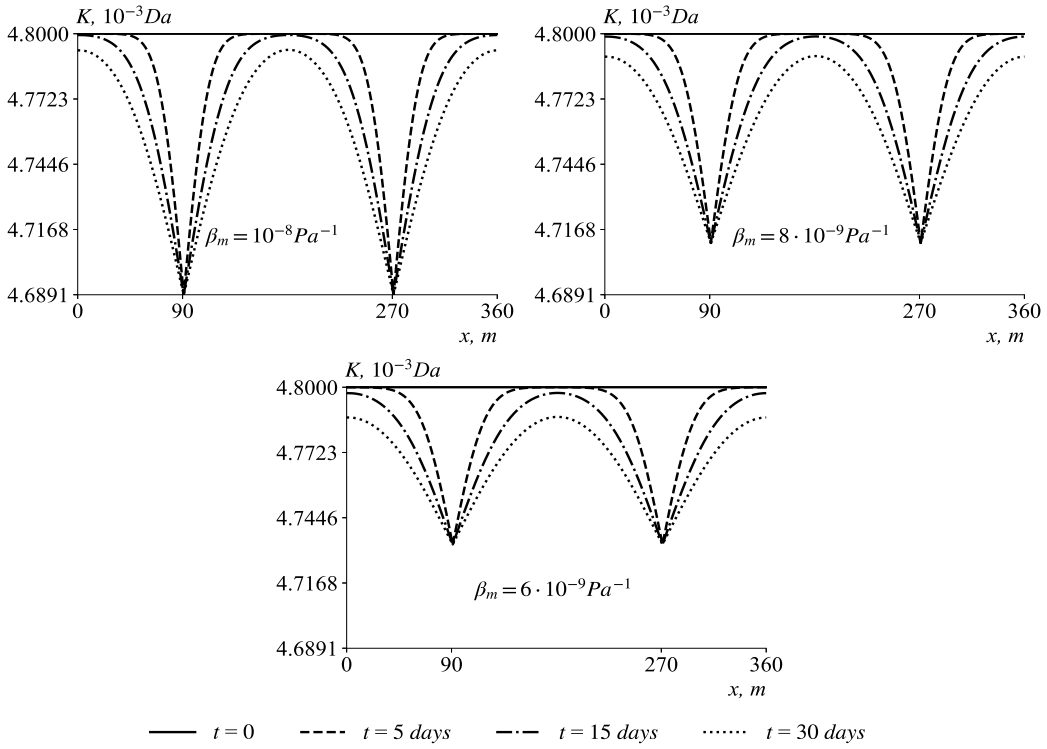


FIGURE 3. The effect of change in the compressibility coefficient on the permeability

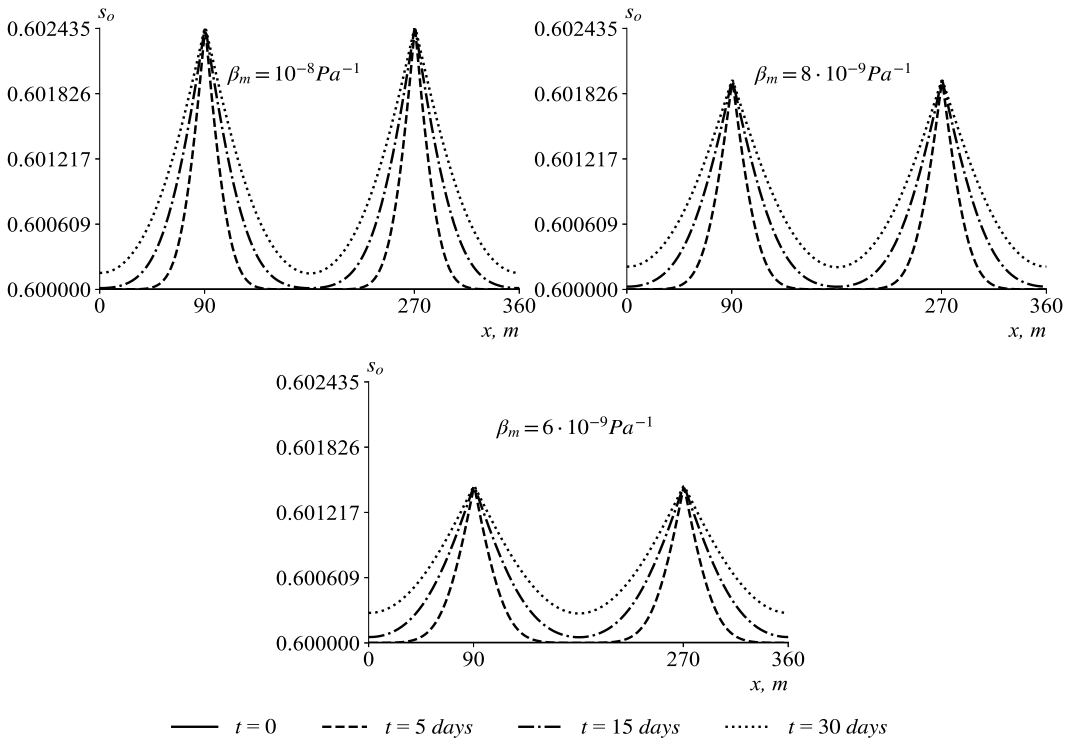


FIGURE 4. Effect of medium compressibility on oil saturation

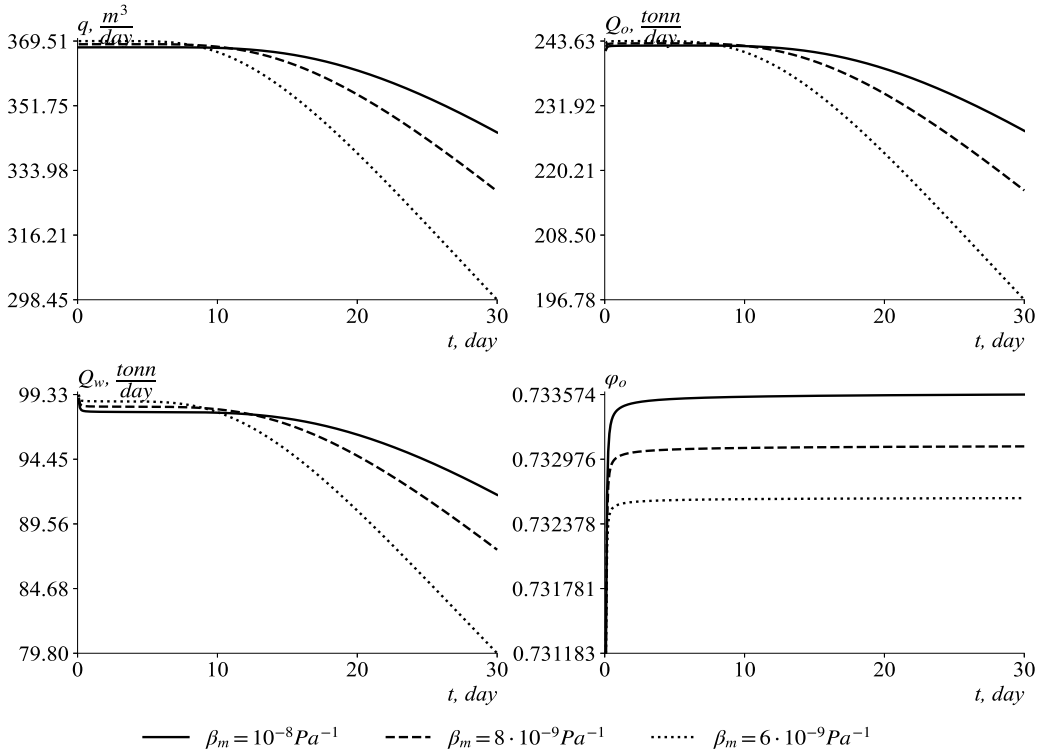


FIGURE 5. Influence of medium compressibility on phase debits and oil phase share

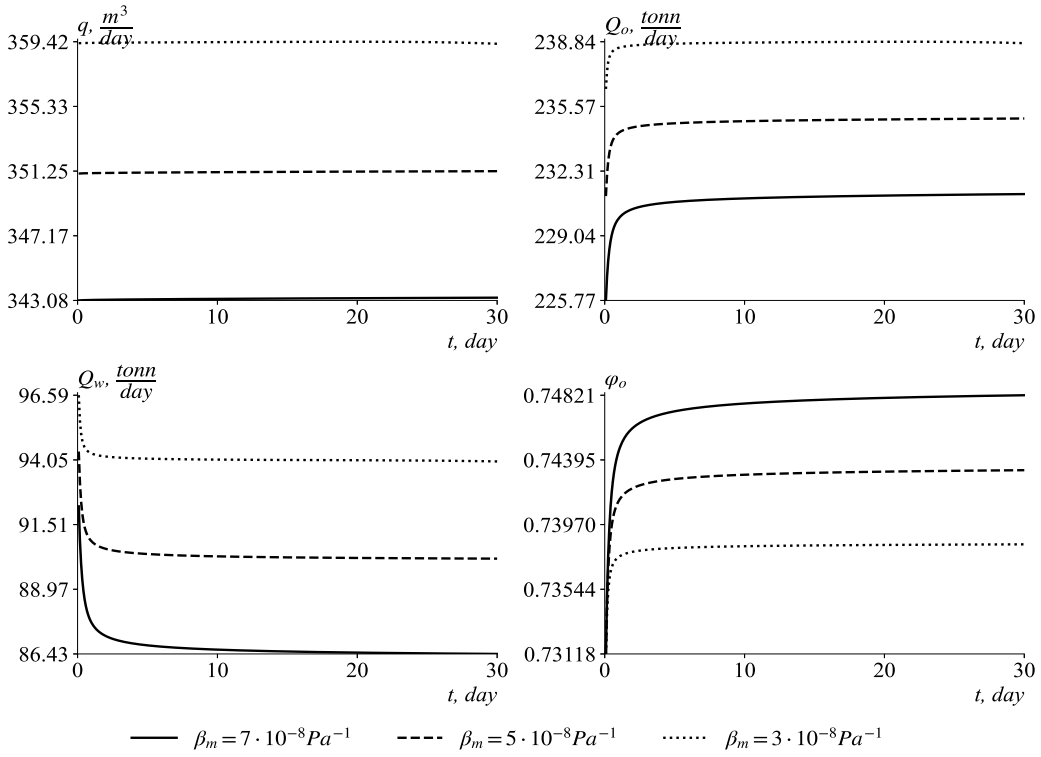


FIGURE 6. Influence of medium compressibility on phase debits and oil phase share

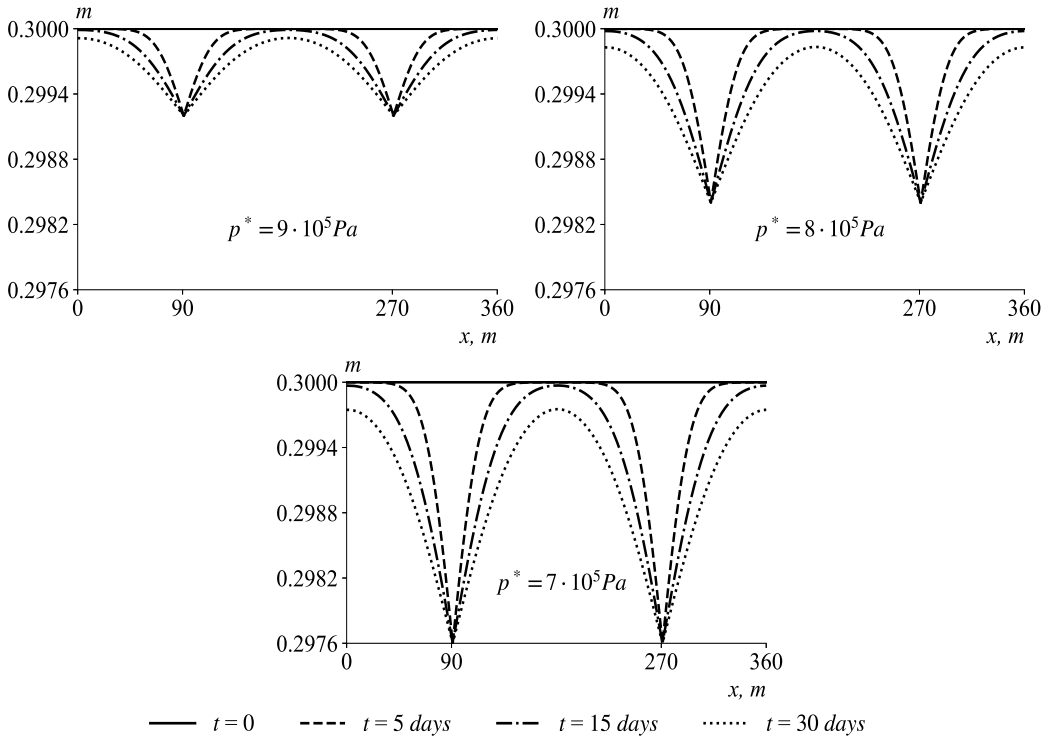


FIGURE 7. The effect of changes in bottomhole pressure on porosity

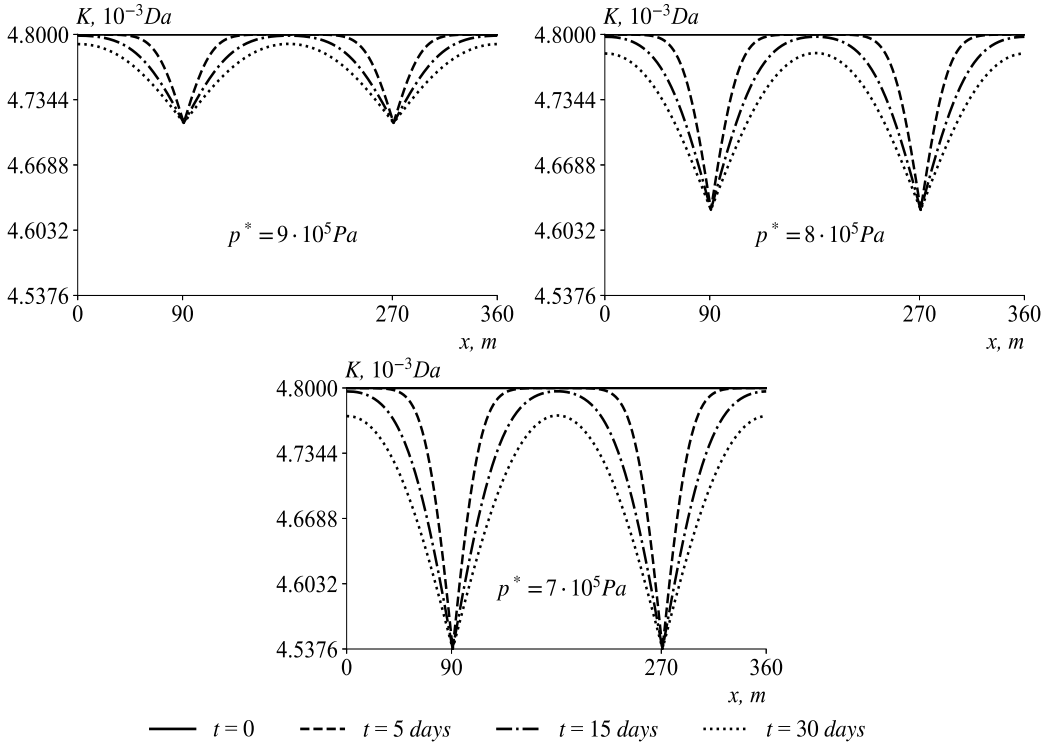


FIGURE 8. Influence of changes in bottomhole pressure on the permeability of the medium

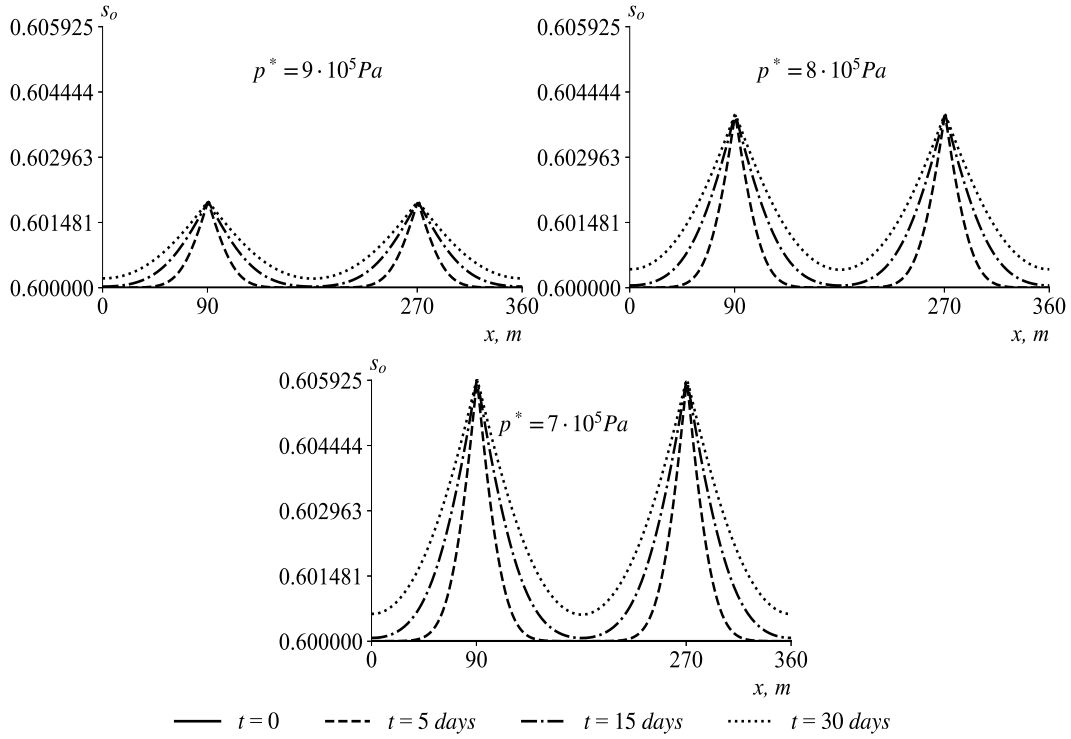


FIGURE 9. Effect of changes in bottomhole pressure on oil saturation

permeability was observed, while oil saturation increased significantly. It was noticed that these changes were especially noticeable in the bottomhole zone.

Fig 10 shows the dynamics of the effect of changes in bottomhole pressure on the oil well, oil and water debits and the share of oil. According to the results obtained, with a decrease in bottomhole pressure, all considered indicators increase significantly.

4. Conclusion

In this paper the process of two-phase filtration in a deformable porous medium mathematically modeled. A suitable problem was posed for the model, which was solved numerically by the large particle method. Based on the results obtained, the influence of compressibility of medium on reservoir pressure, porosity, permeability, oil saturation and oil production was studied. According to obtained results, an increase in the compressibility of the medium leads to a sharp decrease in the porosity and permeability of the medium in the bottomhole zone. As a result, the process of reservoir pressure decrease slows down. The oil saturation increases with the increase in the compressibility of the medium. At relatively low values of the compressibility coefficient of the medium, with its decrease, it was observed that the oil well and oil production first slightly increased, and then sharply decreased, and at relatively large values they increased evenly. It was observed that the proportion of oil decreased with increasing medium compressibility and increased with time. At the same time, the effect of changing bottomhole pressure on the filtration process. With a decrease in bottomhole pressure, the permeability and porosity of the medium decreased, while oil saturation and the share of oil increased.

(1) system of equations is solved on the initial and boundary conditions using large particles method and obtained results are analyzed.

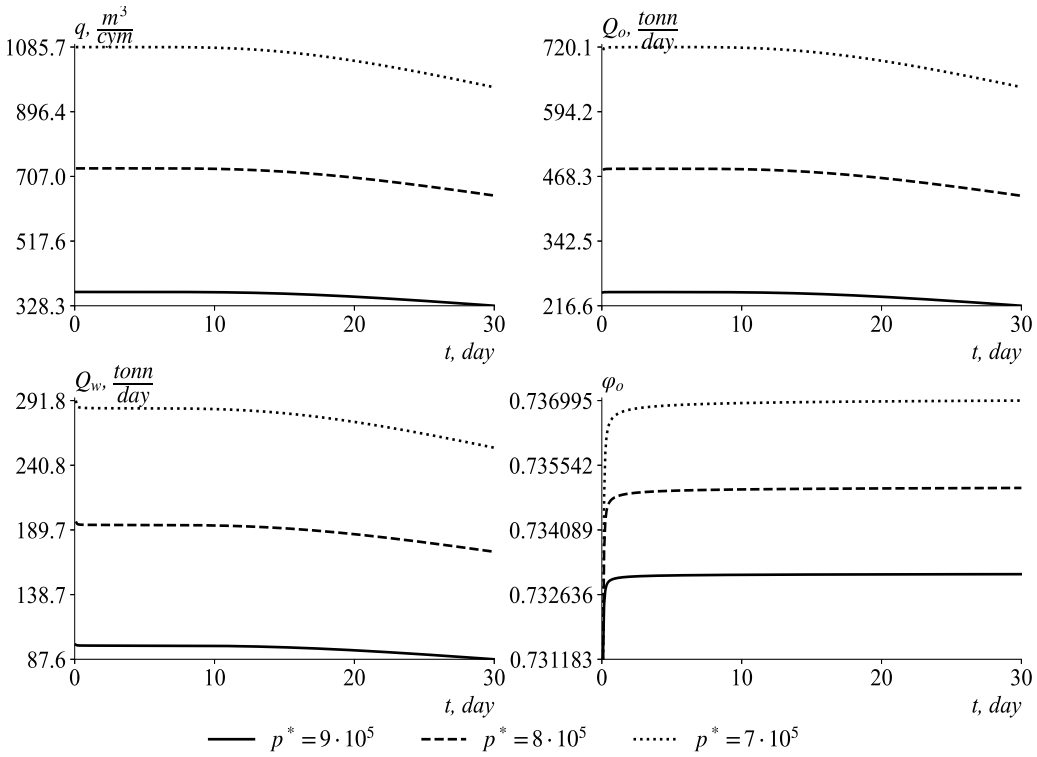


FIGURE 10. Influence of bottomhole pressure change on phase debits and phase shares

REFERENCES

[1] Azhikhanov N.T., Zhumagulov B.T., Masanov Zh.K., Bekbolatov A.B., Finite element mode ling of fluid filtration in a deformable porous medium, *IOP Conf. Series: Journal of Physics: Conf. Series*, 1425 (2020) 012137.

[2] Francisco J.C., Modeling Multiphase Flow Through and Around Multiscale Deformable Porous Materials, *A dissertation of PhD*, USA, Princeton University 2021.

[3] Samimi S., Pak A., A three-dimensional mesh-free model for analyzing multi-phase flow in deforming porous media, *Meccanica*, Vol.51, No.3, 2015, Pp.517-536.

[4] Lo W.C., Sposito G., Majer E., Immiscible two-phase fluid flows in deformable porous media, *Advances in Water Resources*, Vol.25, 2002, Pp.1105-1117.

[5] Sciarra G., Formulation of a phase field model of multiphase flow in deformable porous media, *E3S Web of Conferences*, Vol.9, 2016, 16006.

[6] Romenski E., Belozerov A.A., Peshkov I.M., Conservative formulation for compressible multiphase flows, *Quarterly of Applied Mathematics*, Vol.74, No.1, 2015, Pp.113-136.

[7] Ehlers W., Graf T., Ammann M., Deformation and localization analysis of partially saturated soil, *Comput Methods Apple Mech Eng*, Vol.193, 2004, pp.2885-2910.

[8] Lewis R.W, Sukirman Y., Finite element modeling of three-phase flow in deforming saturated oil reservoirs, *Int J Numer Anal Methods Geomech*, Vol.17, 1993, Pp.577-598.

[9] Khoei A.R., Mohammadnejad T., Numerical modeling of multiphase fluid flow in deforming porous media: a comparison between two- and three-phase models for seismic analysis of earth and rockfill dams, *Comput Geotech*, Vol.38, 2011, pp.142-166.

[10] Romenski E., Reshetova G., Peshkov I., Computational Model for Compressible TwoPhase Flow in Deformed Porous Medium, *Computational Science and Its Applications ICCSA 2021 21st International Conference Cagliari*, Italy, September 13-16, 2021 Proceedings, Part I.

[11] Carrillo F.J., Bourg I.C., Modeling multiphase flow within and around deformable porous materials: A Darcy-Brinkman- Biot approach, *Water Resources Research*, Vol.57, 2021, e2020WR028734.

[12] Perepechko L., Romenski E., Reshetova G., Kireev S., Perepechko Y., Modeling the multiphase flows in deformable porous media, *MATEC Web of Conferences*, Vol.115, 2017, 05004.

[13] Shchipanov A.A., Pressure Transient Analysis of Deformable Reservoirs, *Journal of Engineering Physics and Thermophysics*, Vol. 83, No. 2, 2010. Pp 250-262.

[14] Khuzhayorov B.Kh., Kholiyarov E.Ch., Inverse Problems of Elastoplastic Filtration of Fluid in A Porous Medium, *Journal of Engineering Physics and Thermophysics*, Vol.80, No.3, 2007, Pp 517-525.
[15] Burnashev V.F., Viswanathan K.K., Kaytarov Z.D., Mathematical Modeling of MultiPhase Filtration in a Deformable Porous Medium, *Computation*, Vol.112, No.11, 2023.
[16] Basniev K.S., Kochina I.N., Maksimov V.M., *Underground hydromechanics: A textbook for universities*, M .: Nedra, 1993, 416p.
[17] Aziz H., Settari E., *Mathematical modeling of reservoir systems. Second edition*, M.: ANO IKI, 2004, 416p.
[18] Bulgakova G.T., Baizigitova A.V., Sharifullin A. R., Model of matrix acid treatment of carbonates: influence of sediment on the dissolution process, *Ufa: UG ATU*, Vol.35, No.2, 2009, Pp.256-264.
[19] Belotserkovsky O.M., Davydov Yu.M., *Method of large particles in gas dynamics*, M. Nauka, 1982, 392p.
[20] Makhmudov J.M, Usmonov A.I., Kuljanov J.B., The problem of filtration and solute transport in a two-zone porous medium, *Cite as: AIP Conference Proceedings 2637*, 2022, 040020.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

D-ОПТИМАЛЬНЫЕ ПЛАНЫ В СЛУЧАЕ ФУНКЦИЙ ХААРА

АБИЛМАЖИН АДАМОВ[1], МАРС ГАББАСОВ[1], АЗАМАТ КУСЕБАЙ[1]

[1]Евразийский национальный университет им.Л.Н. Гумилева, Казахстан
e-mail: adam1955@mail.ru, gmarson19@gmail.com

В статье рассматривается проблема проверки заданного дискретного плана на D-оптимальность. При этом в качестве базисных функций рассматриваются кусочно-постоянные функции – функции Хаара и формулируется теоремы о распределения точек оптимального плана для D-критериев оптимальности.

Линейные по параметрам регрессионные модели широко используются на практике для описания различного рода зависимостей [2, 3, 6]. При этом целью эксперимента является нахождения оценок неизвестных параметров модели и проверка оптимальности выбранных точек распределения плана по некоторым критериям [2, 4].

Известно,что большое число алгоритмов метода Монте-Карло и планирования эксперимента основано на выборе некоторого вероятностного распределения ρ, заданного на измеримом пространстве X [1,5].

Рассмотрим линейную по параметрам регрессионную модель, в которой предполагается, что результат измерения $y(x_j)$ в точке $x_j \in X$ представляется в виде [1, 3]:

$$y_j = y(x_j, \omega_j) = \sum_{i=1}^{n} \theta_i \phi_i(x_j) + \epsilon(x_j, \omega_j), j = 1, 2, ..., N, \tag{1}$$

где $\phi(x) = \{\phi_1(x), \phi_2(x), ..., \phi_n(x)\}^T$ - вектор базисных функций на X, в нашем случае функции Хаара, θ_i - неизвестные параметры, $\epsilon(x_j, \omega_j)$ - случайные ошибки, X - множество планирования. Об ошибках предполагаются стандартные допущения [3]:

$$E\epsilon(x) = 0, E\epsilon^2(x) = \sigma^2(x) = const < \infty, E\epsilon(x_i)\epsilon(x_j) = 0, i \neq j$$

где E обозначает математическое ожидание.

Регрессионную модель (1) также можно написать в виде $Y = F\theta + \epsilon$, где

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}, F = \begin{pmatrix} \phi_1(x_1), & \dots & \phi_n(x_1) \\ \phi_1(x_2), & \dots & \phi_n(x_2) \\ \dots, & \dots & \dots \\ \phi_1(x_N), & \dots & \phi_n(x_N) \end{pmatrix} = \begin{pmatrix} \phi^r(x_1) \\ \phi^r(x_2) \\ \vdots \\ \phi^r(x_N) \end{pmatrix}, \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix}.$$

Требуется оценить неизвестные параметры θ и их дисперсии. Проверить на D-оптимальность найденный план, т.е. найти оптимальную плотность

$$\rho^* = \underset{\rho \in P}{argmin}(det[D(\rho)]) \tag{2}$$

Систему функций Хаара на множестве X с заданной (вероятностной) мерой μ определим следующим образом [4]. Разобьем множество X с мерой $\mu(X) = 1$ на 2^m непересекающихся подмножеств $d(m, s), (1 \leq s \leq 2^{m-1}, m = 1, 2, ...,)$ равной меры μ. Подмножества $d(m, s), (s = 1, 2, ..., 2^{s-1})$, определим равенством

$$d(m, s) = \left[\frac{s-1}{2^{m-1}}, \frac{s}{2^{s-1}}\right),$$

где s меняется от 1 до 2^{m-1}, а $m = 1, 2, ...$ (конечно, при $s = 2^{m-1}$ будем считать $d(m, s)$ замкнутым также и справа).

Легко видеть, что при каждом m

$$d(m, 1) \cup d(m, 2) \cup ... \cup d(m, 2^{m-1}) = [0, 1].$$

Систему функций Хаара $\chi_{ms}(x)$ удобно определять группами: группа с номером m содержит 2^{m-1} функций $\chi_{ms}(x)$, $s = 1, 2, ..., 2^{m-1}$, определяемые равенствами

$$\chi_{ms}(x) = \begin{cases} 2^{\frac{m-1}{2}}, & \text{при } x \in d(m+1, 2s-1), \\ -2^{\frac{m-1}{2}}, & \text{при } x \in d(m+1, 2s), \\ 0, & \text{при } x \notin d(m, s). \end{cases} \tag{3}$$

Пусть $k_j (j = 1, 2, ..., n)$ - количество точек, принадлежащих подмножеству $d(m, j)$ множества χ и $\sum_{j=1}^{n} k_j = N$.

Как известно [3], если матрица системы нормальных уравнений $F^T F$ невырождена, то оценка метода наименьших квадратов имеет вид

$$\widehat{\theta} = (F^T F)^{-1} F^T Y \tag{4}$$

а дисперсия

$$D\widehat{\theta} = \sigma^2 (F^T F)^{-1}$$

где

$$F^T F = \begin{pmatrix} (\phi_1, \phi_1) & (\phi_1, \phi_2) & \dots & (\phi_1, \phi_n) \\ (\phi_2, \phi_1) & (\phi_2, \phi_2) & \dots & (\phi_2, \phi_n) \\ \dots & \dots & \dots & \dots \\ (\phi_n, \phi_1) & (\phi_n, \phi_2] & \dots & (\phi_n, \phi_n) \end{pmatrix}$$

,

$$(\phi, \psi) = \sum_{j=1}^{N} \phi(x_j)\psi(x_j), x_j \in X.$$

Матрица $M = F^T F$ называется информационной матрицей для модели (1). Матрица $D\theta = \sigma^2 M^{-1}$ - дисперсионной.

Сначала для простоты исследования рассмотрим случаи $m = 1, 2$, то есть интервал $[0, 1]$ разбивается соответственно на 1 и 2 подмножества и регрессионная модель содержит 2 и 4 неизвестных параметра.

В случае $m = 1$ функции Хаара имеют вид

$$\phi_1(x) = 1, \phi_2(x) = \begin{cases} +1, & \text{при } x \in [0, \frac{1}{2}), \\ -1, & \text{при } x \in [\frac{1}{2}, 1]. \end{cases} \tag{5}$$

Пусть k_1 и k_2 - количества точек, выбранных соответственно из подмножеств $[0, \frac{1}{2})$ и $[\frac{1}{2}, 1]$ и $k_1 + k_2 = N$. Информационная матрица имеет вид

$$F^T F = \begin{pmatrix} (\phi_1, \phi_1) & (\phi_1, \phi_2) \\ (\phi_2, \phi_1) & (\phi_2, \phi_2) \end{pmatrix} = \begin{pmatrix} k_1 + k_2 & k_1 - k_2 \\ k_1 - k_2 & k_1 + k_2 \end{pmatrix}.$$

Определитель этой матрицы равен $4k_1k_2$, следовательно, для ее невырожденности необходимо и достаточно выполнение условий $k_1 \geq 1, k_2 \geq 1$.

$$(F^T F)^{-1} = \frac{1}{4} \begin{pmatrix} \frac{1}{k_1} + \frac{1}{k_2} & \frac{1}{k_1} - \frac{1}{k_2} \\ \frac{1}{k_1} - \frac{1}{k_2} & \frac{1}{k_1} + \frac{1}{k_2} \end{pmatrix}, F^T Y = \begin{pmatrix} \sum_{i=1}^{k_1} y_i + \sum_{i=k_1+1}^{k_1+k_2} y_i \\ \sum_{i=1}^{k_1} y_i - \sum_{i=k_1+1}^{k_1+k_2} y_i \end{pmatrix}.$$

Следовательно, оценка неизвестных параметров $\theta = (\theta_1, \theta_2)$ имеет вид

$$\widehat{\theta} = (F^T F)^{-1} F^T Y = \begin{pmatrix} \frac{1}{2k_1} \sum_{i=1}^{k_1} y_i + \frac{1}{2k_2} \sum_{i=k_1+1}^{k_1+k_2} y_i \\ \frac{1}{2k_1} \sum_{i=1}^{k_1} y_i - \frac{1}{2k_2} \sum_{i=k_1+1}^{k_1+k_2} y_i \end{pmatrix},$$

а дисперсия этих оценок

$$D\widehat{\theta} = \frac{\sigma^2}{4}\begin{pmatrix} \frac{1}{k_1}+\frac{1}{k_2} & \frac{1}{k_1}-\frac{1}{k_2} \\ \frac{1}{k_1}-\frac{1}{k_2} & \frac{1}{k_1}+\frac{1}{k_2} \end{pmatrix}.$$

В случае $m = 2$ имеем четыре функций Хаара: $\phi_1(x), \phi_2(x)$ из (5) и

$$\phi_3(x) = \begin{cases} +\sqrt{2}, & \text{при } x \in [0, \frac{1}{4}), \\ -\sqrt{2}, & \text{при } x \in [\frac{1}{4}, \frac{1}{2}), \\ 0, & \text{при } x \in [\frac{1}{2}, 1], \end{cases} \phi_4(x) = \begin{cases} +\sqrt{2}, & \text{при } x \in [\frac{1}{2}, \frac{3}{4}), \\ -\sqrt{2}, & \text{при } x \in [\frac{3}{4}, 1], \\ 0, & \text{при } x \in [0, \frac{1}{2}), \end{cases}$$

Пусть k_1, k_2, k_3 и k_4 - количества точек, выбранных соответственно из подмножеств $[0, \frac{1}{4}), [\frac{1}{4}, \frac{1}{2}), [\frac{1}{2}, \frac{3}{4})$ и $[\frac{3}{4}, 1]$ и $k_1 + k_2 + k_3 + k_4 = N$. Информационная матрица F^TF имеет вид

$$\begin{pmatrix} k_1+k_2+k_3+k_4 & k_1+k_2-k_3-k_4 & \sqrt{2}(k_1-k_2) & \sqrt{2}(k_3-k_4) \\ k_1+k_2-k_3-k_4 & k_1+k_2+k_3+k_4 & \sqrt{2}(k_1-k_2) & -\sqrt{2}(k_3-k_4) \\ \sqrt{2}(k_1-k_2) & \sqrt{2}(k_1-k_2) & 2(k_1+k_2) & 0 \\ \sqrt{2}(k_3-k_4) & -\sqrt{2}(k_3-k_4) & 0 & 2(k_3+k_4) \end{pmatrix}.$$

Определитель этой матрицы равен $-256k_1k_2k_3k_4$, следовательно, для ее невырожденности необходимо и достаточно выполнение условий $k_1 \geq 1, k_2 \geq 1, k_3 \geq 1, k_4 \geq 1$. Обратная матрица к информационной матрице $(F^TF)^{-1}$ имеет вид

$$\frac{1}{16}\begin{pmatrix} \sum_{i=1}^4 \frac{1}{k_i} & \frac{1}{k_1}+\frac{1}{k_2}-\frac{1}{k_3}-\frac{1}{k_4} & \sqrt{2}(\frac{1}{k_1}-\frac{1}{k_2}) & \sqrt{2}(\frac{1}{k_3}-\frac{1}{k_4}) \\ \frac{1}{k_1}+\frac{1}{k_2}-\frac{1}{k_3}-\frac{1}{k_4} & \sum_{i=1}^4 \frac{1}{k_i} & \sqrt{2}(\frac{1}{k_1}-\frac{1}{k_2}) & -\sqrt{2}(\frac{1}{k_3}-\frac{1}{k_4}) \\ \sqrt{2}(\frac{1}{k_1}-\frac{1}{k_2}) & \sqrt{2}(\frac{1}{k_1}-\frac{1}{k_2}) & 2(\frac{1}{k_1}+\frac{1}{k_2}) & 0 \\ \sqrt{2}(\frac{1}{k_3}-\frac{1}{k_4}) & -\sqrt{2}(\frac{1}{k_3}-\frac{1}{k_4}) & 0 & 2(\frac{1}{k_3}+\frac{1}{k_4}) \end{pmatrix}.$$

$$F^TY = \begin{pmatrix} \sum_{i=1}^{k_1} y_i + \sum_{i=k_1+1}^{k_1+k_2} y_i + \sum_{i=k_1+k_2+1}^{k_1+k_2+k_3} y_i + \sum_{i=k_1+k_2+k_3+1}^{k_1+k_2+k_3+k_4} y_i \\ \sum_{i=1}^{k_1} y_i + \sum_{i=k_1+1}^{k_1+k_2} y_i - \sum_{i=k_1+k_2+1}^{k_1+k_2+k_3} y_i - \sum_{i=k_1+k_2+k_3+1}^{k_1+k_2+k_3+k_4} y_i \\ \sqrt{2}(\sum_{i=1}^{k_1} y_i - \sum_{i=k_1+1}^{k_1+k_2} y_i) \\ \sqrt{2}(\sum_{i=k_1+k_2+1}^{k_1+k_2+k_3} y_i - \sum_{i=k_1+k_2+k_3+1}^{k_1+k_2+k_3+k_4} y_i) \end{pmatrix}.$$

Тогда МНК-оценки неизвестных параметров $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ определяемые формулой $\widehat{\theta} = (F^TF)^{-1}F^TY$ имеют вид

$$\begin{pmatrix} \frac{1}{4k_1}\sum_{i=1}^{k_1} y_i + \frac{1}{4k_2}\sum_{i=k_1+1}^{k_1+k_2} y_i + \frac{1}{4k_3}\sum_{i=k_1+k_2+1}^{k_1+k_2+k_3} y_i + \frac{1}{4k_4}\sum_{i=k_1+k_2+k_3+1}^{N} y_i \\ \frac{1}{4k_1}\sum_{i=1}^{k_1} y_i + \frac{1}{4k_2}\sum_{i=k_1+1}^{k_1+k_2} y_i - \frac{1}{4k_3}\sum_{i=k_1+k_2+1}^{k_1+k_2+k_3} y_i - \frac{1}{4k_4}\sum_{i=k_1+k_2+k_3+1}^{N} y_i \\ \frac{\sqrt{2}}{4k_1}\sum_{i=1}^{k_1} y_i - \frac{\sqrt{2}}{4k_2}\sum_{i=k_1+1}^{k_1+k_2} y_i \\ \frac{\sqrt{2}}{4k_3}\sum_{i=k_1+k_2+1}^{k_1+k_2+k_3} y_i - \frac{\sqrt{2}}{4k_4}\sum_{i=k_1+k_2+k_3+1}^{N} y_i \end{pmatrix},$$

а дисперсия этих оценок $D\widehat{\theta} = \sigma^2(F^TF)^{-1}$.

Этот же результат получим другим способом. Для этого введем в рассмотрение систему характеристических функций $f_1(x), f_2(x), f_3(x), f_4(x)$ отрезков $d(2,1) = [0, \frac{1}{4}), d(2,2) = [\frac{1}{4}, \frac{1}{2}), d(2,3) = [\frac{1}{2}, \frac{3}{4}), d(2,4) = [\frac{3}{4}, 1]$ следующим образом:

$$f_i(x) = \begin{cases} 1, & \text{если } x \in d(2,i), \\ 0, & \text{если } x \notin d(2,i). \end{cases} \tag{6}$$

Тогда очевидно, что функции Хаара выражаются через характеристические функции $f_i(x), i = 1, 2, 3, 4$, в следующем виде:

$$\phi_1(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x),$$
$$\phi_2(x) = f_1(x) + f_2(x) - f_3(x) - f_4(x),$$
$$\phi_3(x) = \sqrt{2}(f_1(x) - f_2(x)),$$

$$\phi_4(x) = \sqrt{2}(f_3(x) - f_4(x)),$$

или

$$\phi(x) = Lf(x), \tag{7}$$

где

$$L = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}, \tag{8}$$

$$\phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x))^T, f(x) = (f_1(x), f_2(x), f_3(x), f_4(x))^T.$$

Подставляя (7) в (1) получим

$$Y = \theta^T F + \epsilon = \theta^T LG + \epsilon = \theta^T_{(!)} G + \epsilon, \tag{9}$$

где

$$G = \begin{pmatrix} f_1(x_1), & f_2(x_1) & f_3(x_1) & f_4(x_1) \\ f_1(x_2), & f_2(x_2) & f_3(x_2) & f_4(x_2) \\ \dots, & \dots & \dots & \\ f_1(x_N), & f_2(x_N) & f_3(x_N) & f_4(x_N) \end{pmatrix},$$

$\theta = (\theta_1, \theta_2, \theta_3, \theta_4)^T$ - неизвестные параметры в случае системы Хаара, $\theta_{(1)} = (\theta_1^{(1)}, \theta_2^{(1)}, \theta_3^{(1)}, \theta_4^{(1)})^T$ - в случае системы характеристических функций.

Отсюда имеем

$$\theta^T L = \theta^T_{(1)}, \theta^T = \theta(1)^T L^{-1}, \theta = (L^{-1})^T \theta_{(1)}.$$

Если теперь $\widehat{\theta}_{(1)}$ - МНК-оценка неизвестных параметров при любом плане ξ с дисперсионной матрицей $D\widehat{\theta}_{(1)}$, тогда согласно теореме 1.4 [3] имеем, что оценка $\widehat{\theta} = (L^{-1})^T \theta_{(1)}$ также является МНК-оценкой и дисперсия вычисляется по формуле

$$D\widehat{\theta} = (L^{-1})^T d\widehat{\theta}_{(1)} L^{-1}. \tag{10}$$

Отсюда

$$detD\widehat{\theta} = det(F^T F)^{-1} = (detL)^{-2} detD\widehat{\theta}_{(1)}, \tag{11}$$

что совпадает с ранее полученным результатом, так как $detL = -16$.

Последний способ получения МНК-оценок для системы функций Хаара позволяет распространить получение МНК-оценок в случае функций Хаара на общий случай. Учитывая тот факт, что функции Хаара являются линейными комбинациями ступенчатых (характеристических) функций типа (6), то более общий случай, когда количество базисных функций больше четырех, можно изучить с помощью полученных ранее результатов для ступенчатых функций [5].

В общем случае произвольного m мы имеем $n = 2^m$ функций Хаара, заданных на интервале $[0, 1]$, который разбит на 2^m подинтервалов $d(m, s), s = 1, 2, ..., 2^m$, одинаковой длины. Рассмотрим систему характеристических функций $f_s(x)$, определенную равенствами

$$f_i(x) = \begin{cases} 1, & \text{если } x \in d(m, i), \\ 0, & \text{если } x \notin d(m, i). \end{cases} \tag{12}$$

Тогда базисные функции Хаара можно выразить в виде (7), где

$$\phi(x) = (\phi_1(x), \phi_2(x), \dots, \phi_n(x))^T, f(x) = (f_1(x), f_2(x), \dots, f_n(x))^T.$$

а матрица L имеет вид

$$L = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & 1 & \dots & -1 & -1 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 2 & 2 & 2 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -2^{\frac{m-1}{2}} & 0 \\ 0 & 0 & 0 & 0 & \dots & 2^{\frac{m-1}{2}} & -2^{\frac{m-1}{2}} \end{pmatrix}. \quad (13)$$

и обладает свойством $L^{-1} = \frac{1}{2^m} L^T$.

Подставляя (7) в (1) получим

$$Y = \theta^T F + \epsilon = \theta^T LG + \epsilon = \theta^T_{(!)} G + \epsilon, \quad (14)$$

где

$$G = \begin{pmatrix} f_1(x_1), & f_2(x_1) & \dots & f_n(x_1) \\ f_1(x_2), & f_2(x_2) & \dots & f_n(x_2) \\ \dots, & \dots & \dots & \\ f_1(x_N), & f_2(x_N) & \dots & f_n(x_N) \end{pmatrix},$$

$\theta = (\theta_1, \theta_2, \dots, \theta_n)^T$ - неизвестные параметры в случае системы Хаара, $\theta_{(1)} = (\theta_1^{(1)}, \theta_2^{(1)}, \dots, \theta_n^{(1)})^T$ - в случае системы характеристических функций.

Отсюда имеем

$$\theta^T L = \theta^T_{(1)}, \theta^T = \theta(1)^T L^{-1}, \theta = (L^{-1})^T \theta_{(1)}.$$

Если теперь $\widehat{\theta}_{(1)}$ - МНК-оценка неизвестных параметров при любом плане ξ с дисперсионной матрицей $D\widehat{\theta}_{(1)}$, тогда согласно теореме 1.4 [3] имеем, что оценка $\widehat{\theta} = (L^{-1})^T \theta_{(1)}$ также является МНК-оценкой и дисперсия вычисляется по формуле (10), а определитель по формуле (11).

Так как определитель информационной матрицы $M(\widehat{\theta}_{(1)})$ в случае ступенчатых функций лекго вычисляется и равен [5]

$$detM(\widehat{\theta}_{(1)}) = n^n k_1 k_2 ... k_n, \quad (15)$$

то данный определитель достигает своего максимального значения при $k_1 = k_2 = ... = k_n$.

Таким образом, если число точек N кратно числу неизвестных параметров $n = 2^m$, то для решения задачи справедлива следующая теорема.

Теорема 1. Пусть $X = [0, 1], n = 2^m, \{\phi_i(x)\}(i = 1, 2, ..., n)$ - система функций Хаара. Рассмотрим линейную регрессионную модель (1) $Y = F\theta + \epsilon$. Если количество точек измерения N кратно n, то есть $N = ln$, то определитель (15) достигает своего максимального значения при $k_1 = k_2 = ... = k_n = l$.

Теорема 2. Пусть $X = [0, 1], n = 2^m, \{\phi_i(x)\}(i = 1, 2, ..., n)$ - система функций Хаара. Рассмотрим линейную регрессионную модель (1) $Y = F\theta + \epsilon$. Пусть теперь число точек измерения N не является кратным числу разбиения n, то есть $N = ln + r$, где $0 < r < n$. Тогда в оптимальном плане все k_i отличаются друг от друга не более, чем на единицу т.е.

$$k_1 = k_2 = ... = k_r = l + 1, k_{r+1} = k_{r+2} = ... = k_n = l.$$

Число таких планов равно C_n^r.

Доказательство. Действительно, для этого упорядочим значения $k_1, k_2, ..., k_n$ и получим вариационный ряд

$$k_1 \le k_2 \le ... \le k_n,$$

где

$$k_1k_2...k_n = \frac{1}{n^n} detM(\xi).$$

Пусть $k_n - k_1 > 1$. Выберем новый план, в котором все k_i, за исключением k_1 и k_n остаются прежними, а k_n заменяем на $k_n - 1$, k_1 заменяем на $k_1 + 1$. Для нового плана определитель равен $(k_1 + 1)k_2...k_{n-1}(k_n - 1) > k_1k_2...k_n$ т.е. исходный план неоптимален, поэтому наше предположение неверно. Таким образом, в оптимальном плане все k_i отличаются друг от друга не более, чем на единицу. Число таких планов равно C_n^r. Одним из таких планов является

$$\xi = \begin{pmatrix} x_1 & x_2 & \dots & x_r & x_{r+1} & \dots & x_n \\ \frac{l+1}{N} & \frac{l+1}{N} & \dots & \frac{l+1}{N} & \frac{l}{N} & \dots & \frac{l}{N} \end{pmatrix}.$$

Теорема 2 доказана.

Таким образом, построен D-оптимальный план для системы функций Хаара для линейной по параметрам регрессионной модели.

Авторы выражают благодарность за финансовую поддержку фундаментальных исследований Министерству науки и высшего образования Республики Казахстан (номер гранта 20280990).

Список литературы

[1] Федоров В.В. Теория оптимального эксперимента.М., 1972.
[2] Ермаков С.М. Метод Монте-Карло и смежные вопросы. М. Наука, 1975 г.
[3] Ермаков С.М., Жиглявский А.А. Математическая теория оптимального эксперимента. М. Наука, 1987 г.
[4] Соболь И.М., Многомерные квадратурные формулы и функции Хаара. М.: Наука, 1969 г.
[5] Adamov A.A.,Kozybaev D.K., Kussebay A.D – optimal plans in the case of a piecewise constant function. International Scientific Jornal Mathematical Modeling, YEAR YI, ISSUE 4/2022, Sofia, Bulgaria, 103-105

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ПРИМЕНЕНИЕ МЕТОДОВ МАШИННОГО ОБУЧЕНИЯ ДЛЯ АНАЛИЗА ДАННЫХ СОЦИАЛЬНО-ЭКОНОМИЧЕСКИХ ОПРОСОВ

ГУЛЬБАНУ РЫСМЕНДЕЕВА[1]

[1]Казахский национальный исследовательский технический университет имени К.И. Сатпаева, Алматы, Казахстан
e-mail: g.rysmendeyeva@satbayev.university

1. Введение

Для обеспечения содержания информационных систем принятия решений в процессе управления активами физических лиц требуется разработка математических моделей сложных социальных систем. Изучение ожиданий молодежи в социально-экономических вопросах имеет большое значение для понимания будущего развития государства, для выработки стратегии в области социальной политики. Важным приоритетом на протяжении жизненного цикла личности является брак, для устойчивости которого важно материальное благополучие. Взросление тесно связано с решением жилищной проблемы. Стратегической целью университета является подготовка высокооплачиваемых специалистов, способных развивать страну и поддерживать благополучие своей семьи.

Цель данной работы изучение факторов, важных для взросления и благополучия. В работе с помощью методов машинного обучения исследуются социально-экономические проблемы с точки зрения студентов первого курса университета. Изучается влияние различных факторов на принятие решения относительно ожидаемого возраста вступления в брак и решения жилищной проблемы, ожидаемого дохода от трудовой деятельности. Планирование ожидаемого дохода является одним из шагов алгоритма планирования семейного благосостояния [1-3].

2. Методология

2.1. **Сбор и подготовка данных.** В данной работе используются модели обучения с учителем для размеченных данных и модель без учителя для кластеризации студентов. Они помогают выделить факторы, разделяющие студентов на кластеры и классы. Необходимые для моделей обучения данные собираются путем анонимного опроса студентов с помощью онлайн анкетирования через Google формы.

Одной из основных задач является систематизация и обработка данных опроса. Данные анкетных форм требуют предварительной подготовки для проведения анализа данных, выбора признаков для обучения моделей, оценки эффективности прогнозирования. Предварительная обработка данных опроса осуществляется методами интеллектуального анализа данных с помощью языка программирования Python и таблиц Excel. Объем данных – 90 анкет студентов начальных курсов разных специальностей, добровольно заполнивших анкеты. Получены разноообразные данные: структуированные, полуструктуированные, неструктуированные. Традиционно прогнозирование базируется на цифрах, показателях, коэффициентах с использованием статистики и математического моделирования. В последнее время, как правило, используют методы машинного обучения (ММО), поскольку использование неструктуированных данных позволяет делать прогнозы быстрее и глубже [4-7].

Подготовка и ранжирование данных включает очистку и организацию исходных данных в консолидированный формат, чтобы результирующий набор данных подходил для дальнейшего анализа и обучения модели. Качество данных влияет на обучение выбранной модели МО. Поскольку данных немного, то удаление неправильно заполненных анкет не производилось. Пропущенные, пустые ответы заменялись минимальными, максимальными или средними значениями. Данные, выходящие за три стандартных отклонения можно было удалить из набора данных, как например ожидаемый доход в 25 млн. тенге в месяц, однако по утверждению студентов такой ежемесячный доход существует. Поскольку размер данных относительно небольшой, то выходящие за три стандартных отклонения данные остались в наборе данных.

Чтобы данные были пригодны для анализа, проводится их оцифровка. Агрегирование схожих данных в одну переменную не производилось, хотя часть вопросов были на оценку ценностных факторов семейного благополучия, а остальные относились к материальным и временным факторам. Например, возраст женитьбы, возраст приобретения своего жилья, отношение к работе, супругу, доходу. Для агрегирования либо выделения факторов, которые не оказывают влияния и не связаны с выходными данными требуются дополнительные экспертные знания. Внешнюю экспертизу не привлекали, поэтому все собранные ответы оставили без агрегирования или удаления. Полученные переменные относятся к разным типам: упорядоченные, непрерывные, категориальные. Например, доход – непрерывный, выбор жить или нет с родителями – категориальный, оценка степени важности недвижимости – дискретный и упорядоченный. В результате преобразования текстовых данных в цифровые данные получили 90 наблюдений и 56 признаков.

Полученные данные не только разного типа, но и лежат в разных диапазонах значений. Нормализация чувствительна к выбросам и используется, когда распределение данных неизвестно. Стандартизация менее чувствительна выбросам, так как зависит от среднего значения и стандартного отклонения. В данной работе данные были стандартизированы по формуле

$$x_{norm} = (x - \mu)/\sigma$$

2.2. **Предварительный анализ данных.** Визуализация данных в виде гистограмм позволяет сделать предварительный анализ отношения студентов к интересующим вопросам. На рис. 1 видно, что большинство современных молодых людей предпочитают вступать в брак после того, как купят дом, машину и обустроятся. Они предпочитают заключить брак в 23 года либо 28 лет. Меньшинство хотели бы заключить брак в возрасте до 21 года либо после 33. На рис. 2 показано, что большинство студентов хотят зарабатывать не ниже 400 тыс. тенге в месяц. Немногие поддерживают доход в размере 150-300 тыс. тенге. Среди студентов есть те, кто хотят зарабатывать более 1 млн. тенге. На рис. 3 видно, что подавляющее большинство считают, что квартира есть самое важное имущество, которое может быть нажито супругами во время брака. На рис. 4 представлен опрос студентов о том, на что тратит семья большую часть своего дохода. По результатам опроса подавляющее большинство студентов считают, что это покупка недвижимости.

3. Результаты расчетов

3.1. **Метод ближайших соседей.** Диаграмма рассеяния и метод ближайших соседей используются для понимания связей между двумя признаками. На рис. 5-8 показаны результаты, полученные по выборочным признакам, указанным в таблице 1. На рис. 5 четких связей между признаками $x52$ и $x53$ не видно. На рис. 6 разделение проходит по ожидаемому возрасту вступления в брак 27 лет. На рис. 7 деление происходит относительно мнения, что накопление на депозите неэффективны и затраты на покупку недвижимости – не самые



Рис. 1. Распределение количества студентов по возрасту заключения брака



Рис. 2. Распределение количества студентов по ожидаемому доходу

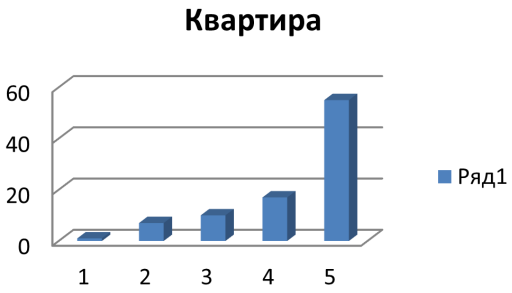


Рис. 3. Распределение количества студентов в отношении совместного имущества супругов



Рис. 4. Распределение количества студентов в отношении крупных затрат семьи

крупные семейные затраты. На рис. 8 разделение мнений происходит относительно пункта, что самое важное – это передать наследникам здоровые гены. На рис. 9-10 видно, что границы разделения принимают сложные формы.

Таблица 1. Список отдельных признаков

Вопрос анкеты	Признак	Интервалы значений
Оцените долгосрочные обязательства семьи	$x1$ =покупка недвижимости	$\{1,2,3,4,5\}$ 1-незначительные, 5-самые крупные
Оцените долгосрочные обязательства семьи	$x2$ =финансирование образование детей	$\{1,2,3,4,5\}$ 1-незначительные, 5-самые крупные
Оцените способ выполнения долгосрочных финансовых обязательств семьи	$x6$ =регулярные взносы на депозит	$\{1,2,3,4,5\}$ 1-малоэффективный, 5-самый эффективный

Оцените по степени важности, что бы вы хотели получить/передать в наследство?	$x29$ =здоровые гены	$\{1, 2, 3, 4, 5\}$ 1-самые незначительные, 5-самые крупные
Инфляция делает непрактичными любые планы семьи на 30-40 лет в тенге. Оцените единицу измерения стоимости.	$x37$ =Продукты питания (хлеб, вода)	1-самая нестабильная, 5-самая стабильная
Планируете ли вы обращаться за материальной помощью?	$x49$	$\{0, 1\}$ 0-нет, 1-да
Готовы ли вы делиться с супругом/супругой доходами?	$x50$	$\{1, 2, 3\}$ 0-нет, 1-да, 3-не знаю
В каком возрасте планируете заключить брак?	$x51$	$[20, 50]$
Сколько родители тратят в среднем в месяц на содержание студента?	$x52$	Выше 20 тыс. тг.
В каком возрасте планируете решить жилищный вопрос своей семьи?	$x53$	$[20, 50]$
Если считаете, что помощь материальная будет необходима, то к кому обратитесь?	$x54$	$\{1, 2, 3, 4\}$
Какой доход в тенге от трудовой/предпринимательской деятельности планируете получать в месяц?	$x55$	Выше 100 тыс. тг.

3.2. **Кластеризация.** Кластеризация методом k-средних используется для определения $k-$групп студентов со схожими ответами на вопросы анкеты. В этом случае заранее неизвестна структура разделения в отличие от классификации по заранее размеченному признаку. Был использован метод локтя, помогающий выбрать оптимальное количество кластеров для задачи кластеризации. На рис. 11 показан график динамики метрики расстояния между кластерами, из которого следует, что оптимально разбить студентов на три кластера с учетом всех 56 признаков. На рис. 12 приводится двумерная визуализация трех кластеров по двум первым признакам, так как невозможно отобразить на графике 56 признаков. На рис. 12 показаны центройды трех кластеров по координатам x = признак $x1$ и y = признак $x2$ (см. Таблица 1). Можно заключить, что студенты, которые считают финансирование образование детей как незначительные затраты попадают в первый кластер. Остальные, кто

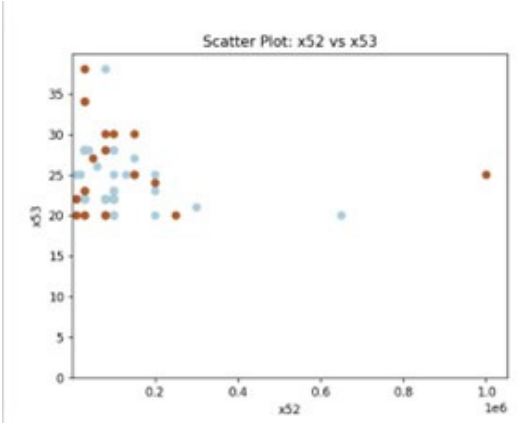


Рис. 5. Карта рассеяния двух признаков $x52$ и $x53$

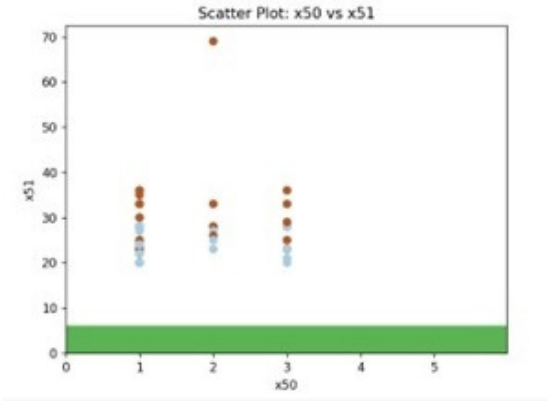


Рис. 6. Карта рассеяния двух признаков $x50$ и $x51$.

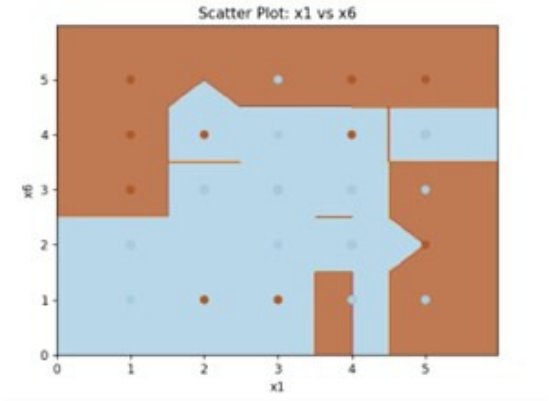


Рис. 7. Деление на два класса по признакам $x1$ и $x6$

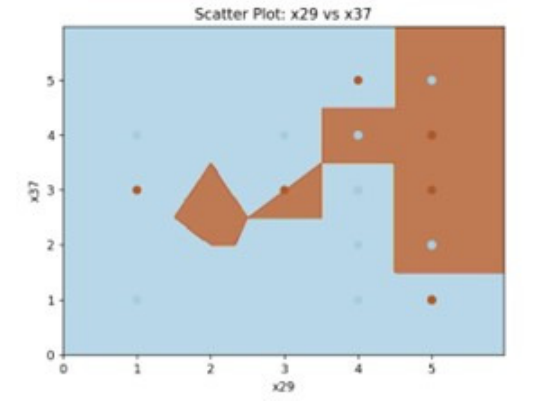


Рис. 8. Деление на два класса по признакам $x29$ и $x37$

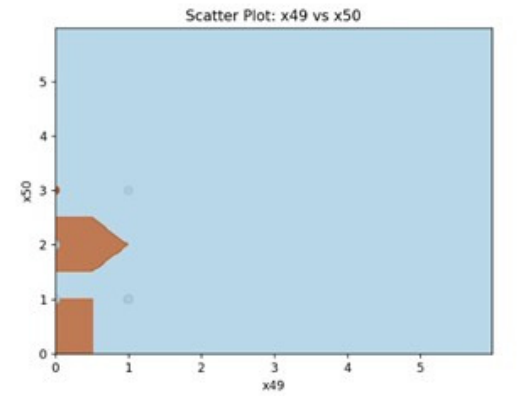


Рис. 9. Деление на два класса по признакам $x49$ и $x50$

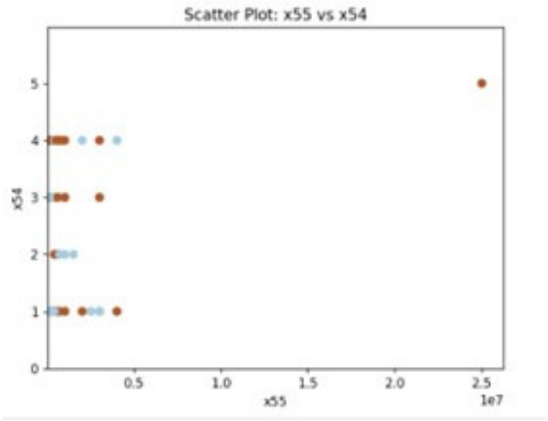


Рис. 10. Карта рассеяния двух признаков $x54$ и $x55$

с этим не согласен, делятся на две группы по другому признаку. Так второй кластер оценивает затраты на покупку недвижимости по шкале 3 и ниже, третий кластер – по шкале 4-5.

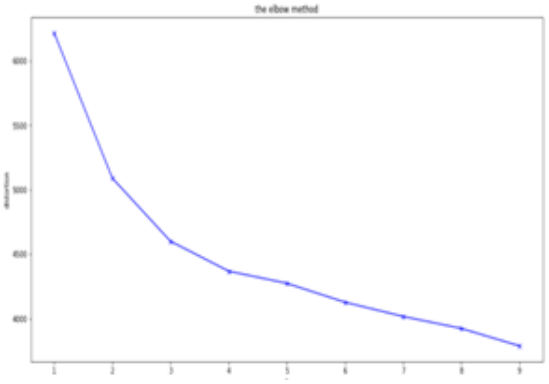


Рис. 11. Метод локтя



Рис. 12. Диаграмма рассеяния трех кластеров по признакам $x1$ и $x2$

3.3. **Классификация.** Выбор метода обучения зависит от задачи классификации, типа данных и размера данных. Большинство методов машинного обучения предполагают обучение на структурированных данных, поэтому неструктурированные данные на подготовительном этапе были обработаны и организованы в структурированный формат. В нашем наборе данных 90 наблюдений и 56 признаков, поэтому было решено использовать несколько методов, а результаты сравнить для изучения преимуществ каждого метода. В данной работе важнейшим фактором семейного благополучия считается ожидаемый доход студента. Поэтому в качестве выходных данных модели классификации был принят прогноз ожидаемого дохода студента. Модель прогнозирования дохода студента может быть использована в процессе обучения и подготовки высокооплачиваемых специалистов. Собранные из анкет данные ожидаемого дохода были размечены и разбиты на два класса: метка 1, если ожидаемый доход не менее 500 тыс. тг. и метка 0, если ниже 500 тыс. тг. Как видно на рис. 13 такое разбиение делит студентов на два почти равных класса.

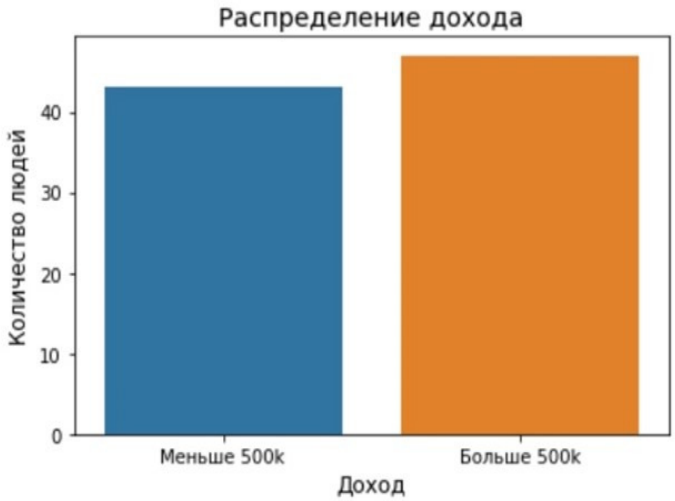


Рис. 13. Выходные данные

В работе использовали три метода классификации: логистическая регрессия, нейронные сети (NN) и метод SVM. Для применения методов классификации 80% данных использовались в качестве обучающих данных, 20% – тестовых данных. Результаты прогнозов на

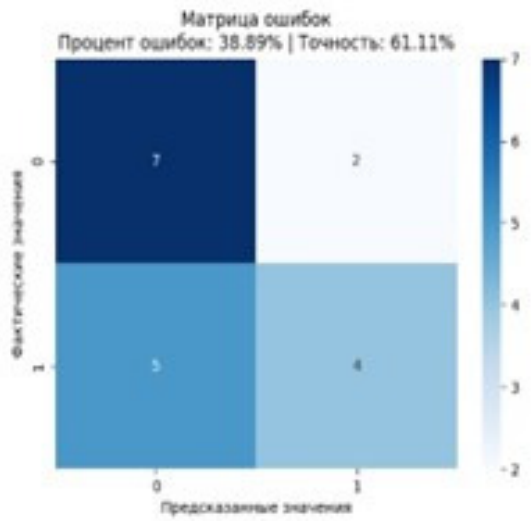


Рис. 14. Матрица ошибок логистической регрессии

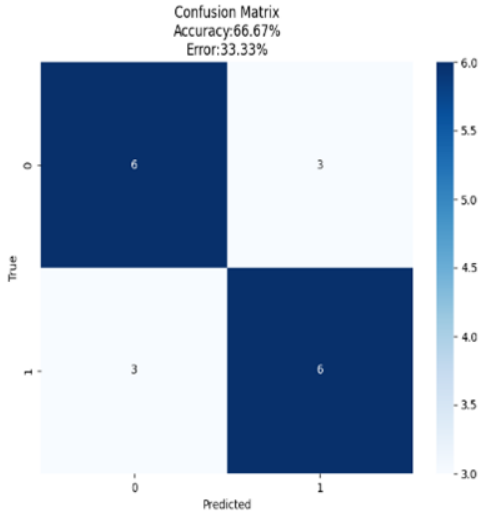


Рис. 15. Матрица ошибок нейросетевого метода

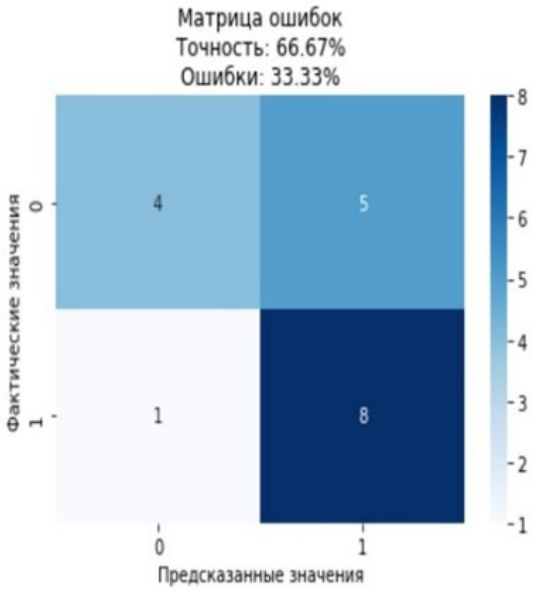


Рис. 16. Матрица ошибок, метода SVM

тестовых данных представлены в матрице ошибок (рис. 14-16). Для тестовых данных анализ ошибок показал, что логистическая регрессия лучше работает на уменьшение ошибки I-рода, а метод SVM лучше в уменьшении ошибки II-го рода. Метод NN является оптимальным.

В таблице 2 даны показатели эффективности классификации каждого метода, из которой видно, что с учетом размера выборки, разнородных типов данных, соотношения количества наблюдений и признаков методы NN и SVM дают лучшие результаты, чем логистическая регрессия. Более того, метод NN дает сбалансированные показатели эффективности в смысле ошибок I-рода и II-го рода, что можно объяснить, тем, что в наборе данных количество наблюдений превышало количество признаков. В дополнение большинство признаков

имело низкую вариативность, так как принимало значения из ограниченного множества $\{1, 2, 3, 4, 5\}$. Таким образом, точность прогноза модели обучения прогнозу ожидаемого дохода по данным опроса составил 67%. Это выше, чем просто угадывание 50%, но недостаточно высокий результат.

Таблица 2. Показатели эффективности методов классификации

Показатели точности	Логистическая регрессия	Нейросетевой метод	Метод SVM
Precision	0.67	0.67	0.62
Recall	0.44	0.67	0.89
F1 score	0.53	0.67	0.73
Accuracy	0.61	0.67	0.67

Таким образом, что в работе проводится сравнительный анализ эффективности методов классификации: логистической регрессии, нейронных сетей, опорных векторов – для конкретной задачи классификации студентов, где выходной признак – ожидаемый доход, а входные признаки – ответы на социально-экономические вопросы анкеты. Полученные модели позволяют изучать паттерны поведения и проводить классификацию и кластеризацию студентов по ожидаемому доходу в зависимости от ожидаемого возраста вступления во взрослую жизнь, финансовых возможностей родителей студентов и других факторов.

Ключевые слова: машинное обучение, социальные системы, жизненный цикл, принятие решения, модель прогнозирования, кластеризация, классификация.

AMS Subject Classification: Computer science.

Список литературы

[1] Рысмендеева Г.С., Моделирование возраста взросления в зависимости от экономических факторов с использованием методов машинного обучения, *International Satbayev Conference*, Том 4, 2023, С. 148-160.

[2] Рысмендеева Г.С., Разработка математических моделей для информационной системы принятия решений в процессе управления активами, *Вестник КазНИТУ*, No. 3, 2021, С. 65-75.

[3] Рысмендеева Г.С., Разработка визуальных моделей информационной системы принятия решений в процессе управления активами физических лиц, *Вестник КазНИТУ*, No. 5, 2020, С. 434-439.

[4] Zakharova I.G., Machine learning methods of providing informational management support for students' professional development, *OBRAZOVANIE I NAUKA-EDUCATION AND SCIENCE*, Vol. 20, No. 9, 2018, pp. 91-114

[5] Dhruvil Shah, Devarsh Patel, Jainish Adesara, Pruthvi Hingu, Manan Shah, Integrating machine learning and blockchain to develop a system to veto the forgeries and provide efficient results in education sector, *Visual Computing for Industry Biomedicine and Art*, Vol. 4, No. 1, Article number: 18, 2021.

[6] Shuo-Chang Tsai, Cheng-Huan Chen, Yi-Tzone Shiao, Jin-Shuei Ciou, Trong-Neng Wu, Precision education with statistical learning and deep learning: a case study in Taiwan, *International Journal of Educational Technology in Higher Education*, Vol. 17, No. 12, 2020.

[7] Zubarev Andrey, Bekirova Olga, Analysis of Bank Default Factors in 2013-2019, *Ekonomicheskaya politika*, Vol. 15, No. 3, 2020, pp. 106-133.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ЭТАПЫ РЕШЕНИЯ ЗАДАЧ МЕТОДОМ АНАЛИЗА ИЕРАРХИЙ В ПРОГРАММЕ MPRIORITY 1.0

ДИЛФУЗА АКАБИРХОДЖАЕВА[1]

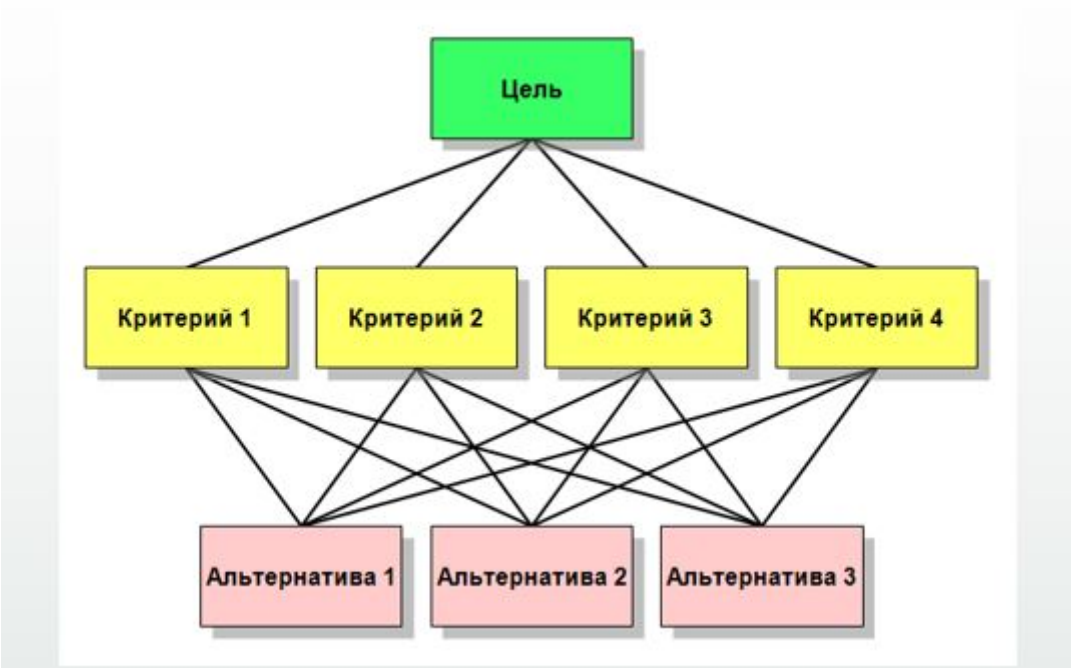
[1]Университет мировой экономики и дипломатии, Ташкент, Узбекистан
e-mail: dakabirxodjayeva@uwed.uz

Аннотация. Процессы принятия решений лежат в основе любой целенаправленной деятельности. Необходимость принимать решения в условиях неопределенности возникает во всех областях техники, экономики и социальной жизни. Поэтому необходимо стремиться к оптимальному использованию имеющейся информации и, взвесив все возможные варианты решений, постараться найти среди них лучший.
Метод анализа иерархий (МАИ) разработан американским ученым Т. Саати [1]. Он обеспечивает с помощью простых и обоснованных правил решение многокритериальных задач, которые содержат качественные и количественные факторы, при этом количественные факторы могут иметь разную размерность. МАИ используется для решения слабоструктурированных и неструктурированных проблем.
Метод анализа иерархий (МАИ) — математический инструмент системного подхода к сложным проблемам принятия решений. МАИ не предписывает лицу, принимающему решение, какого-либо «правильного» решения, а позволяет ему в интерактивном режиме найти такой вариант (альтернативу), который наилучшим образом согласуется с его пониманием сути проблемы и требованиями к её решению.
МАИ широко используется на практике и активно развивается учеными всего мира. МАИ позволяет понятным и рациональным образом структурировать сложную проблему принятия решений в виде иерархии, сравнить и выполнить количественную оценку альтернативных вариантов решения. Метод Анализа Иерархий используется во всем мире для принятия решений в разнообразных ситуациях: от управления на межгосударственном уровне до решения отраслевых и частных проблем в бизнесе, промышленности, здравоохранении и образовании. Метод основан на декомпозиции задачи и представлении ее в виде иерархической структуры. Это позволяет включить в иерархию все знания по решаемой проблеме. В результате решения определяется численно выраженная относительная степень взаимодействия элементов в иерархии. Решение задачи с помощью МАИ выполняется поэтапно.

Ключевые слова: многокритериальные задачи, иерархическая структура, критерии, матрицы парных сравнений, альтернативы, глобальные приоритеты.
Предметная классификация AMS: 93

1. Первый этап

Предусматривает представление проблемы в виде иерархии [2]. В простейшем случае иерархия строится начиная с цели, которая помещается в вершину иерархии, через промежуточные уровни, на которых размещаются критерии и от которых зависят последующие уровни, к самому низшему уровню, который содержит перечень альтернатив. Так, первый шаг МАИ— построение иерархической структуры, объединяющей цель выбора, критерии, альтернативы и другие факторы, влияющие на выбор решения.
Иерархическая структура — это графическое представление проблемы в виде перевернутого дерева, где каждый элемент, за исключением самого верхнего, зависит от одного или более выше расположенных элементов. Иерархические структуры используются для лучшего понимания сложной реальности: мы раскладываем исследуемую проблему на составные части; затем разбиваем на составные части получившиеся элементы и т.д. Так, иерархические структуры, используемые в МАИ, представляют собой инструмент для качественного моделирования сложных проблем.



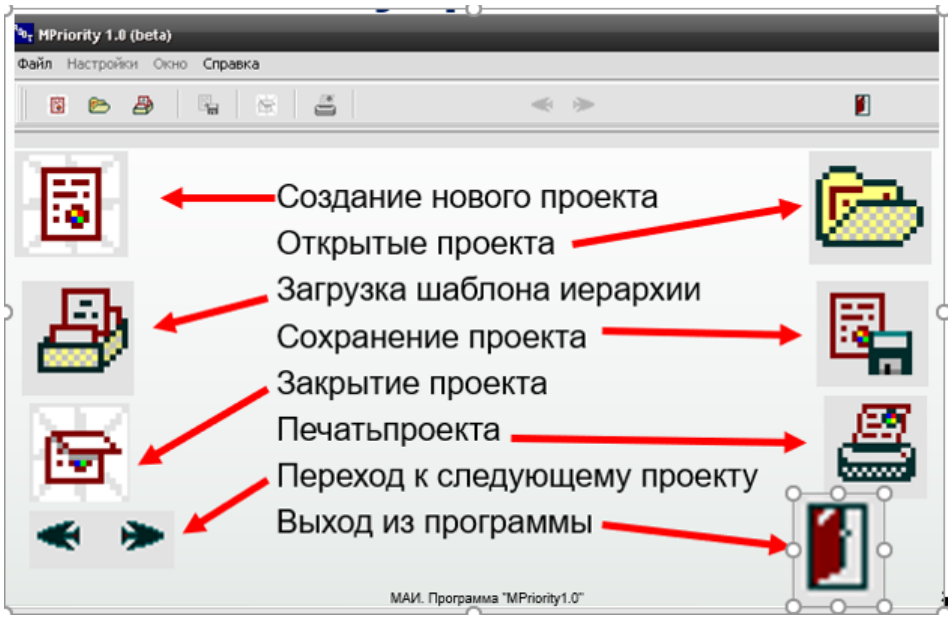
При проведении подобного анализа приходит понимание всей сложности и многогранности исследуемого предмета. Таким образом, построение такой структуры помогает проанализировать все аспекты проблемы и глубже вникнуть в суть задачи. Рассмотрим проблему выбора дома и её решение в программе MPRIORITY 1.0. Ярлык данной программы имеет вид:



Интерфейс программы представляется таким образом:



Назначение каждого инструмента указано на следующем рисунке:



Боковая панель управления программы выглядит следующим образом:



Кнопки на боковой панели управления имеют назначения:

 - Ориентация иерархии (вертикальная - горизонтальная)

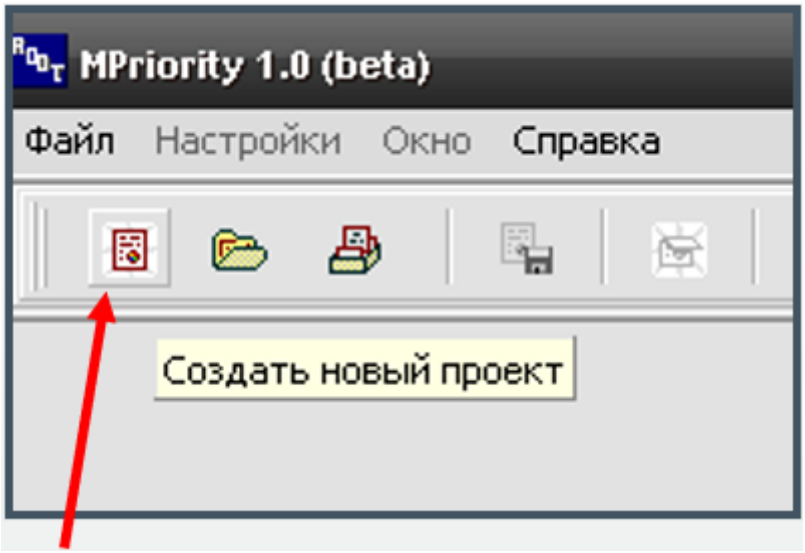
 - Режим редактирования

 - Режим эксперта

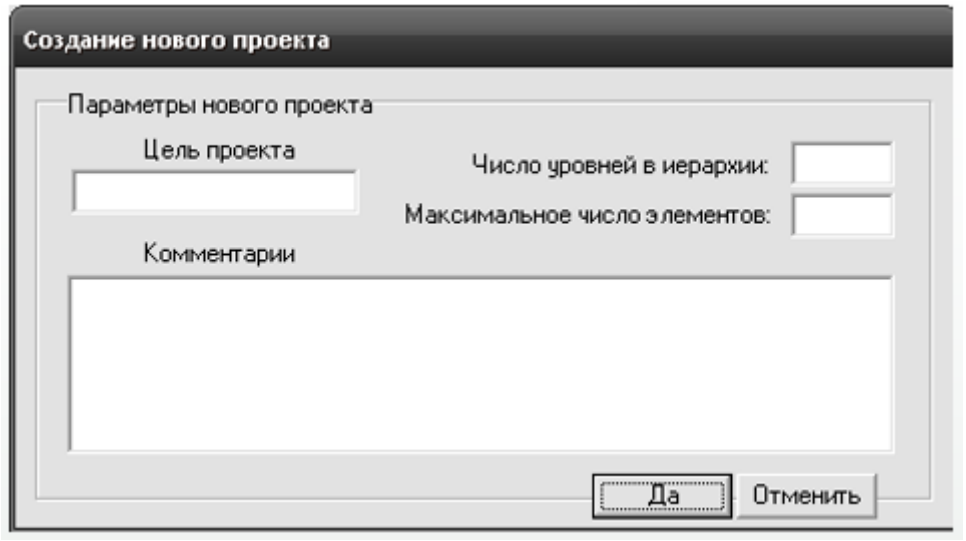
 - Показ результатов

 - Копирование проекта

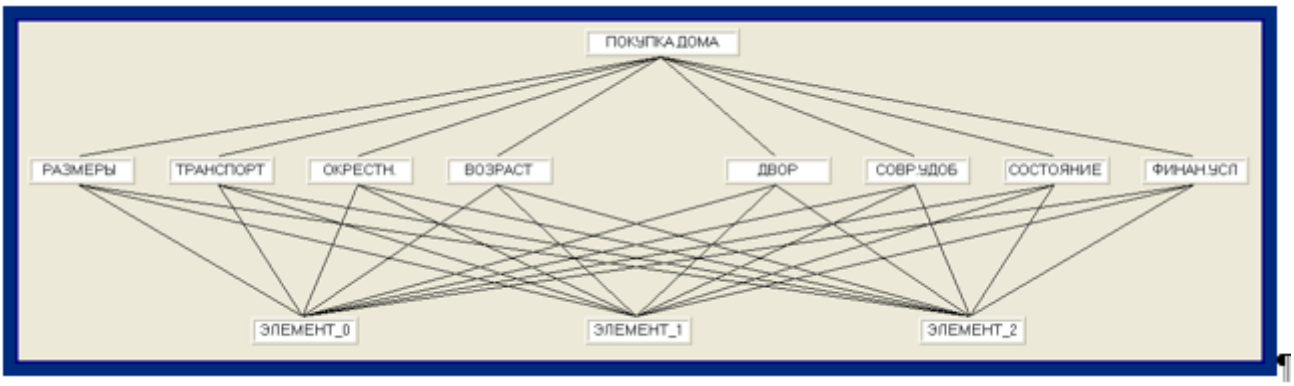
Для ввода данных задачи нажимаем кнопку «Создать новый проект»:



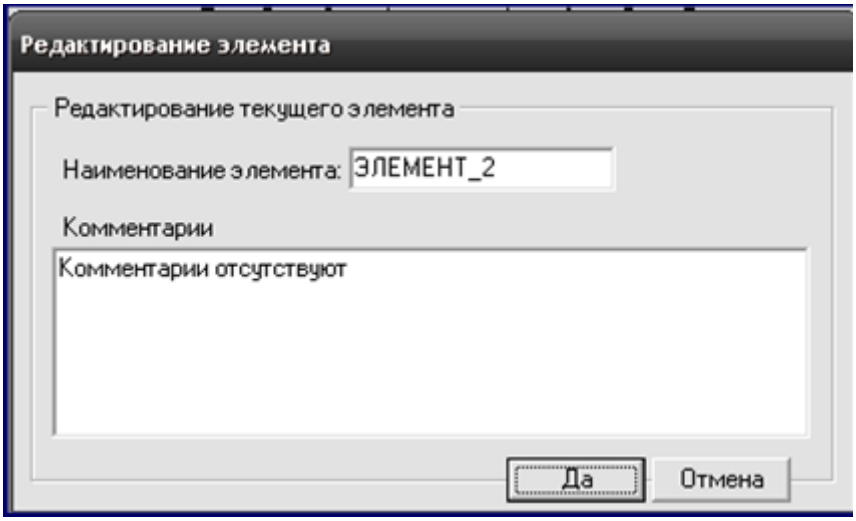
Ввод данных осуществляется в следующем окне:



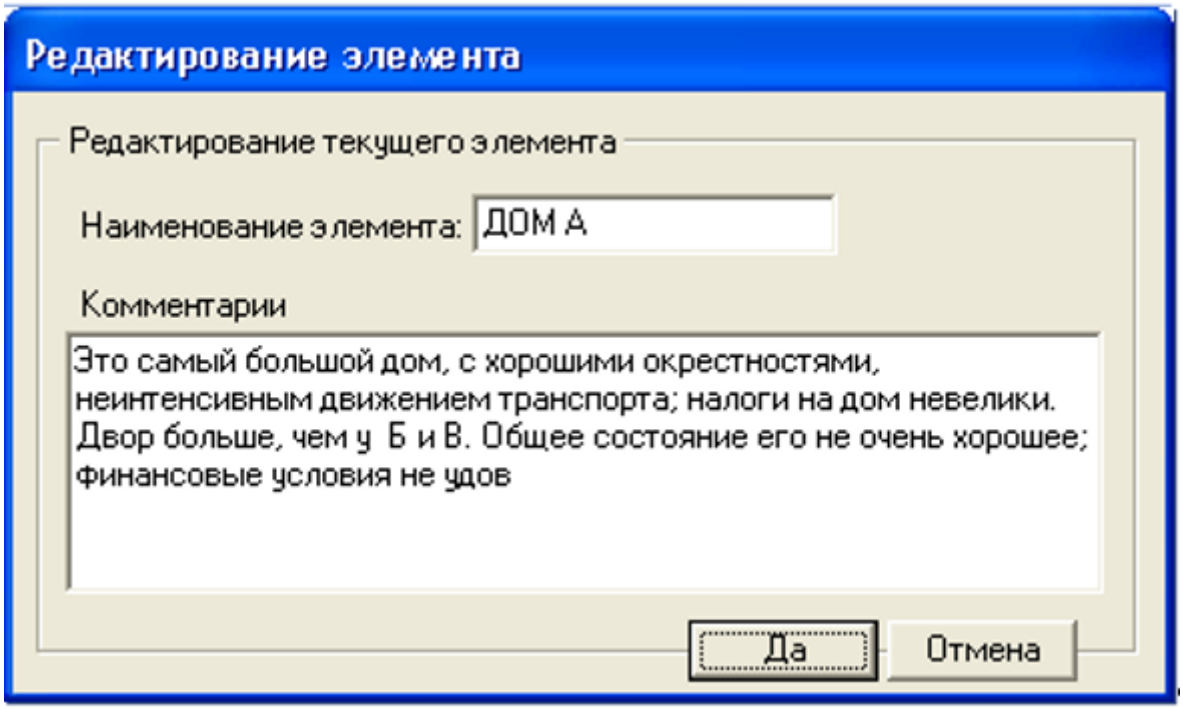
В данном окне мы вводим название проекта, определяем число уровней иерархии задачи и устанавливаем максимальное количество элементов в уровнях.



Для ввода информации по альтернативам нажимаем дважды левую кнопку мыши по каждому элементу третьего уровня. Открывается следующее диалоговое окно:



В данном окне мы вводим информацию о доме-кандидате:

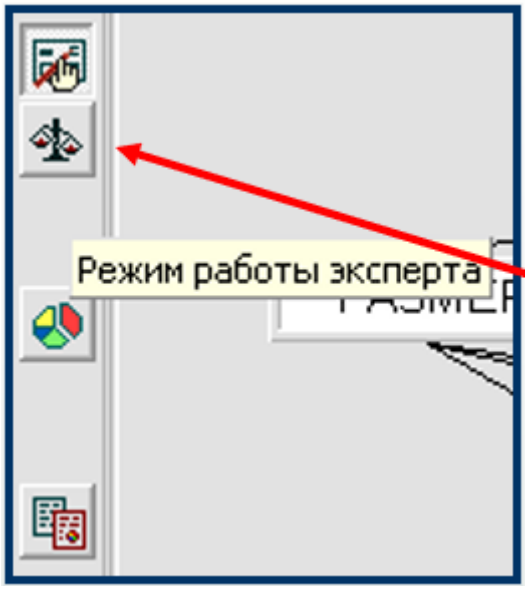


2. Второй этап.

На этом этапе необходимо установить приоритеты критериев и оценить каждую альтернативу по критериям для выбора из них наиболее важной. То есть, после иерархического воспроизведения проблемы возникает вопрос: как установить приоритеты критериев и оценить каждую из альтернатив по критериям, выявив самую важную их них.
В МАИ факторы (критерии, элементы) сравниваются попарно по отношению к их воздействию («весу» или «интенсивности») на общую для них характеристику.
При проведении попарных сравнений в основном ставятся следующие вопросы при сравнении элементов А и Б:
• какой из них важнее или имеет большее воздействие?
• какой из них более вероятен?
• какой из них предпочтительнее? Представим шкалу интенсивности:

А элемент	сравнение	В элемент
1	равная важность элементов А и В	1
3	умеренное превосходство элемента А над В	1/3
5	значительное превосходство элемента А над В	1/5
7	сильное превосходство элемента А над В	1/7
9	очень сильное превосходство элемента А над В	1/9

Переходим в режим эксперта для заполнения в интерактивном режиме матрицы парных сравнений и определения приоритетов:



Нажимаем дважды левой кнопкой мыши по элементу «ПОКУПКА ДОМА», открывается следующее диалоговое окно:

Работа эксперта

Производим попарные сравнения относительно объекта

ПОКУПКА ДОМА

		1.	2.	3.	4.	5.	6.	7.	8.	Приоритет
1.	РАЗМЕРЫ	1	1	1	1	1	1	1	1	0,125
2.	ТРАНСПОРТ	1	1	1	1	1	1	1	1	0,125
3.	ОКРЕСТН.	1	1	1	1	1	1	1	1	0,125
4.	ВОЗРАСТ	1	1	1	1	1	1	1	1	0,125
5.	ДВОР	1	1	1	1	1	1	1	1	0,125
6.	СОВР.УДОБ	1	1	1	1	1	1	1	1	0,125
7.	СОСТОЯНИЕ	1	1	1	1	1	1	1	1	0,125
8.	ФИНАН.УСЛ	1	1	1	1	1	1	1	1	0,125

СЗ: 8

ИС: 0

ОС: 0

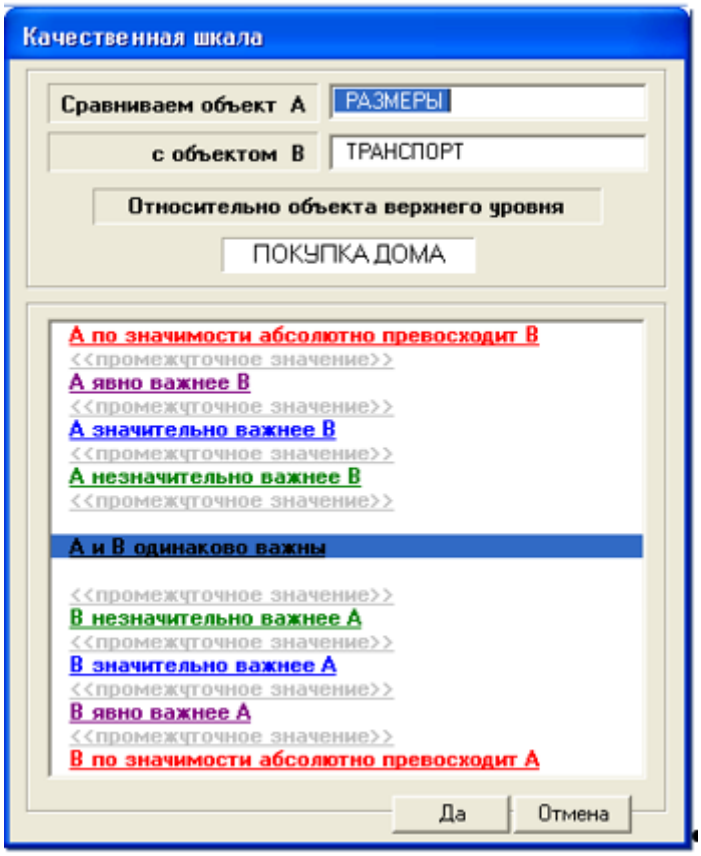
Применить

Закрыть

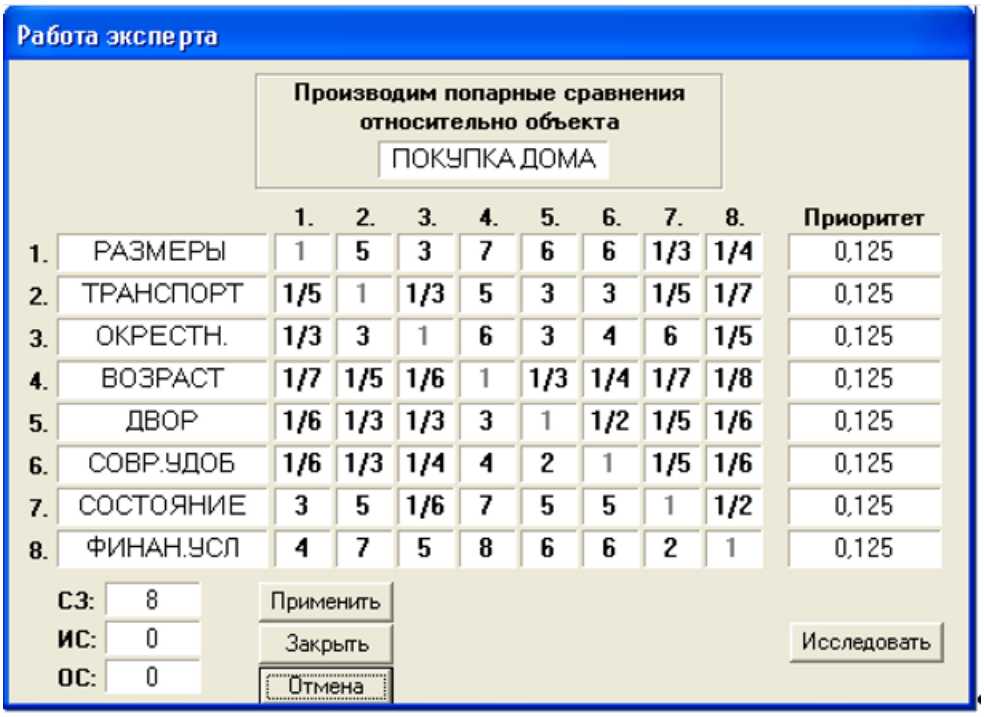
Отмена

Исследовать

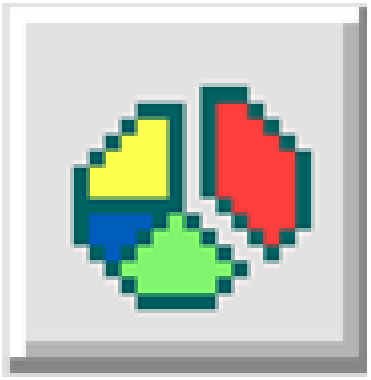
Диагональные клетки не активны и их значения равны 1. Щелкаем дважды левой кнопкой мыши по каждой активной клетке; открывается диалоговое окно; выражаем свои предпочтения, выбрав правильный на наш взгляд вариант ответа.

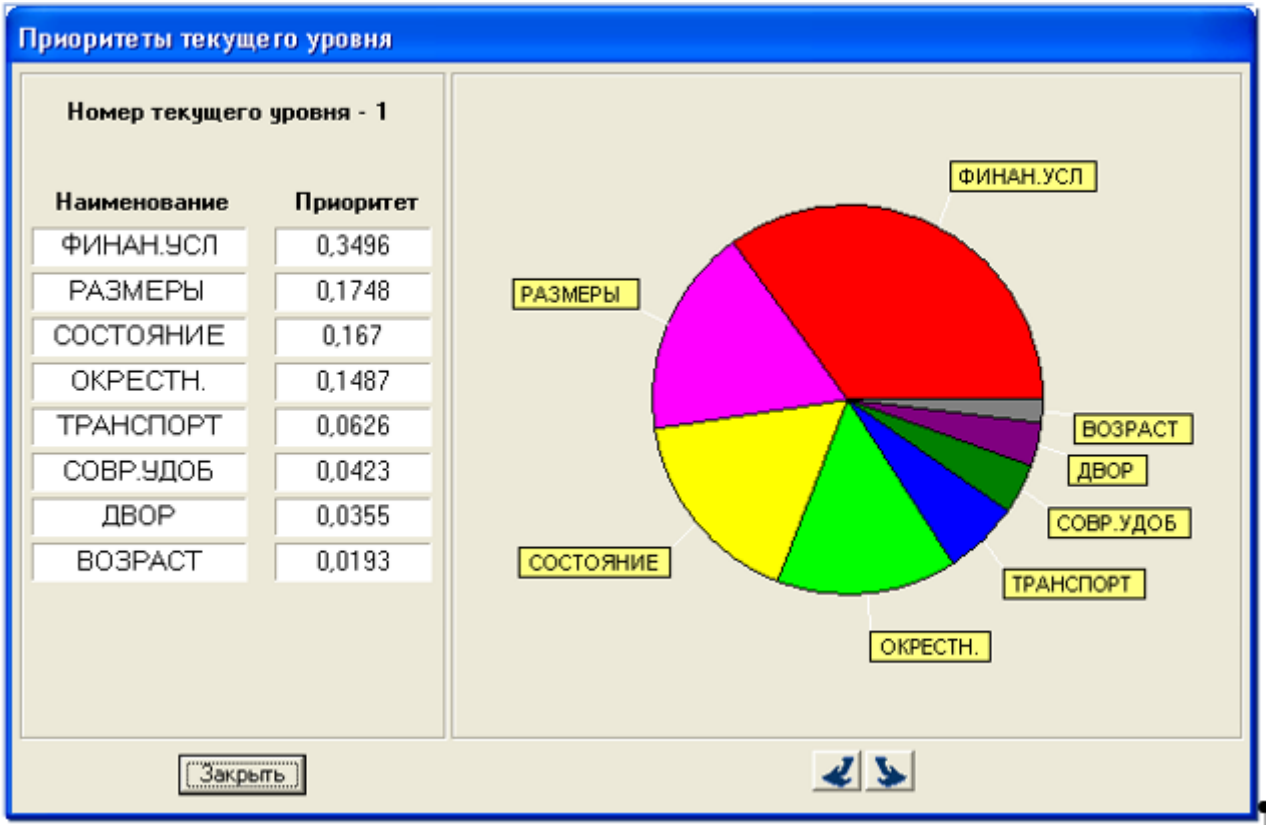


За раз заполняются две симметричные относительно диагонали клетки значениями n и $1/n$.



Затем нажимаем кнопку «Применить». Так, мы получаем количественные приоритеты критериев.
Для отображения локальных приоритетов в виде диаграммы нажимаем кнопку

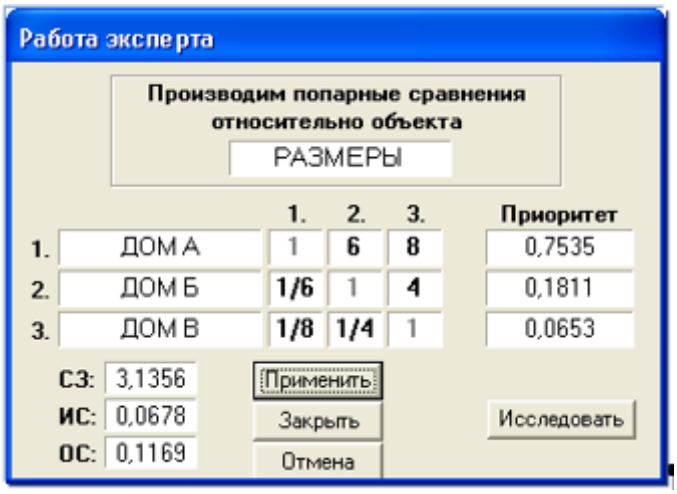

:



3. Третий этап.

После формирования матриц парных сравнений по всем критериям и альтернативам необходимо определить собственные вектора матриц, проверить согласованность матриц с помощью их собственных чисел и провести синтез глобальных приоритетов альтернативных решений относительно основной цели.
Сравнимые попарно элементы – это возможные варианты выбора дома. Сравнивается, насколько более желателен или хорош тот или иной дом для удовлетворения каждого критерия второго уровня.

Сравниваем дома по критерию «размеры», заполняем матрицу парных сравнений. Нажимаем кнопку «Применить». Проверяем значение ОС (отношение согласованности).



Получаем восемь матриц суждений размерностью 3×3, поскольку имеется восемь факторов (критериев) на втором уровне и три дома, которые попарно сравниваются по каждому из факторов (критериев).

Размер дома	*А*	*Б*	*В*	Удобное автобусное сообщение	*А*	*Б*	*В*
А	1	6	8	*А*	1	7	1/5
Б	1/6	1	4	*Б*	1/7	1	1/8
В	1/8	1/4	1	*В*	5	8	1
Окрестности	*А*	*Б*	*В*	Когда построен дом	*А*	*Б*	*В*
А	1	8	6	*А*	1	1	1
Б	1/8	1	1/4	*Б*	1	1	1
В	1/6	4	1	*В*	1	1	1
Двор	*А*	*Б*	*В*	Современное оборудование	*А*	*Б*	*В*
А	1	5	4	*А*	1	8	6
Б	1/5	1	1/3	*Б*	1/8	1	1/5
В	1/4	3	1	*В*	1/6	5	1
Общее состояние	*А*	*Б*	*В*	Финансовые условия	*А*	*Б*	*В*
А	1	1/2	1/2	*А*	1	1/7	1/5
Б	2	1	1	*Б*	7	1	3
В	2	1	1	*В*	5	1/3	1

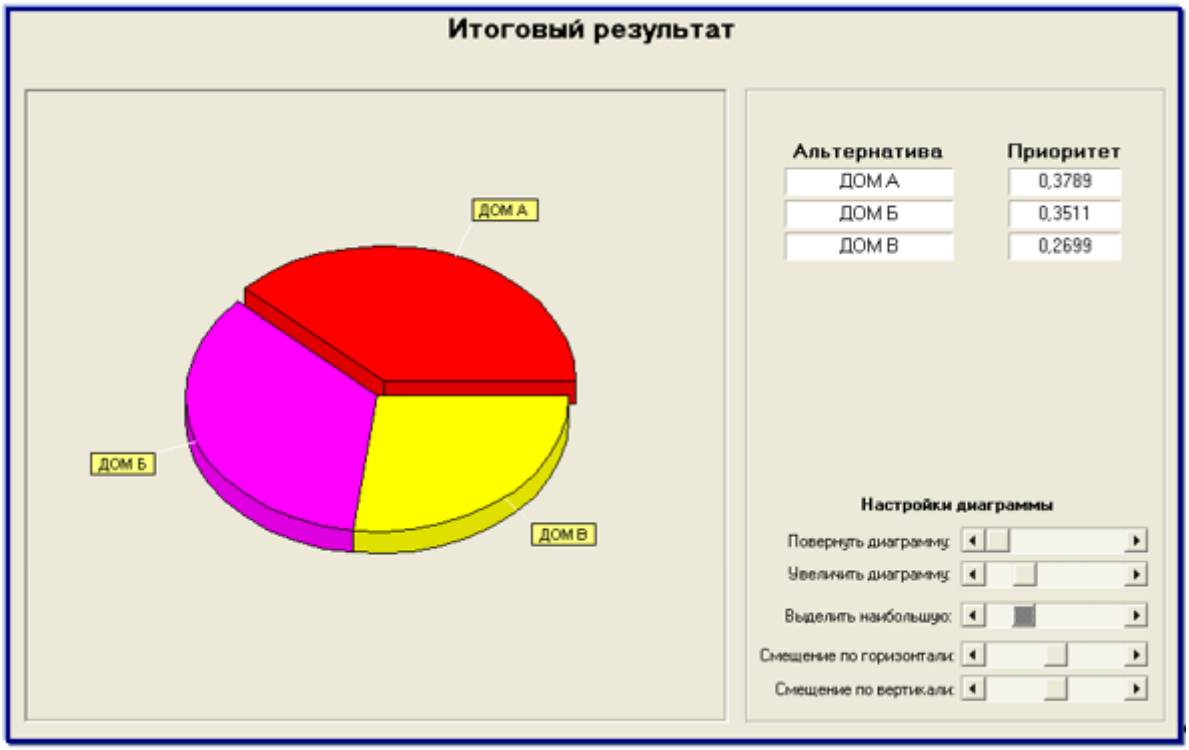
Глобальный приоритет вычисляется путем умножения векторов приоритетов второго уровня на векторы приоритетов третьего уровня, затем результаты складываются вдоль каждой строчки.

	Векторы приоритетов								Глобальные приоритеты
	Размеры (0,174)	Транспорт (0,063)	Окресности (0,148)	возраст (0,019)	двор (0,035)	Сов.удобства (0,042)	Состояние (0,167)	Финан.положение (0,349)	
А	**0,754**	**0,233**	**0,745**	**0,333**	**0,674**	**0,747**	**0,2**	**0,072**	**0,378**
Б	0,181	0,055	0,065	0,333	0,101	0,06	0,4	0,065	0,351
В	0,065	0,713	0,181	0,333	0,226	0,193	0,4	0,278	0,269

Для определения глобальных приоритетов нажимаем кнопку:



Получаем представление конечных результатов:



Оптимальным решением проблемы "Покупка дома"является выбор дома А.
Таким образом, метод анализа иерархий позволяет моделировать психологические особенности принятия экспертных решений в многокритериальных задачах. Предложенная методика раскрывает возможности метода анализа иерархий и довольно просто реализуется в программе MPriority 1.0.

Список литературы

[1] Саати Т., *Принятие решений. Метод анализа иерархий.*, Москва: Радио и связь, 1993, 278 стр.

[2] ЕРШОВА Н. М. , Принятие решений на основе метода анализа иерархий. , / *Вісник Придніпровської державної академії будівництва та архітектури*, Vol.1, No.9 (210), 2015, 39-46 стр.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ДВУСТОРОННИЕ ОЦЕНКИ В МЕТОДАХ ФИКТИВНЫХ ОБЛАСТЕЙ

МАХАТ БУКЕНОВ[1], ЕЛДОС МУХАМЕТОВ[2]

[1]Евразийский национальный университет имени Л. Н. Гумилёва, Астана, Казахстан
[2]НАО "Университет имени Шакакрима города Семей Семей, Казахстан
e-mail: bukenov_m_m@mail.ru

Рассмотрим применение метода фиктивных областей для среды Максвелла. Получены двусторонние оценки по малому параметру α сходимости приближенного решения к точному решению. Рассмотрим постановку динамической вязкоупругой несжимаемой среды, построенной на основе модели Максвелла: в цилиндре $Q = \{D \times [0 \le t \le t_1]\}$ где $D \subset R^3$ ограниченная односвязная область с достаточно гладкий границей γ. Введем обозначение $\gamma_t = \gamma \times [0, t_1]$, вектор-функции деформаций и напряжений: $\bar{\varepsilon} = \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23}\}^T$, $\bar{\sigma} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}\}^T$, здесь символ T означает транспонирование, вектор-функции перемещение и скоростей $\bar{u} = \{u_1, u_2, u_3\}^T$, $\bar{v} = \{v_1, v_2, v_3\}^T$. Следуя работе [1] рассмотрим постановку в скоростях-напряжениях:

$$\frac{\partial \vec{v}}{\partial t} + R\vec{\sigma} = \vec{f} \tag{1}$$

уравнение движения,

$$\frac{\partial \vec{\varepsilon}}{\partial t} - R\vec{v} = 0 \tag{2}$$

соотношение перемещения-деформации.

$$B\frac{\partial \vec{\sigma}}{\partial t} + C\vec{\sigma} = \frac{\partial \vec{\varepsilon}}{\partial t} \tag{3}$$

уравнение состояния для среды Максвелла.

$$div\, \bar{u} = 0 \tag{4}$$

условие несжимаемости среды, при этом надо учесть, что напряжение связаны с функцией давления p:

$$\sigma_{ik} + \delta_{ik}p = 2\mu\varepsilon_{ik}, i, k = 1, 2, 3 \tag{5}$$

Здесь δ_{ik} - символ Кронекера, $\vec{f}$ - вектор массовых сил, $B = B^T$– симметричная, положительно-определенная матрица, зависящая от констант Ламе; $C = C^T$ симметричная, положительно-определенная матрица, зависящая от коэффициента вязкости μ, матрицы B, C – перестановочны, их вид приведен в работе [1]. R - линейный матрично-дифференциальный оператор:

$$R = \begin{pmatrix} \nabla_1 & 0 & 0 & \nabla_2 & \nabla_3 & 0 \\ 0 & \nabla_2 & 0 & \nabla_1 & 0 & \nabla_3 \\ 0 & 0 & \nabla_3 & 0 & \nabla_1 & \nabla_2 \end{pmatrix}^T, \quad R^* = -R^T, \quad \nabla_1 = \frac{\partial}{\partial x_i}, i = 1, 2, 3.$$

Система уравнение (1) - (5) преобразуется к следующему виду:

$$B\frac{\partial^2 \vec{\sigma}}{\partial t^2} + C\frac{\partial \vec{\sigma}}{\partial t} = -RR^*\vec{\sigma} + \vec{F}, \tag{6}$$

здесь $\vec{F} = R\vec{f}$, удовлетворяет начальным условиям $\vec{\sigma}\,(x,\,0) = \vec{q}\,(x)$,

$$\frac{\partial \vec{\sigma}}{\partial t}\,(x,\,0) = \vec{p}\,(x)\,, \tag{7}$$

и граничным условиям:

$$\sum_{k=1}^{3} \vec{\sigma}_{ik}\,(x,\,t)\,n_k = 0, (x,\,t) \in \gamma_t, \tag{8}$$

Здесь $n_k = (n_1,\,n_2,\,n_3)^T$ - вектор нормали к γ. Обозначим задачу (1)-(8) задачи I. В работе [2] показано стационирование решения задачи I к решению статической упругой задачи:

$$R^t\bar{\sigma}^y(x) + \bar{F}(x) = 0, \quad \bar{\sigma}^y = B\bar{\varepsilon}^y(x), \quad \sum_{k=1}^{3} \bar{\sigma}^y_{ik}(x)n_k = 0, \quad x \in \gamma \tag{9}$$

В [2] получена оценка близости решений задачи I и задачи (9).

$$\|\bar{\sigma} - \bar{\sigma}^y\| \le e^{-\beta t}\,\|\bar{\sigma}\,(x,\,0) < \bar{\sigma}^y\,(x)\|\,,$$

где $\beta > 0$ постоянная. Для задачи I верна следующая теорема.

Теорема 0.1. *Пусть* $\bar{\sigma}\,(x,\,t) \in W_2^{2,1}\,(Q)\,,\ \vec{q}\,(x) \in W_2^1\,(D)\,,\ \vec{p}\,(x) \in L_2\,(D),\ \vec{F}\,(x,\,t) \in L_2\,(Q)$, *тогда существует решение задачи I и верна оценка,*

$$\|\vec{\sigma}\,(x,\,t)\|_{\vec{W}_2^{2,1}(Q)} \le C_1 \left(\left\|\vec{F}\right\|_{L_2(Q)} + \|\vec{p}\|_{L_2(D)} + \int\limits_0^{t_1} \|\vec{q}\|_{\vec{W}_2^1(D)} dt \right).$$

Доказательство аналогичному доказательству теоремы в работе [3].
В соответствии с методом фиктивных областей [3], [4], [5], [6] дополним исходную область D, областью D_1 до составной области $D_0 = D \cup D_1$ с границей Γ,$\Gamma_t = \Gamma \times [0,\,t_1]$, $D_1 = D_1 \times [0,\,t_1]$ и построим вспомогательную задачу

$$L_\alpha\vec{\sigma}^\alpha = \vec{F}, \quad (x,\,t) \in Q, \quad L_\alpha\vec{\sigma}^\alpha = 0, \quad (x,\,t) \in Q_1,$$

$$\sum_{k=1}^{3} (\bar{\sigma}^\alpha)_{ik} n_k = 0, \quad (x,\,t) \in \gamma_t \quad \bar{\sigma}^\alpha\,(x,\,0) = 0, \quad x \in D_1 \tag{10}$$

$$\bar{\sigma}^\alpha\,(x,\,0) = \vec{q}\,(x)\,, \quad x \in D \quad \frac{\partial \bar{\sigma}^\alpha}{\partial t}\,(x,\,0) = 0, \quad x \in D_1$$

$$\frac{\partial \bar{\sigma}^\alpha}{\partial t}\,(x,\,0) = \vec{p}\,(x)\,, \quad x \in D \quad \sum_{k=1}^{3} \bar{\sigma}^\alpha_{ik}\,(x,\,t)\,n_k = 0, \quad (x,\,t) \in \Gamma_t$$

здесь

$$L\vec{\sigma} \equiv B\frac{\partial^2\vec{\sigma}}{\partial t^2} + C\frac{\partial\vec{\sigma}}{\partial t} + A\vec{\sigma}, \quad A\vec{\sigma} = -RR^*\vec{\sigma}$$

$$L_\alpha\vec{\sigma}^\alpha \equiv B\frac{\partial^2\vec{\sigma}^\alpha}{\partial t^2} + C\frac{\partial\vec{\sigma}^\alpha}{\partial t} + a^\alpha A\vec{\sigma}, \quad a^\alpha = \begin{cases} 1, & x \in D \\ \alpha^{-2}, & x \in D_1 \end{cases}$$

$\alpha > 0$- малый параметр.
На кривой разрыва коэффициентов γ_t ставим условия согласования

$$\vec{\sigma}^\alpha|^+_{\gamma_t} = \vec{\sigma}^\alpha|^-_{\gamma_t}\,, \quad \left.\frac{\partial\vec{\sigma}^\alpha}{\partial N}\right|^+_{\gamma_t} = \frac{M}{a}\left.\frac{\partial\vec{\sigma}^\alpha}{\partial n}\right|^-_{\gamma_t}. \tag{11}$$

Знаки $''+''$ или $''-''$ означают стремления предельного значения функции изнутри или извне к границе γ_t. Параметр M принимает значение -1 или -1. Введем в расмотрения следующие ряды:

$$S_1 = \sum_{k=0}^{\infty} \alpha^k \vec{V}_k, \text{в} \quad Q, \quad S_2 = \sum_{k=1}^{\infty} \alpha^k \vec{W}_k, \text{в} \quad Q_1, \tag{12}$$

Если подставить (12) в (10), то получим соотношения для определения $\vec{V}_k$ и $\vec{W}_k$:

$$L_\alpha \vec{V}_0 = \vec{F}, \quad (x,\, t) \in Q, \quad L_\alpha \vec{W}_1 = 0, \quad (x,\, t) \in Q_1,$$

$$\sum_{k=1}^{3} (V_0)_{ik} n_k = 0, \quad (x,\, t) \in \gamma_t \quad \vec{W}_1 (x,\, 0) = 0, \quad x \in D_1,$$

$$V_0 (x,\, 0) = \vec{q}(x), \quad x \in D, \quad \frac{\partial \vec{W}_0}{\partial t} (x,\, 0) = 0, \quad \frac{\partial \vec{V}_0}{\partial t} (x,\, 0) = \vec{p}(x), \quad x \in D, \tag{13}$$

$$\sum_{k=1}^{3} \left(\vec{W}_1\right)_{ik} (x,\, t)\, n_k = 0, \quad (x,\, t) \in \Gamma_t, \quad \frac{\partial \vec{W}_1}{\partial n} = M \frac{\partial \vec{V}_0}{\partial N}, \quad (x,\, t) \in \gamma_t.$$

и для $k \geq 1$

$$L_\alpha \vec{V}_k = \vec{F}, \quad (x,\, t) \in Q, \quad L_\alpha \vec{W}_{k+1} = 0, \quad (x,\, t) \in Q_1,$$

$$\sum_{k=1}^{3} (V_0)_{ik} n_k = 0, \quad (x,\, t) \in \gamma_t, \quad \vec{W}_{k+1} (x,\, 0) = 0, \quad x \in D_1$$

$$V_k (x,\, 0) = 0, \quad x \in D, \quad \frac{\partial \vec{W}_{k+1}}{\partial t} (x,\, 0) = 0, \quad x \in D_1,$$

$$\frac{\partial \vec{V}_k}{\partial t} (x,\, 0) = 0, \quad x \in D, \quad \sum_{i=1}^{3} \left(\vec{W}_{k+1}\right)_{n_i} n_i = 0, \quad (x,\, t) \in \Gamma_t, \quad \vec{W}_k = \vec{V}_k, \quad (x,\, t) \in \gamma_t.$$

Функции $\vec{V}_k \in \vec{W}_2^{2,1}(Q)$, $k = 0, 1, ...$, $\vec{W}_k \in \vec{W}_2^{2,1}(Q_1)$, $k = 1, 2, ...$

Теорема 0.2. *Если α_0 таково, что $0 < \alpha < \alpha_0$, то ряды S_1, S_2 абсолютно сходятся в $\vec{W}_2^{2,1}(Q)$ и в $\vec{W}_2^{2,1}(Q_1)$, и верны равенства*

$$\vec{\sigma}^\alpha = S_1, \quad (x,\, t) \in Q, \quad \vec{\sigma}^\alpha = S_2, \quad (x,\, t) \in Q_2, \tag{14}$$

где $\vec{\sigma}^\alpha$ - решения задачи (10).

Доказательство. Ищем очевидные априорные оценки

$$\left\|\vec{W}_k\right\|_{\vec{W}_2^{2,1}(Q_1)} \leq C_2 \left\|\frac{\partial \vec{W}_k}{\partial n}\right\|_{W_2^{1/2,1}(\gamma_t)} \leq C_2 \left\|\frac{\partial \vec{V}_{k-1}}{\partial N}\right\|_{W_2^{1/2,1}(\gamma_t)} \leq C_2 C_3 \left\|\vec{V}_{k-1}\right\|_{W_2^{1/2,1}(\gamma_t)}, \tag{15}$$

где C_2, C_3 константы зависят от областей D_2, D_3 и не зависят от α.
Покажем сходимость рядов S_1 в $\vec{W}_2^{2,1}(Q)$ и S_2 в $\vec{W}_2^{2,1}(Q_1)$ имеем $\left\|\vec{V}_k\right\|_{\vec{W}_2^{2,1}(Q)} \leq C_4 \left\|\vec{V}_k\right\|_{W_2^{3/2,\,1}(\gamma_t)} = C_4 \left\|\vec{W}_k\right\|_{W_2^{3/2,\,1}(\gamma_t)} \leq C_4 C_5 \left\|\vec{W}_k\right\|_{W_2^{2,\,1}(Q_1)}$, используя (8),(15), получим

$$\left\|\vec{V}_k\right\|_{\vec{W}_2^{2,1}(Q)} \leq C_6 \left\|\vec{V}_{k-1}\right\|_{W_2^{3/2,\,1}(Q)}, \quad k \geq 1$$

$$\left\|\vec{V}_0\right\|_{\vec{W}_2^{2,1}(Q)} \leq C_1 \left(\left\|\vec{F}\right\|_{L_2(Q)} + \|\vec{p}\|_{L_2(D)} + \int_0^{t_1} \|\vec{q}\|_{\vec{W}_2^1(D)} dt \right), \tag{16}$$

где $C_6 = C_2 C_3 C_4 C_5$.
Полагая $\alpha < \alpha_0 = C_6^{-1}$, получим ряд S_1, абсолютно сходится в $\vec{W}_2^{2,1}(Q)$ и соответственно

ряд S_2 абсолютно сходится в $\vec{W}_2^{2,1}(Q_1)$. Умножая (13) для $\vec{V}_k$ и $\vec{W}_k$ на α_k, и суммируя по k, имеем

$$LS_1 = \vec{F}, \quad (x,\, t) \in Q, \quad L_\alpha S_2 = 0, \quad (x,\, t) \in Q_1,$$
$$S_1(x,\, 0) = \vec{q}(x), \quad x \in D, \quad \frac{\partial S_1}{\partial t}(x,\, 0) = \vec{p}(x), \quad x \in D, \tag{17}$$
$$S_2(x,\, 0) = 0, \quad x \in D_1, \quad \frac{\partial S_2}{\partial t}(x,\, 0) = 0, \quad x \in D_1$$
$$S_1 = S_2, \quad (x,\, t) \in \gamma_t, \quad \frac{\partial S_2}{\partial n} = \frac{M}{\alpha}\frac{\partial S_1}{\partial N},$$
$$(x,\, t) \in \gamma_t, \quad S_2(x,\, t) = 0, \quad (x,t) \in \Gamma_t.$$

Отсюда получаем $\vec{\sigma}^\alpha = S_1$ в Q, $\vec{\sigma}^\alpha = S_2$ в Q_1, если $0 < \alpha < \alpha_0$. Из доказательства этой теоремы вытекает справедливость следующего удтверждения

$$\left\|\vec{\sigma} - \vec{\sigma}_+^\alpha\right\|_{\vec{W}_2^{2,1}(Q)} \le C_7\alpha, \quad \left\|\vec{\sigma} - \vec{\sigma}_-^\alpha\right\|_{\vec{W}_2^{2,1}(Q)} \le C_8\alpha. \tag{18}$$

Здесь $\vec{\sigma}_+^\alpha = \vec{\sigma}^\alpha$, при $M = 1$, $\vec{\sigma}_-^\alpha = \vec{\sigma}^\alpha$ при $M = -1$. $\vec{\sigma}$ – решение задачи I, C_7, C_8 зависят от областей D и не зависят от α. Далее можно сформулировать теорему дающую двусторонние оценки.

Теорема 0.3. *Если $0 < \alpha < \alpha_0$, $\vec{\sigma}$ – решение задачи I, $\vec{\sigma}_+^\alpha$, $\vec{\sigma}_-^\alpha$ – решение задачи (10) для $M = 1$, и $M = -1$. Тогда верна оценка*

$$\left\|\vec{\sigma} - \frac{1}{2}\left(\vec{\sigma}_+^\alpha + \vec{\sigma}_-^\alpha\right)\right\|_{\vec{W}_2^{2,1}(Q)} \le C_9\alpha^2, \tag{19}$$

$$\vec{\sigma}^\alpha = S_1, \quad (x,\, t) \in Q, \quad \vec{\sigma}^\alpha = S_2, \quad (x,\, t) \in Q_1, \tag{20}$$

$\vec{\sigma}$ – решение задачи I.
Доказательство. В силу теоремы 1, имеем

$$\vec{\sigma}_+^\alpha = \sum_{k=0}^{\infty}\alpha^k\vec{V}_k^+, \quad (x,\, t) \in Q, \quad \vec{\sigma}_-^\alpha = \sum_{k=1}^{\infty}\alpha^k\vec{W}_k^+, \quad (x,\, t) \in Q_1, \tag{21}$$

здесь $\vec{V}_k^+$, $\vec{W}_k^+$ - решение (10) при $M = 1$. Кроме того имеет место

$$\vec{\sigma}_-^\alpha = \sum_{k=0}^{\infty}\alpha^k\vec{V}_k^-, \quad (x,\, t) \in Q, \quad \vec{\sigma}_+^\alpha = \sum_{k=1}^{\infty}\alpha^k\vec{W}_k^-, \quad (x,\, t) \in Q_1, \tag{22}$$

здесь $\vec{V}_k^-$, $\vec{W}_k^-$ - решение (10) при $M = -1$.
Получаем $\vec{V}_0^+ \equiv \vec{V}_0^- \equiv \vec{\sigma}$ - решение задачи I. Введем обозначения $\vec{W}_1 = \vec{W}_1^+ + \vec{W}_1^-$, функция $\vec{W}_1$ удовлетворяет следующей задачи:

$$L_\alpha\vec{W}_1 = 0, \quad (x,\, t) \in Q_1, \quad \frac{\partial\vec{W}_1}{\partial n} = 0, \quad (x,\, t) \in \gamma_t,$$

$$\vec{W}_1(x,\, 0) = 0, \quad \frac{\partial\vec{W}_1}{\partial t}(x,\, 0) = 0, \quad x \in D_1, \quad \vec{W}_1(x,\, t) = 0, \quad (x,\, t) \in \Gamma_t,$$

отсюда получим $\vec{W}_1 = 0$, или $\vec{W}_1^+ = -\vec{W}_1^-$.
Далее введем $\vec{V}_1 = \vec{V}_1^+ + \vec{V}_1^-$, функция $\vec{V}_1$ удовлетворяет задаче

$$L_\alpha\vec{V}_1 = 0, \quad (x,\, t) \in Q, \quad \frac{\partial\vec{V}_1}{\partial t}(x,\, 0) = 0, \quad x \in D,$$
$$(x,\, t) \in \gamma_t, \quad \vec{V}_1(x,\, t) = 0, \quad \vec{V}_1(x,\, 0) = 0, \quad (x,\, t) \in \Gamma_t,$$

откуда имеем $\vec{V}_1 = 0$, *или* $\vec{V}_1^+ = -\vec{V}_1^-$. *Последовательно вводя* $\vec{W}_2 = \vec{W}_2^+ + \vec{W}_2^-$, $\vec{V}_2 = \vec{V}_2^+ + \vec{V}_2^-$, *получим* $\vec{W}_2^+ = \vec{W}_2^-$, $\vec{V}_2^+ = \vec{V}_2^-$, *продолжая этот процесс придем* $\vec{V}_k^+ = \vec{V}_k^-$, *если к – четно,* $\vec{V}_k^+ = -\vec{V}_k^-$, *если к – нечетно (22). Используя (22) подставив (21),(20), имеем*

$$\vec{\sigma}_+^\alpha = \vec{\sigma} + \alpha\vec{V}_1^+ + \alpha^2\vec{V}_2^+ + ... \tag{23}$$

$$\vec{\sigma}_-^\alpha = \vec{\sigma} - \alpha\vec{V}_1^+ + \alpha^2\vec{V}_2^+ + ... \tag{24}$$

Применяя разложения (22), а также оценку (19) при $0 < \alpha < \alpha_0$, *получим*

$$\left\| \vec{\sigma} - \frac{1}{2}\left(\vec{\sigma}_+^\alpha + \vec{\sigma}_-^\alpha\right) \right\|_{\vec{W}_2^{2,1}(Q)} \le \alpha^2 \left\| \vec{V}_2^+ + \alpha^2\vec{V}_4^+ + ... \right\|_{\vec{W}_2^{2,1}(Q)} \le C_8\alpha^2 \left\| \vec{V}_0^+ \right\|_{\vec{W}_2^{2,1}(Q)} \le C_9\alpha^2.$$

здесь $C_8 = C_6^2$, *таким образом для* $x \in D$, $0 < \alpha < \alpha_0$ *имеем двустороннюю оценку*

$$O\left(\alpha^2\right) + \min\left(\vec{\sigma}_+^\alpha, \vec{\sigma}_-^\alpha\right) \le \vec{\sigma} \le \max\left(\vec{\sigma}_+^\alpha, \vec{\sigma}_-^\alpha\right) + O\left(\alpha^2\right).$$

Такая же оценка верна и для несжимаемой среды Кельвина-Фойгта. Для решения вспомогательной задачи использовать, алгоритм, предложенной в работе [7-9].

Ключевые слова: метод фиктивных областей, скорости-напряжения, малый параметр.

Предметная классификация AMS: 65M85.

Список литературы

[1] Bukenov M.M., Azimova D.N., Estimates for maxwell viscoelastic medium "in tension-rates *Eurasian mathematical journal*, Vol.10, No.2, 2019, pp.30-36.

[2] Пацюк В.И., Стационирование динамических процессов в вязкоупругих средах*Дис. канд. физ.-мат.-наук*, 1982.

[3] Букенов М.М., Постановка динамической задачи линейной вязкоупругости в скоростях напряжений, *Сиб. журн. вычисл. матем., 8:4 (2005), 289–295*, Vol.8, No.4, 2005, pp.289-295.

[4] Коновалов А.Н., Метод фиктивных областей в задачах фильтрации двухфазной несжимаемой жидкости с учетом капиллярных сил, *Численные методы механики сплошной среды. Новосибирск*, Vol.3, No.5, 1972, pp.52-67.

[5] Коновалов А.Н., Об одном варианте метода фиктивных областей, *Некоторые проблемы вычислительной и прикладной математики*, 1972, 191-199pp.

[6] Коновалов А.Н., Метод фиктивных областей в задачах кручения. *Численные методы механики сплошной среды. Новосибирск*, Vol.1, No.2, 1973, pp.109-115.

[7] Орунханов М.К., Смагулов Ш.С., Теории метода фиктивных областей, *Численные методы механики сплошной среды*, Vol.13, No.2, 1982, pp.125-137.

[8] Chertova.K., Locally two-sided approximate solutions in parabolic problems, *Bull. Nov. Comp. Center, Num. Anal.*, No.6, 1994, pp.37-42.

[9] Букенов М.М., Малые параметры в алгоритмах задач теории упругости, *Дис. канд. физ.-мат.-наук*, 1986.

Математиканы оқыту әдістемесі және математика тарихы

Methods of teaching mathematics and the history of mathematics

Методика преподавания математики и история математики

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

DIFFERENTIAL EQUATIONS. SHERLOCK HOLMES AND OTHER APPLICATIONS

ANARKUL URDALETOVA[1], SYRGAK KYDYRALIEV[2]

[2]Kyrgyz-Turkish Manas University, Bishkek, Kyrgyz Republic
[1]American University of Central Asia, Bishkek, Kyrgyz Republic
e-mails: anarkul.urdaletova@manas.edu.kg, kydyraliev_s@auca.kg

Abstract. A sultan sent his son to study clairvoyance. He tried his best and succeeded in doing so. Once the sultan took a gemstone ring in his hand and said: "My son, tell me what I have in my hand?" He answered: "It is round, it is a mineral, and it has a hole in the middle." The sultan said: "You named the attributes right, tell me what is this object?" After long deliberation, he answered: "A millstone". The Sultan said: "You established so many exact features with the power of knowledge and learning, but you did not have enough reason to understand that a millstone can not fit in my hand, and I can not take it in my hand." That's how scientists these days "split" a hair in the sciences, but do not know what is important and closest to them [1].
To ensure that our students do not become like this prince, it is necessary to teach them to use mathematics. Since the derivative of the function $y = f(x)$ is the rate of change of y, it is natural that the equations containing derivatives - differential equations are often used to describe the constantly changing reality around us.
Inspiring students, encouraging them to study differential equations, and teaching them to use this tool is a very useful and interesting objective. In our opinion, word problems can play an important role in addressing this issue. Some of these problems, such as the story of Sherlock Holmes and others, are presented in this paper.

Keywords: differential equations, application of differential equation, a new approach to solving differential equations, mathematics teaching, the story of Sherlock Holmes.

AMS Subject Classification: 34A05, 00A09.

Problem 1

I had intended "The Adventure of the Abbey Grange" to be the last of those exploits of my friend, Mr. Sherlock Holmes, which I should ever communicate to the public. This resolution of mine was not due to any lack of material, since I have notes of many hundreds of cases to which I have never alluded, nor was it caused by any waning interest on the part of my readers in the singular personality and unique methods of this remarkable man. The real reason lay in the reluctance which Mr. Holmes has shown to the continued publication of his experiences. So long as he was in actual professional practice the records of his successes were of some practical value to him, but since he has definitely retired from London and betaken himself to study and bee-farming on the Sussex Downs, notoriety has become hateful to him, and he has peremptorily requested that his wishes in this matter should be strictly observed. It was only upon my representing to him that I had given a promise that "The Adventure of the Second Stain" should be published when the times were ripe, and pointing out to him that it is only appropriate that this long series of episodes should culminate in the most important international case which

he has ever been called upon to handle, that I, at last, succeeded in obtaining his consent that a carefully guarded account of the incident should, at last, be laid before the public.

So begins one of the last stories of the famous writer Sir Arthur Conan Doyle about the great detective Sherlock Holmes. While reading this story, you can learn that one day Holmes was approached by the British Prime Minister and the Secretary of State for European Affairs. They asked to find a letter, the publication of which could lead to a large international conflict. Immediately after the start of the search, it was found that certain Eduardo Lucas, who with great probability could be involved in the stealing of the letter, had been killed the day before. The police arrested John Mitton, Lucas's valet. After a while the valet was released as innocent, the murder was solved, and the letter as a result of Sherlock Holmes' efforts was returned.

More details about the case can be found in the published version of the short story "The Adventure of the Second Stain" [2].

By a twist of fate, we recently came across a draft version of this story, which described the reasons for the arrest and the subsequent release of the valet in detail. Given the great interest in everything connected with Sherlock Holmes, we decided to publish this part of the story condensed in the publication.

Lucas' body was found at 8 pm. That evening the valet was visiting his friends and left them at 7 pm.

Immediately after the discovery of the body, a police expert measured the temperature of the body - it was equal to 33°C, and the temperature of the air was equal to 10°C.

According to the law of Sir Isaac Newton, the body temperature changes at a rate proportional to the difference between body temperature and ambient temperature. That is, if the body temperature at time t is denoted by $x(t)$, there is a differential equation

$$\frac{dx}{dt} = -K\left(x - 10\right),$$

where K is a constant of proportionality.

In this case, if the time the body was found was considered as the initial time $t = 0$, then $x(0) = 33$. Also, the police used the fact that the body temperature was measured again half an hour after the discovery of the body and it was $30.6°C$, so

$$x(0.5) = 30.6.$$

So there was a boundary value problem

$$\frac{dx}{dt} = -K\left(x - 10\right), x(0) = 33, x(0.5) = 30.6. \tag{1}$$

Dividing the variables

$$\frac{dx}{x - 10} = -Kdt,$$

and integrating, they got:

$$ln(x - 10) = -Kt + ln(C).$$

Then, from the initial condition

$$x(0) = 33 : ln(33 - 10) = -K \cdot 0 + ln(C),$$

i.e.

$$ln\left(\frac{x - 10}{23}\right) = -Kt. \tag{2}$$

Further, since $x(0.5) = 30.6$, substituting these values into equation (2) they found that $K = 0.22$.

Then,

$$ln\left(\frac{x - 10}{23}\right) = -0.22t.$$

Hence, using the fact that the temperature of a living person is $36.6°C$, they identified the time of the murder:

$$ln\left(\frac{36.6-10}{23}\right) = -0.22t \quad \Rightarrow t = -0.661.$$

That is, in their opinion, the murder was committed $0.661 \cdot 60$ =(converting hours into minutes) = 40 minutes before the body was discovered.

The police decided that in 20 minutes Mitton could get to the house and commit murder. Therefore, Mitton was arrested.

Pursuing the investigation, Holmes found that the police lost sight of the very important fact - according to the weather service, in the evening atmospheric temperature dropped quickly: if the time the body was found was considered as the initial time $t = 0$, then the temperature change could be described as a function $a(t) = 10e^{-t}$. He also learned that a short period of time after the first temperature measurement another one was carried out, and as a result, he received a rate of change of body temperature at the initial time: $(-4.6°C)$.

Thus, Holmes found that rather than the problem (1), there was a problem

$$\frac{dx}{dt} = -K\left(x - 10e^{-t}\right), \tag{3}$$
$$x(0) = 33, x'(0) = -4.6.$$

Using the initial conditions:

$$-4.6 = -K(33 - 10e^{-0}),$$

Holmes identified the value of the coefficient $K : K = 0.2$, and wrote equation (3) as a linear differential equation of the first order

$$\frac{dx}{dt} + 0.2x = 0.2 \cdot 10e^{-t}.$$

His solution:

$$x(t) = -2.5e^{-t} + Ce^{-0.2t}.$$

Condition $x(0) = 33$ allows him to find the value of the constant C, and write the corresponding particular solution of the equation

$$x(t) = -2.5e^{-t} + 35.5e^{-0.2t}. \tag{4}$$

Substituting the value of $x(t) = 36.6$ in (4), Sherlock Holmes showed that $t = -1$, i.e. the murder took place at the time the valet said goodbye to his friends, and therefore he can not be Lucas' killer. Police released the valet, and after a while, as you can read in the published story, Holmes identified the actual killer.

Note

Substituting the value of $x(t) = 36.6$ in (4), we obtain the equation

$$36.6 = -2.5e^{-t} + 35.5e^{-0.2t}. \tag{5}$$

In order to solve equation (5) it is necessary to use calculus. Carrying out the necessary actions, we can find that equation (5) has two roots (-1.02) and (-1.56). Fortunately, the second root is not contrary to the conclusions of Sherlock Holmes.

Problem 2 (from the integral equations to differential equations)

Find curves with the following property: if at any point of the curve we draw straight lines, parallel to the coordinate axes until meeting with the axes, the area of the resulting rectangle is divided by the curve in the ratio 3:1. [3]

We start our solution by establishing that the required curves should be increasing in the first and third quadrants, and decreasing in the second and fourth quadrants. Otherwise, a curve can pass under or over the rectangle and will not divide it.

Further, the curves must pass through the origin of the coordinates. Otherwise, taking a point sufficiently close to the coordinate axis can make the area ratio arbitrarily large (small).

For definiteness, we shall assume that the rectangle is located in the first quadrant.

If we selected a point on the curve with coordinates $(x; y)$, then the area of a rectangle is equal to xy, and part of it under the graph of the curve has the area

$$\int_0^x ydx.$$

Therefore, the equation for the desired curves is

$$0.25xy = \int_0^x ydx. \tag{6}$$

Differentiating integral equation (6), we obtain the differential equation

$$0.25y + 0.25xy' = y.$$

Collecting similar terms and multiplying both sides by 4, we obtain a differential equation with separable variables

$$xy' = 3y. \tag{7}$$

Dividing the variables in the equation (7):

$$\frac{dy}{y} = 3\frac{dx}{x};$$

integrating:

$$ln(y) = 3ln(x) + ln(C);$$

and exponentiating, we get the answer:

$$y = CX^3.$$

Note that in other quadrants the order of the figures in the divided rectangle can change. This leads to the answer

$$x = Cy^3.$$

Problem 3

A swimmer swims across the river with width $2a$ at a speed v_s, directed perpendicularly to the vector v_r - the vector of the river flow rate. Determine the trajectory of the swimmer in the river, as well as the point at which he comes out to the opposite bank. [4]

Place Cartesian axes so that the Y-axis corresponds to the direction of the river flow and the starting place of the swimmer has coordinates $(-a; 0)$. Next, assume that the river flow rate is defined by the function

$$v_r = v_0 cos\left(\frac{\pi x}{2a}\right),$$

i.e. we assume that the highest flow rate v_0 is in the middle of the river and it diminishes as it gets closer to each bank.

Then, the speed of the swimmer is determined by the vector with coordinates $(v_s;\ v_r)$ and occurs at an angle k, where

$$tg(k) = v_s/v_r.$$

Since $tg(k)$ is the slope of the tangent to the motion trajectory $y(x)$, there is a differential equation

$$y' = v_r/v_s,$$

i.e.

$$y' = v_0 cos\left(\frac{\pi x}{2a}\right)/v_s. \tag{8}$$

In particular, assuming that the width of the river ($2a$) is 1 km, the speed v_0 is 9 km/h, the speed v_s is 3 km/hour, equation (8) takes the form

$$y' = 3cos(\pi x).$$

Integrating, we obtain

$$y = \frac{3}{\pi} sin(\pi x) + C.$$

Start condition $y(-0.5) = 0$, allows us to find the value of C and, therefore, to define a function that specifies the trajectory of the swimmer in the river:

$$y = \frac{3}{\pi} sin(\pi x) + \frac{3}{\pi}.$$

From this, we can determine that the swimmer must reach the opposite bank about 1.9 km downstream of the river:

$$y(0.5) = \frac{3}{\pi} sin\frac{\pi}{2} + \frac{3}{\pi} = 1.9 \ \ km.$$

Problem 4

Determine the demand function $q = D(p)$ that for all values of the price p has the same coefficient of demand elasticity.

By definition, the coefficient of price elasticity of demand e is calculated by the formula [5]

$$e = q'(p)\frac{p}{q}.$$

Therefore, problem 4 is reduced to solving the equation

$$q'(p)\frac{p}{q} = k,$$

where k is a constant.

Dividing the variables

$$\frac{dq}{q} = \frac{kdp}{p},$$

we integrate:

$$ln(q) = k \cdot ln(p) + ln(C),$$

and getting rid of the logarithms, we get $q = Cp^k$.

So, if the demand function is of the form

$$q = Cp^k,$$

where C is an arbitrary positive number, and k is a negative number (signs of C and k are determined by the economic meaning of the problem), then at any price the coefficient of demand elasticity is equal to k. [5]

Problem 5

Find the equation of a family of curves that intersect each parabola given by the equation $y = Cx^2$ at the right angle. [3]

Consider an arbitrary point $(x_0; y_0)$. For a parabola passing through this point $y_0 = C(x_0)^2$. Therefore,

$$C = \frac{y_0}{(x_0)^2},$$

and the equation of this parabola

$$y = \frac{y_0}{x_0^2} x^2.$$

Its derivative at the point $(x_0; y_0)$ – slope of the tangent is

$$y'(x_0) = \frac{2y_0}{x_0}.$$

Therefore, the slope of the tangent to the curve, which is perpendicular to the parabola

$$y = \frac{y_0}{x_0^2}x^2,$$

is

$$-\frac{x_0}{2y_0}.$$

Consequently, a family of curves intersecting each parabola is defined by the equation $y = Cx^2$ at the right angle and satisfies the differential equation

$$y' = -\frac{x}{2y}.$$

Its solution is

$$x^2 + 2y^2 = A.$$

So we get that the parabola given by the equation $y = Cx^2$ and ellipses with center at the origin of coordinates given by the equation

$$x^2 + 2y^2 = A$$

intersect at right angles.

Problem 6

0.5 thousand people listened to the speech of the leader of the Momba-Yumba tribe, in which he ordered to cancel corruption. After 1 hour 2 thousand people knew about it. How many people will know about it after 2 hours, if the tribe has 10 thousand people?

We have the equation

$$\frac{dx}{dt} = kx(t)\left(10 - x(t)\right). \tag{9}$$

Divide the variables:

$$\frac{dx}{x(10-x)} = kdt,$$

and integrate:

$\int \frac{dx}{x(10-x)} =$ (use the method of undetermined coefficients to break the integral into two simple integrals) =

$$\int \frac{0.1dx}{x} + \int \frac{0.1dx}{10-x} = 0.1(ln|x| - ln|10-x|) + ln(C_1).$$

Then,

$$ln\frac{x}{10-x} = 10kt + ln(C)$$

and

$$\frac{x}{10-x} = Ce^{10kt}. \tag{10}$$

Substituting the initial conditions in (10), we obtain

$$\frac{0.5}{10-0.5} = C \Rightarrow C = \frac{1}{19},$$

and

$$\frac{2}{10-2} = \frac{1}{19}e^{10k} \Rightarrow e^{10k} = \frac{19}{4}.$$

Thus, the number $x(t)$ of those who got the information at time t is given by the equation

$$\frac{x}{10-x} = \frac{1}{19}\left(\frac{19}{4}\right)^t.$$

In particular, in 2 hours:

$$\frac{x}{10-x} = \frac{1}{19}\left(\frac{19}{4}\right)^2 \Rightarrow \frac{x}{10-x} = \frac{19}{16} \Rightarrow 16x = 190 - 19x \Rightarrow 35x = 190,$$

approximately 5429 people will hear the news.

The equation of the form (9) is common for modeling various processes. Its solution is called a logistic curve. In this section, we look at another example [7] for such equations. A special quality of this examination is an unorthodox approach to integration.

Problem 7

Trout are bred in one of the Scottish lakes. In the absence of fishing, trout reproduction is described by the equation

$$\frac{dx}{dt} = 0.1x(t)(1 - 0.01x(t)), \ \ for \ \ x > 0, \ \ and \ \ for \ \ x = 0. \tag{11}$$

where $x(t)$ is the number of trout in thousands at time t.

In order to solve the equation, we divide the variables:

$$\frac{dx}{x(1-0.01x)} = 0.1dt.$$

In order to integrate

$$\int \frac{dx}{x(1-0.01x)},$$

we divide the numerator and denominator by x^2 and introduce the substitute $1/x = y$. Then

$$\int \frac{dx}{x(1-0.01x)} = \int \frac{-dy}{(y-0.01)} = -ln\,|y - 0.01| + C.$$

The result is a solution of the equation

$$-ln\,|1/x - 0.01| + C = 0.1t$$

or

$$x(t) = \frac{1}{Ae^{-0,1t} + 0.01}.$$

Suppose that at the initial time, there were 20 thousand trout in the lake. Then, the constant of integration A is equal to 0.04, and the number of trout expressed by the formula

$$x(t) = \frac{100}{4e^{-0.1t} + 1}$$

after a long period of time will stabilize at 100 thousand.

Now suppose that the expected fishing is at a constant rate of 2.1.

Then there is an equation

$$\frac{dx}{dt} = 0,1x(1 - 0,01x) - 2.1.$$

In order to integrate this equation, we first remove the parentheses on the right-hand side, and find the roots of the quadratic equation:

$$0.1x - 0.001x^2 - 2.1 = 0,$$

and factorize the quadratic trinomial:

$$0.1x - 0.001x^2 - 2.1 = -0.001(x - 70)(x - 30).$$

As a result, we obtain the equation

$$\frac{dx}{dt} = -000.1(x - 70)(x - 30).$$

In order to use the technique, which allowed us to integrate equation (11), we substitute the variable $z = x - 70$ and obtain the equation

$$\frac{dz}{dt} = -000.1z(z + 40).$$

We will solve it like equation (11): divide the variables, then divide the numerator and denominator by z^2, substitute $1/z = y$, integrate, and obtain

$$y + 40 = Ce^{0.001t}.$$

Returning from y to z, and from z to x, we get

$$x(t) = 70 + \frac{40}{Ce^{0.04t} - 1}.$$

We consider two cases.

In the first case, assume that the initial population contains 20 thousand trout. Then the corresponding particular solution is

$$x(t) = 70 + \frac{40}{0.2e^{0.04t} - 1}.$$

Since the denominator is continuous, negative for t = 0, and positive for large values of the argument, at some point it becomes a very small negative value. That is, at some point in time the value of livestock is negative. This means that at this rate of fishing at some point there will be no trout in the lake. Solving the equation

$$0 = 70 + \frac{40}{0.2e^{0.04t} - 1},$$

we can determine that this moment will come at $t = 19.05$.

In the second case, assume that $x(0) = 50$. Then

$$x(t) = 70 - \frac{40}{e^{0.04t} + 1}.$$

The obtained result shows that under these conditions the number of trout eventually stabilizes at 70 thousand.

References

[1] Sultanov Sh., Sultanov K., *Omar Khayyam*, Molodaia Gvardiia, 1987, 320 p.

[2] Conan Doyle A., *Stories*, Khudozhestvennaya literatura, 1982, 368 p.

[3] Filippov A.F., *Collection of problems on differential equations*, Nauka, 1973, 128 p.

[4] Guterman, M.M., Nitecki, Z.H., *Differential Equations. A Fist Course*, Saunders College Publishing, 1988, 689 p.

[5] Harshbarger R.J., Reinolds J.J., *Calculus with Applications*, Heath and Company, 1993, 640 p.

[6] Kydyraliev S.K., Urdaletova A.B. , *Mathematical models in control theory and operations research*, AUPKR, 2010, 208 p.

[7] Bailey D., *Mathematics in Economics*, McGraw-Hill, 1998, 485 p.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

THE SIGNIFICANCE OF INTERPRETATION IN THE DEVELOPMENT OF SPATIAL THINKING

GULBADAN MATIEVA, GULNISA BORBOEVA

Osh State university, Osh, Kyrgyzstan
e-mail: gborboeva@oshsu.kg

Abstract. The article considers the role of interpretation in development of spatial thinking of students of mathematics in formation of mathematical concepts.The concepts of algebraic and geometric interpretation are introduced. It is noted that the table form of representation of algebraic and geometric interpretations of the same mathematical concept shows its formation, as well as allowing us to see the relationship between abstract mathematical concepts and real objects and situations in space. The article establishes the importance of interpretation in the development of spatial thinking, namely that interpretation influences the development of perception and analysis of spatial relations: plays a decisive role in solving spatial problems; develops spatial intuition; plays an important role in the creation of internal visual representations and images; improves our ability to perceive, analyse and interact with the environment; develops unconventional thinking and the ability to see objects differently than they are presented; allows us to see hidden connections and potential in spatial data, which helps our ability to generate new ideas and solutions.The practical significance of the study lies in the development of a table containing algebraic and geometric interpretations of mathematical concepts. The study carried out a pedagogical experiment and observation. The study conducted in this article can be used for the development of spatial thinking in the process of formation of mathematical concepts.

Keywords: interpretation, mathematical concepts, spatial thinking, mathematics, mathematics students.

1. Introduction

Nowadays, the global changes are so rapid that it is difficult to predict what knowledge, skills and competencies today's learners should possess in order to have a successful future. It becomes difficult to find answers to questions about what to teach and how to teach students in preparation for their future, which is characterized by great variability.

It is known that in the life of modern society, education plays one of the main roles, as it is the main source of generation, improvement and development of human capital, and therefore the resource of socio-economic development of the country, progress in society and the well-being of the individual. Significant changes are taking place in the field of education in the world and in Kyrgyzstan due to the constant and rapid updating of technologies, mainly information, and the processes of transformation in the social sphere.

It is now recognized that the leading role in achieving sustainable development is to be played by education, which is explicitly referred to in many UN documents as a decisive factor for change.

Sustainable development is a development that meets the present's needs without jeopardizing future generations' ability to meet their own needs.

Like all countries worldwide, the Kyrgyz Republic has made a commitment to achieving the Sustainable Development Goals (SDGs) by 2030. The goals of sustainable development are

included in State policy and are reflected in the National Development Strategy of the Kyrgyz Republic for 2018-2040.

The fourth of the 17 Sustainable Development Goal (SDGs 4) is devoted to education and is formulated as 'Ensuring inclusive and equitable quality education and promoting lifelong learning for all'.

The development of sustainable solutions to the development needs of both peoples and the planet can be seen as central to education efforts. Education enables people to understand the nature and scope of sustainable development challenges; it provides an opportunity to develop the critical, innovative and creative approaches needed to find new, more effective solutions to global problems; it allows people to understand the essence of those powerful factors that determine an unsustainable lifestyle; and it can help people develop the self-confidence, organizational skills and optimism that will enable them to act individually and collectively for a sustainable future.

At the moment, in order to realize this goal, it is necessary to direct the educational process to the development of skills of the 21st century, which has led to a change in the State educational standard of our republic. Developing the skills of the 21st century falls on the shoulders of teachers, who themselves must meet the requirements of the modern paradigm of education. One such skill is mathematical literacy, which shows "the ability of a person to think mathematically, formulate, apply and interpret mathematics to solve problems in a variety of practical contexts [2]. A person with such skills is trained by a teacher trained in four areas (psychological and pedagogical, subject, methodological, technological) in a specially organized pedagogical system. However, this is not enough to develop the spatial thinking of the students themselves. Other methods of developing this type of thinking are therefore necessary.

In the works of foreign scientists D. Clements, T. Lowry, M. Batisst, I. Resnick, D. Harris, N. Newcomb, I. S. Yakimansky, I. Y. Kaplonovich, etc., psychological and pedagogical aspects of the development of spatial thinking of school students have been studied.

Research on the development of spatial thinking of future teachers is carried out in the works with references to [3], [4], [5].

The research of the general system of methodical training of the future teacher of mathematics is reflected in the works of Kyrgyz scientists (I.B. Bekboev, A.A. Akmatkulov, J.M. Baisalov, K.M. Torogeldiyeva, Sh. Aliev. M. Altybaeva, K.T. Turdubaeva, G.K. Kazieva, etc.). However, the problem of development of this kind of thinking of future teachers of mathematics in the two level system of education of our republic is underexplored.

The paper [6] highlights the importance of a concrete-inductive method for introducing geometric concepts in the development of spatial thinking of future math teachers. In the article referring to [7] the aim of this study is to identify the effects of formalism in teaching on primary and secondary school mathematics teacher trainees' algebraic and geometric interpretations of the notions of linear dependency/independency. Quantitative research methods are drawn in order to determine differences in success levels between algebraic and geometric interpretations of the linear dependency/independency of vectors presented in twoand three-dimensional space. On the other hand, qualitative research methods were utilized in order to investigate thinking modes involved in the geometric interpretation of the same notion.

Research objective. The article considers the significance of geometric and algebraic interpretations for the development of spatial thinking of future teachers of mathematics in the process of formation of geometric concepts.

2. Materials and Methods

In the modern information world, the word "interpretation" is used quite often. However, the meaning of the term varies depending on the area under consideration.

The concept of interpretation can be broadly or narrowly interpreted. Interpretation in a narrow sense is linguistic cognitive activity primarily of the individual, revealing in its results the

subjective understanding of the object of interpretation. In the broad sense of the word interpretation is understood as almost any imaginative operation aimed at obtaining new knowledge of the collective or individual level. Various functions of human thinking (schematization, classification, categorization, generalization, concretization, etc.) - essence of interpretation [8]. It is in this sense that the interpretation is considered in this paper.

Whatever form the interpretation takes, it is intimately linked to understanding, as it serves as its basis. From the fact that in the process of understanding the individual himself attributes meaning to the object, it does not follow that every understanding is equally acceptable. The interpretation of an object is always hypothetical and can be revised.

The primary goal of understanding is to give meaning to the object of knowledge. The basis of understanding, i.e. the source that provides us with interpretations, is the individual semantic context, which is a system of interrelated semantic units.

The main focus in the knowledge of mathematical objects and processes, as well as in the knowledge of realities, is on interpretation. It provides an opportunity to understand the meaning, properties and relationships of the mathematical object or process.

Interpretation in mathematics is a set of meanings assigned in one way or another to elements (expressions, formulas, symbols, etc.) of a natural science or abstract deductive theory. In the same cases, when such "comprehension" are subject to the elements of this theory, they also speak about the interpretation of symbols, formulas, etc.

In mathematics, two kinds of interpretation are mainly considered: algebraic and geometric, which contribute to better understanding and application of mathematics in different fields, and which allow solving different mathematical problems using different approaches.

Algebraic interpretation is a way of representing and understanding mathematical concepts using algebraic structures and operations. In algebraic interpretation, mathematical objects and operations can be represented using algebraic expressions, equations, and systems of equations. This allows abstracting from specific geometric objects and viewing their properties and relations in a more general algebraic context. Geometric interpretation is an approach to understanding mathematical concepts and methods through geometric representations and illustrations. Rather than using abstract symbols and formulas, geometric interpretation allows visualization and interaction with mathematical objects and relationships using geometric shapes, graphs and spatial representations. This interpretation is widely used in various fields of mathematics, including linear algebra, geometry, analysis, and probability theory. Thus, in linear algebra, geometric interpretation allows the representation of vectors and their operations as directed segments and geometric transformations. Geometric interpretation is used in geometry to represent the properties and relationships between geometric shapes, such as lines, planes, and polytopes.

Geometric interpretation aids in visualizing and intuitively understanding mathematical concepts, which can be advantageous in solving problems, formulating hypotheses, conducting proofs, and generalizing mathematical context. It also associates abstract ideas with concrete and visual representations.

3. Main results and analysis

Let us examine various interpretations of the basic concept of linear algebra - matrix.

The algebraic interpretation of the matrix is related to the operations and properties of the algebra. The matrix is a rectangular table of numbers ordered in rows and columns. Each element of the matrix can be a number or a symbol, and matrices can be folded, multiplied, and invertible when certain conditions are met. Algebraically, the matrix can be interpreted as a linear transformation that maps vectors from one space to another. This allows the use of matrices to solve systems of linear equations, to find eigenvalues and vectors, and to describe linear maps and transformations. Thus, an algebraic interpretation of the matrix allows it

to be used to study and analyze various algebraic structures and operations related to linear transformations and equation systems.

The geometric interpretation of the matrix is related to the representation of the matrix as a linear transformation of vectors in space. Suppose we have a matrix A of dimension $m \times n$. Each column of this matrix can be viewed as a vector in n dimensional space. Then, by applying matrix A to a vector, we get a new vector that is a linear combination of the columns of matrix A. In other words, the geometric interpretation of a matrix is that it describes a linear transformation of space in which vectors can be squashed, stretched, rotated, or flipped. The properties of the transformations depend on the values of the matrix elements. Thus, the matrix A of the rotation or reflection changes the orientation of the vectors. The diagonal matrix scales the vectors along the coordinate axis. Thus, it is established that the geometric interpretation of the matrix allows understanding its influence on the shape and orientation of vectors in space.

The tabular method of describing these concepts contributes to the understanding of the concept under consideration, which is necessary for solving various problems. The table below contains the definition, algebraic and geometric interpretations of the geometric concept of "plane rotation" (Table 1).

Table 1. Interpretations of "Plane Rotation"

Concep t	Definition	Algebraic interpretation	Geometric interpretation
Plane rotation	is a transformation in which all points in the plane rotate around a certain point O, called the center of rotation, to a certain angle θ.	$M(\theta) = \begin{pmatrix} \cos\theta - \sin\theta \\ \sin\theta \end{pmatrix}$	x_x

The interpretations of other geometric concepts are discussed in table 2.

Table 2. Interpretations of geometric concepts

So, interpretations of the mathematical concept in the form of the above proposed table are one of the ways of determining its formation in a student of mathematics.

The most important prerequisite for the formation of ideas about objects is to teach students how to look at and remember details and reproduce them from memory, as well as to master the techniques that students need to create images when reading a drawing based on the perception of the plane image [9].

We conclude that activities demonstrating the links between geometry and algebra should be an important part of the curriculum. The study of topics from both geometric and algebraic perspectives should be a constant part of any curriculum in mathematics.

In physiology, it is observed that the right hemisphere of the human brain is responsible for intuition, while the left hemisphere is responsible for logic. The interaction of the two hemispheres is necessary for controlling the brain and transmitting information from one to the other. The stronger the connection between the two hemispheres of the brain, the higher the human intellectual development, memory, perception, attention, imagination, speech, thinking. The learning process involves a distinction between perception and understanding of information. Hence, the geometric interpretation of mathematical concepts and, conversely, the algebraic interpretation of geometric concepts contributes to the formation of concepts and development of the above psychic processes, including spatial thinking, what is necessary for a successful life and work of the person in the sphere of his activity.

The relationship between spatial thinking and advances in mathematics in development is consistent, predictable and evolves over time [10], [11]. Fortunately, spatial thinking can be learned and improved with practice at any age [12], and the development of spatial thinking is causally linked to an improved understanding of mathematics [13]. So, spatial thinking, the

ability to mentally represent and transform objects and their relationships, is thought to be so closely related to mathematics [14], [15].

Interpretation plays an important role in the development of spatial thinking, which is stand as the ability to perceive, analyze and manipulate spatial objects and representations.

We focus on two major components of spatial reasoning. Spatial visualization involves mentally creating and manipulating images of objects in space, from fixed or changing perspectives on the objects, so that one can reason about the objects and actions on them, both when the objects are and are not visible. Property-based spatial analytic reasoning decomposes objects into their parts using geometric properties to specify how the parts or shapes are related, and, using these relationships, operates on the parts. Spatial analytic reasoning generally employs concepts such as measurement, congruence, parallelism, and isometries to conceptualize spatial relationships [16].

In the context of spatial thinking, interpretation refers to the ability to analyse, understand, and give meaning and meaning to visual or conceptual images.

We note several advantages in which interpretation affects the development of spatial thinking:

- perception and analysis of spatial relations: interpretation allows us to recognize and understand the spatial relationships between objects, such as distance, direction, size and form. The ability to interpret these relationships helps us to construct internal ideas about space and use them for navigation and manipulation;
- solving spatial problems: interpretation plays a crucial role in solving spatial problems such as designing and assembling objects, drawing and **??**designing.To accomplish the required task, we must interpret information about the form, position, orientation, and relationships between the objects;
- development of spatial intuition: interpretation helps us to develop spatial intuition, which is an unconscious understanding of spatial relations and possibilities of manipulation. The more we are experienced in interpreting and understanding spatial patterns, the better we develop our ability to quickly and accurately assess spatial situations and make appropriate decisions;
- visualization and presentation: well-developed ability to interpret visual images allows us to create and manipulate mental models of objects and space, which promotes the development of creativity, problem solving and analytical thinking;
- cognitive process: the development of creativity and innovative thinking is linked to the improvement of our abilities in perception, analysis, and interaction with the environment through interpretation. Spatial problems often require unconventional thinking and the ability to see objects differently than they are presented. Interpretation allows us to see hidden connections and potential in spatial data, contributing to our ability to generate new ideas and solutions. In general, interpretation fosters spatial thinking, which is crucial for resolving problems in various fields like science, engineering, design, architecture, and many others.

4. Conclusion

The interpretation of mathematical concepts develops spatial imagination and the ability to solve spatial problems. It helps to represent and manipulate objects in the mind, draw conclusions about their properties and predict the results of actions. In addition to this definition, the algebraic and geometric interpretation of the same mathematical concept in the table helps to better understand this concept, reveal its properties, which plays an important role in the development of spatial thinking, establishing understanding and interaction with spatial objects and situations.

References

[1] Keiner, M. (2005). History, definition (s) and models of sustainable development. ETH Zurich.

[2] Ernest, P., Skovsmose, O., Van Bendegem, J. P., Bicudo, M., Miarka, R., Kvasz, L., & Moeller, R. (2016). The philosophy of mathematics education. Springer Nature.

[3] Korkmaz, H. İ., & Tekin, B. (2020). Investigating Spatial Thinking Skills of Prospective Preschool Teachers. Journal of Theoretical Educational Science, 13(1), 191-204.

[4] Lieban, D., & Lavicza, Z. (2017). Geometric Modeling as Spatial Thinking Approach among Prospective Teachers. In Electronic Proceedings of the 22nd Asian Technology Conference in Mathematics.

[5] Vasilenko, A. V. (2013). Methodical training of future teachers for the development of spatial thinking of students by means of information technologies. RUDN Journal of Informatization in Education, (2), 59-65.

[6] G.M. Borboeva (2022). Geometrialyk tushunuktordu kiiruudo konkrettuuinductivduu method matematik-studenttin meykindik oy jugurtuusun kalyptandyruu karajaty katary [The concrete-inductive method of introducing geometrical concepts as a means of shaping the spatial thinking of a maths-student] Science. Education. Technique. - Osh: KUU - No. 3 (75). 165-169.

[7] Ertekin, E., Solak, S., & Yazici, E. (2010). The effects of formalism on teacher trainees' algebraic and geometric interpretation of the notions of linear dependency/independency. International Journal of Mathematical Education in Science and Technology, 41(8), 1015-1035.

[8] Larionova E.V., Bukova O.M., Professional education, 7, 15-19 (2018).

[9] Resnick, I., Harris, D., Logan, T., & Lowrie, T. (2020). The relation between mathematics achievement and spatial reasoning. Mathematics Education Research Journal, 32, 171-174.

[10] Resnick, I., Newcombe, N. S., & Jordan, N. C. (2019). The relation between spatial reasoning and mathematics achievement in children with mathematics learning difficulties. In P. Rasanen (Ed.), International handbook of mathematical learning difficulties.

[11] Lowrie, T., Logan, T., & Hegarty, M. (2019). The influence of spatial visualization training on students' spatial reasoning and mathematics performance. Journal of Cognition and Development, 20(5), 729-751.

[12] Cheng, Y.L., & Mix, K.S. (2014). Spatial training improves children's mathematics ability. Journal of cognition and development, 15(1), 2-11.

[13] Lowrie, T., Resnick, I., Harris, D., & Logan, T. (2020). In search of the mechanisms that enable transfer from spatial reasoning to mathematics understanding. Mathematics Education Research Journal, 32, 175-188.

[14] Battista, M. T., Frazee, L. M., & Winer, M. L. (2018). Analyzing the relation between spatial and geometric reasoning for elementary and middle school students. In Visualizing mathematics: The role of spatial reasoning in mathematical thought (pp. 195-228). Cham: Springer International Publishing.

[15] Lieban, D., & Lavicza, Z. (2017). Geometric Modeling as Spatial Thinking Approach among Prospective Teachers. In Electronic Proceedings of the 22nd Asian Technology Conference in Mathematics.

[16] Burte, H., Gardony, A. L., Hutton, A., & Taylor, H. A. (2020). Elementary teachers' attitudes and beliefs about spatial thinking and mathematics. Cognitive research: principles and implications, 5(1), 1-18.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

THE THEORETICAL BASIS AND PRACTICAL IMPLEMENTATION METHODOLOGICAL TRAINING OF MATHEMATICS TEACHERS IN AN INNOVATIVE MANNER

GULNAR ZHAKYPBEKOVA

South Kazakhstan State Pedagogical University, Shymkent, Kazakhstan
e-mail:jakipbekoba.gulnar@gmail.com

Abstract. Methodical science teaching mathematics covers all stages of mathematical education. Lesson methodology has affected the work of mathematics teachers and academics. Hence the importance of methodical preparation of teachers of mathematics and their means of methodic knowledge. Modern mathematics teacher must know the components of the methodology, the methodology of mathematics and learn the methodology of scientific research, learning theory and its applications. This will require the need for special analysis methodology of teaching mathematics. The urgency of this challenge is increased by the ideas of innovation research.

Keywords: Methodology of mathematics, methodology, methodology, teacher training, pedagogical innovation, methodological training of mathematics teachers in an innovative manner.

AMS Subject Classification: 68

1. Introduction

The article deals with the issues of theoretical justification and practical implementation of improving the methodological training of a mathematics teacher on an innovative basis. The science of mathematics teaching methodology covers all the steps of mathematical education. For teachers of higher educational institutions and school teachers, it is becoming a daily necessity to engage in the methodology of teaching the subject. Therefore, the methodological training of a mathematics teacher and its improvement on an innovative basis was considered as an urgent problem.

Modern time requirements modern subject teachers should know the methodological components of teaching the subject, master the science of methodology and its applications through scientific research. This indicates the need for a special study of the methodology for teaching mathematics. The relevance of this complex problem increases due to the solution of scientific research work from the point of view of the idea of innovation. On the basis of the state program for the development of education and science in the country for 2016-2019, updated educational programs were introduced to schools [1].

In connection with the introduction of updated educational programs, the approaches to training and the assessment system have also radically changed. This showed the subject teachers the importance of using modern innovations to harmonize the updated educational program, pedagogical approaches, and assessment system. The state program for 2020-2025 sets the task "ensuring the high status of the teacher's profession, modernization of pedagogical education" [2].

At the Republican meeting on the topic "mathematical education in schools: state, problems, proposals", held in Karaganda in 2022, the minister of Education and science of the Republic of

Kazakhstan A.Aimagambetov said: "the president set very important tasks for the development of education and science. Today, an important issue is the development of technical specialties in the country. It is known that the basis for the development of technical specialties is mathematics. Therefore, in schools, special attention should be paid to the training of personnel in the updated programs of mathematics, professional development of specialists and the content of school textbooks. Work has already begun for this purpose. Groups consisting of experienced teachers, methodologists and university specialists have been formed. Since 2023, students will study mathematics according to the updated program" [3].

The participants of the meeting decided that "it is necessary not only to increase the interest of students in mathematics, but also to increase the professional competence of teachers". Within the framework of the concept of "improving professional competence", the issue of improving the methodological training of subject teachers on an innovative basis occupies a special place. The tasks set are aimed at improving the methodological training of mathematics teachers on an innovative basis in accordance with the requirements of the time and the effective application of systematic and active approaches to the educational process in the context of the transition to the updated content of education.

That is, each teacher seeks to improve his methodological system and apply it in practice. Because their main goal is to ensure the development of students as competitive individuals in the global educational space. These issues indicate that the issues of theoretical justification and practical implementation of improving the methodological training of a mathematics teacher on an innovative basis are relevant today. Indeed, in today's age of technology, the education sector needs to focus on improving the methodological training of a mathematics teacher on an innovative basis.

This is because mathematical science was the beginning of human culture at all stages of the development of society. Mathematics is not only the basis of the Natural Sciences, an important part of scientific and technological progress, the key to knowing the environment, but also occupies an important place in the development of the individual. It is also known that mathematical knowledge is extremely necessary in all spheres of society, in everyday life.

The development of mathematical knowledge at all stages is constantly developing in connection with social changes in society, updating educational programs, pedagogical concepts, improving the methodology of the discipline, and methodological training of specialists. The current stage will focus on such issues as changes in educational values, a turn to science, the logical nature of mathematics, updates in the education system, innovations in the research direction.

That is, improving the methodological training of a teacher in teaching mathematics on an innovative basis, focusing first on the concepts and research methods of teaching school mathematics, the humanization of education, the orientation of teaching to the individual and the results of research of the applied direction of mathematics. For this, system analysis and active teaching approaches in the presentation of materials in the content of the subject of mathematics are proposed.

The spiral presentation of the content of the discipline, highlighting the main components in it, establishing connections between them will be effective due to System Analysis and active teaching methods. The orientation of students to creative, inventive activities in teaching mathematics should form a value attitude of the teacher to continuing education and ensure the readiness of the teacher to effectively use rapidly developing innovative technologies in the classroom. To do this, a mathematics teacher must be able to use digital educational resources for new educational technologies and methods. Within the framework of the Digital Kazakhstan program adopted in the country, the use of digital technologies in the educational process is important for the formation of modern knowledge that corresponds to the self-development of a new type of person [4].

The use of digital educational resources in teaching mathematics has a positive effect on the development of mathematical thinking of students in the construction of graphs, transformation of spatial figures, translation from one language to another. Due to the use of computers, other modern technologies in teaching mathematics, the visibility and effectiveness of training increases. This reflects the desire of a mathematics teacher in his professional activities to bring the potential level of students into line with world standards, improving methodological training based on digital educational resources on an innovative basis.

2. Literature review

Since the second half of the last century, the methodology of teaching mathematics began to take shape as a science in itself. He studied the subject, form of methodological science (Sarantsev G. I.), methodically substantiated the subject of methods of teaching mathematics (Nugmonov M.), introduced the methodological system of teaching mathematics (Pyshkalo a.m.), considered the philosophical problems of mathematics (I. Lakatos), studied the problems of improving mathematical education (A. Pardala). Training of teachers in the conditions of informatization of Education (Lapchik M. P.), the formation of professional competence of teachers in the use of information technology tools (Shmakova A. P.).

Research related to the methodological substantiation of a number of problems of the methodology of teaching mathematics in Kazakhstan (Abylkasymova A. E., Kagazbaeva A. K., Kozhabaev K. G., Kenesh A. S., Mubarakov A., Rakhimbek D., etc.) and the use of information technologies in the professional activities of the teacher (Fedotova E. L., Kenesbayev M. S., Bidaibekov E. Y., etc.) appeared.

In connection with the introduction of the updated educational program in schools, a lot of work has been done in the direction of "advanced training and retraining of teachers". Most of the teachers of the school took courses at the National Center for advanced training "Orleu".

A number of studies were conducted in this direction (Kibataeva N. K., Orakova A. Sh., Kusherbaeva M. R.). For example, Kibataeva N. K. in her research pointed out that "the innovative nature of the ongoing changes in the education of the country is directly related to the new format of professional development, which is the main tool for updating Kazakhstan's education." Also, scientific and methodological seminars, intra-school courses, trainings, etc.were organized in schools [5].

However, there are also enough problems that need to be solved, one of which is to improve the methodological training of a mathematics teacher on an innovative basis. In the works of G. I. Sarantsev and A.M. Pyshkalo, it is believed that the initial provisions that determine the features of the methodology for teaching traditional mathematics should include the laws of Personality Development, the results of learning and the problems of developing the individual abilities of the student.

Because, according to the requirements of today's time, the focus should be on the self-development of the student. Self-development of the individual at the stage of development of modern information technologies requires a radical revision of the goals, content, form and methods of teaching in the discipline at a new modern level. Modern education is the education of a future citizen of society, a person who can search and read all his life, while modern education should develop today's student as a subject of cognitive activity.

As the basis of the problem under consideration, ways to improve the methodological training of a mathematics teacher on an innovative basis are indicated. In the general case, the components of the methodological system of teaching mathematics include the purpose of teaching, the activities of the teacher, the activities of students and the results of teaching. It allows us to identify the forms of Organization of training and their relationships as an integral part of the methodological system of training and its system-forming parts. Here, joint educational direct and feedback between the teacher and students formed an inseparable connection in the methodological system.

Currently, B. Bloom's taxonomy is widely used in all educational fields. B. Bloom's taxonomy consists of basic levels in the form of knowledge, understanding, Application, Analysis, compilation, evaluation. For example, in the works of B. Bloom, active learning covers three main areas of learning, which are called knowledge, skills, attitude. This taxonomy of learning can be understood as the "goal of the learning process". For example, in mathematics, "know" is the memorization of the studied material, "understand" is the transformation of the expression in the educational material, the translation of verbally transmitted information into a mathematical formula, that is, the translation from one language to another. The student's brief presentation of the educational material, explanation or prediction of ways to solve the problem, prediction of the expected result also belong to the level of "understanding". "Application" means the ability to apply the studied educational material in real and new conditions. "Analysis" means dividing the structure of the educational material into separate structures so that it becomes clear. "Synthesis" - defines the integration of elements to obtain a whole that has novelty. For example, actions aimed at solving a problem are considered in the form of a plan for solving it or a set of generalized connections. "Assessment" - differs in the assessment of the essence of the educational material for a specific purpose.

In this approach, learning is organized on the basis of close communication between the teacher and students and takes into account the individual abilities of each person involved in the learning process. Students acquire new knowledge through joint search, Discussion, Analysis, accumulation of information on the content of training. This method leads students to solve complex problems on the way to the goal.

3. Materials and methods

In teaching mathematics, there are traditional forms of organizing classes, such as classes, lectures, seminars, conferences, laboratory classes, practical classes, workshops, electives, excursions, circles, olympiads, creating projects, industrial practice, independent work, consulting, exams. Today's goal is to improve the quality of classes on an innovative basis through the effective use of active methods.

Active Education expects radical changes in the modern direction from the activities of the teacher. That is, active teaching approaches provide training on an innovative basis. Active learning is based on the relationship and interaction between teacher and student, student and student. When using innovative methods, students develop creative abilities through cognitive activity, competition, the ability to work in a team, open expression of views in solving problems, the ability to prove their thoughts [6].

Active learning methods include problem-based learning, heuristic conversation, discussion method, brainstorming, round table, business games, etc.

Problem-based learning is organized to develop the mind in order to solve the problem. The method is used for the purpose of independent learning of students, creative thinking, increasing cognitive activity. An important stage here is to create a problem situation. When a problem situation arises, "difficulties" arise for students. In solving it, through their creative activity, they create research, search, learn to think productively. This increases their interest in the subject.

In problem-based learning, the teacher does not offer ready-made rules or solutions to the problem. He asks problematic questions in the direction of solving the problem and leads the students to find the necessary rules themselves. Students will find the rules themselves. For example, a teacher, explaining the Pythagorean theorem, can set the following problem situation. "The ancient Greek mathematician Pythagoras walked in Egypt and took a three-piece rope with a length of 3,4,5 to draw a right angle on the ground. To make the desired triangle, you need to connect the ends of the pieces of rope. Pythagoras built a triangle. Then he wondered what property of a triangle was used in the method used by Pythagoras to create a right angle". The teacher instructed the students to find this property of the Triangle.

The heuristic narrative method originates from Socrates ' teaching method "heuristics" (in the sense of inventing, finding, discovering). The psychological nature of this method is collective thinking or conversation as a search for an answer to the problem posed, finding a solution.

The discussion method is characterized as a free analysis of theoretical problems. This method is organized by asking pre-oriented questions. The discussion method is often used in group work in classes. Sometimes it can also be used in a theoretical lesson.

The "brainstorming" method is not yet fully formed in the educational process. This method is often used in a scientific direction, in economic management. The main rule in the method is that participants are given complete freedom to express their thoughts. Their reasoning is not discussed, not tested. In this method, participants feel more relaxed.

The "round table" method is usually organized for discussion purposes. This method is often used in the field of politics and science. This method is organized in order to solve theoretical problems with the participation of specialists from various fields.

The "business game" method is used to engage future specialists in organizational, managerial activities in their professional field.

These methods are effective when not only the teacher is active, but also the students. Currently, in addition to the listed methods, many methods are used in the lesson: presentations, case technologies, didactic Games, project method, "basket method", etc. For example, presentations are the easiest and most affordable method used in the classroom. It contains information such as lesson content, independent work of students, projects. Case technologies have been used in teaching since the last century. It is based on analyzing specific situations and finding solutions. Didactic games are used in situations such as quiz questions, travel games, etc. The basket-method is based on simulating the situation [7].

In general, the lesson is still the main form of Organization of training. The difference between modern lessons from the traditional lessons of the past is that the teacher organizes independent types of work, using modern teaching methods, aimed at the full assimilation of the content of the subject, taking into account the cognitive activity of each student and the creativity of each. This is due to the fact that today in the educational process it is considered relevant to use teaching methods that develop students ' ability to independently acquire knowledge, accumulate the necessary information, make predictions, make judgments and draw conclusions. In this case, the teacher must be able to use effective educational activities that allow him to effectively organize educational activities. For this purpose, it is important to be able to optimally use methods aimed at Active Learning based on innovative teaching technologies.

Among the active methods, observation, practice, Discussion, Project actions bring positive results. For example, students create conditions for the development of reflection and assess their level of knowledge by observing their actions, evaluating them, correlating learning results with the set goal. At the same time, the teacher's activity is not limited to explaining the content of the lesson, he is recognized as a person who teaches new knowledge based on creative actions, directing it to the independent assimilation of students. He systematically plans the structure of each lesson, taking into account the stages of the lesson, the connections between the stages. For example, the structure of the lesson may depend on the didactic goal, the volume of educational content, the age of students, the specifics of the class [8].

When planning the structure of the lesson, instead of the lesson in the section, depending on the type of lesson, the teacher should make sure that the students are not overloaded or, on the contrary, there is a lack of material, take into account in advance the content of the lesson and the relationship between the previous lessons. In other words, it acts as an organizer, coordinating activities such as goal setting, planning, forecasting, control, adjustment, evaluation, regulation in accordance with the training stages of the lesson.

The main stages of the lesson, which characterize the various actions of the teacher and students in each lesson, can be combined with the interpretation of a new lesson, learning,

repetition, testing of knowledge, skills, homework, etc., and the structure of the lesson can be described as follows.

1.Motivational phase:

- updating the content of education in achieving the goals of the lesson;
- create problem situations to achieve the goal;
- setting the learning goal of the lesson, that is, knowing new concepts, definitions, etc., generalizing the lesson, etc.

2. Stage of educational activity:

- identify ways to achieve the educational goal;
- drawing up an action plan;
- implementation of the created plan.

3. Stage of control and evaluation of educational activities:

- comparison of the results of educational activities with the evaluation criterion;
- assessment of the quality of the results of independent work.

4. Lesson approval period. Homework. Conclusion.

Drawing up a short-term lesson plan, the teacher:

- setting educational tasks based on the correlation of known and unknown, mastered and not mastered, so that students willingly accept the educational goals proposed by them or formulate them themselves;
- create a problem situation, mobilize students' attention and knowledge to solve it;
- encourage students to develop a plan for ways to achieve their goals;
- forecast of expected results;
- make adjustments and additions to the action plan, methods used by students in case of inconsistency of the expected results and learning outcomes;
- help students assess the quality and level of assimilation of educational material.

Above, the general direction of the structure of the lesson for the updated educational program was indicated. The main feature of the lesson here is the development of students as individuals, forming their learning activities. There are several important aspects of organizing these classes.

The first aspect is motivational goal setting. The purpose of the current lesson is clear and the expected learning outcomes are predicted in advance. The effectiveness of the lesson is determined by the results of training. The result of the lesson does not depend on the student's academic performance, the amount of material mastered, but on the activities, abilities of students, the ability to apply knowledge, implement their projects, in a word, universal educational activity. The new educational goals of the lesson include those that the students themselves formulate and understand that they are important to them. Such approaches in the lesson focus not on memorizing knowledge, but on the ability to apply the acquired knowledge in life.

Second aspect: active methods. In the updated educational program, a new meaning of the lesson is revealed. The new essence of the lesson is to solve the problems posed by students through their own cognitive activity. The nature of the problem lesson can be considered as a rejection of reproductive methods in traditional lessons. When using this method, students should be allowed to express their thoughts freely. It is then that students develop problem-solving skills and develop themselves in a creative direction. Such classes are distinguished by the effective use of active methods of discussion, dialogue, business Games, role-playing games, attacks on the "brain", etc. The use of many modern pedagogical technologies in the formation of students ' educational activities, such as critical thinking, design activities, research work, discussion, collective and independent thinking activities, has a positive effect. But it is better for the teacher to effectively use the methods used as intended [9].

The teacher, together with students, conducts work in the direction of searching, analyzing and studying the scientific content of the knowledge that they master. Such classes are described as classes organized in a modern direction. At this stage, the main thing will be the educational

content, and the student will be perceived by the teacher as the creator of the educational content. These actions lay the foundation for developmental learning technology. The orientation of the lesson to developmental learning strengthens the student's confidence that he feels free in the environment, that he is a full participant in the environment. At the same time, the teacher plays the role of the organizer of the planned educational activities.

Having planned the lesson, the teacher draws up a technological map of the lesson. When using active teaching methods in mathematics lessons, it is very important that the teacher offers students new ideas and tasks for their development. It is very effective here to promote students 'dialogue and participation in discussions, to form students' independent thinking and research skills, to organize differentiated learning, project work [10].

In mathematics lessons, methods of argumentation, problematic, creative, critical thinking, communication, etc.are often used. These methods are considered the main ones in the development of students. In pedagogy, within the framework of morality and humanization, democratization of the education system, ideas about the development of the personality of the student, the personality of the teacher began to be expressed more and more often. In improving the methodological training of a teacher on an innovative basis, attention is paid to the formation of a culture of mastering digital technology and modern technologies, system analysis of the content of the discipline [11].

So, in the educational process, it is very important to improve the methodological training of a teacher in mathematics on an innovative basis in the training of specialists with a high level of professional knowledge and internal culture, the ability to make optimal decisions, predict the course of development of a particular situation.

4. Results and Discussion

Depending on the content and features of the updated educational program, institutions for advanced training and retraining of teaching staff have done a lot of work. However, in the process of working with the updated educational program, a number of circumstances have been observed that determine the need to improve the methodological training of school mathematics teachers on an innovative basis. For teachers accustomed to the traditional teaching system over the years, the adoption of the experience of developed countries, updating pedagogical thinking, changes in the assessment system, the introduction of innovations and leadership, interaction with international practices, insufficient skills in the use of digital technologies, language goals of classes, etc.caused a number of psychological "difficulties".

Such problems were solved in stages and systematically. Each stage of the study of the problem under discussion was based on the results of the previous stage and the results of improving the methodological training of a mathematics teacher on an innovative basis [7].

The problem of improving the methodological training of a mathematics teacher on an innovative basis was solved by modeling, system analysis, Special Research Methods in subject methods, knowledge of the methodological foundations of pedagogy and psychology, clarification of General Provisions up to methodological recommendations. The individual results were tested experimentally. This mainly concerned the use of real learning situations, business and role-playing games, educational projects, and modern teaching methods.

Improving the methodological training of a mathematics teacher in his professional activity on an innovative basis refers to the science of the methodology of teaching mathematics. Methodically, search expresses the main nature of the professional activity of a mathematics teacher. The teacher must ensure the methodology of scientific research, system analysis, adaptation to various changes, forecasting learning outcomes, assessing the situation, choosing technologies, their effective use.

To do this, the teacher must conduct a systematic analysis of the materials included in the content of Mathematics within the framework of the updated educational program and justify it in an innovative way in accordance with the methods of students ' actions. Here, special attention

is paid to the tasks in independent work. They allow you to master educational material, evaluate modern options for its presentation, evaluate the results of Educational Projects, Business games. That is,

- the modern lesson should be personality-oriented, individual in nature;
- in the lesson, the priority is given to the independent work of students;
- the action of using active methods is implemented;
- each lesson is aimed at revealing the individual abilities of students, developing them;
- the teacher performs the duties of a guide to the educational process, an assistant to students.

The teacher improves his methodological training, studying the theory of teaching mathematics, the application of the theory in practice, the methodology of its teaching. In addition, he conducts experiments on methodological analysis and evaluation of the results in the use of modern teaching methods. Improving the methodological training of a mathematics teacher on an innovative basis was determined by dialectics, modeling, system analysis and active methods.

Special research methods include the concept of pedagogical and psychological research, the systematic presentation of the objects under study, their properties and relationships, the integrated use of System Analysis and active teaching methods, the formulation of methodological recommendations, the analysis of experimental results.

5. Conclusion

To improve his methodological training on an innovative basis, a mathematics teacher must know the content of the discipline, the methodology of teaching the discipline, its relationship with other sciences, the directions of development of methodological science, the types of experiments. It is based on the ability to independently obtain information, find ways to rationally solve problems, critically analyze the knowledge gained and apply it in solving new tasks in their professional activities. It should not be limited to ready-made education when presenting educational material to students, but rather direct them to the intellectual development of the student.

This is due to the mathematical knowledge of teachers and the level of improvement of methodological training on an innovative basis. The demands of the time have changed the approach to the current lesson. The level of the teacher is reflected in the lessons he conducts. Their high-level classes form such valuable qualities of students as knowledge of the world, disclosure of abilities, development as a person, step into new knowledge, self-knowledge, self-development and motivation, interest, professionalism, initiative. Therefore, the teacher should be able to plan the traditional structure of the lesson with his creative level searches based on modern innovation, combine the content of educational material, technologies for its presentation in the framework of the use of active teaching methods.

Thus, the theory and methodology of teaching mathematics, its application and improvement of methodological training of teachers in teaching mathematics on an innovative basis, modern teaching methods allow us to effectively build, manage the process of teaching mathematics, and obtain results in accordance with the planned goals.

References

[1] Gosudarstvennaya programma razvitiya obrazovaniya i nauki Republic of Kazakhstan na 2016-2019 gody, *[Elektronnyy resurs] URL: https: //adilet.zan.kz/ kaz/docs/ U1600000205 (data obrasheniya: 14.07.2023).*

[2] Gosudarstvennaya programma razvitiya obrazovaniya i nauki Republic of Kazakhstan na 2020-2025 gody *[Elektronnyy resurs] URL: https://adilet.zan.kz/kaz /docs/P1900000988 (data obrasheniya: 14.07.2023).*

[3] A.Aimagambetov [Elektronnyy resurs] *URL: 2023-zhyldan-bastap-oqushylar-matematika-panin-zhanartylgan-bagdarlama-boiynsha-oqidy//https://bilim-all.kz/article/17321.* 2023.

[4] On the approval of the State Program "Digital Kazakhstan" Resolition of the Republic of Kazakhstan dated December 12*Elektronnyy resurs] URL: https://adilet.zan.kz/kaz/dogs/P17000008727*, 2017.

[5] Kibataeva N.K. V focuse "Orleu": innovatsionnyy opyt obucheniya pedagogov, *Munitsipal'noye obrazovaniye: innovatsii i eksperiment.*, №4, 2015, pp.31-34.

[6] Jakypbekova G.T., Aldeshova S.E., Seisenvek M.A. Formation of professional competencies of teachers-specialists in the use of digital educational funds. *Abai Kazakh National Pedagogical University, Bulletin of the series "Physical and mathematical sciences", - Almaty*, №1, 2019, pp.258-263.

[7] Jakypbekova G.T., Sanyndyq Q.B. Bolashaq ınformatıka mamandarynyń elektrondyq bilim bery resyrstaryn qoldanydagy kasibı quzirettilikterin qalyptastyry. *Mejdynarodnoı naychno-praktıcheskoı jyrnal "Teorıa ı praktıka sovremennoı naykı"*, №6 (72), 2021, Rossıa, g.Saratov, pp.273-280.

[8] Yrakova E.A. Kasiptik bilim bery juesinin ınovasıalyq bilim bery tehnologıalary. *Qazirgi pedagogıkalyq bilim bery maseleleri.*, №71-4, 2021, pp.304-307.

[9] G.Seidaliyeva, G.Seidaliyeva, D. Ydyrysbayev, G. Zhakypbekova, B. Sydykhov. *Adoption of Distance Education and Mobile Technology by University Students, DOI: https://doi.org/10.3991/ijim.v16i23.36205 iJIM International Journal of Interactive Mobile Tehnologies, Scopus iJIM.*, Vol. 16, №23, 2022, pp.4-16.

[10] Chikova, O.A. Digital transformation of the content of pedagogical education. *Domentic and foreign pedagogy.*, Vol. 2, №3(73), 2020, pp.22-39.

[11] Blinov, V. I. Digital didactics: fashion trend or new science? *Professional education. Capital.* №3, 2019, pp.27-32.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

PEDAGOGICAL FEATURES OF DEVELOPING SOFT SKILLS IN THE PROCESS OF TRAINING FUTURE MATHEMATICIANS

KAMALBEK BERKIMBAEV[1], DMITRY ERMAKOV[2], ZHULDYZ AMANTAY[2], ARSLAN KURBANMAGOMEDOV[2]

[1]International Kazakh-Turkish University named after Khoja Ahmed, Turkestan, Republic of Kazakhstan
[2]People's Friendship University of Russia named after Patrice Lumumba (RUDN University) Moscow, Russian Federation
e-mail: amantay_zh@pfur.ru

Abstract. The modern multidimensionally transforming world poses challenges of a new reality for individuals. In recent times, there has been increasing significance given to universal socio-cultural competencies, commonly known as soft skills. This article discusses the peculiarities of thinking, learning, and professional activities that influence the development of flexible skills in the process of training future mathematicians at the university. The results of a survey conducted among mathematics students from higher educational institutions in Kazakhstan (n=118) and Russia (n=181) are presented. It is shown that the development of flexible skills is important for mathematics students in both general cultural aspects (communication, cooperation, social responsibility, emotional intelligence, resilience, etc.) and professional aspects (creativity, critical thinking, decision making, solving complex, and multidimensional problems). Similar models are implemented in the higher education systems of the Republic of Kazakhstan and the Russian Federation, which are generally close to the "4C's", with a primary focus on cognitive and social flexible skills and a lesser emphasis on emotional and volitional skills. Well-known active learning technologies can be applied for this purpose. Among the differences, there is a greater attention to retraining in the first case and to time and resource management in the second case, which may be associated not only with pedagogical but also with ethnocultural features. There are greater divergences in the aspirations for the development of flexible skills, which are significantly higher for Russian students. Apparently, young Russians associate their future professional careers and adaptation in the job market more with universal skills, while Kazakhs associate it more with professional competencies.

Keywords: education, Kazakhstan, mathematics, Russia, soft skills, student, university.

AMS Subject Classification: 97-06

As soon as I touch mathematics, I forget everything in the world again
Sofya Vasilyevna Kovalevskaya

1. Introduction

The multidimensionally transforming world presents challenges for individuals in both the present and the future [1]. Consequently, socio-cultural competencies, also known as soft skills, are gaining increasing importance. Unlike hard skills, which are specific to a particular subject area, soft skills have a universal and interdisciplinary nature. They encompass thinking patterns, communication abilities, creativity, self-awareness, empathy, and success in various activities such as education, professional endeavors, and adaptation to society as a whole, among others [2].

At the World Economic Forum, leading soft skills have been defined as a set of non-specialized, career-relevant skills that contribute to successful participation in the workplace, high productivity, and are transferable across different professional domains [3]. One well-known framework is the "4C's" model (collaboration, communication, creativity, critical thinking) [4]. However, these skills are not only essential for the labor market; they also hold significant socio-pedagogical value in general, serving as a necessary condition for adapting to a world that was recently referred to as VUCA (Volatile, Uncertain, Complex, and Ambiguous), and now as BANI (Brittle, Anxious, Non-Linear, Incomprehensible) [5].

As known, mathematics (from ancient Greek $\mu\alpha\theta\eta\mu\alpha\tau\iota\kappa\alpha$ - study) explores the properties and relationships among abstract entities such as numbers, figures, or symbols. Around 70,000 years ago, early humans developed systems for tracking the quantity of animals in a herd. Evidence of such reasoning can be found in the Egyptian civilization of the 5th millennium BCE, as well as in Ancient India and Ancient China. The Assyrian-Babylonian cultures of Mesopotamia developed their own symbolic system, which gave rise to the Hellenistic tradition. The works of Pythagoras, Thales, Eratosthenes, Archimedes, transmitted through the Roman Empire to Christian and Muslim societies, are still known today and form the foundation of mathematics [6,7] which the French philosopher René Descartes defined as the "science of order and measure," and Galileo Galilei called the "language of nature" [8].

Being an important tool for understanding, analyzing, and representing objective reality, mathematics plays a crucial role not only in science but also in the worldview of individuals and society as a whole [9,10,11]. The potential of mathematics as an educational discipline (in schools and universities) can be harnessed to develop flexible skills such as systematic and critical thinking (holistic perception and abstraction, analysis and synthesis, comparison, generalization, classification, and systematization, the ability to ask questions, argue, and assess the validity and novelty of ideas), creativity (the ability to generate ideas and solutions), and more. While these skills may not have a direct connection to mathematical science, they are necessary for team collaboration in the development and practical implementation of scientific research and applied projects in this field. Effective communication, coordination, and cooperation (the ability to express and interpret thoughts orally and in writing, interact efficiently with others) as well as emotional intelligence (the ability to understand one's own emotions and their causes, recognize and empathize with others' emotions, and manage emotions) are also essential in this regard.

2. Problem Statement

A participant on a mathematics forum posed a question for discussion: "Those of you who have a degree in mathematics or spent multiple years studying higher rigorous mathematics, what are the benefits in your way of thinking? My professor told us in his first lecture that studying mathematics will turn our brains upside down (not literally I'm probably translating it badly) and change our way of thinking. I wonder what exactly he means" [12]. In response, as an example, a list of "soft" skills compiled by the University of Warwick's Mathematics Institute (United Kingdom) is provided [13]: - Analytical Abilities – clarity of thought, attention to details, complex reasoning, logical and argumentative skills, counter-arguments. - Problem Solving – problem formulation, comprehensive examination, confidence in solutions, seeking help when needed. - Communication Skills – assimilating and conveying information, effective listening, report preparation, group presentations. - Useful Habits – thoroughness and meticulousness in work, time management and meeting deadlines, working under pressure, independent work and collaboration to address common problems. - Beneficial Personal Qualities – determination, perseverance, creativity, self-confidence, intellectual rigor. Below is a list of important soft skills for mathematicians compiled by artificial intelligence (neural network YaGPT) [14]: - Communication skills: Mathematics is a field of knowledge that requires interaction with others. Mathematicians must be able to express their thoughts clearly and accurately, listen

and understand others, and communicate effectively orally and in writing. - Analytical abilities: In mathematics, like any other field of knowledge, it is important to analyze information and draw conclusions based on the data obtained. Mathematicians must be capable of critical thinking, identifying problems, and finding solutions. - Organizational skills: Working in the field of mathematics can be highly complex and demanding in terms of time and resources. Mathematicians need to be able to plan their time, organize their work, and manage projects and teams effectively. - Creativity: Mathematics requires creative thinking and the ability to find unconventional solutions. Mathematicians must think outside the box, come up with new ideas and approaches to problem-solving. - Teamwork: Mathematics is a collaborative science, so mathematicians need to work well in teams and collaborate with others to achieve common goals. - Self-discipline: Mathematics requires high levels of concentration and perseverance. Mathematicians need to have strong self-discipline and be able to work on tasks for extended periods of time. - Presentation skills: Mathematicians often work with large amounts of information that needs to be presented orally or in writing. They must be able to clearly and coherently express their thoughts and substantiate their ideas. - Adaptability and flexibility: Mathematics can deal with various types of problems and situations, and mathematicians need to be adaptable and flexible in order to successfully navigate different scenarios.

Education plays a significant role in the development of soft skills. Therefore, the aim of this study is to identify the pedagogical features of developing soft skills in undergraduate and graduate students - future mathematicians - studying at universities in the Republic of Kazakhstan (L.N. Gumilyov Eurasian National University, Al-Farabi Kazakh National University, Korkyt Ata Kyzylorda University, H.A. Yassawi International Kazakh-Turkish University, Taraz Regional University named after M.Kh.Dulaty, etc.) and in the Russian Federation (People's Friendship University of Russia named after Patrice Lumumba, RUDN) in the relevant degree programs ($\ll Mathematics \gg$, $\ll AppliedMathematicsandInformatics \gg$, $\ll MathematicsandComputerScience \gg$) and who hold the respective citizenship. Of particular interest is the identification of similarities and differences associated with the methodological and ethnocultural characteristics of mathematics education in these countries, tracing back to European (Pythagoras, Euclid, Isaac Newton, Rene Descartes, Leonhard Euler, Gottfried Wilhelm Leibniz, David Hilbert, Nikolai Lobachevsky, Andrey Kolmogorov, etc.) and Eastern (Omar Khayyam, Al-Biruni, Al-Khwarizmi, Al-Farabi, Muhammad al-Ulugh Beg, Umirzak Sultan-Galiyev, etc.) traditions.

3. Methodology and Research Results

Methodology and research results The main method used is surveying, which allows for the identification of respondents' attitudes towards the studied phenomenon [15]. The survey, conducted in the second semester of the 2022/2023 academic year, involved 118 Kazakhstani students (males - 37.3%, females - 61.0%, unspecified - 1.7%; average age 21.2 years) and 181 Russian students (males - 43.6%, females - 56.4%; average age 18.8 years). When comparing the average age, it should be noted that education in Kazakhstani schools usually lasts for 12 years before university enrollment, while in Russian schools it typically lasts for 11 years.

The percentages of Kazakhstani students who were familiar and unfamiliar with the concept of soft skills were 78.0% and 22.0%, respectively, while for Russian students, the percentages were 53.6% and 46.4% respectively. Awareness in the first case is higher. This may be related to the fact that in the Russian educational discourse, along with the concept of soft skills, universal competencies are quite often mentioned (they are included in federal state educational standards) and include skills such as critical thinking, communication, and teamwork [16]. The State Compulsory Standard for Higher Education in Kazakhstan, approved by the Ministry of Science and Higher Education of the Republic of Kazakhstan, includes the concept of "skills" alongside competencies (communication is represented in the aspect that interests us) [17].

Currently, Russia is undergoing a reform of multi-level higher education, while Kazakhstan follows the Bologna Process, to which it joined later.

The assessment by students of the degree of development of various soft skills during their university education (the actual state, "Really"), as well as their corresponding preferences (desired state, "Desirable"), on a 3-point scale: not specially formed/not formed - 1; partly - 2; deeply – 3) are presented in "Table" 1. The list of skills proposed for assessment was compiled by us earlier based on materials from international reports [18].

This section should capture fundamental and new works (not less than 10 works) of foreign authors, published in English on investigated subjects, the analysis of the given works from the point of view of their scientific contribution, as well as blanks in research that your article supplements to.

Abundance of unrelated references as well as inappropriate judgments about the author's own achievements and self-reference on the author's previous works are INADMISSIBLE.

Soft Skills	Really (1)			Desirable (1)			$\Delta = (2) - (1)$	
	KZ	Russia	p	KZ	Russia	p	KZ	Russia
An adequate perception of criticism is really desirable	2,11	1,95	*	2,29	2,48	**	0,18	0,52
Cognitive flexibility	2,16	2,29	-	2,27	2,50	***	0,11	0,21
Communicating	2,29	2,47	-	2,41	2,65	***	0,12	0,18
Collaborating	2,21	2,36	-	2,34	2,51	*	0,13	0,15
Coordinating with others	2,15	2,13	-	2,36	2,48	-	0,21	0,35
Creativity	2,31	2,25	-	2,42	2,56	-	0,11	0,31
Critical thinking	2,22	2,19	-	2,40	2,58	*	0,18	0,39
Complex problem solving	2,19	2,34	-	2,50	2,56	-	0,31	0,22
Emotional intelligence	2,16	2,23	-	2,42	2,57	**	0,25	0,34
Forming your own opinion	2,21	2,34	-	2,41	2,56	-	0,19	0,22
Judgment and decision making	2,25	2,36	-	2,35	2,58	***	0,09	0,22
Motivating, training and developing others	2,23	2,15	-	2,30	2,45	-	0,07	0,29
Negotiation	2,25	2,33	-	2,31	2,57	***	0,05	0,24
People management	2,11	2,09	-	2,36	2,49	-	0,25	0,40
Retrain	2,10	1,91	**	2,25	2,14	-	0,14	0,23
Risk taking	2,19	2,28	-	2,33	2,51	*	0,14	0,23
Resource management	2,22	2,44	**	2,39	2,59	**	0,17	0,15
Service orientation	2,06	1,99	-	2,26	2,45	*	0,20	0,46
Social responsibility	2,25	2,34	-	2,41	2,51	-	0,16	0,18
Stress management	2,17	2,06	-	2,44	2,56	-	0,28	0,22
Time management	2,14	2,34	**	2,42	2,56	-	0,28	0,22
Working under uncertainty	2,25	2,19	-	2,39	2,47	-	0,14	0,28
Average	2,19	2,23	-	2,36	2,52	-	0,17	0,29

Note: $*$–$p \leq 0.1$; ** – $p \leq 0.05$; *** – $p \leq 0.01$. The Mann–Whitney U test (IBM SPSS Statistics) was used to assess the differences between the samples.

It should be noted that the assessment ratings for individual skills of students in Kazakhstani and Russian universities differ somewhat when the average values are similar. In the first case, more attention is paid to creativity (2.31) and working in conditions of uncertainty (2.25) in the educational process. In the second case, the focus is on communication (2.47), cooperation and decision-making (2.36), and forming one's own opinion (2.34). With a higher degree of reliability, there are differences in the perception of criticism (respectively, 2.11 and 1.95; p $\leq$

Forms and methods of training	evaluation	
	Kazakhstan	Russia
Contests Olympics	3,00	2,63
Multimedia training	3,21	2,97
Individual (special) discipline	3,32	3,36
Separate tasks in different subjects	3,26	3,15
Single topics / modules in different disciplines	3,31	3,24
Practices, internships	3,72	3,65
Testing	3,07	2,88
Trainings, games, case studies	3,52	3,52
Learning projects	3,47	3,40
E-learning, virtual learning	3,38	2,88

0.1) and relearning (2.10 and 1.91), which, along with customer orientation (2.06 and 1.99), are least expressed. Time management (2.14 and 2.34; $p \leq 0.05$) also shows some differences.

In both cases, there are higher wishes for the development of soft skills (0.17 and 0.29 on average, respectively), especially among Russian students. The most reliable differences can be observed in the ratings of the significance of adequate perception of criticism (2.29 and 2.48; $p \leq 0.05$), negotiation skills (2.31 and 2.57; $p \leq 0.01$), communication (2.41 and 2.65; $p \leq 0.01$), resource management (2.22 and 2.44; $p \leq 0.05$), critical thinking (2.40 and 2.58; $p \leq 0.1$), emotional intelligence (2.42 and 2.57; $p \leq 0.05$), which generally hold leading positions. There are also differences in the ratings of risk-taking (2.19 and 2.28; $p \leq 0.1$), decision-making (2.35 and 2.58; $p \leq 0.01$), as well as the less significant customer orientation (2.26 and 2.454; $p \leq 0.1$) and task switching ability (2.27 and 2.50; $p \leq 0.01$).

Regarding the ways of developing soft skills in the educational process, the opinions of students from both countries (on a 5-point scale: not at all effective - 1; not very effective - 2; quite effective - 3; very effective - 4; the most effective - 5) are almost the same ("Table 2").

4. Results and Discussion

Let's consider the pedagogical peculiarities of developing flexible skills in mathematics students according to their traditional division into three groups - cognitive, social, and emotional-volitional [19].

Regarding the first group, it is important to note mathematical thinking as a whole, which is associated with the manifestation of objective but abstract laws that depend not on the nature or personality of the thinking individual, but on the system of signs and relationships [20]. It represents one of the most complex and ancient models of representing reality, originating in prehistoric times [21,22].

Currently, both professional scholars and popular publications and blogs consider mathematical thinking as a distinct form of thinking. In general, it is defined as the ability to abstract from a specific situation and work with symbols, operations, and formulas [23]. Within it, in turn, several types of thinking are distinguished, which are designated differently: "philosopher", "analyst" and "geometer" [24]; visual, analytical, conceptual [25]; visual (representation of numbers and operations in a visual form – graphs, diagrams, etc., switching between different representations, establishing relationships and patterns, solving complex problems), analytical (system and logical analysis, splitting complex problems into simple ones and solving them separately, focusing on details, not on the big picture, deductive reasoning and conclusions), creative (generating new and non-standard solutions, using connections between different concepts, perceiving difficulties and mistakes as opportunities for development).

It is also noted that each person has his own style of mathematical thinking [26], which is related to individual traits (some people prefer working with numbers, while others prefer

geometric shapes, some can quickly calculate complex formulas mentally, while others may need a calculator, etc.) [27]. "Mathematical thinking is more than being able to do arithmetic or solve algebra problems. In fact, it is possible to think like a mathematician and do fairly poorly when it comes to balancing your checkbook" [29, pp. 59-61].

Thinking of this kind is important even for those who do not have a desire to use mathematics in their career. There is no need to impose complex abstractions or force memorization of theorem proofs that will never be needed in life. However, there should also not be prejudice towards mathematics and mathematicians [29]. Mathematics is "an atomic-powered prosthesis that you attach to your common sense, vastly multiplying its reach and strength," which allows for a correct understanding of the world and avoiding mistakes [30, p. 6].

Creativity also holds immense importance in mathematical work [31,32]. Many people might think that mathematicians think like computers, coldly adhering to strict formal logic and methodically moving from one action to another like robots. However, mathematicians themselves often describe their breakthroughs as an intuitive manifestation of contradictory and paradoxical ideas that come into contact from different contexts. Uncertainties and conflicts do not hinder but rather stimulate the development of mathematics. Creativity sometimes combines seemingly incompatible perspectives as complementary aspects. Therefore, mathematics is deeply creative activity, not just a set of formalized rules and results [33].

The use of computers, actively complementing human thinking at present [34], does not deprive mathematical activity of creativity. There are increasingly more parallels drawn between mathematics and art [35], including in the field of education (for example, the STEAM approach - Science, Technology, Engineering, Art, Mathematics [36, 37]). It is worth noting that the primary role of mathematics as a foundation for both science and art (as well as the development of geometric theory of perspective) was proclaimed for the first time by the Italian architect L. B. Alberti in the mid-15th century [38].

According to the data obtained from our respondents, when it comes to mathematics education, the development of purely cognitive soft skills is not a prominent priority. This is undoubtedly related to the need for not only professional but also general cultural preparation in higher education. However, both countries show a high level of training in decision-making skills. Kazakhstani universities are characterized by the development of creativity, and it is desirable to pay closer attention to solving complex and comprehensive problems. The preferences of Russian students are connected with critical thinking.

Moving on to the discussion of social skills, it can be presumed that our obtained results generally correspond to the well-known "4C's" model, as communication and cooperation are among the leaders in the evaluations of students from both countries, to a lesser extent - coordination. The development of these skills is likely primarily associated with the cultural function of higher education. It is worth noting the high ratings for social responsibility, while customer orientation receives one of the lowest ratings, which is mainly characteristic of business (although Russian students also express significant desires, particularly regarding negotiation skills training).

However, it should be noted that although traditionally mathematics has not been associated with the need for direct interpersonal interaction, teamwork, etc., these trends are now intensifying in solving scientific problems (communication between mathematicians, research freedom and responsibility, popularization of mathematics, etc. [39], including the development of specialized technological tools [40]). This is particularly noticeable in the process of learning mathematics [41, 42, 43, 44].

Emotions play an important role in mathematical cognition and learning. As known from psychology and neurobiology, they are connected to cognitive processes involved in performing mathematical operations, manifest in attitudes towards problem-solving, achievements in mathematics, and are also related to professional identity [45].

Such views have emerged relatively recently, and in the first half of the 20th century, a different position prevailed. For example, J.M. Gillette argued that the abstract nature of mathematical

truths makes them distant from everyday experience [46]. Following this reasoning, it is logical to conclude that emotions and mathematics are fundamentally separate. G. Polya proposed integrating emotions into the analysis of mathematical activity, especially regarding problem-solving. He also claimed that "teaching to solve problems is education of the will" [47, p. 94], suggesting that students should feel the struggle with internal and external psychological obstacles while seeking solutions. Around the same time, studies on negative emotional manifestations, such as math anxiety, emerged. Math anxiety is defined as a sense of tension and unease that hinders problem-solving [48]. There is also the phenomenon of "mathematophobia" the experience of fear and aversion towards mathematics [49]. This direction has also played a role in recognizing emotions as an integral component of mathematical activity.

Engaging in mathematics entails a uniqueness that influences not only emotional manifestations but also behavior as a whole (for example: "Now I use mathematics in the same way that high-level yogis use body positions - as something that helps stabilize my mind and body. Not only in a psychological-mental sense, but also purely organizationally. Mathematics doesn't so much self-organize me as it provides a certain foundation on which I rely to engage in other things" [50]).

In this regard, the development of the concept of resilience in relation to mathematics is of interest [51]. The term "mathematical behavior" is also known. Since 1994, a specialized scientific journal on this topic, the "Journal of Mathematical Behavior," has been published, in which mathematicians, psychologists, educators, linguists, and others publish research findings on how people create, sustain, transmit, apply, and comprehend mathematical ideas [52].

The students' preferences regarding the development of emotional intelligence in the educational process stand out in the data obtained from both countries. Kazakhstani students specifically mentioned time management and stress management, while Russian students emphasized resource management. It is worth noting that the development of emotional and volitional skills in the actual educational process appears to be the least prioritized (except for resource management at RUDN University, which may be related to the specific educational program). Adequate perception of criticism and relearning are given the least significance, which is unlikely to be agreed upon in terms of both professional mathematical and general cultural preparation.

Thus, soft skills can be considered as important components of the preparation of future mathematicians. Corresponding pedagogical technologies play a significant role in this regard. As mentioned above, respondents highly appreciate the potential of active forms and methods of learning (practices, trainings, games, educational projects), which are known to be effective for education as a whole [53], as well as for mathematics in particular [54]. To a lesser extent, individual (specialized) disciplines/modules or specific topics/tasks in different disciplines, multimedia, electronic, virtual learning, testing, as well as competitions and Olympiads, are noted. In the case of creative competitions, it should be noted that they generally have a positive impact on the development of subject-specific skills as well as transdisciplinary skills. However, it is possible that not all students have the opportunity and experience to participate in them [55,56,57].

5. Conclusion

The results obtained by us, as well as their comparison with data from other similar studies, allow us to conclude that the development of fsoft skills is important for mathematics students both in terms of general cultural aspects (communication, cooperation, social responsibility, emotional intelligence, resilience, etc.) and in terms of professional aspects (creativity, critical thinking, decision-making, complex problem solving). In the higher education systems of Kazakhstan and the Russian Federation, similar models are implemented, which are overall close to the "4C's" approach, focusing primarily on cognitive and social soft skills, and to a lesser extent on emotional and volitional skills. Well-known active learning technologies can be used for this purpose.

Among the differences, it is worth noting a more careful attention to work in conditions of uncertainty and retraining in the first case, as well as time and resource management in the second, which may be related not so much to pedagogical as to ethno-cultural features (for example, religious ones: Islam is more widespread in Kazakhstan, Christianity in Russia). Greater differences are observed in the desires for the development of flexible skills, which are significantly higher for Russian students. Apparently, young Russians associate their future professional career and adaptation in the labor market more with universal skills, while Kazakhs associate it more with professional competencies. The identified questions can be the subject of further research.

The research was funded by the Russian Science Foundation grant No. 23-28-01367, available at https://rscf.ru/project/23-28-01367

References

[1] Uncertain times, unsettled lives. Shaping our future in a transforming world: Human Development Report 2021/2022. New York, NY: United Nations Development Programme, 2022. ix, 305 p.
[2] White D. The Soft Skills Book: The key difference to becoming highly effective and valued. London, UK : LID Publishing, 2021. 160 p.
[3] The future of jobs. Employment, skills and workforce strategy for the fourth industrial revolution : global challenge insight report. Geneva : World Economic Forum, 2016 157 p.
[4] Tran L. H. N. Soft-skills implementation. A literature review // Building soft skills for employability: challenges and practices in Vietnam. Oxford, UK : Routledge, 2020 P. 18–40.
[5] Kraaijenbrink J. What BANI Really Means (And How It Corrects Your World View) (22.06.2022). URL: https://www.forbes.com/sites/jeroenkraaijenbrink/2022/06/22/what-bani-really-means-and-how-it-corrects-your-world-view/?sh=53045e5f11bb.
[6] Burton D. The history of mathematics: an introduction. New York : McGraw Hill, 2010. 816 p.
[7] Alvi U. What is mathematical thinking? (13.01.2021). https://whatmaster.com/mathematical-thinking.
[8] Courant R., Robbins H. What Is Mathematics? An Elementary Approach to Ideas and Methods. Oxford: Oxford University Press, 1996. 592 p.
[9] Bochner S.The role of mathematics in the rise of science. Princeton, NJ : Princeton University Press, 1966. x, 386 p.
[10] Kline M. Mathematics in Western Culture. Oxford : Oxford University Press, 2008. 516 p.
[11] Yadav S. Role of Mathematics in the Development of Society // International Journal of Research and Analytical Reviews. 2019. Vol. 6. № 4. P. 296–298.
[12] What Soft-Skills does a mathematician possess? URL: https://www.reddit.com/r/math/comments/n7ylln/what_softskills_does_a_mathematician_possess
[13] Transferable Skills. URL: https://warwick.ac.uk/fac/sci/maths/currentstudents/ughandbook/general/overview/transferableskills.
[14] YaGPT. https://dialogs.yandex.ru/store/skills/9cad1648-pomogi-pridumat.
[15] Amantay Zh. A., Ermakov D. S. Socio-pedagogical features of the formation of soft skills in the Republic of Kazakhstan // Advances in social science, education and humanities research. 2021. Vol. 555 P. 17–22.
[16] Portal of Federal state educational standards of higher education. URL: https://fgosvo.ruhttps://fgosvo.ru. (In Russian).
[17] On the approval of the state mandatory standards of higher and postgraduate education (Order of the Minister of Science and Higher Education of the Republic of Kazakhstan dated 20.07.2022, № 2). URL: https://adilet.zan.kz/kaz/docs/V2200028916. (In Kazakh).
[18] Amantay Zh. A., Ermakov, D. S. "Soft skills" model of a modern university graduate opinions of students and teachers // European proceedings of social and behavioural sciences. 2020. Vol. 104 P. 30–35.
[19] White D. The Soft Skills Book: The key difference to becoming highly effective and valued. London, UK : LID Publishing, 2021. 160 p.
[20] Waismann F. Introduction to mathematical thinking: the formation of concepts in modern mathematics. New York : F. Ungar Pub. Co, 1951. x, 260 p.
[21] Burton D. The history of mathematics: an introduction. New York : McGraw Hill, 2010. 816 p.
[22] White D. The Soft Skills Book: The key difference to becoming highly effective and valued. London, UK : LID Publishing, 2021. 160 p.
[23] Alvi U. What is mathematical thinking? (13.01.2021). https://whatmaster.com/mathematical-thinking.
[24] What types of mathematical thinking exist (31.05.2023). https://traveltimes.ru/en/what-types-of-mathematical-thinking.

[25] The Legacy of Felix Klein / eds. H.-G. Weigand, W. McCallum, M. Menghini,M. Neubrand, G.Schubring. Springer : Cham, Switzerland, 2019. 225 p.
[26] Burton L. Mathematicians and their Epistemologies – and the Learning of Mathematics // European Research in Mathematics Education / ed. I. Schwank. Osnabrück : Forschungsinstitut für Mathematikdidaktik, 1999. Vol. I. P. 87–102.
[27] Huincahue J., Borromeo-Ferri R., Reyes-Santander P., Garrido-Véliz V. Mathematical Thinking Styles – The Advantage of Analytic Thinkers When Learning Mathematics // Education Sciences. 2021. Vol. 11 № 6. P. 289.
[28] What types of mathematical thinking exist (31.05.2023). https://traveltimes.ru/en/what-types-of-mathematical-thinking.
[29] Devlin K. Introduction to Mathematical Thinking. Palo Alto, CA,2012. 102 p.
[30] Krantz S. G. Essentials of Mathematical Thinking (Textbooks in Mathematics). Boca Raton, FL: Chapman and Hall / CRC Press,2017. 352 p.
[31] Ellenberg J. How Not to Be Wrong: The Power of Mathematical Thinking. London : Penguin Publishing Group, 2015. 480 p.
[32] Sriraman B. Mathematical creativity: psychology, progress and caveats // ZDM (Zentralblatt für Didaktik der Mathematik) Mathematics Education. 2017. Vol. 49. № 971–975.
[33] Kolmogorov A. N. Mathematics – science and profession. Moscow : Nauka, 1988. 285 p. (In Russian).
[34] Byers W. How Mathematicians Think: Using Ambiguity, Contradiction, and Paradox to Create Mathematics. Princeton, NJ : Princeton University Press, 2010. 425 p.
[35] Davies B. Whither Mathematics? // Notices of the American Mathematical Society. 2005. Vol. 52. № 11. P. 1350–1356.
[36] Arias-Alfonso A. F., Franco C. A. The Creative Act in the Dialogue between Art and Mathematics // Mathematics. 2021. Vol. 9. № 13. P. 1517.
[37] Yang B. From STEM to STEAM: The Connections and Fostering of Creativity in STEAM // Lecture Notes in Education Psychology and Public Media. 2023. Vol. 2. P. 441-446.
[38] Needles T. STEAM Power: Infusing Art Into Your STEM Curriculum. International Society for Technology in Education, 2020. 180 p.
[39] Wootton D. The Invention of Science: A New History of the Scientific Revolution. New York : Harper Perennial, 2016. 784 p.
[40] Greuel G.-M. Mathematics Between Research, Application, and Communication // Raising Public Awareness of Mathematics. New York: Springer, 2012. P. 359–386.
[41] Kontorovich, I., Koichu, B., Leikin, R., Berman, A. (2012). An exploratory framework for handling the complexity of mathematical problem posing in small groups. Journal of Mathematical Behavior,31(1), 149–161.
[42] Wichelt L. Communication: A Vital Skill of Mathematics. Kearney, NE: University of Nebraska-Lincoln, 2009. 32 p.
[43] Pinto C. M. A., Mendonça J., Nicola S. DrIVE-MATH Project: Case Study from the Polytechnic of Porto, PT // Open Education Studies. 20224(1):1-20.
[44] Johnston B. 'Soft' skills identified by students who peer-led mathematics computing workshops // ANZIAM Journal. 2019. Vol. 61. P. C104-C118. DOI: 10.21914/anziamj.v61i0.15034.
[45] Abdu, R., Schwarz, B. Split up, but stay together: Collaboration and cooperation in mathematical problem solving. Instructional Science. 48, 313–336 (2020). doi:10.1007/s11251-020-09512-7.
[46] Xolocotzin Eligio U. An Overview of the Growth and Trends of Current Research on Emotions and Mathematics // Understanding Emotions in Mathematical Thinking and Learning. New York: Academic Press, 2017. P. 3–41. doi:10.1016/b978-0-12-802218-4.00001-7.
[47] Gillette, J. M. (1901). The relation of emotion to mathematical belief. Psychological Review, 8(6), 602–606.
[48] Polya, G. 1945. How to solve it: A new aspect of mathematical method. Princeton, NJ: Princeton University Press. 288 p.
[49] Richardson, F. C., Suinn, R. M. (1972). The mathematics anxiety rating scale: Psychometric data. Journal of Counseling Psychology, 19(6), 551.
[50] Gough, M. F. (1954). Why failures in mathematics? Mathemaphobia: Causes and treatments. The Clearing House: A Journal of Educational Strategies, Issues and Ideas, 28(5), 290–294.
[51] Vladimir Voevodsky: Mathematics as a method of stabilizing the mind (08.11.2017). https://elementy.ru/nauchno-populyarnaya_biblioteka/izbrannoe/433832/Vladimir_Voevodskiy_matematika_kak_metod_stabilizatsii_razuma. (In Russian).
[52] Lee C., Johnston-Wilder S. The Construct of Mathematical Resilience // Understanding Emotions in Mathematical Thinking and Learning. New York : Academic Press, 2017. P. 269–291.
[53] The Journal of Mathematical Behavior. URL: https://www.sciencedirect.com/journal/the-journal-of-mathematical-behavior.

[54] Silberman M. L. Active Learning: 101 Strategies to Teach Any Subject. Hoboken, NJ : Prentice Hall, 1996. 189 p.
[55] Pinto C. M. A., Mendonça J., Nicola S. DrIVE-MATH Project: Case Study from the Polytechnic of Porto, PT // Open Education Studies. 20224(1):1-20.
[56] Falk de Losada M., Taylor P. J. Perspectives on mathematics competitions and their relationship with mathematics education. ZDM Mathematics Education 54, 941–959 (2022). doi:10.1007/s11858-022-01404-z.
[57] Kenderov P. Competitions and mathematics education // Proceedings of the International Congress of Mathematicians. Madrid, Spain: European Mathematical Society, 2006. P. 1583–1598.
[58] Zemlyakova I. V., Chebun'kina T. A. The role and place of mathematical Olympiads in the system of training students of higher educational institutions. Vestnik of Kostroma State University. Series: Pedagogy. Psychology. Sociokinetics. 2020. Vol. 26. № 2. P. 206–210 (In Russian). DOI: 10.34216/2073-1426-2020-26-2-206-210.
[59] Experimental Study of the Modified High-Strength Coarse-Grained Concrete / G. E. Okolnikova, G. E. Grishin, A. K. Kurbanmagomedov, D. A. Bronnikov // System technologies. – 2019. – No. 2(31). – P. 25-31.
[60] External impacts on structure and relaxation properties of thermotropic liquid crystal copolyester / G. S. Zhukova, Z. R. Radjabov, A. K. Kurbanmagomedov, G. V. Boykova // IOP Conference Series: Materials Science and Engineering : 3, Applied and Fundamental Research Dedicated to the 75th Anniversary of Professor Abdul-Hamid Mahmoudovich Bisliyev. – Grozny, Chechnya, 2020. – P. 012074. – DOI 10.1088/1757-899X/905/1/012074. – EDN KZGHYJ.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

THE ROLE OF CONSTRUCTIVE LEARNING OF MATHEMATICS IN EDUCATION

KARLYGASH NYSHANBAYEVA[1], ALDANAZAR AMIRBEKULY[1]

[1]South Kazakhstan State Pedagogical University, Shymkent, Kazakhstan
e-mail:amirbekuly@inbox.ru

The general task that forms the basis of the educational process is the education of a comprehensively trained and competitive young generation. A comprehensively trained person or a competent person is a person who embodies the highest human qualities, moral values. The upbringing of people with these qualities requires determining what, why and how to teach young people, and it is based not on mechanical memory, but on theoretical (critical) thinking, on increasing the demands for intellectual activity. Among the available theories of learning, constructive learning is the most useful, of great educational and vital importance. In terms of its significance, constructive learning is included in a group of theories that are based on the creation of knowledge by the synthesis of new ideas with existing knowledge, that is, the knowledge system does not appear in a passive form, but on the basis of the learner's active life experience. Constructivism considers epistemology as its basis - a system of philosophical thoughts about the nature of knowledge and study. Constructiveness brings to the fore its basic principle - knowledge cannot be obtained by a passive method, students must be active in the learning process. Therefore, students, in accordance with their age, should be given independence so that they take an active part in the learning process. Learning, as an act of will, consists of such processes as appropriation (assimilation), adaptation (accommodation) and ordering (equilibration) and goes through the corresponding stages of development. In this case, readiness for any form of study depends on the stages of the student's intellectual development.

In the course of cognitive constructive learning with the help of intellectual operations, the student moves freely, relying on previously acquired knowledge, creates a system of new knowledge in the integrity scheme. Thus, constructive learning is operational and creative in nature. Students, carrying out logical actions with information received from textbooks, from students and other sources, form new knowledge, moving from the particular to the general, or vice versa, from the general to the particular, due to their inner instinct, ability to comprehend and assimilate. This means that constructivism stands on the basis of pedagogical philosophy, the main idea of which is not to transfer knowledge to the student in a finished form. Therefore, in the course of the educational process, a pedagogical environment should be created that provides students with the design of their knowledge and self-development. At the same time, from the standpoint of general approaches, constructivism should be understood as the construction of the environment throughout our life in our own understanding. That is why, the understanding of the world by each of the trainees, depending on their own worldview, ideas and faith, is distinguished by its originality.

The pedagogical meaning and significance of constructive learning lies in the student's transition from a passive listener to the position of a shaper of his own world of knowledge. This transition is made with the help of the teacher, taking into account the real capabilities of the student. All this creates ample opportunities for achieving success in the educational process

and working in an optimal mode, which ensures the development of both the teacher and the trainees.

Constructive learning of mathematics plays a crucial role in education. It emphasizes active engagement and problem-solving, enabling students to develop a deep understanding of mathematical concepts and skills.

Here are some key roles of constructive learning in mathematics education:

1. Encourages critical thinking and problem-solving skills: Constructive learning promotes students' ability to analyze and solve complex mathematical problems. It encourages them to think critically, make connections between concepts, and apply mathematical principles in real-life contexts.

2. Fosters conceptual understanding: Constructive learning focuses on helping students develop a conceptual understanding of mathematical concepts rather than memorizing formulas or procedures. It encourages them to construct meaning by connecting new knowledge with their existing understanding.

3. Promotes active engagement: Constructive learning promotes hands-on activities, group discussions, and collaborative learning, which actively engage students in the learning process. It allows them to explore mathematical concepts and develop their own strategies to solve problems.

4. Develops mathematical reasoning skills: Constructive learning helps students develop their ability to reason mathematically. Through engaging in activities that require logical reasoning and justification, students learn to analyze patterns, make inferences, and communicate their mathematical thinking effectively.

5. Builds a positive attitude towards mathematics: Constructive learning creates a positive and engaging learning environment. It helps students view mathematics as an enjoyable and meaningful subject, reducing math anxiety and increasing students' motivation to learn.

6. Enhances transferability of knowledge: Constructive learning encourages students to apply their mathematical knowledge and skills to real-life situations. This helps them see the relevance of mathematics in everyday life and develops their ability to transfer their knowledge to new contexts.

Constructive learning of mathematics promotes a deep understanding of mathematical concepts, enhances problem-solving skills, and fosters a positive attitude towards the subject. It equips students with the necessary skills and knowledge to excel in mathematics and beyond.

The role of constructive learning of mathematics in education is to promote meaningful and active learning experiences that enable students to build on their prior knowledge and construct their own understanding of mathematical concepts. Instead of simply memorizing and applying formulas and procedures, students are encouraged to engage with the subject matter, explore multiple strategies, and develop problem-solving skills.

Constructive learning of mathematics involves hands-on activities, problem-solving tasks, and group discussions, which help students connect mathematical concepts to real-world situations and apply them in different contexts. This approach emphasizes the process of learning rather than just the end result, fostering a deeper understanding of mathematical principles and their applications. By actively constructing their understanding, students are able to develop critical thinking skills, logical reasoning abilities, and an appreciation for the beauty and relevance of mathematics. They learn to think flexibly, make connections between different mathematical ideas, and apply their knowledge in novel situations. Moreover, constructive learning of mathematics helps students develop a positive attitude towards the subject. It encourages them to see themselves as capable learners and problem solvers, boosting their self-confidence and motivation. By providing opportunities for exploration and discovery, students become more engaged and invested in their own learning, leading to improved academic performance and long-term retention of mathematical concepts.

Traditional math teaching is characterized by a teacher-centered approach where the teacher delivers content and students passively receive information. It often involves memorization, drill and practice exercises, and a focus on finding the correct answer. The emphasis is on computation skills, rote learning, and following a set of rules or formulas. Constructive math teaching, on the other hand, is more student-centered and focuses on actively involving students in constructing their own understanding of mathematical concepts. It encourages critical thinking, problem-solving skills, and mathematical reasoning. Instead of simply memorizing formulas or procedures, students are encouraged to think deeply about why mathematical concepts work the way they do.

In constructive math teaching, teachers aim to facilitate discussion and collaboration among students. Students are encouraged to share their ideas, explain their thinking processes, and engage in meaningful conversations about mathematical concepts. This approach promotes a deeper understanding of math by allowing students to explore different strategies and perspectives. Additionally, constructive math teaching often incorporates real-world applications of mathematics to help students see the relevance of what they are learning. It involves tasks that require higher-order thinking skills such as analyzing data sets or solving open-ended problems.

Overall, while traditional math teaching focuses on specific rules and procedures to reach an answer, constructive math teaching aims to develop conceptual understanding and problem-solving skills in students.

Some key differences between traditional math teaching and constructive math teaching are:

- Approach: Traditional teaching relies heavily on direct instruction and rote memorization of procedures, while constructive teaching focuses on students actively constructing their own knowledge through inquiry-based learning.

- Problem-solving: Constructive teaching emphasizes problem-solving as an integral part of learning mathematics, whereas traditional teaching tends to focus more on procedures and algorithms.

- Student engagement: Constructive teaching encourages active participation, collaboration, and discussion among students, whereas traditional teaching often involves passive learning with the teacher as the primary source of information.

- Conceptual understanding: Constructive teaching places a greater emphasis on understanding mathematical concepts rather than just memorizing formulas or algorithms. Students are encouraged to make connections between different math topics.

- Mistakes as learning opportunities: Constructive teaching views mistakes as valuable learning opportunities that can lead to deeper understanding, whereas traditional teaching often treats mistakes as something to be avoided.

In a word, constructive math teaching aims to develop students' critical thinking skills, problem-solving abilities, and conceptual understanding of mathematics by engaging them in active exploration and discovery.

Traditional math teaching and constructive math teachings are two different approaches to teaching mathematics to students. Here are the main differences between the two:

1. Focus on procedural versus conceptual understanding: Traditional math teaching often focuses primarily on teaching students procedures and algorithms for solving mathematical problems. Students learn and memorize formulas, rules, and steps to follow without necessarily understanding the underlying concepts.

In contrast, constructive math teaching emphasizes developing conceptual understanding alongside procedural skills. It encourages students to explore mathematical concepts, make connections, and develop their own understanding of mathematical principles. It prioritizes problem-solving, reasoning, and critical thinking over memorization of algorithms.

2.Teacher-centered versus student-centered approach: Traditional math teaching tends to be more teacher-centered, with the teacher playing the central role in instruction. The teacher presents information and directs students through lectures and exercises.

Constructive math teaching, on the other hand, is more student-centered. It involves active student participation in exploring mathematical ideas through hands-on activities, collaborative problem-solving, and open-ended tasks. The teacher acts as a facilitator or guide who supports students' learning by providing appropriate challenges and asking probing questions.

3. Emphasis on right answers versus process-based learning: Traditional math teaching tends to value finding the correct answers quickly. There is often an emphasis on rote memorization of formulas and getting the "right" answer rather than understanding or explaining why an answer is correct. In constructive math teaching, the emphasis is less on finding correct answers immediately and more on problem-solving processes. Students are encouraged to explain their thinking, justify their solutions, and explore multiple solution strategies or representations. Mistakes are seen as opportunities for learning rather than failures.

4. Individual work versus collaborative work: Traditional math teaching often involves individual work with minimal interaction or collaboration among students. Students may solve problems independently or compete against one another for grades.

In constructive math teaching, collaboration is encouraged as an essential component of learning mathematics. Students are encouraged to work in groups, discuss and share their ideas, and learn from one another. Communication and collaboration skills are considered an integral part of mathematical understanding. Constructive math teaching aims to foster a deeper understanding of mathematics while developing critical thinking skills and problem-solving abilities, whereas traditional math teaching focuses more on procedural fluency and getting the right answers.

Constructive learning of mathematics plays a crucial role in education by promoting active, meaningful, and student-centered learning experiences. It equips students with the skills and knowledge needed to thrive in an increasingly complex and changing world, where mathematical literacy is essential for success in various fields and everyday life.

One of the main benefits of constructive learning in mathematics is that it promotes a deep understanding of mathematical concepts. Instead of simply memorizing formulas and procedures, students are encouraged to explore and discover mathematical principles on their own. This hands-on approach allows them to make connections, solve problems, and develop a conceptual understanding of mathematics. This understanding is essential for long-term retention and the ability to apply mathematical knowledge to real-life situations. Constructive learning also fosters a positive attitude towards mathematics. Many students struggle with math because they find it abstract and disconnected from their everyday lives. Through constructive learning, math is presented in a meaningful and relevant context, making it more relatable and interesting for students. This approach helps to reduce math anxiety and increases students' motivation and engagement with the subject. Another benefit of constructive learning is that it promotes critical thinking and problem-solving skills. In traditional math education, students are often given a set of pre-determined problems to solve. However, in constructive learning, students are encouraged to explore open-ended problems and find multiple solutions. This approach develops their problem-solving abilities, creativity, and analytical thinking. It also prepares students to tackle real-world challenges that may not have straightforward solutions.

Constructive learning of mathematics also encourages collaboration and communication among students. Students are often encouraged to work in groups, discuss strategies, and explain their thinking process to their peers. This collaborative environment fosters teamwork, communication skills, and the ability to justify one's reasoning. It also provides opportunities for students to learn from each other and gain different perspectives on mathematical concepts.The role of constructive learning in mathematics education is to provide students with a solid foundation in mathematical knowledge and skills. It helps them to develop a deep conceptual understanding, positive attitudes, problem-solving abilities, and collaboration skills. By engaging students actively in the learning process, constructive learning promotes lifelong learning and prepares students for future success in mathematics and beyond.

There are numerous scholars who have conducted research on the constructive learning of mathematics. Some prominent names in this field include:

- Jean Piaget: Piaget was a Swiss psychologist who made significant contributions to the study of cognitive development, including the understanding of how children construct mathematical knowledge.

- Lev Vygotsky: Vygotsky, a Soviet psychologist, is known for his sociocultural theory of development. He emphasized the role of social interactions and cultural influences in the construction of mathematical understanding.

- Seymour Papert: Papert was a mathematician, computer scientist, and educator who co-founded the MIT Media Lab. He is well-known for his work on constructivist learning theories, particularly in the context of mathematics education.

- David H. Jonassen: Jonassen is a professor of instructional technology and educational psychology. He has conducted extensive research on problem-solving, cognitive tools, and constructivist learning environments in the context of mathematics education.

-Paul Cobb: Cobb is a professor of mathematics education at Vanderbilt University. His research focuses on the social and cultural aspects of learning mathematics, particularly in relation to students' understanding of mathematical discourse and the construction of mathematical knowledge.

-Anna Sfard: Sfard is a professor of mathematics education and known for her work on the concept of mathematical discourse and the roles of language in the construction of mathematical knowledge.

-Erik De Corte: De Corte is a Belgian psychologist and educational scientist who has conducted research on the development and improvement of mathematical problem-solving skills through constructive learning approaches.

- Jo Boaler: Boaler is a professor of mathematics education at Stanford University. Her research explores effective instructional approaches that promote constructive learning in mathematics, such as problem-solving and mathematical reasoning.

- F.D.Bunyatova: Constructive learning in mathematics refers to a teaching approach in which students actively construct new mathematical knowledge and understanding through hands-on activities, problem-solving, and collaboration. It emphasizes the importance of meaningful learning experiences that connect mathematical concepts to real-world applications.

-N.P. Shatalova: Constructive learning of mathematics refers to a teaching and learning approach that emphasizes the active involvement of students in the construction of mathematical knowledge. This approach is based on the idea that students learn best when they actively engage with mathematical concepts and ideas, rather than passively receiving information from the teacher.

In constructive learning of mathematics, students are encouraged to explore and discover mathematical concepts through hands-on activities, problem-solving tasks, and collaborative discussions. They are given the opportunity to make connections between different mathematical ideas and apply them in real-life situations. The role of the teacher in constructive learning is to facilitate and support students' learning processes. Rather than simply providing answers or solutions, the teacher guides students in their exploration, asks thought-provoking questions, and encourages them to reflect on their thinking. One key aspect of constructive learning is the use of scaffolding techniques. Scaffolding involves providing temporary support to students to help them bridge the gap between their current understanding and the desired learning outcomes. This support can take different forms, such as providing examples, prompts, or cues, and gradually reducing them as students develop their understanding. Constructive learning of mathematics also emphasizes the importance of metacognition, or thinking about thinking. Students are encouraged to reflect on their learning processes, monitor their understanding, and identify strategies that work best for them. This metacognitive awareness helps students

become more independent learners and develop a deeper understanding of mathematical concepts.Constructive learning of mathematics promotes an active, student-centered approach to learning that fosters deep understanding, critical thinking, and problem-solving skills. It encourages students to construct their mathematical knowledge through exploration and collaboration, ultimately leading to better retention and application of mathematical concepts.

These scholars, among others, have contributed significantly to the understanding of how children learn and construct mathematical knowledge. Their research has informed the development of instructional practices and theories aimed at enhancing mathematical learning experiences.

Constructive mathematics teaching refers to an approach to math education that focuses on actively engaging students in the process of constructing their own mathematical understanding. It emphasizes problem-solving, exploration, and collaboration, rather than memorization and passive learning. One of the key roles of constructive mathematics teaching is to improve knowledge acquisition by promoting a deeper and more meaningful understanding of mathematical concepts. This approach encourages students to actively engage with mathematical ideas, make connections between different topics, and develop their own strategies for solving problems. As a result, students are more likely to retain and apply their acquired knowledge in a variety of contexts. Constructive mathematics teaching also fosters critical thinking skills in students. By encouraging them to explore different ways of solving problems and justify their reasoning, this approach helps students develop logical thinking abilities that can be applied beyond the realm of mathematics. These critical thinking skills are essential not only for success in higher-level math courses but also for problem-solving in real-world situations. Additionally, constructive mathematics teaching promotes a positive attitude towards math and enhances motivation for learning. This approach emphasizes that everyone has the potential to understand and excel in mathematics through effort and persistence. By providing opportunities for success and celebrating student achievements, it helps build confidence and self-efficacy in mathematics.

Furthermore, constructive mathematics teaching encourages collaborative learning environments where students interact with their peers. Through group discussions, cooperative activities, and sharing multiple solution strategies, students learn from and with each other. This collaborative approach enhances knowledge acquisition by promoting shared understanding, fostering social interactions around math concepts, and providing opportunities for different perspectives.

Constructive mathematics teaching plays a crucial role in improving knowledge acquisition by fostering deep conceptual understanding, critical thinking skills, motivation for learning math, and collaboration among students. By taking an active role in constructing their own mathematical understanding, students are better equipped to acquire complex mathematical knowledge that is transferable to various contexts.

There are several stages involved in implementing constructive mathematics teaching:

- Planning: This stage involves setting clear objectives, selecting appropriate learning resources and activities, and considering the needs and abilities of students.

- Introduction: In this stage, the teacher introduces the mathematical concepts and skills to the students. They may use concrete materials, real-life examples, or visual aids to make the concepts more understandable.

- Exploration: This stage focuses on allowing students to explore the mathematical ideas and concepts on their own. The teacher encourages students to ask questions, make connections, and engage in problem-solving activities.

- Collaboration: During this stage, students work together in small groups or pairs to discuss, share ideas, and solve problems. The teacher facilitates group discussions and provides guidance when needed.

- Reflection: After students have completed their activities or solved problems, they reflect on their learning experiences. They discuss their strategies, highlight their achievements, and identify areas for improvement.

- Consolidation: In this stage, the teacher summarizes the main ideas and concepts learned during the lesson. They may provide additional explanations, examples, or demonstrations to reinforce the learning.

- Assessment: The final stage involves assessing students' understanding and skills. The assessment can be formative, such as through observation, questioning, and class discussions, or summative, such as through tests, quizzes, or projects.

Throughout these stages, the teacher encourages critical thinking, reasoning, and problem-solving skills in students. They also promote a positive learning environment where mistakes are seen as opportunities for learning and growth.

The role of constructive learning in mathematics education is to empower students to become active participants in their own learning process. By engaging with mathematical concepts in a meaningful way, students are more likely to develop a deep understanding of math and acquire problem-solving skills that can be applied beyond the classroom. Constructive learning promotes critical thinking, collaboration, metacognition, and student agency - all essential elements for successful mathematics education.

Constructive learning of mathematics plays a vital role in education. It is an approach to learning that emphasizes active engagement, critical thinking, and problem-solving skills. This article discusses the importance of constructive learning in mathematics education and how it benefits students.

Here are some strategies for promoting constructive learning in mathematics:

- Hands-on manipulatives: Provide concrete materials like blocks, dice, or counters that students can manipulate to explore mathematical concepts. This helps to make abstract concepts more tangible and understandable.

- Problem-solving tasks: Present students with open-ended, challenging problems that require them to apply mathematical concepts and strategies to find a solution. Encourage critical thinking, reasoning, and creativity in their problem-solving approaches.

- Cooperative learning: Foster collaboration and teamwork by assigning group tasks or projects. Students can work together to solve problems, discuss strategies, and provide feedback to one another. This promotes discussion, communication, and the sharing of different perspectives.

- Real-world connections: Help students see the relevance of mathematics in everyday life by connecting mathematical concepts to real-world situations. This can be done through practical examples or by exploring how mathematics is used in various professions or industries.

- Technology integration: Utilize technology tools, such as graphing calculators, interactive simulations, or online resources, to enhance students' engagement and understanding of mathematical concepts. These tools can provide visual representations and facilitate interactive learning experiences.

- Reflection and metacognition: Encourage students to reflect on their own learning processes and strategies. By thinking critically about their own thinking (metacognition), students can develop a deeper understanding of the mathematical concepts and become more independent learners.

- Differentiated instruction: Recognize the diverse needs and abilities of students and provide differentiated instruction to cater to their individual learning styles and levels. This may involve providing additional support for struggling students or extending challenges to more advanced learners.

Constructive learning in mathematics involves creating an active, student-centered learning environment where students can explore, discover, and construct their mathematical knowledge through meaningful and engaging experiences.

The future of constructive mathematics learning holds great promise. With advancements in technology and a deeper understanding of how students learn, there are several key trends that are likely to shape this field in the coming years:

- Personalized Learning: The traditional one-size-fits-all approach to teaching mathematics is becoming obsolete. In the future, constructive mathematics learning will be increasingly personalized, with tailored content and instruction that meet individual students' needs and interests.

- Blended Learning: Blending traditional classroom instruction with online resources and platforms will become more prevalent in the future of constructive mathematics learning. This hybrid model allows for more flexibility in pacing and gives students the opportunity to reinforce their understanding through digital tools.

- Collaboration Tools: The future of constructive mathematics learning will emphasize collaboration among students through online platforms that facilitate virtual group work and peer-to-peer learning. These tools foster discussion and problem-solving skills while also providing opportunities for students to work on projects together regardless of physical location.

- Adaptive Learning Systems: Adaptive learning systems utilize algorithms to create individualized pathways for each student based on their unique strengths, weaknesses, and progress level. As these systems become more sophisticated, they will further personalize the learning experience, helping students build a strong foundation in mathematics.

The future of constructive mathematics learning is exciting and promising. Personalization, blended learning, collaboration tools, and adaptive learning systems are some of the key trends that will revolutionize mathematics education and help students develop a deep understanding and appreciation for the subject.

In conclusion, constructive mathematics teaching is an approach that focuses on promoting active learning, critical thinking, and problem-solving skills among students. This approach engages students in meaningful mathematical tasks that constantly challenge and build upon their existing knowledge and understanding.

By using manipulatives, real-life contexts, and collaborative activities, constructive mathematics teaching encourages students to explore concepts and develop their own strategies for solving problems. This empowers students to take ownership of their learning and develop a deeper understanding of mathematical concepts. Furthermore, constructive mathematics teaching promotes a growth mindset by acknowledging that mistakes are essential for learning. It encourages students to embrace challenges and persevere through difficulties, fostering a positive and resilient attitude towards mathematics. Constructive mathematics teaching helps students see the relevance of mathematics in their daily lives and in real-world contexts. Through hands-on activities, students can apply mathematical concepts to solve real-life problems, further enhancing their understanding of the subject. It is important for teachers to provide a supportive learning environment where mistakes are seen as opportunities for growth and learning. By fostering a growth mindset among students, constructive mathematics teaching helps build students' confidence and motivation in math.

Overall, constructive mathematics teaching provides a solid foundation for students to develop strong mathematical reasoning skills, communication skills, and the ability to apply mathematical concepts in practical contexts. It equips them with the tools necessary for success in not only mathematics but also in real-life situations where critical thinking and problem-solving skills are crucial.

Keywords: constructivism, constructive learning, competent person, epistemology, constructive mathematics teaching.

AMS Subject Classification: 97U30 Teachers' manuals and planning aids (aspects of mathematics)

References

[1] Shatalova N.P. Azbuka konstruktivnogo obucheniya. Monograph. Krasnoyarsk: Nauchno-innovatsionnyy tsentr, 2011

[2] Knyazeva E. N. Constructivist epistemology // Philosophical sciences. 2010, No.11. P.88-103
[3] Bunyatova F. Constructive learning: essence, principle, goals and samples of lessons. Baku, 2008
[4] Sretenskaya D. A. Constructive pedagogy as a basis for improving the quality of education at the university // Saratov Journal of Medical Scientific Research. 2011.V. 7, No. 4. S. 988–991
[5] Lektorsky V.A. Capt. Radical constructivism and constructive realism in epistemiology. //Vopr.Philosophy, M.: 2005, No. 8.
[6] Douglas Bridges, Hajime Ishihara, Michael Rathjen. Handbook of Constructive Mathematics. Cambridge University Press; New edition. May 11, 2023.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

EDUCATIONAL CONDITIONS TO DEVELOP COGNITIVE INTEREST OF TECHNICAL COLLEGE STUDENTS IN LEARNING MATHEMATICS

MAKSAT KEDEIBAYEVA[1], DILBARA KEDEIBAEVA[1], AICHURO EGEMNAZAROVA[1]

[1]Osh State University, Osh, Kyrgyzstan
e-mail: dilbar_63@mail.ru, egemay_65@mail.ru

Abstract. In this article, the author has attempted to solve one of the problems of effective teaching of mathematics to students of non-mathematical specialties of secondary vocational education by conducting a series of works on the theoretical and practical substantiation of the effectiveness of the use of pedagogical conditions: the use of historical elements of mathematics to increase cognitive interest. In order to improve the quality and efficiency of the educational process, the thematic plan of the work program created for students of the technical college of secondary vocational education will be revised and supplemented with historical information on professional issues and topics. Also, as a result of this article, after studying the state educational standard of basic general education, it was said that the use of historical materials in a mathematics lesson would allow fulfilling a number of regulatory requirements Key words: technical college, cognition, historical element, educational process, secondary vocational education, mathematics, professional tasks.

Keywords: technical college, cognition, historical element, educational process, secondary vocational education, mathematics, professional tasks.

Cognitive interest is an essential source for enhancing the effectiveness of teaching in general and mathematics in particular. Therefore, one of the main tasks of a modern teacher in the development of mathematical education in our country is the conscious and internally motivated provision of the process of mathematical education [1, p. 4].

I would like to note the works of certain scholars who studied the problem of increasing cognitive activity; they are: psychologists L. I. Boyovich, A. N. Leontiev and others. Educators: V. A. Slastenin and G. I. Shchukin. Methodologists: T. I. Yerofeyev and M. V. Myachin. However, developing the cognitive interest of students of non-mathematical specialities (technical colleges) in the system of secondary vocational education (SVE) to the study of mathematics and solving motivational problems remains relevant. Also, since secondary vocational education is the most important component of the current educational space of the state, it affects the development of the innovative economy of the country [2. p. 43].

The aim of the article is to theoretically and practically substantiate the effectiveness of the use of pedagogical conditions: historical elements of mathematics to increase the cognitive interest of students of non-mathematical specialities (technical colleges) in the system of secondary vocational education.

Mathematics, as a subject learned in the system of secondary vocational education, is included in the content of the main educational programs of almost all specialities (including technical colleges). The analysis of the subject curriculums has revealed a weak focus on the profession, its content and the lack of applications related to future professional activity. In such conditions, the cognitive interest of future specialists in mathematics decreases.

On the other hand, due to the universality of its methods, which are considered as a means of comprehending processes in various spheres of the surrounding reality, mathematics has sufficient

potential for professional orientation. All this gives grounds for the fact that in the development of students cognitive interest in mathematics, teaching using historical elements of mathematics can and will be expedient, taking into account the principle of professional orientation [2, p. 45].

Having analysed the existing literature, it is possible to characterise the cognitive interest in mathematics of students of non-mathematical specialties of secondary vocational education - technical colleges, as a form of manifestation of cognitive need, and to ensure that students pay attention to understanding the goals of mathematical activity, and the preference of this type of activity over others is expressed in the desire to master mathematics and apply it in future professional activity.

The main stages of enhancing the cognitive interest of students of non-mathematical profiles of secondary vocational education in mathematics (according to G. I. Shchukina) are curiosity, interest, personal cognitive interest, cognitive interest focused on the theoretical model of the profession.

Among the many educational conditions to develop cognitive interest in mathematics among students of non-mathematical specialities (technical colleges) in the system of secondary vocational education, we have identified three, in our opinion, the most effective ones:
- repeated learning of the content of the subject of mathematics, including professional orientation and elements of output and complement of mathematics;
- preferential use of professional mathematical problems and the design method in organization of the process of teaching mathematics;
- teaching using historical elements of mathematics and interactive teaching methods.

As a result of mastering the subject of mathematics, the student should be able to solve practical problems in the field of professional activity. To fulfil the first formulated condition, we have supplemented the content of the subject of mathematics with professional and historical elements of mathematics [5, p.156].

A brief fragment of the thematic plan of the work program created for students of a technical college (speciality) of secondary vocational education is as follows (Table 1).

Table 1

Topic No:	Topic title	Curriculum hours	Classroom hours	Number of teaching hours			
				Theoretical training			Indepe learni stud
				Lectures	Seminars	Practical hours	
1	Introduction. Integers and rational numbers. Real numbers. Historical information on numbers.	13	9	6	3	-	4
1.1	Approximate calculations. Approximate values of the magnitude and error of approximations. Approximate calculations of technical problems in mathematics.	7	5	3	2	-	2
1.2	Root, degree, and logarithm. Historical elements about roots and degrees.	6	4	3	1	-	2
2	The root of the n-degree. A degree with a rational and valid exponent.	23	16	9	7	-	7
2.1	Irrational, power, exponential, logarithmic expressions. Production tasks.	10	7	4	3	-	3
2.2	Straight lines on the plane and in space.	13	9	5	4	-	4
3	The relative position of a straight line and a plane. The concept of perpendicularity and inclination. Cross section of flat surfaces.	13	10	8	2	-	3
3.1	The angle between the planes. Two sharp corners. Problems with production content.	7	5	4	1	-	2

The total number of teaching hours: 200; 155 classroom hours, 45 hours of independent learning of students. According to the state standard, it is necessary to update theoretical and methodological complexes and teaching methods in regulating the systematic and active attitude of students to learning in secondary vocational education. The forms of organization of lessons are

changing; the whole system of education is changing. In a rapidly changing world, the main task of an educational institution is to ensure that every school or college student not only receives a profound education, but also could fully reveal their personal creative abilities, be physically healthy, spiritually rich, intellectually developed, could easily make optimal decisions, create an educational space in which a personality is formed [6, p.164].

The focus is on the development of the students personality, which is supported by projects, researches and creative tasks.

The transformation of science into a productive force has led to the fact that knowledge of many general education subjects (including mathematics) is required not only for the formation of a precise worldview, but also for mastering special knowledge related to the future profession. Such knowledge is an important qualification requirement for employees of many modern professions, essential for successful work [5, p. 154].

The main features of teaching mathematics at a college could be reckoned mathematics, which includes historical elements of mathematics, supplemented by professionally oriented tasks. Profile study of the subject provides timely learning of materials necessary for the development of training programs for qualified workers (employees) in the following ways:
- identification of professionally important material for a particular profession; - determination of the required depth of research; - finding the most suitable place and time to study it; - increasing the amount of time to learning the educational material significant for the professional development of the student, as well as the introduction of additional topics or elements of personal education;
- performing special tasks, laboratory and practical work that allows you to create a real production activity in a learning environment and to form general labour and professional knowledge and skills;
- solving problems of production content;
- synthesis of knowledge and skills in various subjects, performing complex interdisciplinary tasks of production content;
- formation of important professional qualities of a person in the process of teaching specialized subjects [4, p. 4].
Essentially, teaching mathematics at college should be aimed at the implementation of the following objectives:
- mathematics as a universal language of science, as a means of modelling phenomena and processes, forming presentations of and methods of mathematics;
- development of logical thinking, spatial imagination, algorithmic culture, critical thinking at the level necessary for future professional activity, continuing education and self-education;
- mastering mathematical knowledge and skills necessary in everyday life, studying related natural science disciplines at the basic level and disciplines of the professional cycle, obtaining education in areas that do not require in-depth mathematical training;
- education of human culture with the help of mathematics, understanding of the importance of mathematics for scientific and technological progress, attitude to mathematics as part of the culture of mankind by getting acquainted with the history of the development of mathematics, the evolution of mathematical ideas [2, p.47].

In addition to demonstrating knowledge, effective teaching ensures the activation of cognitive activity of all students, stimulates their need for knowledge and awakens interest in the subject. The effectiveness of training is mainly due to solving problems of production content and solving applied problems, performing laboratory and practical work, developing projects, preparing presentations and performing control work.

In teaching the subject, providing information from its historical elements, orientation to the profession helps students develop independent logical thinking by increasing cognitive activity. When forming general educational competencies (general cultural, educational and cognitive, informational, communicative, social and labour, personal self-improvement competencies) in

the educational process, they should be considered in accordance with the general educational and professional competencies of the standard.

For example, while learning Topic 1 on The development of the concept of number in technical universities, one can consider solving problems of the following type:

1. A brief historical summary of the numbers is given.
2. The use of numbers in technology and everyday life is presented by examples.

The history of the origin of natural numbers corresponds to the early society

And, of course, it appeared in its simplest form, but numbers evolved with humanity. Initially, they were used only for counting, measuring something, i.e. they helped people with exactly what they needed in their practical activities. Then the number becomes part of mathematics, and science determines the history of the origin and development of natural numbers.

In the most ancient times, people counted with their fingers. When the fingers of one hand ended, they moved to the fingers of the other hand, and if there were not enough in both hands, they moved to the toes, that is, they did not have the concept of a number, as we used to understand it. Therefore, at that time, if someone boasted that he had two arms and one leg, it meant that he had fifteen chickens, and if he was called a whole person, then he had two arms and two legs [3, p. 22]. Peruvian shepherds (Indians) kept records of animals and crops, tying ropes of various lengths and colours to belts or bundles (Fig. 1) [4].



Fig. 1

Some wealthy people collected such knots in threads of several meters and kept an accounting book. The ancient Sumerians (ancient people) were the first to invent a system of writing numbers. They only used two numbers. The vertical line represented one unit, and the angle formed by two horizontal lines represented ten. These lines were written on wet clay with a sharp stick, then dried and burned, resulting in cuneiform writing (Fig. 2).



Fig. 2

After counting by parts, people invented special symbols called numbers. They began to be used to denote different sizes of different products. Different civilizations have created their own numbers. For example, in the numbering of Ancient Egypt, which appeared 5000 years ago, there were special signs (hieroglyphs) for writing numbers 1, 10, 100, 1000, ... (Fig. 3).

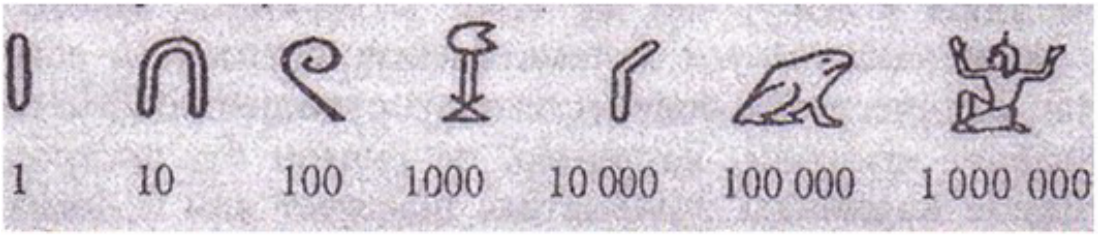


Fig. 3

For example, to describe the integer 23145, it is enough to write two hieroglyphs denoting ten thousand in a row, then three hieroglyphs denoting a thousand, one hundred, four ten and five hieroglyphs - one: (Fig. 4).



Fig. 4

In ancient Greece, the numbers 5, 10, 100, 1000, 10000 were denoted by the letters G, H, X, M, and the number 1 was a vertical curved line. Later the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 200, 300, 400, 500, 600, 700 , 800, 900, 1000, 2000, 3000, 4000, 5000, 6000, 7000, 8000, 9000, 10000 and 20000 began to be denoted by letters of the Greek alphabet. A dash is placed above the letters to distinguish the numbers from the letters. The ancient Indians came up with their own sign for each number. Here is how they looked (Fig. 5).

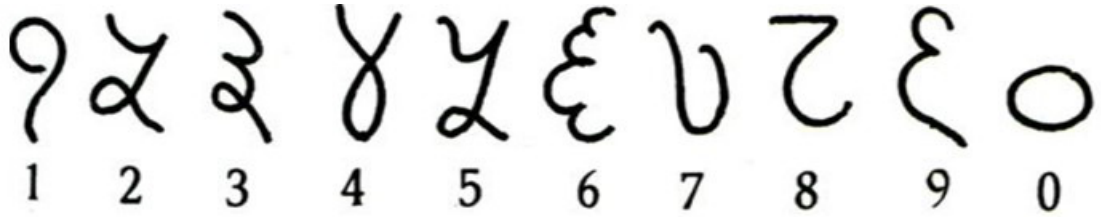


Fig. 5

But India is cut off from the rest of the world thousands of kilometers and high mountains lie on the way. The Arabs were the first outsiders who borrowed numbers from the Indians and brought them to Europe. After a while, the Arabs simplified these icons, they began to look like this (Fig. 6)



Fig. 6

They are similar to most of our numbers. The word number also came to us from the Arabs. The Arabs called zero, or empty, sifra. Since then, the word tsifra (digit) has appeared. However, all ten characters that we use now to write numbers are called numbers: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

1. Arabic numerals of the 10th century.
2. The Spanish figure of 976.

3. French figures of the eighteenth century
4. Gothic figures of 1400.
5. Renaissance figures
6. Modern numbers

With the development of writing, the concept of number developed and expanded. 2. It is possible to increase the activity of college students by setting them production tasks. For example,

a) An auto mechanic built an optimal regulator for pre-injection of diesel fuel into the cylinders of the car engine, which saved 8% of fuel. If an average of 13 kg per hour was used earlier, how much fuel was saved per month (200 working hours) as a result of this adjustment? What is this monetary savings (1 kg of fuel costs 30 roubles)? (Performing work on various types of maintenance) (It is essential to know: perform practical calculations in all areas of production activity).

b) The pulley of the drilling machine consists of 4 stages. Their diameters form an arithmetic progression. The diameters of the largest and smallest pulleys are 17 and 11 cm. Find the diameters of the remaining pulleys.

c) The diameters of 5 pulleys mounted on a common shaft form an arithmetic progression, the outer members of which are 110 and 206 mm. Find the diameter of the second pulley.

Thus, knowledge of history is an important factor in the formation of a worldview. Historical knowledge is aimed at creating historical truth, a scientific picture, an idea of the direction of human society in the process of cognition. Mathematics and history are two inseparable branches of knowledge. Information from the history of mathematics, historical questions bring these two subjects closer together.

History enriches mathematics with humanitarian and aesthetic content, develops imaginative thinking of students. Mathematics, which educates logical and systematic thinking, in turn occupies a worthy place in history and helps to understand it better. Studying the state educational standard of basic general education, we see that the use of historical materials in mathematics lessons allows us to fulfil a number of standard requirements:

1. Development of a worldview in accordance with the current level of progress of knowledge and public practice.

2. Establishment of a conscious, respectful and friendly attitude to the culture, history, religion, traditions and values of the Kyrgyz and peoples of the world.

3. The ability to independently plan ways to achieve objectives, including alternative ones, consciously choose the most effective ways to solve educational and cognitive tasks.

4. The ability to create, use and transform signs and symbols, models and schemes for solving educational and cognitive tasks.

5. The ability to organize educational cooperation and joint activities with the teacher and fellow students, to work individually and in groups.

References

[1] Kaldybaev, S. K. Possibilities of educational technologies in teaching school subjects / S. K. Kaldybaev // Izvestia of the Kyrgyz Academy of Education. Bishkek, 2015. 4 (36). - Pp. 3-8.

[2] Kondaurova, I. K. Development of cognitive interest in mathematics among college students / I. K. Kondaurova. Saratov, 2019.

[3] Bavrin, I. I. Ancient problems./ I. I. Bavrin, E. A. Fribus. M., 1994.

[4] Depnan, I. Ya. History of arithmetic. / I.Ya. Depnan. M.1965

[5] Drobyshev, Yu. A. Historical and mathematical aspect in methodical teacher training / Yu. A. Drobyshev. Monograph. Kaluga, 2004. 156 p. 5.

[6] Drobyshev, Yu. A. History of mathematics: ways of forming knowledge about methods of solving algebraic equations / Yu. A. Drobyshev. Kaluga: Publishing House of K. E. Tsiolkovsky, KSPU, 2004. - 164 p. 6.

[7] Drobyshev, Yu. A. Multilevel historical and mathematical training of a future teacher of mathematics. Diss. Doctor of Pedagogical Sciences, Moscow, 2011. 452 p.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

HUMANITARIAN ASPECTS OF TEACHING INVERSE PROBLEMS FOR DIFFERENTIAL EQUATIONS

VICTOR KORNILOV

Moscow City University, Moscow, Russia
e-mail: kornilovvs@mgpu.ru

The report discusses some humanitarian aspects of teaching inverse problems for differential equations in special elective courses. It is assumed that such students study at senior courses in the areas of training in applied mathematics. Attention is drawn to the fact that it is important not only to teach students mathematical methods of research of applied problems, but also to acquaint them with integrative technologies that allow applying the humanitarian principles of cognition in practice. It is emphasized that it is possible to implement the humanitarian aspects of education on the basis of the development of humanitarian-oriented training sessions, in which the teacher creates situations in which students, after they have studied the applied problem and found its solution, will have to formulate and justify conclusions of a humanitarian nature. The report briefly describes the stages of system design. For clarity, as an example, the study sections and topics are written out, which can make up, taking into account the orientation of students' training, the content of the relevant special courses. Students, attending such special elective courses, develop the ability to apply rational reasoning in solving educational problems, humanitarian analysis of their solutions.

The methods and means of modern world science make it possible in applied research to study and acquire scientific information about numerous physical processes and objects that can occur and be located in various geographical places and conditions, including inaccessibility to the researcher. In such cases, mathematical modeling methods come to the rescue, which make it possible to construct mathematical models of inverse problems for differential equations (hereinafter referred to as IPDE), the methods of solving which are considered in the theory of inverse problems – a scientific direction of applied mathematics that has been actively developing over the past 50 years by Russian and foreign specialists. Such authors as A. S. Alekseev, V. A. Ambartsumyan, A.V. Baev, Y. M. Berezansky, F. D. Gakhov, A.M. Denisov, S. I. Kabanikhin, M. G. Crane, M. M. Lavrentiev, B. M. Levitan, V. A. Marchenko, P. S. Novikov, M. T. Nuzhin, A. I. Prilepko, I. M. Rapoport, V. G. Romanov, L. N. Sretensky, A. N. Tikhonov, L. D. Fadeev, A. Lorenzi, M. Grasselli, M. Yamamoto, M. Hanke, D. Colton, K. Chadan, S. He, I. Lasiecka, P. Yao and many other specialists were crucial in the development of IPDE.

But, in fairness, we note that in different years of the XVII–XIX centuries in engineering sciences, mechanics, geodesy, astronomy, astrophysics, theoretical physics, geophysics, Russian scientists, scientists from other countries of the world studied inverse problems. Among such authors are R. Descartes (1596-1650), F. de Bon (1601-1652), I. Barrow (1630–1677), R. Hooke (1635–1703), G. F. Leibniz (1646–1716), D. K. Bobylev (1842–1917), E. Rutherford (1871–1937), I. A. Iveronov (1867–1916), E. Wichert (1861–1928), G. Gerglotz (1881–1953), D. Ryabushinsky (1882–1962) and other authors.

For example, one of the methods of deducing Hooke's law (discovered in 1660 by the English naturalist Robert Hooke) is based on the study of the inverse problem of mechanics, which

was studied in 1911 by the British physicist Ernest Rutherford in his widely known classical experiment on the scattering of alpha particles. An example of solving the inverse problem of mechanics is the discovery of the law of universal gravitation. The German mathematician, physicist and philosopher Gottfried Wilhelm Leibniz solved the inverse tangent problem, which was previously formulated by the French mathematician Florimond de Bon. This problem of determining curves whose tangents have a given property played an important role in the creation of the theory of differential equations. It is impossible not to mention the merits of Dmitry Konstantinovich Bobylev, a well-known Russian specialist in mathematics, physics and mechanics, in the creation of mathematical methods for finding solutions to the problem. Dmitry Konstantinovich solved inverse problems of determining the coefficients of the mathematical model of elasticity of analytical mechanics. We also note the Russian astronomer, geodesist and gravimetrist Ivan Alexandrovich Iveronov, who was engaged in the study of the inverse geodesic problem.

Currently, in applied research conducted in such scientific fields as seismology, geophysics, astrophysics, laser sounding of natural environments, underwater acoustics, ecological and hydrogeological sounding and many other scientific fields, the technologies of the theory of IPDE are widely used, which allow to obtain new scientific information, to identify cause-and-effect relationships of physical processes and phenomena.

The need for specialists who are able to successfully apply mathematical models of IPDE in applied research is an obvious reason for their training among university students studying in the fields of applied mathematics. Since the beginning of the 70s of the XX century in some universities of Moscow, Novosibirsk, Kazan, Tomsk, Yekaterinburg, Krasnoyarsk and other cities of Russia, at the faculties of applied mathematics and physics and Mathematics faculties, students of 3–4 courses have been taking special courses on IPDE (see, for example, [1, 2, 3]). The content of such special courses is developed by specialists taking into account the areas of training of students.

In the classroom, students get acquainted with applied problems that lead to the need to explore the IPDE; master the methods of solving various IPDE (analytical, approximate). At such special courses, students are introduced to the technologies of studying the correctness (conditional correctness) of the considered mathematical models of the IPDE.

It is appropriate to note that by mastering the theory of IPDE, students realize the applied importance of such causal problems, master the scientific and cognitive potential of IPDE, are convinced of their wide application in industry, medicine, biology, chemistry, physics, etc., as well as in such scientific fields as, for example, the study of the Terrestrial environment, marine natural resources disasters; management of thermonuclear fusion, field geophysics, processing of photographic images, planning and processing of experimental data, etc.

For clarity, let's pay attention to some applied problems that students get acquainted with in such special courses and, which, in the future, are studied in the classroom with the help of mathematical models of IPDE. The teacher can demonstrate to students how, using a mathematical model of the IPDE using a system of ordinary differential equations, to investigate, for example, the processes of chemical kinetics. Or to justify that in the study of an applied problem, for example, the study of numerical data on the thermal radiation of the density of radioactive heat sources measured on the Earth's surface, can be formulated using parabolic equations. Or students can be shown an example of studying the problem of hydrodynamics or the problem of diffusion using mathematical models of IPDE using elliptic equations. Or how to study applied problems of electromagnetic wave propagation using hyperbolic equations.

When teaching IPDE students, it is advisable not only to teach mathematical methods of research of applied problems, but also to acquaint them with nature-like and integrative technologies that allow applying the humanitarian principles of cognition in practice, I implement humanitarian-oriented educational classes on IPDE.

First of all, let us pay attention to the fact that the Russian education system is currently taking practical steps aimed at the humanitarian development of students. This is confirmed, among other things, by scientific publications of scientists, teachers and teachers of various subject areas, in which the authors discuss theoretical and methodological aspects, pedagogical technologies of humanitarian-oriented learning. Among such authors: N.V. Borisova, G.Y. Burakova, Ya.A. Vagramenko, D. N. Gergilev, E.V. Danilchuk, E.V. Donskova, G.V. Dorofeev, A.V. Dorofeeva, T.A. Ivanova, M.S. Kagan, V. I. Kolmakov, S.A. Komissarova, T.N. Mirakova, A.H. Naziev, Yu.N. Pavlovsky, O. I. Savin, G.I. Sarantsev, A.A. Stolyar and other authors.

In confirmation of what has been said, we will pay attention to the opinions of some of them. For example, S.A. Komissarova believes that it is possible to implement humanitarian aspects in natural science education through the organization of cognitive activity of students. And this activity of theirs, in her opinion, should be directed to the holistic development of natural reality. For example, D. N. Gergilev, V. I. Kolmakov, O. I. Savin are convinced that humanization in engineering education should meet the requirements of the engineering profession and post-industrial culture. According to T.A. The humanitarization of mathematical education should include the methodology of scientific research, the history of mathematics and creative mathematical activity, turns into humanitarian knowledge if this potential is understood, comprehended by students and helps in their intellectual activity. Y.N. Pavlovsky believes that developers of mathematical models of physical processes should have a humanitarian understanding of the process at the same level at which it is understood by experts in this field of research and practice.

Now we note that the design of a system of humanities-oriented training sessions on inverse problems for differential equations includes several stages.

Mathematical analysis of the content of IPDE training. For clarity, we will briefly reveal, as one of the options, the contents of the sections of the IPDE offered in such special courses.

1. The section of the IPDE when ordinary differential equations (hereinafter referred to as ODE) are used. Calculation of unknown coefficients of linear ODE and analysis of approaches and scientific ideas leading to a successful solution of ODE. Calculation of unknown coefficients of nonlinear ODE and evaluation of the effectiveness of mathematical methods for their solution. Finding the right part of the ODE and qualitative analysis of the contribution of mathematical methods of their solution to the development of the theory of ODE. IPDE in management theory and analysis of scientific ideas and approaches used to restore the properties of objects.

2. IPDE section, when mathematical physics equations are used. Coefficient IPDE (one-dimensional, two-dimensional, multidimensional). Boundary conditions (one-dimensional, two-dimensional, multidimensional). Evolutionary IPDE (one-dimensional, two-dimensional, multidimensional). One-dimensional problems of hyperbolic type, the use of rational reasoning to find their solution. Two-dimensional and multidimensional hyperbolic types, humanitarian significance in the ecological analysis of the level of environmental safety. Parabolic-type IPDE humanitarian analysis of the solutions found. Identification of the scientific significance of the results obtained in the analysis of the influence of nearby objects on human health. Elliptical-type IPDE, applied analysis of the solutions obtained.

3. Section of the problems of calculating functions by known values of integrals. Applied tasks of computed tomography, qualitative analysis of solutions to identify humanitarian value. Applied problems of finding unknown functions when their spherical averages are known, the scientific value of mathematical methods for finding these solutions in the theory of Volterra integral equations. The tasks of identifying functions when the values of its integrals are known, the analysis of mathematical methods for finding these solutions in the development of the theory of Fredholm integral equations.

4. Section of approximate methods for solving the problem. Difference methods for solving the problem. A method for inverting a difference scheme for solving one-dimensional inverse problems for hyperbolic equations. A difference method for solving the inverse problem for

the heat equation. A difference method for determining the unknown right-hand side of a two-dimensional Poisson equation based on the results of observations at the boundary of the domain. Application of the numerical results of the IPDE solutions in the humanitarian analysis of applied research.

Didactic analysis of the content of IPDE training. Carrying out a didactic analysis, it is necessary to pay attention to what kind of students it is advisable to form scientific knowledge in the learning process; how to structure the educational material more effectively and into which sections to divide it in terms of the sequence of presentation and completeness; what forms and methods of control to carry out; what intra-subject and intersubject connections will need to be implemented. It is obvious that such a didactic analysis is of great importance for the implementation of educational goals at the design stage of humanitarian-oriented classes.

Formation of the IPDE system. When planning training sessions, it is advisable to analyze those inverse tasks that are planned to be included in the training content. From a scientific point of view, the study of physical processes generated by pulsed sources is of great interest, which leads to the need to solve the problem. Therefore, it is advisable for students to offer IPDE in generalized statements. But first of all, it is desirable to explain to students in lecture classes that when conducting applied research, together with continuously distributed quantities (mechanical impulse, heat sources, charge, etc.), the researcher has to deal with concentrated quantities (concentrated pulse, point heat source, point charge, etc.). With the help of generalized functions, it becomes possible, to express in mathematical form, for example, the density of a material point, the density of a point charge, the intensity of an instantaneous point source and other idealized concepts.

Generalized functions are a generalization of the classical concept of a function. Great importance in the creation of the foundations of generalized functions was played by the research of J. Hadamard, S. Bochner, N. M. Gunther, G. R. Kirchhoff, O. L. Cauchy, A. L. Lebesgue, S. D. Poisson, M. Riess, O. Heaviside and other authors. Generalized functions, as historians of science tell us, were used for the first time in scientific research in 1928 by P. A. M. Dirac. In the future, the mathematical theory of generalized functions was developed in the 30s of the XX century by S. L. Sobolev. In the works of N. N. Bogolyubov, A. Vaitman, V. S. Vladimirova, I. M. Gelfand, R. Yost, J. Lere, L. Schwartz, G. E. Shilov and other authors, the theory of generalized functions finds its further development.

At present, generalized functions are widely used in physical and mathematical applied research. Of great interest, for example, are studies of the properties of physical processes and phenomena that are generated by impulsive sources (sources of the "explosion" type). Such impulsive sources can be modeled by the Dirac delta function, a representative of singular generalized functions. The concept of such a singular generalized function can be arrived at if we consider it as the limit of functional sequences that have certain properties and it is defined as a continuous linear functional on the space of differentiable functions. The Dirac delta function allows you to express in mathematical form the spatial density, for example, masses, forces, the intensity of a heat source concentrated at one point or, for example, the distribution of charge on a surface. The Dirac delta function allows its expansion, for example, in systems of functions, in a Fourier series, representation by the Fourier integral, etc. The Dirac delta function is a rightfully effective tool for solving many problems of mathematical physics, it is often used in applied mathematics, quantum mechanics, mechanical engineering, signal processing and other applied sciences.

With such training, students master mathematical methods and technologies for studying various types of such problems. This is the calculation of coefficients or the right parts of differential equations; this is the restoration of boundary conditions; this and finding the geometric characteristics of the contour of the area; this is the restoration of the history of the state of the process under study. Among the mathematical methods that students master are the methods

of Laplace, Poisson, Fourier, convolution, Green, Duhamel, D'Alembert, Kirchhoff, Sobolev, characteristics, Volterra and Fredholm equations and other mathematical technologies.

The attention of students in the classroom is drawn to the specific features of inverse problems, which include non-linearity, non-uniqueness and incorrectness, as well as specificity, which significantly complicate the study of such problems. Students master mathematical methods and technologies for proving the existence, uniqueness and stability (conditional stability) of solving both one-dimensional and two-dimensional and multidimensional inverse problems, identifying and analyzing cause-and-effect relationships.

Among educational inverse problems that can be offered to students for research, inverse problems in generalized formulations can also be noted. These are such formulations of inverse problems in which generalized functions are present, for example, in the equations themselves, or are present in the initial or boundary conditions. The Dirac delta function (or its derivatives) can act as such generalized functions. The presence of generalized functions in the mathematical formulation of the inverse problem implies finding a generalized solution of the inverse problem. Students are introduced to mathematical methods for extracting the singular part of a solution, which allow reducing the original inverse problem in a generalized formulation to an inverse problem for continuous functions. The implementation of such a mathematical technology significantly simplifies the study of the inverse problem, which may consist, for example, in proving the uniqueness or conditional stability of the solution to the inverse problem.

It is important to emphasize that students in the study of IPDE in humanitarian-oriented classes gain experience in logical analysis, humanitarian analysis, physical interpretation of the obtained generalized solutions of inverse problems, mastering the logic of applied mathematics with its methods of proof. And when discussing with the teacher the results obtained on the study of the inverse problem, students also master the scientific style of reasoning, which may consist, for example, of rational reasoning or deductive reasoning, which allow you to get the right result.

For clarity, we note several types of such rational reasoning that can be applied in the study of inverse problems.

1. Application of assertions in the study of inverse problems valid in real cases. They constitute an important feature of the language of applied mathematics, which sees behind each concept a model of a real object.

2. Refinements in the course of studying inverse problems. When studying the inverse problem, additional assumptions can be introduced that will simplify the search for a solution to this inverse problem. For example, simplifications may be made in cases of practical relevance.

3. The use of reasonable analogies in the study of inverse problems. This way of reasoning, based on physical or computational experiments, in some cases allows the statements that take place for one-dimensional inverse problems to be extended to two-dimensional or three-dimensional inverse problems.

4. Proof of the well-posedness of inverse problems, which is based on the consideration of special cases. This may be the case, for example, in cases where deductive proof proves difficult or does not allow it at all.

5. Putting forward working hypotheses about the structure of the solution to the direct problem when searching for a solution to the inverse problem. Such hypotheses take into account the physical interpretation of the studied mathematical model of the inverse problem, taking into account physical laws.

In the process of studying inverse problems for differential equations in generalized formulations, students master new fundamental knowledge on generalized functions. This may be, for example, the study of local properties of generalized functions, it may be the operation of convolution of generalized functions, it may be the operation of differentiation or the operation of integration of generalized functions, it may be, for example, the implementation of regularization of generalized functions or the Fourier or Laplace transform of generalized functions

and other mathematical operations on generalized functions. Moreover, depending on the formulation of the inverse problem, such generalized functions can be both one-dimensional and multidimensional.

In addition, students master mathematical technologies that make it possible to apply the properties of generalized functions to the construction, for example, of fundamental solutions to direct problems, at intermediate stages of the study of inverse problems.

When considering such and other applied problems in special courses, students' attention is focused on the fact that IPDE technologies allow obtaining scientific information, for example, about the ecology of the environment, about the impact on people's health of nearby objects; or, for example, about the thermophysical characteristics of the medium under study, about the distribution density of sources; or, for example, about the properties of the deep layers of the World Ocean, about evolutionary processes and other scientific information about the processes and phenomena being studied, which is difficult or impossible to obtain by conducting physical experiments.

It is important to pay attention to the fact that students gain experience in understanding the physical properties of the object under study, which largely allows them in the future in their professional activities as an applied researcher not to make mistakes in building mathematical models of the IPDE with acceptable geophysical models of the studied media. Otherwise, it is possible to construct an erroneous statement of the inverse problem and, in the future, according to the found solution, incorrectly interpret the results obtained.

In the humanitarian-oriented classes on IPDE the teacher creates situations in which students, after they have studied the IPDE and found its solution, will have to formulate and substantiate conclusions of a humanitarian nature. Such training sessions, obviously, familiarize students with the problem of humanization of mathematical education, as well as with the problem of moral responsibility to society for the negative consequences of the practical implementation of applied research.

Students, attending such special courses on IPDE, form their knowledge not only on IPDE, but also in the disciplines of physical and mathematical profile and natural science. Students expand their knowledge of the surrounding reality, develop a scientific worldview, a humanitarian culture, acquire useful skills in conducting applied research using mathematical models of the IPDE.

Keywords: inverse problems for differential equations, humanitarian aspects of teaching, pedagogical technologies, student.

AMS Subject Classification: 97M10 Modeling and Interdisciplinarity (Aspects of Mathematical Education)

References

[1] Vabishevich P.N., *Computational methods of mathematical physics. Inverse problems and management problems*, M.: University Book, 2019, 478 p.

[2] Kabanikhin S.I., *Inverse and incorrect problems: textbook*, Novosibirsk: Siberian Scientific Publishing House, 2008, 460 p.

[3] Kornilov V.S., *Theory and methodology of teaching inverse problems for differential equations: monograph*, Moscow: Publishing house "OntoPrint", 2017, 500 p.

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ТРАНСФОРМАЦИЯ СИСТЕМЫ ОБРАЗОВАНИЯ НА ОСНОВЕ STEM ПОДХОДА КАК УСЛОВИЕ ПОДГОТОВКИ КОНКУРЕНТОСПОСОБНОГО ЧЕЛОВЕЧЕСКОГО КАПИТАЛА В СОВРЕМЕННОМ МИРЕ

ЖАУМБАЙ КАРАЙЕВ[1], ЖАНАРТАЙ КОБДИКОВА[2]

[1]НАО им.Ы.Алтынсарина, Алматы, Республика Казахстан
[2]Казахская академия спорта и туризма, Алматы, Республика Казахстан
e-mail: karaevkz@mail.ru, e-mail: zhuk53@mail.ru

Всеобщая глобализация и цифровизация, высочайший динамизм технологизации придает совершенно новый темп экономическому и социальному развитию общества в передовых странах. В 2011 году в ФРГ президент Давосского всемирного экономического форума К.Шваб обьявил о наступлении 4-й промышленной революции (Индустрии 4.0), которая характеризуется главным образом, высокотехнологичной цифровой промышленностью [1].

Общество, охватывающее период всеобщей цифровизации и развития технологий, с уровнем индустрии 4.0 мировыми экспертами названо общество 4.0 [1,2,3]. Индустрия 4.0 предполагает новый подход к производству, основанный на массовом внедрении информационных технологий в промышленности, масштабной автоматизации бизнес-процессов и распространения искусственного интеллекта [1]. Развитые страны 21 века (в число которых в ближайшие годы стремится войти Казахстан) характеризуются конкурентоспособной экономикой с Индустрией 4.0, с конкурентоспособной наукой и образованием, отвечающими запросам Общества 4.0. В Индустрии 4.0 экономический рост основан не на природных ресурсах, а на инновациях и конкурентоспособном человеческом капитале [2, 3, 4]. Высокотехнологическое производство Индустрии 4.0 предполагает не только использование, но и развитие процесса цифровизации и роботизации промышленности (использования квантовых компьютеров, искусственного интеллекта, технологий lot, big data и пр), а также динамичного обновления технологий, оборудования, т.е. изобретение инноваций и их внедрение в производство.

Мировой опыт показывает, что общество 4.0 с индустрией 4.0 требуют кардинальной трансформации системы образования. Данная трансформация ставит перед образованием непростые задачи: подготовка подрастающего поколения к жизни в обществе будущего (общество 4.0), которая требует от него особых интеллектуальных способностей, необходимых для специалистов высокотехнологичных производств, сформированных ключевых компетенций и навыков XXI века. Таким образом, Индустрия 4.0, «умная экономика» ставит перед системой образования совершенно новые задачи. Социальным запросом Общества 4.0 является «соответствие образовательной парадигмы к промышленной парадигме Индустрии 4.0». По мнению зарубежных экспертов специалисты Общества 4.0 должны владеть ключевыми образовательными компетенциями и отличаться высокой научной, цифровой и инженерно-технологической подготовкой [2, 3, 4].

Специалисты будущего должны быть готовыми к мобильной адаптации на новые условия, приобретению новых навыков, чтобы успеть за изменениями на рынке труда и динамично развивающимся инновациям техники и технологий. По этому для развития системы образование в быстроменяющихся условиях особо важное значение имеет концептуальная

модель – «обучение на протяжении всей жизни» (Lifelong Learning). Актуализируется интеграция усилий различных форм образования: формального (человек обучается в государственных организациях образования); неформального (наставничество, тренинги, стажировка и т.п.); информального (самообразование с помощью Coursera, You Tube и т.п.).

Анализ исследования ученых показывает [4,5,6,7,8], что Индустрия 4.0 ставит перед высшим образование такие условия, чтобы их рейтинговые показатели были на уровне «Университета 4.0» или не ниже «Университета 3.0». Как известно, «Университет 1.0» – это университет, который ведет качественную образовательную деятельность. «Университет 2.0» – образовательную и научную деятельность. «Университет 3.0» осуществляет образовательную, научную и предпринимательскую деятельность, т.е. выполняет заказ высокотехнологичных производств. «Университет 4.0», кроме перечисленных функций «Университета 3.0», способен решить проблемы современной промышленности за счет изменения концепта самой промышленности, стать лидером развития высокотехнологичных отраслей [6].

В мире чувствуется «запоздалая реакция» системы образования на запросы стремительно развивающихся инновационных технологий и производств по подготовке соответствующих кадров.

В 90-годы американцы столкнулись с серьезной проблемой – при наличии предложений в высокотехнологичных компаниях потенциальные кадры в большинстве своем не обладали достаточной квалификацией.

На стыке двух тысячелетий крупнейшие компании США заявили о несоответствии школьного и вузовского образования запросам динамично развивающихся высокотехнологичных отраслей. Проанализировав ситуацию Национальный научный фонд США в 2001 году предложил ввести STEM-подход для модернизации системы образования Америки [9,10].

Образовательная концепция STEM (Science, Technology, Engineering, Mathematics) представляет собой новые подходы к обучению, основанные на интеграции предметов естественно-математического цикла, информатики, технологии и инженерии в единую систему обучения для решения конкретных задач, взятых из реальной жизни.

Таким образом, STEM образование представляет собой интегрированный подход обучения, который «переплетает» науки, в рамках которого академические научно–технические концепции изучаются в контексте реальной жизни [7,9,11]. Основные составляющие STEM образования – наука, технология и инженерия, также математика и ИКТ охватывают все сферы современной жизни, являются главными индикаторами функционирования и развития Индустрии 4.0.

Данное положение актуализирует значение STEM подхода в подготовке конкурентоспособного человеческого капитала для Индустрии 4.0. Следовательно, STEM образование является по сути ответом на вызовы динамичной индустриально-цифровой эры развития человечества.

Изучив опыт США по трансформации системы образования на основе STEM-подхода, начатого в 2001 году, некоторые страны с развитой экономикой начали внедрять данный тренд в свою сферу образования [2,3,4,8,9,10]. Этот процесс ускорился после объявления К.Швабом о наступлении эпохи Индустрии 4.0 в 2011 году [1,2,3,4,5,8]. Таким образом, факторами обуславливающими внедрение STEM-подхода в образовании, явились: 1) решение Национального научного фонда США в 2001 году; 2) наступление 4-й промышленной революции, обоснованной К.Швабом в 2011 году. В настоящее время США, Англия, Китай, Юж. Корея, Сингапур, Турция и др. развитые страны системно внедряют STEM образование на основе принятых государственных программ. В ФРГ реализуется MINT программа (математика, информатика, наука, технология), курируемой канцлером страны [5,8,9,10].

В странах СНГ пока нет системного подхода к внедрению концепта STEM образования в образовательные стандарты. В России своеобразно реализуется внедрение данного тренда в федеральные государственные образовательные стандарты [11]. Более того, в странах

СНГ отсутствует системное внедрение в практику концептуальных идей деятельностного, компетентностного и цифрового подходов [12].

Начиная с 2014 года в РФ дан приоритет инженерному образованию. В 2019 году разработана новая концепция предметной области «Технология» с учетом требований STEM подхода [13,14].

В Казахстане отсутствует единый подход к реализации идей STEM образования. В некоторых передовых школах (НИШ, БИНОМ и др.) осуществляется внедрение элементов STEM подхода, характеризующиеся, в основном, изучением первоначальных основ «Робототехники», проведением лабораторных работ с использованием ИКТ и применением проектно-исследовательских методов в учебном процессе [5,10].

В последние годы появилось множество научных исследований по этой проблеме за рубежом и в странах СНГ [2,3,4,6,7,8,11]. Изучению общепедагогических основ STEM-образования посвящены работы [2,3,4,6,9]. Научно-методические вопросы обучения предметов STEM-образования рассмотрены в исследованиях [9,10,11,13,14]. Особый интерес в плане нашего исследования представляет работа С.Пейперта [12] по использованию среды ЛОГО в качестве инструмента самостоятельной познавательной деятельности ученика, эффективного средства изучения основ алгоритмизации и программирования в контексте формирования инженерно-технологического мышления.

Однако анализ приведенных и других исследования показывает, что в настоящее время отсутствуют работы, в которых целостно рассматривались бы: 1) психолого-педагогические основы трансформации системы образования с применением концепции STEM-подхода; 2) модернизации теории обучения и теории содержания образования на основе STEM-требований; 3) дидактическая сущность и научно-обоснованные характеристики основных составляющих STEM-образования. Также не раскрыты педагогические возможности современных ИКТ и робототехники как принципиально нового средства в реализации личностно-деятельностного и функционально-компетентностного подходов в формировании инженерно-технологических навыков. Не исследованы научные основы интеграции содержания образования предметов естественно-математического цикла, технологии, робототехники, информатики в контексте требований STEM-подхода. Отсутствует системное исследование вопросов организации предпрофильной подготовки и профильного обучения на основе STEM требований.

Изучая и обобщая опыт зарубежных стран и опыта работы педагогов нашей республики, обосновать актуальность и способы трансформации системы образования РК на основе STEM-подхода, раскрыть дидактическую сущность STEM-образования. Фундаментальные исследования в области современной педагогики и психологии; теория личности и деятельности; теория содержания образования; концепция развивающего обучения и компетентностного подхода; теоретические положения в области педагогики и психологии по проектированию педагогических систем и внедрению результатов педагогических исследований в педагогическую практику. Для решения рассматриваемой дидактической задачи были использованы следующие методы исследования: изучение научной и педагогической литературы; теоретический анализ разработанности рассматриваемых проблем; сравнительно-педагогический анализ государственных стандартов, учебных программ и учебной литературы; научно-методический анализ содержания школьного образования с позиции рассматриваемой проблемы; моделирование учебного процесса; наблюдение; беседа.

Экспертами доказано, что устойчивое развитие государства с конкурентоспособной экономикой и индустрией 4.0 связано главным образом с реформой образования в STEM направлении [7,9].

Это закономерно, так как жесткая конкуренция и тотальная цифровизация всех сфер жизни человечества, в том числе производственной сферы требует постоянного совершенства технологических процессов, вынуждая их подниматься от механизированного уровня

на уровень «высоких технологий». Данная ситуация предполагает изучение основ технических и технологических процессов в школе, используя законов и закономерностей предметов естественно-научного цикла (ЕМЦ), математики и информатики в интегрированном виде. Более того, главное условие конкурентоспособности – «постоянно быть на шаг впереди» предполагает наличие специалистов не только мобильно осваивающее новые технологии, но и умеющие их совершенствовать, то есть инженеров-изобретателей, создающих более эффективные технологии и новые оборудования с инновационной цифровой начинкой.

Следовательно, специалисты будущего уже со школы должны овладеть политехническими компетенциями, изучая предметы «Технология», «Робототехника» и «Информатика» в интеграции с предметами естественно-математического цикла, нацеленными на решение конкретных прикладных задач, то есть должны овладеть навыками современной инженерии.

Основные составляющие STEM-образования – технология и инженерия, как и цифровизация, охватывают все сферы современной жизни, являются индикаторами развития Индустрии 4.0, что актуализирует значение STEM-подхода к подготовке конкурентоспособного человеческого капитала.

Все это требует трансформации среднего образования на основе STEM-подхода, реформирования систем ТиПО и Вузов с учетом требований инженерно-технологического образования. Только в таком случае образовательная парадигма Общества 4.0 будет соответствовать производственной парадигме Индустрии 4.0. Из вышеизложенного определения и сущности STEM-образования видно, что STEM-образование способствует эффективной реализации концептуальных идей деятельностного, компетентностного и цифрового подходов.

Из сущности деятельностного, компетентностного и цифрового подходов и STEM образований следует, что эти подходы концептуально взаимосвязаны и взаимообусловлены. STEM подход является расширением и модификацией интегрированного формата личностно-деятельностного, компетентностного и цифрового подходов в контексте требований инженерно-технологического тренда системы образования, т.е инновационной составляющей новой парадигмы развития данной сферы.

Новая методология образования, основанная на STEM подходе, который интегрирует и расширяет инновационные потенциалы вышеперечисленных подходов, предполагает трансформацию содержательно-методических основ школьного образования. Наше исследование показало, что наряду с классическими принципами отбора содержания (дополнений к принципам формирования инженерной компетенции учащихся), необходимо введение в теорию содержания образования следующих принципов их отбора [12]:

-принцип цифровизации образования, который предполагает при отборе содержания образования (организация обучения) учет дидактических возможностей ИКТ и обоснованного определения педагогических целей их использования;

-принцип соответствия содержания требованиям всех уровней таксономии целей обучения с охватом зон «актуального» и «ближайшего» развития обучаемого. Современное содержание образования должно охватить весь диапазон иерархии целей обучения Б.Блума от «знания» до «создания», уровни усвоения от «ученического» до «творческого», иерархии познавательной деятельности от «репродуктивного» до «продуктивного», весь спектр зон развития по Л.С.Выготскому;

-принцип практической и прикладной направленности, занимательности содержания образования, который предполагает усиление аспектов практической значимости и связи с жизнью учебного материала, формирование прикладного содержания учебных предметов;

-принцип субъектности содержания образования, который предполагает активную продуктивную деятельность ученика и субъект-субъектное отношение учителя и учеников в процессе обучения;

-принцип соответствия содержания образования требованиям STEM-образования (в том числе инженерно-технологического образования), профильного обучения, проектных работ в полифункциональных лабораториях, мастерских и Maker spice зон;

-принцип отбора интегрированного содержания предметов ЕМЦ с прикладно-практическим значением для организации проектных и учебно-исследовательских работ, учет дидактических возможностей робототехнических систем;

-принцип соответствия содержания требованиям инклюзивного образования [12];

Формирование STEM компетенции учеников предполагает, кроме сформированности математической, читательской, естественно-научной и глобальной компетенций, также цифровые и инженерно-технологические навыки. Следовательно, сформированность STEM компетенции определяется результатами не только PISA, но и международных конкурсов по робототехнике и международных соревнований WORLDSKILLS.

В этих исследованиях оценивается способность учащихся, используя знания в соответствующей предметной области, понять и решить проблему, которая лежит вне рамок самой этой области. Таким образом, оценивается уровень функциональной грамотности и сформированности метапредметных навыков учащихся.

Собственно, PISA для того и нужна, чтобы правительства при разработке стратегий в области образования имели четкие ориентиры. Генеральный секретарь ОЭСР Анхель Гурриа отметил, что PISA— значительно больше, чем просто рейтинг, «это показатель того, насколько хорошо система образования той или иной страны готовит молодых людей к завтрашнему дню» [12].

Заметим, что результаты международных конкурсов по робототехнике и WORLDSKILLS показывают на сколько готово образование страны к запросам Индустрии 4.0. Естественно, при этом мы должны учитывать то, что основанием для формирования STEM компетенции является предметные и метапредметные компетенции, в первую очередь, предметов, входящих в состав STEM образования [5].

Вышеизложенные показывают также, что STEM-подход является главным трендом развития современного образования, модификацией деятельностного и компетентностного подходов с учетом требований Индустрии 4.0.

Главная цель STEM подхода – преодолеть свойственную традиционному образованию оторванность от решения практических задач и выстроить понятные ученикам связи между учебными предметами. Это означает развитие и расширение прикладных навыков учащихся, полученных на уроках технологии, используя способы инженерии, инновационный и развивающий потенциал робототехники, программирования, 3D-принтинга, а также предметов естественно-математического цикла. Поэтому STEM – это универсальный практико-ориентированный подход, который позволяет ученикам справляться с учебно-исследовательскими задачами, а также прикладными задачами научных проектов.

Характерная особенность инженерного образования, которая является одним из главных компонентов STEM-образования по формированию инженерных навыков, заключается в прочном есте¬ственнонаучном, математическом и мировоз¬зренческом фундаменте знаний, широте меж¬дисциплинарных системно-интегративных зна¬ний о природе, обществе, мышлении, а так¬же высоком уровне общепрофессиональных и специально-профессиональных знаний [16].

В рамках школьного STEM образования закладываются основы комплексного решения запросов экономики на развитие высокотехнологичных отраслей и естественный интерес учащихся к математике, информатике, инженерии, технологии и, в целом, к технической науке. Существующаяся знаниевая, оторванная от жизни система образования, снижала мотивацию учащихся к изучению предметов естественно-математического цикла, не позволяла развитие креативного и инженерного мышления, что не давало возможность подготовки трудовых ресурсов современного общества.

Таким образом, в основе STEM-подхода лежат четыре принципа: проектная форма организации образовательного процесса, прикладной характер учебных задач, межпредметный характер обучения, охват дисциплин, которые являются ключевыми для подготовки инженера или специалиста по прикладным научным исследованиям: предметы естественнонаучного цикла (физика, химия, биология, география), математики, современные технологии и инженерные дисциплины [9].

Одна из важных задач предмета «Технология», которая является основным компонентом STEM образования – показать учащимся практическую значимость естественнонаучных знаний, возможность их целенаправленного использования в различных технологиях. Осознание обучающимися жизненной, практической ценности «академических» знаний способствует не только развитию мотивации учения, но и раскрывает функциональную значимость понятий, явлений, законов, их использование в технологических процессах, формирует технологическое мышление и технологическую грамотность В основе содержания технологии лежит целостный образ окружающего мира, который преломляется через результат творческой деятельности учащихся.

Таким образом, особенностью содержания технологического образования является его интегрированность. Изучение технологий и технологических процессов основывается на законах и закономерностях математики, физики, химии, биологии и других предметов. Тенденция к интеграции знаний вообще свойственна современному научному сознанию, что выражается не только в тесном взаимодействии традиционных наук, но и в возникновении новых технологий на границе смежных наук – биотехнологии, нанотехнологии, нейротехнологии и др [13]. Поэтому необходимость синтеза знаний, их комплексного усвоения и применения в практической деятельности и жизни человека становится основанием развития межпредметной интеграции на всех уровнях образования. Межпредметные связи в школьном технологическом образовании являются конкретным выражением интеграционных процессов, происходящих сегодня в науке и в жизни общества [13]. Следующим основным компонентом STEM образования является инженерия. Инженерия (инженерное дело) – область технической деятельности, включающая в себя целый ряд специализированных областей и дисциплин, направленная на практическое приложение и применение научных, экономических, социальных и практических знаний с целью обращения природных ресурсов на пользу человека [15].

Цель инженерной деятельности – изобретение, разработка, создание, внедрение, улучшение техники, материалов и процессов. Инженерное дело тесно переплетается с наукой, опираясь на постулаты фундаментальной науки и результаты прикладных исследований. Посредством инженерии обеспечивается связь между фундаментальной науки и отраслями народного хозяйства. Инженерное дело является отраслю научно-технической деятельности. Оно охватывает три вида инженерно-технической деятельности: 1) исследовательская (научно-техническая) деятельность – прикладные научные исследования, технико-экономическое обоснование; 2) конструкторская (проектная) деятельность – это конструирование (проектирование), создание и испытание прототипов технических устройств, разработка технологии их изготовления, подготовка проектной документации; 3) технологическая (производственная) деятельность – деятельность, направленная на внедрение инженерных разработок в практическую деятельность экономических субъектов с их последующим сопровождением (технической поддержкой) [15].

Таким образом, инженерия (инженерное дело) – область человеческой деятельности, дисциплина, профессия, задачей которой является применение достижений науки, техники, использование законов физики и природных ресурсов для решения конкретных проблем, целей и задач человечества. Инженерия предполагает применения научных новшеств в решении реальных прикладных задач, что способствует развитию у специалистов умений

изобретателя, новатора [9]. Такие навыки актуальны в развитии высокотехнологичных производств, умной экономики в жесткой конкурентной среде, так как конкуренция требует динамичного внедрения инноваций в практику.

Инженерное образования в школе, как и его составная часть – технологическое образование, осуществляется за счет интеграции основного и дополнительного образования по всем уровням образования [16].

В начальной школе – пропедевтика (развитие у младшего школьника опыта общения с природой, умения наблюдать и исследовать явления окружающего мира с помощью простых инструментов сбора и обработки данных, формирование базовых навыков работы с материалами, знакомство с принципами проектной деятельности) [16]. Нетрудно заметить, что пропедевтика инженерного образования реализуется изучением предметов «Технология» и «Робототехника».

В основной школе – формирование первоначальных конструкторско-технологических знаний и умений (приобретение опыта применения физических, химических, биологических методов исследования объектов и явлений природы, базовые умения планировать работу, конструировать и моделировать, знакомство с основами 3D моделирования, робототехники, электротехники и электроники, программирования). Реализация инженерного образования в основной школе осуществляется главным образом, посредством изучения предметов «Технология», «Робототехника» и «Информатика и ИКТ» [14]. В профильной школе – профориентация (освоение технологии решения творческих задач, моделирования, конструирования, прототипирования и программирования; овладение основными алгоритмами и опытом проектно-исследовательской инженерной деятельности; участие в инженерных конкурсах и фестивалях) [16]. В профильной школе инженерное образование осваивается посредством использования проектного метода, ориентированного на решение конкретной производственной задачи во взаимодействии школы с вузом и производством. В результате реализации инженерного образования у учащихся формируется инженерное мышление.

Роль STEM в школе – интеграция инновационного потенциала предметов «Технология», «Робототехника» и «Информатика и ИКТ», модернизация содержания предметов естественно-математического цикла с учетом требований инженерно-технологического образования и процесса цифровизации. Опыт внедрения концепции ТРИЗ в отдельных школах показывает, что формирования инженерно-технологических навыков учеников в средней школе возможно и необходимо.

Таким образом, STEM-подход предлагает, прежде всего, системное внедрение в школу нового, инженерно-технологического аспекта образования на базе цифровой платформы [5].

В традиционном образовании предметы ЕМЦ: физика, математика, химия, биология, также информатика изучаются по отдельности, их содержание никак не пересекается друг с другом, оставляя в памяти ученика разрозненные обрывки информации. Предметная область “Технология” позволяет выстроить логические связи между этими дисциплинами, смотреть на окружающий мир глобально, усвоить закономерности глубже.

Если мы готовим наших учеников к жизни после школы, то мы должны позволить им использовать те инструменты, которые в дальнейшем всё равно станут частью их повседневной жизни. Поэтому с целью формирования инженерно-технологических навыков необходимо изучение в школе предметов робототехники, технологии, информатики и ИКТ, проектирования и графики, элективных курсов 3D-моделирования, инженерной графики и др. Таким образом STEM-образование предполагает: 1) переход от сугубо фундаментального подхода изучения предметов ЕМЦ к их изучению в контексте практического применения научных знаний в решении прикладных задач из реальной жизни; 2) усиление прикладных, практических и лабораторно-экспериментальных компонентов содержания предметов ЕМЦ; 3) определение содержание предмета “Технология” посредством интеграции содержания предметов ЕМЦ, информатики и робототехники с целью поэтапного освоения различных технологий, формирования инженерно-технологических навыков;

4) использование робототехники с целью закладывания основ цифровой инженерии и в лабораторно-экспериментальных работах предметов ЕМЦ [9]. При этом STEM-подход как и остальные инновационные тренды в образовании, соответствующие вызовам современного индустриально¬-цифрового периода развития общества, предполагает обоснованную научно¬-методологическую базу и практически апробированную дидактическую платформу для широкого внедрения в учебный процесс.

В работе [13] показана концепция поэтапного введения ученика в инженерно-технологическое знание посредством робототехники следующим образом:

• Первый этап «Играю». 1-4 классы в игровой форме изучают основы алгоритмики, а также учатся собирать и программировать роботов. Используются программы: Scratch, Lego WeDo, Paint, Paint3D. При этом дети исследуют мир, делают первые выводы, осознают связь реального и виртуального.

• Второй этап «Учусь». 5-7 классы изучают программирование ROBOTC, Arduino, а также учатся работать в Unity. При этом используется интегрированное содержание предметов: математика, информатика, физика, химия, рисование. Содержание робототехники включает Unity, Scratch 3, ArduinolDE, RobotC, Lego Mindstorms, схемотехника.

• Третий этап «Создаю». 7-10 классы разрабатывают и создают школьные проекты, участвуют в WorldSkills и иных соревнованиях. При этом используется интегрированное содержание предметов: математика, информатика, физика, химия, биология, черчение. Ученики изучают язык программирования Python. Содержание робототехники включает разработку схем, расширение платы (шилд), разработку и создание роботов: колесного, гусеничного, шагающего.

• Четвертый этап «Изобретаю». 11-12 классы создают научные проекты, участвуют в республиканских и международных конкурсах. При этом реализуется проектная деятельность, преимущественно, в командной работе. Ученики знают, что делать; понимают, как делать; разрабатывают собственные проекты; участвуют в научно-технических конкурсах.

Соревнования как метод организации учебно-познавательной деятельности школьников по робототехнике способствует формированию качеств конкурентоспособной личности. Этот метод опирается на естественные склонности учащегося к лидерству, к соперничеству. В процессе соревнования школьник достигает определенного успеха в отношениях с товарищами и учителем, приобретает новый социальный статус. Соревнование вызывает не только активность, но и формирует у него способность к самоактуализации, которую можно рассматривать как метод самовоспитания, самоопределения и саморазвития.

Таким образом, в основной школе главными составляющими STEM-образования являются предметная область «Технология» и предмет «Робототехника», которые включены в учебный план школы с 1-9 класса по 2 часа в неделю [13,14]. Содержание «Технологии» включает: 1) освоение методов различных технологий: народного ремесла, обработка древесины и металла с использованием ИКТ, 3D-моделирования, САПР и т.п; 2) освоение методов конструирования, проектирования, моделирования, а также креативной технологии; 3)формирование навыков технического труда. Содержание «Робототехники» включает: 1) освоение методов конструирования роботов с использованием ресурсного набора; 2) освоение методов управления роботами (от простого до автоматизированного) посредством изучения и использования языков программирования микроконтроллеров Scratch, Python, Java, C, C++ по мере усложнения программного обеспечения; 3) разработка собственной модели робота, участие на конкурсах с целью формирования инженерно-технологического мастерства. Профильный уровень среднего образования должен включать кроме традиционных профилей, «технологический» профиль, где STEM обучение представляет собой интеграцию предметов естественно-математического цикла, технологии, инженерии и ИКТ в единую систему обучения для решения реальных прикладных задач.

Предпрофильная подготовка и организация профильного обучения в условиях внедрения STEM образования в школу должны проводиться на основе анализа содержания Атласа новых профессий, который также будет перманентно дополнен и переработан в связи с новыми прогнозами появления профессий будущего. Как было отмечено выше, спектр профессий будущего будет определять цифровизация и роботизация, которые стремительно врываются во все сферы деятельности человека. Таким образом, трансформация образования на основе STEM подхода, кардинально модернизирует профориентационную работу в школе, максимально приближая ее к требованиям быстроменяющегося запроса рынка труда. Более того, STEM образование обеспечивает выпускников школ новыми компетенциями, необходимыми для эффективного освоения выбранной профессии будущего. Данное положение предполагает, что Вузы РК должны коренным образом перестроить свою деятельность по определению наиболее перспективных специальностей на основе Атласа новых профессий.

Следовательно, STEM образование с опорой на Атлас новых профессий позволяет реализовать концепцию Skills Technology Foresight (форсайт от англ. Foresight – взгляд в будущее, предвидение), которые отвечают требованиям сферы бизнеса, развития промышленности и госуправления в XXI веке, то есть подготовку конкурентоспособного человеческого капитала.

Анализ передового опыта зарубежных стран показывает актуальность организации системной работы по предпрофильной подготовке учащихся в старших классах основной школы и проведение профильного обучения в старшем звене общеобразовательной школы. Запросы общества будущего (Общество 4.0 с Индустрией 4.0) требуют обязательного наличия технологического профиля в профильном обучении, направленного на формирование инженерно-технологических компетенций учащихся. Данный профиль предполагает разработку интегрированного содержания предметов и элективных курсов, ориентированных на решение прикладных, инженерно-технологических задач с цифровой начинкой. Он также требует сетевого взаимодействия организации образования и производства, формируя тем самым, основу дуального обучения уже в школах.

Профильное обучение в STEM-подходе отличается переходом от предметного обучения к проектному (феноменальному) обучению, направленному на решение реальных прикладных задач из различных сфер жизни (производства, медицины и т.п). Широко используется проектно- исследовательский метод, реализация и защита проекта индивидуально или в командной форме. Полученные в основной школе знания и умения школьников по пяти предметам естественно-математического цикла, технологии и программированию, а также на достаточном уровне сформированные инженерные навыки позволяют на «Технологическом» профиле ставить перед учащимися сложные задания в виде проектов, направленные на решение реальных производственных задач. Эти задачи решаются методом проектов и с помощью сетевого взаимодействия: школа – дополнительное образование – ТиПО – Вуз – производство. Реализуются методы дуального обучения при решении производственных задач на базе производства. При таком профильном обучении реализуется действенная профориентационная робота в школе. Как показывает зарубежный опыт, выпускники профильных классов гарантированно поступают в инженерно-технологические университета 3.0 или 4.0. [18,19]. Сетевое взаимодействие вуза и производства, дуальное обучение обеспечивают выпускнику вуза гарантированное трудоустройство в компании, которые являются партнерами вуза. Обобщая вышеизложенное, можно утверждать, что STEM-подход модернизирует традиционное образование, обеспечивая подлинной реализации деятельностного, компетентностного и цифрового подходов, усиливая инженерно-технологическую (прикладную и практическую) направленность содержания посредством внедрения предметов «Технология», «Робототехника» с преимущественным применением проектного метода. STEM-подход формирует устойчивый мотив учащихся к изучению предметов естественно-математического цикла. Будущее экономического роста во многом зависит от наличия квалифицированных инженерных кадров, начало формирования которых



Рис. 1. Образование и наука, модернизированная на основе STEM подхода – основа устойчивого развития страны.

должно быть положено на уровне средней школы через поддержку и активное внедрение STEM дисциплин. Предварительные опыты показали, что наиболее эффективным дидактическим инструментом реализации STEM образования на практике является технология трехмерной методической системы обучения [5,12].

Трансформация содержания системы среднего образования РК на основе STEM-подхода предполагает наличие в учебном плане нового «технологического» профиля. Данный профиль требует использования преимущественно проектного метода и взаимодействия школ, внешкольных организаций, вузов и производства. Такая взаимосвязь способствует развитию STEM-компетентности обучаемых, формированию прогрессивной кадровой базы страны для индустрии 4.0, которая позволит ей стать экономически развитой и конкурентоспособной страной. Анализ передового опыта зарубежных стран показывает, что только такое образование и наука, модернизированые посредством STEM подхода, являются основой устойчивого развития любого государства в XXI веке (см.рис.1).

Трансформация системы среднего образования на основе STEM подхода требует разработки специального плана реализации данной модернизации. Опыт стран с Индустрией 4.0 по внедрению STEM образования показывает его стратегическое значение по подготовке конкурентоспособного человеческого капитала, который является главным условием создания цифровой экономики, высокотехнологичных производств и вхождения РК в число 50-ти развитых стран мира в ближайшее время.

Ключевые слова: STEM подход, STEM образование, Индустрия 4.0, инженерно-технологические навыки, робототехника, Атлас новых профессий.

Предметная классификация AMS: 97

Список литературы

[1] Шваб К., Четвертая промышленная революция; «ЭКСМО», 2016г.; 138с.

[2] Иванова С.В., Иванов О.Б., Перспективы развития образования в условиях четвертой промышленной революции //Вопросы экономики образования, 2019 г., с.7-29.

[3] Плакитин Ю.А., Плакитина Л.С., Дьяченко К.И., От цифровизации к «индустрии 4.0» и «обществу 5.0» - возможности адаптации угольной промышленности России, //Горная промышленность, 2018г., №5, с.56-61.

[4] Алексанков А. М., Четвертая промышленная революция и модернизация об- разования: международный опыт / А. М. Алексанков // Стратегические приорите- ты. -2017. -№ 1(13). -с. 53-69.
[5] Бейсембаев Г., Караев Ж., Актуальные проблемы трансформации системы среднего образования на основе STEM-похода, Білім-Образование, –№3, –2021г, стр. –33-61.
[6] Ефимов В.С., Лаптева А.В., Университет 4.0: Философско-методологический анализ //Университетское управление: практика и анализ, 2017г., №1, с.16-28.
[7] Методические рекомендации по внедрению STEAM-технологий в образовательную практику; //nsportal.ru, муниципиальный клуб «Учитель года» Саратовской области, 2019г.
[8] Кондаков А., Образование в эпоху четвертой промышленной революции //Вести образования, 2017г.
[9] Пахомов Ю., STEM и STEAM-образование: от дошкольника до выпускника вуза, интрнет-ресурс. Режим доступа: https://pedsovet.org > article > stem-i-steam-obrazovani.
[10] Рамазанов Р.Г., Годунова Е.А., STEM-образование: возможности и перспективы; //Открытая школа №1, 2021г., с.14-17.
[11] Чемеков В.Н., Крылов Д.А., STEM- новый подход к инженерному образованию; Педагогическая наука, г.Йошкар-Ола, 2015г., с.59-63.
[12] Караев Ж.А, Кобдикова Ж.У., Технология трехмерной методической системы обучения: сущность и применение, -Алматы, «Зерде»,2018г., 480С.
[13] Современное технологическое образование. Сб.статей и докладов XXVIмежд.науч.-практич.конф., Москва, 2020г. 290 с.
[14] Концепция преподавания учебного предмета «Технология», Интернет-ресурс. Режим доступа: https://docs.edu.gov.ru/document/c4d7feb359d9563f114aea8106c9a2aa.
[15] Интернет-ресурс, Словари и энциклопедии на Академике. Режим доступа: https://academic.ru/
[16] Инженерное образование в инженерно-технической школе на уровне основного общего образования, Интернет-ресурс. Режим доступа:https: //School 777.spb.ru

VII World Congress of Turkic World Mathematicians
20-23 September 2023, Turkestan, Kazakhstan

ТЕХНОЛОГИЯ ТРЕХМЕРНОЙ МЕТОДИЧЕСКОЙ СИСТЕМЫ ОБУЧЕНИЯ – ЭФФЕКТИВНЫЙ МЕХАНИЗМ ГАРАНТИРОВАННОГО ОБЕСПЕЧЕНИЯ КАЧЕСТВА ОБРАЗОВАНИЯ

ЖАУМБАЙ КАРАЕВ[1], ЖАНАРТАЙ КОБДИКОВА[2]

[1]НАО им.Ы.Алтынсарина, Алматы, Республика Казахстан
[2]Казахская академия спорта и туризма, Алматы, Республика Казахстан
e-mail: karaevkz@mail.ru, e-mail: zhuk53@mail.ru

Реформы системы образования, проведенные в течение последних 30 лет МОН РК, были нацелены на повышение качества образования. Согласно Закону РК "О статусе педагога"(п.1.2, ст.15), основной обязанностью педагогов является обеспечение качества обучения в соответствии с требованиями государственных общеобязательных стандартов. Однако анализ результатов экзамена ЕНТ показывает, что определенная часть учителей не справляется с этой обязанностью, поскольку ежегодно в среднем 25-30% учащихся не достигают порогового уровня, соответствующего самому низкому уровню усвоения "узнавание установленного государственным общеобязательным стандартом образования. Следует отметить, что «общеобязательность» стандарта требует гарантированного усвоения всеми учениками учебного материала предметов среднего образования не ниже уровня "узнавания".

Результаты международных исследований качества образования PISA и TIMSS в Казахстане продолжают оставаться низкими, что свидетельствует о неэффективности часто проводимых реформ, направленных на повышение уровня обучения в школах страны. Выражение "реформа образования"стало отрицательно восприниматься и вызывать раздражение в обществе.

Исследования ученых и наш собственный опыт показывают, что для успешной реформы системы образования, соответствующей современным требованиям развития педагогической науки, необходимо преобразование всей системы образования на основе новой методологии и концепции инновационных подходов, таких как ценностный, личностно-деятельностный, компетентностный, цифровой и STEM-подходы, которые затрагивают основные принципы и закономерности развития данной сферы.

Однако внедрение этих подходов в учебный процесс требует наличия интегрирующего механизма для реализации их инновационного потенциала на практике с гарантированным достижением образовательных целей. Педагогическая технология обучения является таким интегрирующим механизмом, объединяющим потенциалы новых подходов и обеспечивающим их эффективное применение на практике [1,2,3,4,5,6,7,8,9,10,11,12,13,14].

Это требует от разработчиков педагогических технологий решения двуединой задачи: 1) модернизации существующей теории обучения (дидактики) в соответствии с требованиями новой парадигмы и инновационных подходов в образовании; 2) разработки педагогических технологий на основе модифицированной теории обучения.

Учеными-педагогами всего мира гуманистическая парадигма определена как научно-методологическая основа развития современного образования [2,3].

Гуманизм представляет собой систему воззрений, которая признает ценность каждого человека как личности, его право на свободу, счастье, развитие и проявление способностей.

Она рассматривает благо человека как критерий оценки социальных институтов и призывает к равенству, справедливости и человечности в отношениях между людьми. Гуманистическая парадигма образования нацелена на развитие способности "научиться учиться быть открытым для нового и развить потребность в знаниях. Она предлагает новое содержание, подходы, отношения, поведение и педагогический менталитет.

Известно, что концептуальные идеи гуманистической парадигмы развития образования реализуются через инновационные подходы в учебно-воспитательном процессе [2,3,4,5,6]. Эти подходы являются научно-теоретической основой разработки новой теории обучения и современной модели образования.

Внедрение концепции новой парадигмы и подходов в развитии образования, а также разработка интегрирующего их инновационного потенциала технологического подхода требует преобразования существующей знаниевой дидактики, то есть теории обучения. Новая парадигма развития системы образования требует пересмотра принципов и закономерностей обучения, а также принципов отбора содержания, особенно с учетом личностно-деятельностного, компетентностного, цифрового и STEM подходов, где ученик играет активную роль в учебном процессе [2,3,4,5].

При трансформации содержания образования необходимо внедрение новых принципов отбора в теорию содержания, в дополнение к модернизированным классическим принципам. Эти новые принципы включают: -принцип цифровизации образования, который учитывает дидактические возможности информационно-коммуникационных технологий при отборе содержания образования и обоснованное определение педагогических целей их использования; -принцип соответствия содержания требованиям всех уровней таксономии целей обучения Б. Блума, охватывающий зоны "актуального"и "ближайшего"развития обучаемого (метод «обратного дизайна»); -принцип учета педагогических возможностей робототехнических систем с целью реализации концепции STEM образования и других подходов [2,3,4].

Педагогическая технология представляет собой научно обоснованный дидактический алгоритм для разработки урока, реализация которого на практике должна гарантировать успех в обучении для каждого ученика [2,3].

Согласно В.П. Беспалько, диагностическая постановка целей является отправной точкой для разработки педагогических технологий. Он утверждает, что в современной школе и педагогике до сих пор отсутствуют диагностические цели, что является основной причиной застоя в школьной системе и формальной педагогике [1].

Как известно, диагностическая постановка цели обучения определяется результатами обучения, которые проявляются в действиях учащихся и могут быть точно идентифицированы и измерены [5]. Из определения диагностической постановки цели, где упоминаются "результаты обучения, выраженные в действиях учащихся следует следующее: а) обучение должно быть организовано на основе самостоятельной познавательной деятельности учащихся; б) результаты обучения образуют иерархическую структуру, так как деятельность учащихся характеризуется иерархией: репродуктивные, преобразующие и продуктивные действия [1].

Исходя из этой иерархии видов деятельности, учеными были определены соответствующие иерархии результатов обучения. Например, академиком В.П. Беспалько были определены следующие уровни (иерархии) усвоения учебного материала: ученический, алгоритмический, эвристический и творческий уровни усвоения [1,2].

Одной из наиболее известных систем целей обучения, обладающей такими характеристиками, является таксономия целей Б. Блума. Система целей Б. Блума состоит из иерархически связанных компонентов: знание $\rightarrow$ понимание $\rightarrow$ применение $\rightarrow$ анализ $\rightarrow$ синтез $\rightarrow$ оценка [2,3]. Б. Блум предложил таксономию целей обучения, которая представляет собой вертикальную иерархию в рамках всей методической системы обучения. Как известно [5], цель обучения - это конкретно определенный, качественно описанный желаемый результат, который ученик должен достичь в определенный момент времени. Формулировка

цели обучения предполагает наличие механизма (способа), позволяющего проверить соответствие достигнутого результата поставленной цели. Качество обучения определяется соответствием между целью и результатом. Следовательно, цель и результат должны быть представлены, измерены, описаны в одних и тех же единицах и параметрах [5].

В.П.Беспалько определяет диагностически поставленные цели как иерархически упорядоченные результаты ученической деятельности, представленные в виде уровней усвоения (ученический, алгоритмический, эвристический и творческий).

Для реализации диагностически поставленных целей обучения необходимо иметь иерархически структурированное содержание образования, поскольку развивающие действия учащихся направлены на освоение учебного материала с иерархической систематизацией. Мы называем такое «деятельностное содержание» "трехмерным содержанием так как оно имеет вертикальную (уровневую) структуру, которая позволяет достичь всего спектра целей в таксономии обучения. Важно отметить, что иерархическая природа содержания не только определяет его структурно-содержательную основу, но также усиливает процессуальные и развивающие деятельностные аспекты, которые в контексте "знаниевого"содержания остаются незаметными и почти не функционируют. Иерархическая структура образовательного содержания имеет научное обоснование в работах И.Я.Лернера и Х.Табы [2,3].

Наше исследование показало, что не только цели и содержание, но и все компоненты методической системы обучения (цели, содержание, методы, формы и средства обучения) образуют иерархию и находятся в уровневой взаимосвязи [2,3,5,6]. Следует отметить, что структура первых двух элементов (цели, содержание) методической системы образует строгую иерархию, в то время как иерархическая структура остальных элементов является "производной"от них.

Мы назвали такую методическую систему обучения трехмерной, так как ее элементы образуют иерархическую и многоуровневую структуру.

"Трехмерность"отражает наличие вертикальной (высотной) иерархии в каждом из компонентов методической системы обучения: цели, содержание, методы, формы и средства обучения. Таким образом, трехмерная методическая система обучения является научно-обоснованным расширением идеи Б.Блума о таксономии целей обучения на все компоненты методической системы.

Только трехмерная методическая система может соответствовать требованиям развивающего обучения, личностно-деятельностного и компетентностного подходов. Через самостоятельную познавательную деятельность, в процессе последовательного выполнения заданий на разных уровнях, начиная с уровня "знание"и заканчивая уровнем "создание ученик развивает исследовательские навыки, умения самостоятельно получать и применять знания, что способствует формированию функциональной грамотности. Заметим, что формирование предметных компетенций у учащихся происходит, когда их уровень усвоения достигает или превышает уровень "применения".

Учебные цели	ОПИСАНИЕ УЧЕБНЫХ ЗАДАНИЙ
Знание	**Простые задания (в одно действие) на:** - знание фактов, основных понятий, правила и принципов, процедуры, терминов; - распознание; - воспроизведение; - выявление; - перечисление и описание; - сопоставление и различение; - распределение; - решение типовых задач в стандартной ситуации; - типовые расчеты; - проведение простейших опытов по инструкции.
Понимание	**Простые и составные задания на: -**понимание фактов, правил и принципов; - сравнение; - выявление связи; - выделение главного; - объяснение причин и следствий; - интерпретацию материала (объяснение, краткое изложение своими словами) – интерпретацию схем, графиков и диаграмм; - преобразование словесного материала в математические выражения; - решение типовых задач в нестандартной ситуации.
Применение	**Составные задания на:** - использование изученного материала в новых ситуациях – применение правил, методов, понятий, принципов, законов, теорий в практических ситуациях: - использование или применение информации и идеи для решения какой-либо проблемы; - указание схематически; - моделирование; - упорядочение; - доказательство; - проведение аналогий; - решение расчетных и экспериментальных задач, содержащих подзадачи с явными связями между ними; - проведение лабораторных, экспериментально-исследовательских работ, решение учебно-исследовательских и проектных задач (с интегрированным внутрипредметным и межпредметным содержанием).
Создание	**Составные задания на:** - обобщение; - моделирование; - абстрагирование, -написание эссе, сочинения; - творческий перенос знаний; - выдвижение и подтверждение гипотез; - аргументацию; - разработку плана проведения эксперимента, проведение аналогий при решении жизненных проблем, - установление связей, взаимного влияния; - неалгоритмический поиск решения проблем; - анализ и интерпретация результатов исследований; - оценивание логики построения материала в виде письменного текста; - оценивание соответствия выводов имеющимся данным; - оценивание значимости того или иного продукта деятельности, исходя из внешних критериев качества; - прогнозирование; - решение нетиповых задач, в том числе связанных с реальными жизненными ситуациями; - решение расчетных и экспериментальных задач, содержащих подзадачи с неявными связями, защита проектных работ (с теоретической и прикладной направленностью, предполагающих использование интегрированного межпредметного содержания из одной предметной области или цикла предметов). Решение олимпиадных задач по предметам естественно-математического цикла с **высоким уровнем сложности.** Конструирование модели роботов, требующих цифровых и инженерно-технологических навыков высокого порядка

Таблица 1: Система требований к разработке развивающих учебных заданий



Рис. 1. Дидактическая матрица

Реализация развивающего обучения осуществляется через последовательное выполнение учениками уровневых заданий трехмерного содержания, переходящих от репродуктивного уровня к продуктивному.

В использовании трехмерной методической системы обучения ключевую роль играет разработка уровневых развивающих заданий на основе таксономии целей Б.Блума, уровней усвоения В.П.Беспалько и характеристик качества знания соответствующего уровня.

В таблице 1 представлены научно обоснованные требования к отбору содержания уровневых развивающих заданий.

Наше исследование показало, что структура уровней мотивации, умений, качества знания и усвоения также имеет научно обоснованную иерархическую структуру [2,3,6].

Таблицу их взаимосвязи с трехмерной методической системой обучения, компоненты которой связаны между собой и по вертикали, и по горизонтали, мы назвали– дидактической матрицей (см. рисунок1).

Дидактическая матрица предоставляет возможность структурировать образовательный процесс, охватывая его мотивационный, содержательно-методический и контрольно-оценочный компоненты. В соответствии с трехмерной методической системой обучения, ее остальные компоненты тоже приобретают трехмерную структуру, которая подразумевает динамику образовательного процесса, направленную вверх. Традиционное знаниецентрическое обучение в основном реализуется на первых двух уровнях дидактической матрицы. Качество обучения определяется соотношением между поставленными целями и достигнутыми результатами. Очевидно, что дидактическая матрица полностью охватывает зону "ближайшего развития"ученика.

Поскольку мыслительные операции, такие как анализ, синтез, сравнение, выделение главного, обобщение и другие, применяются на различных уровнях усвоения (знание, понимание, применение и т.д.) с разной степенью сложности, мы включили в дидактическую матрицу – уточненный вариант таксономии целей Б.Блума, разработанный Л.Андерсоном и Д.Кратволем. Мы объединив "анализ "синтез"и "оценку сформулировали цель – "создание". Создание (творчество), которое не было включено в предыдущую таксономию, представляет самый высокий уровень в новой версии. Цель «создание» включает интегрированную характеристику целей «анализ», «синтез», «оценка», а также предполагает на основе творческой созидательной деятельности получение новой продукции, открытие нового

(субъективного или объективного), создание запланированного объекта. Для выполнения творческих заданий учащиеся генерируют, планируют и создают [2,3].

По нашему мнению, только такая таксономия целей (знание, понимание, применение, создание) адекватно соответствует иерархии уровня усвоения (результата обучения), предложенной академиком В.П.Беспалько, и способствует объективной (критериальной) оценке результатов обучения.

Выполнение заданий на первых двух уровнях (знание, понимание) способствует формированию прочной основы фундаментальных знаний по предмету, за что традиционно ставятся оценки "4"и "5". Выполнение заданий этих уровней, а также уровня "применение создает основу предметной компетенции и функциональной грамотности. Содержание заданий на уровне "применение"включает в себя интеграцию предметов внутри одной предметной области, которая должна быть усвоена не ниже "алгоритмического"уровня. Это может представляться в виде лабораторных работ, учебно-экспериментальных заданий, которые предполагают применение интегрированного содержания для исследования явлений. Интегрированное содержание, необходимое для решения проектных и учебно-исследовательских задач, составляет основу содержания уровней "применение"и "создание". При этом содержание заданий на уровне "создание"представляет собой интегрированное содержание предметов, необходимых для решения поставленных исследовательских (проектных) задач, которое ученики должны освоить не ниже "эвристического"уровня.

Выполнение заданий на уровне "создание"формирует метапредметные компетенции и инженерно-технологические навыки [4].

В работе [5] доказано, что качество обучения характеризуется как соотношение цели и результата, как меры достижения целей при том, что цели (результаты) заданы только диагностично и спрогнозированы в зоне ближайшего развития ученика. Дидактическая матрица позволяет, исходя из этого, наглядно и иерархически расположить качество знаний учащихся [2,3,6] (см. рис. 1).

Традиционно оценка качества знаний учащихся осуществляется на основе приближенной оценки "на глаз". Ученики часто получают хорошие оценки за правильное повторение и пересказ учебного материала. Оценка качества обучения определяется количеством учеников, получивших таким образом высокие оценки. Такое необъективное оценивание является результатом отсутствия четкой постановки целей обучения и объективной оценки различных аспектов качества знаний. Дидактическая матрица позволяет определить качество знания на основе критериев и осуществить объективную оценку.

Группа исследователей под руководством М.М. Поташника предлагает следующее определение "качества знания"[5]: это соотношение между поставленной диагностической целью и результатами, достигнутыми учениками в ходе самостоятельной познавательной деятельности. Например, если целью является узнавание, то результатом учения является ученическое усвоение. Схематично это можно представить так: Цель (знание) - Соотношение (правильность, полнота = качество знания) - Результат (ученический уровень усвоения). Соотношение следующих целей в их иерархии и их результатов характеризируется множеством качеств знаний, имеющих интегральные свойства, которые включают свойства качеств знаний предыдущих уровней. Например, Цель (понимание) - Соотношение [действенность, осознанность] - Результат (алгоритмический уровень усвоения). Далее, Цель (применение) - Соотношение[систематичность, глубина, оперативность, гибкость] - Результат (эвристический уровень усвоения). Цель (создание) - Соотношение [системность, прочность] - Результат (творческий уровень усвоения).

Следует отметить, что качество знаний формирует иерархию, где каждый последующий уровень включает в себя качество предыдущего уровня. Таким образом, иерархическое представление целей и результатов позволяет четко, по критериям и объективно оценивать качество знаний. При этом критериями оценивания служат описания целей обучения, так как диагностично поставленная цель является четко описанным желаемым результатом.

В развивающем обучении процесс формирования мотивации имеет иерархическую структуру. Мотивация возникает при постановке задачи, содержащей противоречие, удивление или недоумение, что побуждает человека к разрешению задачи. Постепенное продвижение ученика по дидактической матрице обеспечиваются мотивами соответствующего уровня. В этом контексте мы опираемся на концепцию иерархии потребностей А. Маслоу, согласно которой человек стремится к обеспечению среды (группы, класса), уважению к себе (стремление к лидерству) и достижению своего полного потенциала (творческое самовыражение) [15].

Таким образом, дидактическая матрица является синтезом всех компонентов дидактического процесса и методической системы обучения, представленных в иерархическом виде. Она позволяет отобразить эти компоненты во взаимосвязи, динамике и охватывает мотивационные, содержательно-методические, оценочные, процессуальные и развивающие аспекты учебного процесса.

Под технологией трехмерной методической системой обучения понимается педагогическая система, основанная на дидактической матрице (включая трехмерную методическую систему). Она позволяет проводить исследовательскую работу учащихся, интегрировать коллективные и индивидуальные формы обучения, а также гарантировать достижение результата обучения для каждого ученика, который может быть объективно оценен по критериям.

Дидактическая матрица является моделью ТТМСО, проектом деятельностно-развивающего образовательного процесса.

Технология трехмерной методической системы обучения интегрирует инновацонный потенциал двух основных трендов педагогических технологий: технологии, ориентированной на исследование и технологии, ориентированной на достижение результата.

В синектической части ТТМСО применяются активные и интерактивные методы обучения. Наш опыт также подтверждает эффективность использования технологии критического мышления при решении проектных и исследовательских задач в синектической части ТТМСО, которая состоит из трех этапов: вызова, осмысления и рефлексии [2,3].

Структура ТТМСО может показаться сложной на первый взгляд. Однако исследовательская часть ТТМСО полностью реализуется при решении учебно-исследовательских и проектных задач. На практике во многих случаях в этой части ТТМСО используются методы мозгового штурма и постановки проблемных задач, где дети в коллективе погружаются в процесс исследования изучаемой проблемы. Затем учитель предоставляет ученикам развивающие уровневые задания для закрепления знаний, полученных в первой части и самостоятельного изучения учебного материала, углубляясь в его суть в процессе выполнения уровневых заданий, двигаясь вверх по лестнице дидактической матрицы. Выполнение заданий уровня «создание» предполагает приоритетное использование исследовательских умений учащихся, приобретенных в синектической части технологии, а также опорного знания на уровне «применение», полученного в части, ориентировнной на результат. Таким образом, реализация цели «создание» требует интеграции дидактического потенциала обеих частей ТТМСО.

Взаимодополняющие друг друга две части ТТМСО способствуют успешному достижению главной задачи в современной педагогике – формированию предметных и метапредметных компетенций, которые отражают высокий уровень качества обучения, соответствующий мировым стандартам знаний.

Отметим, что при реализации ТТМСО, относительно результата обучения каждая последующая ступень иерархии сужается, поэтому логико-структурная модель ТТМСО приобретает вид пирамиды (см. рисунок 2) [2,3].

Гуманистическая парадигма образования и здоровьесберегающий подход к обучению предъявляют дидактике XXI века задачу изменения концептуальных основ педагогической квалиметрии.

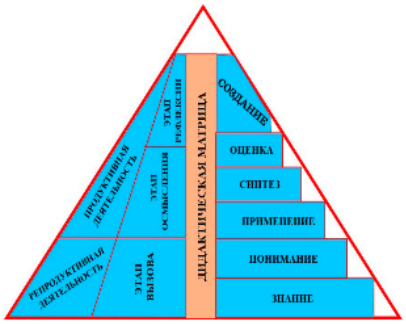


Рис. 2. Логико-структурная модель ТТМСО

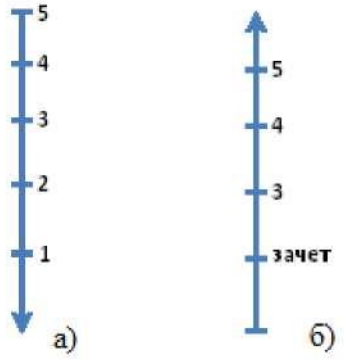


Рис. 3. а) оценка методом вычитания; б) оценка методом сложения

В ТТМСО используется новая концепция оценивания, которая позволяет не только объективно оценивать учебные достижения учащихся, но и стимулирует успешность. ТТМСО предполагает переход от принципа "оценка-наказание за ошибки"к принципу "оценка-стимул к успеху". Метод оценивания "вычитания"заменяется методом "сложения".

В рамках применения педагогической технологии, основанной на трехмерной методической системе обучения, используется метод оценивания "сложения где точкой роста служит минимальный уровень общего образования – «знание».

Достижение этого уровня является обязательным для каждого учащегося. Переход от метода "вычитания"к методу "сложения"осуществляется введением единицы измерения "зачет"(см.рис.3).

Для оценки деятельности учащихся, достигших минимального уровня обязательной подготовки, используется отметка "зачтено"или "не зачтено". Отличие "незачета"от оценки "двойка"заключается в том, что незачет подлежит пересдаче в случае негативного результата.

Зачет проводится по каждой теме, а их содержание подбирается таким образом, чтобы обязательные результаты обучения были представлены полностью. Зачет считается пройденным, если ученик выполнил все задания обязательного уровня, обеспечивая качество знаний уровня «знание» (правильность и полнота знаний).

При применении ТТМСО "зачет"присваивается за выполнение заданий ученического уровня [2,3]. Выполнив задания ученического уровня в своем темпе, каждый ученик получает возможность последовательного выполнения заданий алгоритмического, эвристического и творческого уровней, получая соответствующую оценку в зависимости от достигнутого уровня освоения.

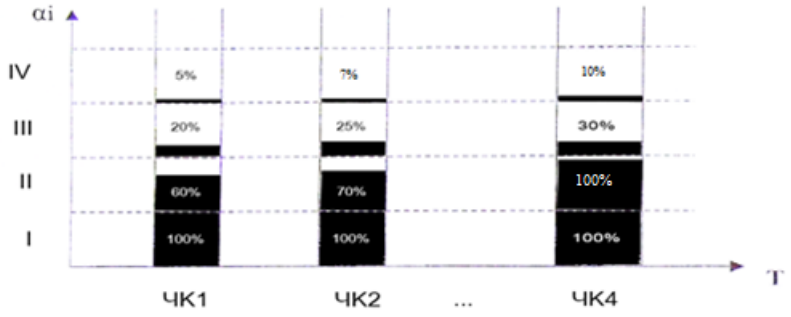


Рис. 4. Учебные успехи класса по четвертям учебного года

Успехи учеников при выполнении заданий отмечаются в "прозрачном журнале"с помощью знаков "+". Самостоятельно (после одобрения учителем) ученики отмечают свои достижения на "прозрачном журнале"[2,3].

Использование "прозрачного журнала"создает позитивную мотивацию и наглядно фиксирует учебные достижения учеников по мере выполнения ими уровневых заданий. В классе на краю доски вывешивается специально подготовленная таблица "прозрачного журнала"[2,3].

В процессе фиксации выполненных заданий каждым учеником в журнале, они могут наблюдать за продвижением друг друга и получать информацию о степени завершения уровневых заданий. Это позволяет видеть динамику прогресса учеников в освоении трехмерного содержания. В процессе последовательного освоения учебного материала, самостоятельное обучение принимает форму игры, где учащиеся соревнуются в выполнении многочисленных уровневых заданий.

Оценивание, которое стимулирует к новым достижениям, создает благоприятную и комфортную образовательную среду.

Таким образом, самая стрессовая часть учебного процесса – контрольно-оценочный этап становится торжественной частью УВП, характеризующейся поощрением и наградой за успешный труд, исключая негативные последствия традиционного оценивания, такие как «боязнь неуспеха» и «синдром отличника».

В результате применения этой конструктивной модели обучения, ученики чувствуют себя уверенно и активно участвуют в дискуссиях, свободно выражая свои мнения. После этого оценки учеников заносятся в официальный классный журнал.

Применение интерактивных методов, последовательное усложнение уровневых заданий и стимулирующее оценивание способствуют активной деятельности и высокому внутреннему мотиву учеников в рамках ТТМСО.

При использовании этой технологии цели первых двух уровней (знание, понимание) достигаются на 100%, а показатели уровней "применение"и "создание"динамически растут (см.рис.4) [2,3]. Такая результативность обусловлена строгой последовательностью усложнения уровневых заданий, высоким внутренним мотивом и активностью учащихся, а также использованием стимулирующего метода оценивания.

При использовании педагогической технологии обучения, основанной на трехмерной методической системе, можно оценить результаты учебной деятельности всего класса в течение одного учебного года, которые вяглядят следующим образом по определенному предмету (см. рис.4).

Эти результаты представлены на рисунке 4, который позволяет не только оценить процентное соотношение усвоения материала, но и сравнить эти показатели по четвертям, увидеть их рост и развитие. Наш практический опыт подтверждает, что ТТМСО имеет высокий

развивающий потенциал и эффективное технологическое свойство, которое обеспечивает гарантированные учебные успехи всех учеников не ниже уровня "понимания а также динамичный рост количества учеников, освоивших уровни "применение"и "создание"в течение одного учебного года.

Ключевые слова: Инновационные подходы к развитию образования, новые принципы отбора содержания образования, трехмерная методическая система обучения, дидактическая матрица, технология трехмерной методической системы обучения, критериальная система и стимулирующий подход оценивания.

Предметная классификация AMS: 97

Список литературы

[1] Беспалько В.П. Слагаемые педагогической технологии. – М.: Педагогика, 1989г., - 192 с.
[2] Караев Ж.А., Кобдикова Ж.У., Актуальные проблемы модернизации педагогической системы на основе технологического подхода. -Алматы, Зерде, 2014г., -312 стр.
[3] Караев Ж.А., Кобдикова Ж.У. Технология трехмерной методической системы обучения: сущность и применение. -Алматы, Зерде, 2018г., -480 стр.
[4] Караев Ж.А., Бейсембаев Г.Б., Мазбаев О.Б., Дидактические вопросы развития системы образования на основе STEM подхода,//Білім-Образование, -2022г.- № 1. -с.5-15.
[5] Поташник М.М. и др. Управление качеством образования. –М.: Педагогическое общество России, 2000 г. – 448 с.
[6] Караев Ж.А. Активизация познавательной деятельности учащихся в условиях применения компьютерной технологии обучения. // Автореферат докт. диссерт., Алматы, 1994 г., 36с.
[7] Чошанов М.А. Гибкая технология проблемно-модульного обучения. - М.: Народное образование. - 1996. - 157с.
[8] Кларин М.В. Развитие «педагогической технологии» и проблемы теории обучения. // Советская педагогика. - 1984 г. - №4. - С. 117-122.
[9] Селевко Г.К. Современные образовательные технологии: Учебное пособие. - М.: Народное образование. - 1998. - 256 с.
[10] Космодемьянская И.А. Разработка вопросов педагогической тех- нологии в США. // Советская педагогика.- №10. - 1973.-С.132-140.
[11] Кларин М.В. Инновационные модели обучения в зарубежных педагогических поисках. -М.: Арена. - 1994. - 232 с.
[12] Караев Ж.А., Кобдикова Ж.У. Оценка деятельности учащихся в условиях применения педагогической технологии обучения. // Вестник высшей школы Казахстана. - 1998 г. - № 5.- С. 82-88.
[13] Третьяков П.И., Сенновский И.П. Технология модульного обучения в школе. - М.: Новая школа. -1997.
[14] Караев Ж.А. Трехмерная методическая система обучения - основа формирования функциональной грамотности учащихся // Международный журнал экспериментального образования. - №11. - 2013 г., С. 19-25.
[15] Маслоу А. Мотивация и личность. 3-е изд. – СПБ.: Питер, 2011 г.-352 с.

МАЗМҰНЫ
CONTENTS
СОДЕРЖАНИЕ

Ықтималдықтар теориясы және математикалық статистика
Probability theory and mathematical statistics
Теория вероятностей и математическая статистика

On asymptotically properties of estimates of the relative-risk function by strong mixing under random censoring
Nurlan Dushatov, Rustamjon Muradov 19

On estimation of survival function in Cox model under random right censoring and robust inference
Rustamjon Muradov, Abduraxim Abdushukurov, Abhik Ghosh, Ayanendranath Basu 24

On estimation of multivariate survival function with random censoring in the present of prognostic factors
Rustamjon Muradov, Abduraxim Abdushukurov, Biswabrata Pradhan 32

On Cramer-Rao type inequality in competing risks model under random censoring from unobservation intervals, efficiency and sequential estimation
Sabitbek Erisbaev 42

The method of accompanying distributions in estimating the proximity of sequences of probability measures
Shakir Formanov, Nursadyk Akanbay, Lola Sharipova 52

On asymptotic expansion for a certain class of statistics in generalized urn models
Sherzod Mirakhmedov 62

Локальная асимптотическая нормальность экспериментов при неоднородном случайном цензурировании интервалом ненаблюдения
Абдурахим Абдушукуров, Наргиза Нурмухамедова 73

Қаржы-экономикалық жүйелерді ықтималдық модельдеуде графтарды қолдану
Айгуль Мырзашева, Нургуль Шаждекеева, Айнагуль Адиева 87

Параболическая интерполяция таблиц смертности для дробных возрастов
Фархат Назарбаев 94

Кері және қисынсыз есептер
Inverse and ill-posed problems
Обратные и некорректные задачи

On the regularization of the solution of a nonlinear operator equation of the first kind in Hilbert space
Askerbek Saadabaev, Izat Usenov 99

Regularization and stability of solutions of system of linear integral fredholm equations of the first kind on a semi-axis
Avyt Asanov, Zuurakan Kadenova, Damira Bekeshova 107

Integration of the periodic Hunter-Saxton equation with a self-consistent source
Bazar Babajanov, Dilshod Atajonov 114

Solving of some inverse problem of epidemiology by stochastic nature-like methods
Sergei Kabanikhin, Maktagali Bektemesov, Zholaman Bektemessov, Zhuldyz Almatykyzy 122

Regularization of linear machine learning problems
Sergey Kabanikhin, Liu Shuang 127

Algorithm for solving the inverse problem for the Helmholtz equation on "discretization-optimization"
Syrym Kasenov, Aigerim Tleulesova 135

Regularisation of ill-posed algebraic systems
Yang Yang 140

Обратные задачи уравнения переноса излучении в многозонной области
Амангельды Сариев, Айнагуль Шыганакова 144

О линейной обратной задаче с нелокальной краевой условии периодического типа для уравнения смешанного типа второго рода второго порядка в неограниченном параллелепипеде
Бийбиназ Сипатдинова, Бахтиёр Халхаджаев, Сироҗиддин Джамалов 152

Численные алгоритмы решения обратной задачи восстановления младшего коэффициента уравнения субдиффузии
Мурат Султанов, Владимир Мисилов, Еркебулан Нурланулы 163

О линейной двухточечной обратной задаче для многомерного волнового уравнения с полунлокальными граничными условиями
Шохрух Худойкулов, Курбаниёз Мамбетсапаев, Сироҗиддин Джамалов, Шерзод Юсупов 171

Оңтайлы басқару
Optimal control
Оптимальное управление

The nonlinear differential game in the case of the l–catch
Ulmasjon Soyibboev, Mohisanam Turgunboyeva 183

Математикалық және компьютерлік модельдеу
Mathematical and computer modeling
Математическое и компьютерное моделирование

Investigation of the theory of own and forced oscillatory processes of different types of flat elements
A. Seitmuratov, Z. Yergalauova, S. Menlikhozhayeva, A.Ibrayeva, G. Yensebayeva 193

On estimation of distribution function by presmoothed relative-risk function
Abdurahim Abdushukurov, Sukhrob Bozorov 202

Mathematical model of optimal management of the customs clearance process

Abdusobir Saidov, Tohir Abdurakhmonov 212

Algebraic methods for solving recognition problems with non-crossing classes
Anvar Kabulov, Alimdzhan Babadzhanov, Islambek Saymanov 228

Models, methods and algorithms for monitoring environmental impact on agricultural production
Anvar Kabulov, Ibrokhimali Normatov, Islambek Saymanov, Abdussattar Baizhumanov, Inomjon Yarashov 234

Study of solutions to a class of Volterra integral equations with a singular point
Asan Baizakov, Aida Bekturova 243

Construction of high order mathematical models of the magic square and their computer simulation
Asan Baizakov, Zamirgul Kazakbaeva 247

Communicating processes simulation in agile management for student teams projects of software development
Elena Dudysheva 251

Complex rays and thick lenses
Elman Hasanoglu 260

Numerical solution of anomalous filtration problem in a porous medium
Jamol Makhmudov, Azizbek Usmonov, Jakhongir Kuljanov 263

The method of summing divergent series by reducing to a repeated series
Madina Teshaeva 274

Mathematical model of the problem of optimizing production considering rental acreage by the criterion of maximum income
Mayramkan Asankulova, Gulsunai Iskandarova 277

Numerical study of the temperature value at nodal points of a complex-shaped body
Nurlan Temirbekov, Beket Kenzhegulov, C. Kenzhegulova, A. Yakupova 283

A mathematical model of two-phase filtration in a porous medium taking into account its deformation
Vladimir Burnashev, Zokhidjon Kaytarov 289

D-оптимальные планы в случае функций Хаара
Абилмажин Адамов, Марс Габбасов, Азамат Кусебай 302

Применение методов машинного обучения для анализа данных социально-экономических опросов
Гульбану Рысмендеева 308

Этапы решения задач методом анализа иерархий в программе MPRIORITY 1.0
Дилфуза Акабирходжаева 316

Двусторонние оценки в методах фиктивных областей
Махат Букенов, Елдос Мухаметов 328

Математиканы оқыту әдістемесі және математика тарихы
Methods of teaching mathematics and the history of mathematics
Методика преподавания математики и история математики

Differential equations. Sherlock Holmes and other applications
Anarkul Urdaletova, Syrgak Kydyraliev 334

The significance of interpretation in the development of spatial thinking
Gulbadan Matieva, Gulnisa Borboeva 342

The theoretical basis and practical implementation methodological training of mathematics teachers in an innovative manner
Gulnar Zhakypbekova 348

Pedagogical features of developing soft skills in the process of training future mathematicians
Kamalbek Berkimbaev, Dmitry Ermakov, Zhuldyz Amantay, Arslan Kurbanmagomedov 357

The role of constructive learning of Mathematics in education
Karlygash Nyshanbayeva, Aldanazar Amirbekuly 367

Educational conditions to develop cognitive interest of technical college students in learning mathematics
Maksat Kedeibayeva, Dilbara Kedeibaeva, Aichuro Egemnazarova 376

Humanitarian aspects of teaching inverse problems for differential equations
Victor Kornilov 382

Трансформация системы образования на основе STEM подхода как условие подготовки конкурентоспособного человеческого капитала в современном мире
Жаумбай Карайев, Жанартай Кобдикова 388

Технология трехмерной методической системы обучения – эффективный механизм гарантированного обеспечения качества образования
Жаумбай Карайев, Жанартай Кобдикова 399

